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TIME-VARYING AND ADAPTIVE INTEGRAL CONTROL OF INFINITE-DIMENSIONAL REGULAR LINEAR SYSTEMS WITH INPUT NONLINEARITIES*

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Abstract. Closing the loop around an exponentially stable, single-input, single-output, regular linear system—subject to a globally Lipschitz, nondecreasing actuator nonlinearity and compensated by an integral controller with time-dependent gain $k(t)$—is shown to ensure asymptotic tracking of a constant reference signal r, provided that (a) the steady-state gain of the linear part of the system is positive, (b) the reference value r is feasible in an entirely natural sense, and (c) the function $t \mapsto k(t)$ monotonically decreases to zero at a sufficiently slow rate. This result forms the basis of a simple adaptive control strategy that ensures asymptotic tracking under conditions (a) and (b).

Key words. adaptive control, infinite-dimensional regular systems, input nonlinearities, integral control, saturation, robust tracking

AMS subject classifications. 34D05, 93C25, 93D05, 93D09, 93D15, 93D21

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1. Introduction. The paper has, as a precursor, the article [9] which contains an extension, to infinite-dimensional systems with input nonlinearities, of the well-known principle (see, for example, [5], [13], and [17]) that closing the loop around a stable, linear, finite-dimensional, continuous-time, single-input, single-output plant, with transfer function $\mathbf{G}(s)$ compensated by a pure integral controller $\mathbf{C}(s) = k/s$, will result in a stable closed-loop system that achieves asymptotic tracking of arbitrary constant reference signals, provided that $|k|$ is sufficiently small and $\mathbf{G}(0)k > 0$. In particular, in [9] it is shown that the above principle may remain valid if the plant to be controlled is a single-input, single-output, continuous-time, infinite-dimensional, regular (as defined in section 2 below) linear system subject to an input nonlinearity ϕ. More precisely, if ϕ is globally Lipschitz and nondecreasing, if $\mathbf{G}(0) > 0$, and if the constant reference signal r is feasible (in the sense that $[\mathbf{G}(0)]^{-1}r$ is in the closure of the image of ϕ), then there exists $k^* > 0$ such that, $\forall k \in (0, k^*)$, the output $y(t)$ of the closed-loop system (shown in Figure 1) converges to r as $t \to \infty$. Therefore, if a (regular) plant is known to be stable, if the input nonlinearity is of the above class, if $\mathbf{G}(0) \neq 0$, and if the sign of $\mathbf{G}(0)$ is known (in principle, the latter information can be obtained from plant step response data), then the problem of tracking feasible signals r by low-gain integral control reduces to that of tuning the gain parameter k. In a nonadaptive, linear, finite-dimensional context, one such controller design approach ("tuning regulator theory"[5]) has been successfully applied in process control (see, for example, [4] and [14]). Furthermore, the problem of tuning the integrator gain adaptively has been addressed recently in a number of papers: see, for example, [3] and [15], [16] for the finite-dimensional case (with input constraints treated in [15]), and [10], [11], [12] for the linear infinite-dimensional case.

The present paper addresses aspects of adaptive tuning of the integrator gain

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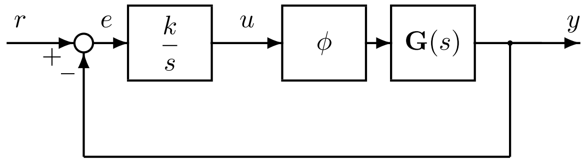

FIG. 1. *Low-gain control with input nonlinearity.*

for infinite-dimensional regular linear systems (with transfer function $\mathbf{G}$), subject to input nonlinearities ϕ of the same class as considered in its precursor [9]. In [9], the constant-gain case is treated; there, the existence of a value $k^* > 0$, with the property that asymptotic tracking of feasible reference signals r is ensured for every *fixed* gain $k \in (0, k^*)$, is established. Let k^{**} denote the supremum of all such k^*. In [9], it is shown that $k^{**} \geq \kappa^*/\lambda$, where $\lambda > 0$ is a Lipschitz constant for ϕ and κ^* denotes the supremum of all numbers $\kappa > 0$ such that

$$1 + \kappa \operatorname{Re} \frac{\mathbf{G}(s)}{s} \geq 0 \quad \forall\ s \text{ with} \operatorname{Re} s > 0.$$

For lower bounds and formulae for κ^* in terms of plant data, we refer to [8]. In general, k^{**} is a function of the plant data and so, in the presence of uncertainty, may fail to be computable. In such cases, it is natural to consider time-dependent gain strategies $t \mapsto k(t) > 0$ capable of attaining sufficiently small values. Theorem 3.8 has the following flavor: if $k(\cdot)$ monotonically decreases to zero *sufficiently slowly*, then asymptotic tracking of feasible reference signals is achieved. The practical utility of this result is limited insofar as the gain function is selected a priori: no use is made of the instantaneous output information $y(t)$ from the plant to update the gain. Utilizing the available output information, Theorem 3.13 establishes the efficacy of the simple adaptive gain strategy

$$k(t) = \frac{1}{l(t)}, \qquad \text{where} \quad \dot{l}(t) = |r - y(t)|, \;\; l(0) = l_0 > 0,$$

and shows that, if the reference signal r is such that $[\mathbf{G}(0)]^{-1} r \in \operatorname{im} \phi$ is not a critical value of ϕ, then the monotone function $t \mapsto k(t) > 0$ converges to a positive limit as $t \to \infty$.

2. Preliminaries on regular linear systems. We assemble some fundamental facts pertaining to regular linear systems and tailored to later requirements; the reader is referred to [20], [21], [22], [23], [24], and [9] for full details. This section is prefaced with the remark that the class of regular linear infinite-dimensional systems is rather general: it includes most distributed parameter systems and all time-delay systems (retarded and neutral) which are of interest in applications. Although there exist abstract examples of well-posed, infinite-dimensional systems that fail to be regular, the authors are of the opinion that any physically motivated, well-posed, linear, continuous-time, autonomous control system is regular.

First, some notation: for a Hilbert space H and any $\tau \geq 0$, $\mathbf{R}_\tau$ denotes the operator of right-shift by τ on $L^p_{\mathrm{loc}}(\mathbb{R}_+, H)$, where $\mathbb{R}_+ := [0, \infty)$; the truncation operator $\mathbf{P}_\tau : L^p_{\mathrm{loc}}(\mathbb{R}_+, H) \to L^p(\mathbb{R}_+, H)$ is given by $(\mathbf{P}_\tau u)(t) = u(t)$ if $t \in [0, \tau]$ and $(\mathbf{P}_\tau u)(t) = 0$ otherwise; for $\alpha \in \mathbb{R}$, we define the exponentially weighted L^p-space $L^p_\alpha(\mathbb{R}_+, H) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}_+, H) |\ f(\cdot) \exp(-\alpha\, \cdot) \in L^p(\mathbb{R}_+, H)\}$; $\mathcal{B}(H_1, H_2)$ denotes the space of bounded linear operators from a Hilbert space H_1 to a Hilbert space H_2; for $\alpha \in \mathbb{R}$, $\mathbb{C}_\alpha := \{s \in \mathbb{C} \,|\, \operatorname{Re} s > \alpha\}$; the Laplace transform is denoted by $\mathcal{L}$.

Well-posed systems. The concept of a well-posed linear system was introduced by Weiss [24]. An equivalent definition can be found in [19].

DEFINITION 2.1. *Let U, X, and Y be real Hilbert spaces. A well-posed linear system with state space X, input space U, and output space Y is a quadruple $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$, where $\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on X; $\mathbf{\Phi} = (\mathbf{\Phi}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to X such that, $\forall\, \tau, t \geq 0$,*

$$\mathbf{\Phi}_{\tau+t}(\mathbf{P}_\tau u + \mathbf{R}_\tau v) = \mathbf{T}_t \mathbf{\Phi}_\tau u + \mathbf{\Phi}_t v \quad \forall\; u, v \in L^2(\mathbb{R}_+, U)\,;$$

$\mathbf{\Psi} = (\mathbf{\Psi}_t)_{t\geq 0}$ is a family of bounded linear operators from X to $L^2(\mathbb{R}_+, Y)$ such that $\mathbf{\Psi}_0 = 0$ and, $\forall\, \tau, t \geq 0$,

$$\mathbf{\Psi}_{\tau+t} x_0 = \mathbf{P}_\tau \mathbf{\Psi}_\tau x_0 + \mathbf{R}_\tau \mathbf{\Psi}_t \mathbf{T}_\tau x_0 \quad \forall\; x_0 \in X;$$

and $\mathbf{F} = (\mathbf{F}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to $L^2(\mathbb{R}_+, Y)$ such that $\mathbf{F}_0 = 0$ and, $\forall\, \tau, t \geq 0$,

$$\mathbf{F}_{\tau+t}(\mathbf{P}_\tau u + \mathbf{R}_\tau v) = \mathbf{P}_\tau \mathbf{F}_\tau u + \mathbf{R}_\tau(\mathbf{\Psi}_t \mathbf{\Phi}_\tau u + \mathbf{F}_t v) \quad \forall\; u, v \in L^2(\mathbb{R}_+, U).$$

For an input $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ and initial state $x_0 \in X$, the associated state function $x \in C(\mathbb{R}_+, X)$ and output function $y \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ of $\mathbf{\Sigma}$ are given by

$$x(t) = \mathbf{T}_t x_0 + \mathbf{\Phi}_t \mathbf{P}_t u, \tag{1a}$$

$$\mathbf{P}_t y = \mathbf{\Psi}_t x_0 + \mathbf{F}_t \mathbf{P}_t u. \tag{1b}$$

$\mathbf{\Sigma}$ is said to be *exponentially stable* if the semigroup $\mathbf{T}$ is exponentially stable:

$$\omega(\mathbf{T}) := \lim_{t\to\infty} \frac{1}{t} \ln \|\mathbf{T}_t\| < 0.$$

$\mathbf{\Psi}_\infty$ and $\mathbf{F}_\infty$ will denote the unique operators $X \to L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ and $L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$, respectively, satisfying

$$\mathbf{\Psi}_\tau = \mathbf{P}_\tau \mathbf{\Psi}_\infty, \quad \mathbf{F}_\tau = \mathbf{P}_\tau \mathbf{F}_\infty \quad \forall\; \tau \geq 0. \tag{2}$$

If $\mathbf{\Sigma}$ is exponentially stable, then the operators $\mathbf{\Phi}_t$ and $\mathbf{\Psi}_t$ are uniformly bounded; $\mathbf{\Psi}_\infty$ is a bounded operator from X into $L^2(\mathbb{R}_+, Y)$, and $\mathbf{F}_\infty$ maps $L^2(\mathbb{R}_+, U)$ boundedly into $L^2(\mathbb{R}_+, Y)$. Since $\mathbf{P}_\tau \mathbf{F}_\infty = \mathbf{P}_\tau \mathbf{F}_\infty \mathbf{P}_\tau \,\forall\, \tau \geq 0$, $\mathbf{F}_\infty$ is a *causal* operator.

Regularity. Weiss [20] has established that, if $\alpha > \omega(\mathbf{T})$ and $u \in L^2_\alpha(\mathbb{R}_+, U)$, then $\mathbf{F}_\infty u \in L^2_\alpha(\mathbb{R}_+, Y)$ and there exists a unique holomorphic $\mathbf{G} : \mathbb{C}_{\omega(\mathbf{T})} \to \mathcal{B}(U, Y)$ such that

$$\mathbf{G}(s)(\mathcal{L}u)(s) = [\mathcal{L}(\mathbf{F}_\infty u)](s) \quad \forall\; s \in \mathbb{C}_\alpha,$$

where $\mathcal{L}$ denotes Laplace transform. In particular, $\mathbf{G}$ is bounded on $\mathbb{C}_\alpha$ $\forall\; \alpha > \omega(\mathbf{T})$. The function $\mathbf{G}$ is called the *transfer function* of $\mathbf{\Sigma}$.

$\mathbf{\Sigma}$ and its transfer function $\mathbf{G}$ are said to be *regular* if there exists a linear operator D such that

$$\lim_{s\to\infty,\, s\in\mathbb{R}} \mathbf{G}(s)u = Du \quad \forall\; u \in U,$$

in which case, by the principle of uniform boundedness, it follows that $D \in \mathcal{B}(U, Y)$. The operator D is called the *feedthrough operator* of $\mathbf{\Sigma}$.

Generating operators. The generator of $\mathbf{T}$ is denoted by A with domain $\mathrm{dom}(A)$. Let X_1 be the space $\mathrm{dom}(A)$ endowed with the graph norm. The norm on X is denoted by $\|\cdot\|$, whilst $\|\cdot\|_1$ denotes the graph norm. Let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|(\lambda I - A)^{-1}x\|$, where $\lambda \in \varrho(A)$ is any fixed element of the resolvent set $\varrho(A)$ of A. Then $X_1 \subset X \subset X_{-1}$ and the canonical injections are bounded and dense. The semigroup $\mathbf{T}$ can be restricted to a C_0-semigroup on X_1 and extended to a C_0-semigroup on X_{-1}. The exponential growth constant is the same on all three spaces. The generator on X_{-1} is an extension of A to X (which is bounded as an operator from X to X_{-1}). We shall use the same symbol $\mathbf{T}$ (respectively, A) for the original semigroup (respectively, its generator) and the associated restrictions and extensions. With this convention, we may write $A \in \mathcal{B}(X, X_{-1})$. Considered as a generator on X_{-1}, the domain of A is X.

By a representation theorem due to Salamon [19] (see also Weiss [22, 23]), there exist unique operators $B \in \mathcal{B}(U, X_{-1})$ and $C \in \mathcal{B}(X_1, Y)$ (the *control operator* and the *observation operator* of $\mathbf{\Sigma}$, respectively) such that, $\forall\, t \geq 0$, $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ and $x_0 \in X_1$,

$$\mathbf{\Phi}_t \mathbf{P}_t u = \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau \quad \text{and} \quad (\mathbf{\Psi}_\infty x_0)(\mathrm{t}) = C \mathbf{T}_{\mathrm{t}} x_0.$$

B is said to be *bounded* if it is so as a map from the input space U to the state space X; otherwise, B is said to be *unbounded*. C is said to be *bounded* if it can be extended continuously to X; otherwise, C is said to be *unbounded*. If $\mathbf{T}$ is exponentially stable, then there exist constants β, $\gamma > 0$ such that, $\forall\, t \geq 0$, $u \in L^2(\mathbb{R}_+, U)$, and $x_0 \in X_1$,

$$\|\mathbf{\Phi}_t \mathbf{P}_t u\| = \left\| \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau \right\| \leq \beta \|u\|_{L^2(0,t;U)}, \tag{3}$$

$$\|\mathbf{\Psi}_\infty x_0\|_{L^2(0,t;Y)} = \left(\int_0^t \|C \mathbf{T}_\tau x_0\|^2 d\tau \right)^{1/2} \leq \gamma \|x_0\|, \tag{4}$$

wherein, with slight abuse of notation, we write $\|u\|_{L^2(0,t;U)}$ and $\|\mathbf{\Psi}_\infty x_0\|_{L^2(0,t;Y)}$ to denote $\|\mathbf{P}_t u\|_{L^2(\mathbb{R}_+;U)}$ and $\|\mathbf{P}_t \mathbf{\Psi}_\infty x_0\|_{L^2(\mathbb{R}_+;U)}$, respectively. The *Lebesgue extension* of C was adopted in [23] and is defined by

$$C_L x_0 = \lim_{t \to 0} C \frac{1}{t} \int_0^t \mathbf{T}_\tau x_0\, d\tau,$$

where $\mathrm{dom}(C_L)$ is the set of all those $x_0 \in X$ for which the above limit exists. Clearly $X_1 \subset \mathrm{dom}(C_L) \subset X$. Furthermore, for any $x_0 \in X$, we have that $\mathbf{T}_t x_0 \in \mathrm{dom}(C_L)$ for almost all (a.a.) $t \geq 0$ and

$$(\mathbf{\Psi}_\infty x_0)(t) = C_L \mathbf{T}_t x_0 \quad \text{a.a. } t \geq 0. \tag{5}$$

If $\mathbf{\Sigma}$ is regular, then for any $x_0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, the functions $x(\cdot)$ and $y(\cdot)$, defined by (1a), satisfy the equations

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{6a}$$

$$y(t) = C_L x(t) + Du(t) \tag{6b}$$

for a.a. $t \geq 0$ (in particular $x(t) \in \mathrm{dom}(C_L)$ for a.a. $t \geq 0$). The derivative on the left-hand side of (6a) has, of course, to be understood in X_{-1}. In other words, if we consider the initial-value problem (6a) in the space X_{-1}, then for any $x_0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, (6a) has unique strong solution (in the sense of Pazy [18, p. 109]) given by the variation of parameters formula

$$t \mapsto x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau. \tag{7}$$

It has been demonstrated in [20] that if $\mathbf{\Sigma}$ is regular, then $(sI - A)^{-1}BU \subset \mathrm{dom}(C_L)$ $\forall\, s \in \varrho(A)$, and the transfer function $\mathbf{G}$ can be expressed as

$$\mathbf{G}(s) = C_L(sI - A)^{-1}B + D \qquad \forall\, s \in \mathbb{C}_{\omega(\mathbf{T})},$$

which is familiar from finite-dimensional systems theory. The operators A, B, C, and D are called the *generating operators* of $\mathbf{\Sigma}$.

Two technical lemmas. In essence, part (a) of the following lemma provides an estimate, in the $L^2(\mathbb{R}_+, X)$ topology, for the solution of the initial-value problem (6a) with initial data $x_0 \in X$ and input $u \in L^2(\mathbb{R}_+, U)$, part (b) asserts that the solution is in $L^\infty(\mathbb{R}_+, X)$ whenever $x_0 \in X$ and $u \in L^\infty(\mathbb{R}_+, U)$, whilst part (c) establishes that the initial-value problem, again with initial data $x_0 \in X$, has a convergent-input, convergent-state property. Parts (a) and (c) constitute Lemma 2.2 of [9]; proof of part (b) is implicit in the argument establishing Lemma 2.2 of [9].

LEMMA 2.2. *Let (A, B, C, D) be the generating operators of an exponentially stable regular system $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$.*

(a) *There exist constants $\alpha_0, \alpha_1 > 0$ such that $\forall\, (x_0, u) \in X \times L^2(\mathbb{R}_+, U)$, the solution $x(\cdot)$ of the initial-value problem* (6a) *satisfies*

$$\|x\|_{L^2(\mathbb{R}_+, X)} \leq \alpha_0 \|x_0\| + \alpha_1 \|u\|_{L^2(\mathbb{R}_+, U)}.$$

(b) $\forall\, (x_0, u) \in X \times L^\infty(\mathbb{R}_+, U)$, *the solution $x(\cdot)$ of the initial-value problem* (6a) *satisfies*

$$x \in L^\infty(\mathbb{R}_+, X).$$

(c) *If $u \in L^\infty(\mathbb{R}_+, U)$ and $\lim_{t\to\infty} u(t) = u_\infty$ exists, then, $\forall\, x_0 \in X$, the solution $x(\cdot)$ of the initial-value problem* (6a) *satisfies*

$$\lim_{t\to\infty} \|x(t) + A^{-1}Bu_\infty\| = 0.$$

The next lemma shows that, for finite-dimensional U and Y, the impulse response of an exponentially stable regular system with bounded B (or bounded C) is the sum of a weighted L^1-function and a point mass at 0.

LEMMA 2.3. *Let (A, B, C, D) be the generating operators of an exponentially stable regular system $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$. Assume that either B or C is bounded.*

(a) *There exists $\alpha < 0$ such that, $\forall\, u \in U$,*

$$\mathcal{L}^{-1}(\mathbf{G}(\cdot)u - Du) \in L^1_\alpha(\mathbb{R}_+, Y).$$

(b) *If $U = \mathbb{R}^m$ and $Y = \mathbb{R}^p$, then*

$$\mathcal{L}^{-1}(\mathbf{G}) \in L^1_\alpha(\mathbb{R}_+, \mathbb{R}^{p\times m}) + (\mathbb{R}^{p\times m})\,\delta_0,$$

where δ_0 denotes the unit point mass at 0.

Proof. Suppose that B is bounded, and set $\mathbf{G}_0(s) := \mathbf{G}(s) - D = C(sI - A)^{-1}B$. Fix $u \in U$ and choose $(b_n) \subset X_1$ such that $\lim_{n\to\infty} \|Bu - b_n\| = 0$ (such a sequence exists by denseness of X_1 in X). Consequently, $\mathbf{\Psi}_\infty Bu$, $\mathbf{\Psi}_\infty b_n \in L^2(\mathbb{R}_+, Y)$, and

$$\lim_{n\to\infty} \|\mathbf{\Psi}_\infty Bu - \mathbf{\Psi}_\infty b_n\|_{L^2(\mathbb{R}_+,Y)} = 0.$$

Hence, $\forall\ s \in \mathbb{C}_0$,

$$\mathbf{G}_0(s)u = \lim_{n\to\infty} C(sI - A)^{-1} b_n = \lim_{n\to\infty} \int_0^\infty (\mathbf{\Psi}_\infty b_n)(t)e^{-st}\, dt = [\mathcal{L}(\mathbf{\Psi}_\infty Bu)](s).$$

Note that, by exponential stability, $\mathbf{\Psi}_\infty Bu \in L^2_\beta(\mathbb{R}_+, Y)$ for some $\beta < 0$, and hence $\mathbf{\Psi}_\infty Bu \in L^1_\alpha(\mathbb{R}_+, Y)\ \forall\ \alpha \in (\beta, 0)$, which yields part (a) in the case of bounded B. This result together with the duality between admissible control and observation operators (see [23]) yields part (a) in the case of bounded C. Part (b) is an immediate consequence of part (a). ∎

3. Integral control of regular systems with input nonlinearities. The problem of tracking constant reference signals r will be addressed in a context of uncertain single-input ($u(t) \in \mathbb{R}$), single-output ($y(t) \in \mathbb{R}$) linear systems, having a nonlinearity ϕ in the input channel:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\phi(u(t)), \quad x(0) = x_0 \in X, \\ y(t) &= C_L x(t) + D\phi(u(t)). \end{aligned} \tag{8}$$

We consider quadruples (A, B, C, D), with C having Lebesgue extension C_L, of class $\mathcal{R}$ defined below.

DEFINITION 3.1. *Let $\mathcal{R}$ denote the class of quadruples (A, B, C, D) which are the generating operators of a regular linear system $\mathbf{\Sigma}$, with state space X, input space $\mathbb{R}$, output space $\mathbb{R}$, and transfer function $\mathbf{G}$, satisfying*

(a) $\mathbf{\Sigma}$ *is exponentially stable*; (b) $\mathbf{G}(0) > 0$.

The following property of $\mathcal{R}$ is readily verified.

PROPOSITION 3.2. *If $(A, B, C, D) \in \mathcal{R}$, then $(A + \varepsilon I, B, C, D) \in \mathcal{R}\ \ \forall \varepsilon > 0$ sufficiently small.*

Admissible input nonlinearities are those functions ϕ of class $\mathcal{N}$ defined below.

DEFINITION 3.3. *Let $\mathcal{N}$ be the class of functions $\phi : \mathbb{R} \to \mathbb{R}$ with the properties* (a) *ϕ is monotone nondecreasing;* (b) *ϕ satisfies a global Lipschitz condition (with Lipschitz constant λ), that is, for some λ, $|\phi(u) - \phi(v)| \le \lambda|u - v|\ \forall\, u, v \in \mathbb{R}$.*

As an example of $\phi \in \mathcal{N}$, consider the input nonlinearity in Figure 2 below.

Let $(\varepsilon_n) \subset (0, \infty)$ be a sequence with $\varepsilon_n \downarrow 0$ as $n \to \infty$. For each $\phi \in \mathcal{N}$, define $\phi^\diamond : \mathbb{R} \to \mathbb{R}$ by

$$\phi^\diamond(\xi) = \limsup_{n\to\infty} \frac{\phi(\xi + \varepsilon_n) - \phi(\xi)}{\varepsilon_n}.$$

Note that $0 \le \phi^\diamond(\xi) \le \lambda\ \ \forall\ \xi \in \mathbb{R}$, where λ is a Lipschitz constant for ϕ. Moreover, as the (pointwise) upper limit of a sequence of continuous functions, $\phi^\diamond$ is Borel measurable, and so its composition $\phi^\diamond \circ u$ with a Lebesgue measurable function u is Lebesgue measurable; furthermore, by the same argument as used in proving Lemma 3.5 of [9], a chain rule applies to such compositions, which we now record.

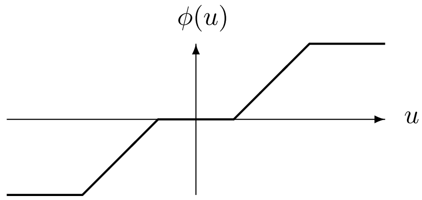

FIG. 2. *Nonlinearity with saturation and dead zone.*

PROPOSITION 3.4. *Let $\phi \in \mathcal{N}$ and let $u : \mathbb{R}_+ \to \mathbb{R}$ be absolutely continuous. Then, $\phi \circ u$ is absolutely continuous and*

$$(\phi \circ u)'(t) = \phi^\diamond(u(t))\dot{u}(t) \quad \text{for a.a. } t \in \mathbb{R}_+.$$

The Clarke [2] directional derivative $\phi^o(u; v)$ of $\phi \in \mathcal{N}$ at u in direction v is given by

$$\limsup_{\substack{w \to u \\ h \downarrow 0}} \frac{\phi(w + hv) - \phi(w)}{h}.$$

Define $\phi^-(\cdot) := -\phi^o(\cdot; -1)$. By upper semicontinuity of ϕ^o, ϕ^- is lower semicontinuous. By definition of $\phi^\diamond$ and monotonicity of $\phi \in \mathcal{N}$ (with Lipschitz constant λ), we have

$$0 \le \phi^-(u) \le \phi^\diamond(u) \le \lambda \quad \forall\, u. \tag{9}$$

$u \in \mathbb{R}$ is said to be a *critical point* (and $\phi(u)$ is said to be a *critical value*) of ϕ if $\phi^-(u) = 0$.

3.1. Integral control with time-varying gain. Let $(A, B, C, D) \in \mathcal{R}$, $\phi \in \mathcal{N}$, and $k \in L^\infty(\mathbb{R}_+, \mathbb{R})$. We denote, by $r \in \mathbb{R}$, the value of the constant reference signal to be tracked by the output $y(t)$. In Proposition 3.6 of [9], it is shown that the following condition is necessary for solvability of the tracking problem: $[\mathbf{G}(0)]^{-1} r \in \operatorname{clos}(\operatorname{im} \phi)$. Reference values r satisfying this condition are referred to as *feasible*.

We will investigate integral control action

$$u(t) = u_0 + \int_0^t k(\tau)[r - C_L x(\tau) - D\phi(u(\tau))]\, d\tau,$$

with time-varying gain $k(\cdot)$, leading to the following nonlinear system of differential equations:

$$\dot{x}(t) = Ax(t) + B\phi(u(t)), \quad x(0) = x_0 \in X, \tag{10a}$$
$$\dot{u}(t) = k(t)[r - C_L x(t) - D\phi(u(t))], \quad u(0) = u_0 \in \mathbb{R}. \tag{10b}$$

For $a \in (0, \infty]$, a continuous function

$$[0, a) \to X \times \mathbb{R}, \quad t \mapsto (x(t), u(t))$$

is a *solution* of (10) if $(x(\cdot), u(\cdot))$ is absolutely continuous as a $(X_{-1} \times \mathbb{R})$-valued function, $x(t) \in \operatorname{dom}(C_L)$ for a.a. $t \in [0, a)$, $(x(0), u(0)) = (x_0, u_0)$, and the differential equations in (10) are satisfied almost everywhere (a.e.) on $[0, a)$, where the derivative in (10a) should be interpreted in the space X_{-1}.[1]

An application of a well-known result on abstract Cauchy problems (see Pazy [18, Theorem 2.4, p. 107]) shows that a continuous $(X \times \mathbb{R})$-valued function $(x(\cdot), u(\cdot))$ is a solution of (10) if and only if it satisfies the following integrated version of (10):

$$x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B\phi(u(\tau))\, d\tau, \tag{11a}$$

$$u(t) = u_0 + \int_0^t k(\tau)[r - C_L x(\tau) - D\phi(u(\tau))]\, d\tau. \tag{11b}$$

The next result asserts that (10) has a unique solution: the proof is contained in the appendix.

LEMMA 3.5. *Let $(A, B, C, D) \in \mathcal{R}$, $\phi \in \mathcal{N}$, $k \in L^\infty(\mathbb{R}_+)$, and $r \in \mathbb{R}$. For each $(x_0, u_0) \in X \times \mathbb{R}$, there exists a unique solution $(x(\cdot), u(\cdot))$ of (10) defined on $\mathbb{R}_+$.*

In [9], the constant-gain case is considered in the context of systems $(A, B, C, D) \in \mathcal{R}$ with input nonlinearities $\phi \in \mathcal{N}$: there, the existence of a value $k^* > 0$, with the property that asymptotic tracking of feasible reference signals r is ensured for all fixed gains $k \in (0, k^*)$, is established. However, k^* is, in general, a function of the plant data and so, in the presence of plant uncertainty, may fail to be computable in practice. In such circumstances, one might be led naïvely to consider a time-dependent gain strategy $t \mapsto k(t) > 0$ with $k(t)$ approaching zero as t tends to infinity.

The main result of this section is contained in the following two theorems which confirm the validity of the above naïvety provided that the gain approaches zero sufficiently slowly. In particular, Theorem 3.6 proves that if $t \mapsto k(t) > 0$ is chosen to be bounded and monotone decreasing to zero, then the unique solution of (10) is such that both $x(\cdot)$ and $\phi(u(\cdot))$ converge. The essence of Theorem 3.8 is the assertion that if, in addition, r is feasible and $k(\cdot)$ approaches zero sufficiently slowly, then $\phi(u(\cdot))$ converges to the value $\phi_r := [\mathbf{G}(0)]^{-1} r$, thereby ensuring asymptotic tracking of r.

THEOREM 3.6. *Let $(A, B, C, D) \in \mathcal{R}$, $\phi \in \mathcal{N}$, and $r \in \mathbb{R}$. Let $k : \mathbb{R}_+ \to (0, \infty)$ be a bounded, monotone function with $k(t) \downarrow 0$ as $t \to \infty$. $\forall\ (x_0, u_0) \in X \times \mathbb{R}$, the unique solution $(x(\cdot), u(\cdot))$ of (10) satisfies*

(a) *$\lim_{t\to\infty} \phi(u(t))$ exists and is finite, and*

(b) *$\lim_{t\to\infty} \|x(t) + A^{-1}B\phi^*\| = 0$, where $\phi^* := \lim_{t\to\infty} \phi(u(t))$.*

Proof. Let $(x_0, u_0) \in X \times \mathbb{R}$ be arbitrary. Let λ be a Lipschitz constant for $\phi \in \mathcal{N}$ (and so $0 \le \phi^\diamond(u) \le \lambda \ \ \forall\ u \in \mathbb{R}$). By Lemma 3.5, there exists a unique solution of (10) on $\mathbb{R}_+$. We denote this solution by $(x(\cdot), u(\cdot))$ and introduce new variables by writing $\phi_r = [\mathbf{G}(0)]^{-1} r$ and defining

$$z(t) := x(t) + A^{-1}B\phi(u(t)), \quad v(t) := \phi(u(t)) - \phi_r \quad \forall\ t \in \mathbb{R}_+. \tag{12}$$

By regularity, it follows that $z(t) \in \mathrm{dom}(C_L)$ for a.a. $t \in \mathbb{R}_+$. Moreover, by Proposition 3.4, $\dot v(t) = \phi^\diamond(u(t))\dot u(t)$ for a.a. $t \in \mathbb{R}_+$. Since (z, v) is absolutely continuous as

[1] Being a Hilbert space, $X_{-1} \times \mathbb{R}$ is reflexive, and hence any absolutely continuous $(X_{-1} \times \mathbb{R})$-valued function is a.e. differentiable and can be recovered from its derivative by integration; see [1, Theorem 3.1, p. 10].

an $(X_{-1} \times \mathbb{R})$-valued function, we obtain by direct calculation

$$\dot{z} = Az - k\phi^\diamond(u)A^{-1}B(C_L z + \mathbf{G}(0)v), \tag{13a}$$
$$z(0) = x_0 + A^{-1}B\phi(u_0),$$
$$\dot{v} = -k\phi^\diamond(u)(C_L z + \mathbf{G}(0)v), \tag{13b}$$
$$v(0) = \phi(u_0) - \phi_r.$$

We claim that there exist positive constants γ_1, γ_2, and σ_1 such that, $\forall\ t, s$ with $\sigma_1 \leq s \leq t$,

$$\int_s^t |C_L z|\,|k\phi^\diamond(u)v| \leq \gamma_1 \|z(s)\| \left(\int_s^t k^2\phi^\diamond(u)v^2\right)^{1/2} + \gamma_2 \int_s^t k^2\phi^\diamond(u)v^2. \tag{14}$$

In order to prove (14), let us first estimate $\int_s^t |C_L z|^2$. For notational convenience, write $w = \phi^\diamond(u)\,[C_L z + \mathbf{G}(0)v]$. As a solution of (13$a$), $z(\cdot)$ satisfies

$$z(\tau) = \mathbf{T}_{\tau-s}z(s) - A^{-1}\int_s^\tau \mathbf{T}_{\tau-\xi}Bk(\xi)w(\xi)\,d\xi$$

$\forall\ s$ with $0 \leq s \leq \tau$. Invoking (4) and (5) and noting that $C_L A^{-1}$ maps X boundedly into $\mathbb{R}$, there exist constants $\alpha_0, \alpha_1 > 0$ such that

$$\int_s^t |C_L z(\tau)|^2 d\tau \leq \alpha_0 \|z(s)\|^2 + \alpha_1 \int_s^t \left\| \int_s^\tau \mathbf{T}_{\tau-\xi}Bk(\xi)w(\xi)d\xi \right\|^2 d\tau \tag{15}$$

$\forall\, 0 \leq s \leq t$. By Lemma 2.2 (part (a)), interpreted in the context of the initial-value problem

$$\dot{\zeta} = A\zeta + Bkw, \quad \zeta(s) = 0,$$

we have

$$\left(\int_s^t \left\| \int_s^\tau \mathbf{T}_{\tau-\xi}Bk(\xi)w(\xi)d\xi \right\|^2 d\tau\right)^{1/2} \leq \alpha_2 \left(\int_s^t |kw|^2\right)^{1/2}$$

for some constant α_2. Therefore, by (15) and monotonicity of k, it follows that, for some constants $\alpha_3, \alpha_4 > 0$,

$$\left(\int_s^t |C_L z|^2\right)^{1/2} \leq \alpha_3 \|z(s)\| + k(s)\alpha_4 \left(\int_s^t |\phi^\diamond(u)|^2 |C_L z|^2\right)^{1/2}$$
$$+ \alpha_4 \mathbf{G}(0) \left(\int_s^t |k\phi^\diamond(u)v|^2\right)^{1/2} \quad \forall\ 0 \leq s \leq t. \tag{16}$$

Fix $\sigma_1 > 0$ such that $\delta := k(\sigma_1)\alpha_4\lambda < 1$. Then,

$$k(s)\alpha_4 \left(\int_s^t |\phi^\diamond(u)|^2 |C_L z|^2\right)^{1/2} \leq \delta \left(\int_s^t |C_L z|^2\right)^{1/2} \quad \forall\ \sigma_1 \leq s \leq t,$$

and so, by (16),

$$\left(\int_s^t |C_L z|^2\right)^{1/2} \leq \beta_1 \|z(s)\| + \beta_2 \left(\int_s^t k^2\phi^\diamond(u)v^2\right)^{1/2} \quad \forall\ \sigma_1 \leq s \leq t, \tag{17}$$

with $\beta_1 = \alpha_3/(1-\delta)$ and $\beta_2 = \alpha_4 \mathbf{G}(0)\sqrt{\lambda}/(1-\delta)$. We may now deduce that, $\forall\ t, s$ with $\sigma_1 \le s \le t$,

$$\int_s^t |C_L z|\,|k\phi^\diamond(u)v| \le \left(\int_s^t |C_L z|^2\right)^{1/2} \left(\int_s^t |k\phi^\diamond(u)v|^2\right)^{1/2}$$
$$\le \beta_1\sqrt{\lambda}\|z(s)\| \left(\int_s^t k^2\phi^\diamond(u)v^2\right)^{1/2} + \beta_2\sqrt{\lambda}\int_s^t k^2\phi^\diamond(u)v^2,$$

which is (14) with $\gamma_1 = \beta_1\sqrt{\lambda}$ and $\gamma_2 = \beta_2\sqrt{\lambda}$. By (13b), for a.a. $t \ge 0$,

$$v(t)\dot v(t) = -k(t)\mathbf{G}(0)\phi^\diamond(u(t))v^2(t) - k(t)\phi^\diamond(u(t))v(t)C_L z(t), \tag{18}$$

and hence

$$v(t)\dot v(t) \le -k(t)\mathbf{G}(0)\phi^\diamond(u(t))v^2(t) + |C_L z(t)|\,|k(t)\phi^\diamond(u(t))v(t)|.$$

Integrating this inequality and using (14) and monotonicity of k yields, $\forall\ t, s$ with $\sigma_1 \le s \le t$,

$$v^2(t) \le v^2(s) + 2\gamma_1\sqrt{k(s)}\|z(s)\| \left(\int_s^t k\phi^\diamond(u)v^2\right)^{1/2}$$
$$+ 2\int_s^t (k\gamma_2 - \mathbf{G}(0))k\phi^\diamond(u)v^2. \tag{19}$$

By positivity of $\mathbf{G}(0)$ and monotonicity of $k(\cdot)$, there exists $\sigma \ge \sigma_1$ such that, $\forall\, \tau \ge \sigma$, $(k(\tau)\gamma_2 - \mathbf{G}(0)) \le -\frac{1}{2}\mathbf{G}(0) < 0$. Therefore, it follows from (19) that

$$0 \le v^2(\sigma) + 2\gamma_1\sqrt{k(\sigma)}\|z(\sigma)\| \left(\int_\sigma^t k\phi^\diamond(u)v^2\right)^{1/2} - \mathbf{G}(0)\int_\sigma^t k\phi^\diamond(u)v^2 \quad \forall\, t \ge \sigma,$$

and so

$$\int_\sigma^\infty k\phi^\diamond(u)v^2 < \infty. \tag{20}$$

Moreover, by (14) we deduce that

$$\int_\sigma^\infty |C_L z|\,|k\phi^\diamond(u)v| < \infty. \tag{21}$$

Combining (18), (20), and (21) shows that there exists a number $\nu \in \mathbb{R}_+$ such that

$$\lim_{t\to\infty} v^2(t) = v^2(\sigma) + 2\lim_{t\to\infty}\int_\sigma^t v\dot v = \nu,$$

whence assertion (a) of the theorem. Assertion (b) now follows by Lemma 2.2 (part (c)). ∎

Let $\mathcal{M}$ denote the space of finite signed Borel measures on $\mathbb{R}_+$.

LEMMA 3.7. *Let $(A, B, C, D) \in \mathcal{R}$, $\phi \in \mathcal{N}$, and $(x_0, u_0) \in X \times \mathbb{R}$. Assume that $\mathcal{L}^{-1}(\mathbf{G}) \in \mathcal{M}$. Let $k : \mathbb{R}_+ \to (0,\infty)$ be bounded and such that $\int_0^t k =: K(t) \to \infty$ as $t \to \infty$. Let $r \in \mathbb{R}$ be feasible, that is, $\phi_r := [\mathbf{G}(0)]^{-1}r \in \operatorname{clos}(\operatorname{im}\phi)$. Let $(x(\cdot), u(\cdot)) : \mathbb{R}_+ \to X \times \mathbb{R}$ be the unique solution of* (10).

If $\lim_{t\to\infty}\phi(u(t))$ exists and is finite, then the following statements hold:

(a) $\lim_{t\to\infty} \phi(u(t)) = \phi_r$,
(b) $\lim_{t\to\infty} \|x(t) + A^{-1}B\phi_r\| = 0$,
(c) $\lim_{t\to\infty}[r - y(t) + (\mathbf{\Psi}_\infty x_0)(t)] = 0$, *where* $y(t) = C_L x(t) + D\phi(u(t))$,
(d) *if* $\phi_r \in \operatorname{im}\phi$, *then* $\lim_{t\to\infty} \operatorname{dist}(u(t), \phi^{-1}(\phi_r)) = 0$, *and*
(e) *if* $\phi_r \in \operatorname{int}(\operatorname{im}\phi)$, *then* $u(\cdot)$ *is bounded.*

Proof. By hypothesis, there exists $\phi^* \in \mathbb{R}$ such that $\lim_{t\to\infty} \phi(u(t)) = \phi^*$. The essence of the proof is to show that $\phi^* = \phi_r$. Setting

$$y_0(t) = (\mathbf{\Psi}_\infty)(x_0)(t), \quad y_1(t) = (\mathcal{L}^{-1}(\mathbf{G}) \star \phi(u))(t),$$

we have

$$\dot{u}(t) = k(t)[\mathbf{G}(0)(\phi_r - \phi^*) - y_0(t) - (y_1(t) - \mathbf{G}(0)\phi^*)].$$

Seeking a contradiction, suppose that $|\phi_r - \phi^*| \neq 0$. Since $\lim_{t\to\infty} \phi(u(t)) = \phi^*$ and $\mathcal{L}^{-1}(\mathbf{G}) \in \mathcal{M}$, it follows that $\lim_{t\to\infty} y_1(t) = \mathbf{G}(0)\phi^*$ (see [7, Theorem 6.1, part (ii), p. 96]). Let $s > 0$ be such that

$$|y_1(t) - \mathbf{G}(0)\phi^*| \leq \tfrac{1}{2}\mathbf{G}(0)|\phi_r - \phi^*| \quad \forall\, t \geq s.$$

As a consequence we obtain

$$\begin{aligned} -\tfrac{1}{2}k(t)\mathbf{G}(0)|\phi_r - \phi^*| - k(t)|y_0(t)| &\leq \dot{u}(t) - k(t)\mathbf{G}(0)(\phi_r - \phi^*) \\ &\leq \tfrac{1}{2}k(t)\mathbf{G}(0)|\phi_r - \phi^*| + k(t)|y_0(t)| \quad \forall\, t \geq s. \end{aligned} \tag{22}$$

Since $\phi_r \neq \phi^*$, either $\phi_r > \phi^*$ or $\phi_r < \phi^*$. If $\phi_r > \phi^*$, then

$$\tfrac{1}{2}k(t)\mathbf{G}(0)(\phi_r - \phi^*) - k(t)|y_0(t)| \leq \dot{u}(t) \quad \forall\, t \geq s,$$

which, on integration, yields

$$\tfrac{1}{2}\mathbf{G}(0)(K(t) - K(s))(\phi_r - \phi^*) - \int_s^t k(\tau)|y_0(\tau)|d\tau \leq u(t) - u(s) \quad \forall\, t \geq s.$$

By exponential stability, $y_0 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$, and thus $y_0 \in L^1(\mathbb{R}_+, \mathbb{R})$, which in turn implies that $ky_0 \in L^1(\mathbb{R}_+, \mathbb{R})$. Since $K(t) \to \infty$ as $t \to \infty$, we conclude that $u(t) \to \infty$ as $t \to \infty$, whence the contradiction

$$\phi_r \leq \sup\,\phi = \lim_{t\to\infty} \phi(u(t)) = \phi^* < \phi_r.$$

If $\phi_r < \phi^*$, then

$$\dot{u}(t) \leq k(t)|y_0(t)| - \tfrac{1}{2}k(t)\mathbf{G}(0)|\phi_r - \phi^*| \quad \forall\, t \geq s,$$

which, on integration, yields $u(t) \to -\infty$ as $t \to \infty$ and the contradiction

$$\phi_r < \phi^* = \lim_{t\to\infty} \phi(u(t)) = \inf\phi \leq \phi_r.$$

Therefore, we may conclude that $\lim_{t\to\infty} \phi(u(t)) = \phi_r$, which is assertion (a). Assertion (b) follows from Lemma 2.2, part (c); assertion (c) is a consequence of assertion (a), together with the identity

$$r - y(t) + (\mathbf{\Psi}_\infty x_0)(t) = \mathbf{G}(0)\phi_r - (\mathcal{L}^{-1}(\mathbf{G}) \star \phi(u))(t),$$

and the fact that $\lim_{t\to\infty}(\mathcal{L}^{-1}(\mathbf{G})\star\phi(u))(t) = \mathbf{G}(0)\phi_r$.

To prove assertion (d), let $\phi_r \in \operatorname{im}\phi$ and suppose that the claim is false. Then there exists a sequence $(t_n) \subset \mathbb{R}_+$ with $\lim_{n\to\infty} t_n = \infty$ and $\varepsilon > 0$ such that

$$\operatorname{dist}(u(t_n), \phi^{-1}(\phi_r)) \geq \varepsilon \quad \forall\, n. \tag{23}$$

If the sequence $(u(t_n))$ is bounded, we may assume without loss of generality that it converges to a finite limit u_∞. By continuity of ϕ and assertion (a), we have that $\phi(u_\infty) = \phi_r$, and thus $u_\infty \in \phi^{-1}(\phi_r)$. This contradicts (23). So, suppose that $(u(t_n))$ is unbounded. Without loss of generality, we may then assume that $\lim_{n\to\infty} u(t_n) = \infty$. By monotonicity and assertion (a), it follows that $\phi_r = \sup\phi$. Since $\phi_r \in \operatorname{im}\phi$, there exists ξ^* such that $\phi(\xi^*) = \phi_r = \sup\phi = \max\phi$. By monotonicity of ϕ, $\phi(\xi) = \phi_r = \max\phi \ \ \forall\, \xi \geq \xi^*$. In particular, we see that $u(t_n) \in \phi^{-1}(\phi_r)$ for all sufficiently large n, which contradicts (23).

Now, assume that $\phi_r \in \operatorname{int}(\operatorname{im}\phi)$ and, for contradiction, suppose that assertion (e) is false. Then there exists a sequence $(t_n) \subset (0,\infty)$ with $t_n \to \infty$ and $|u(t_n)| \to \infty$ as $n \to \infty$. Without loss of generality, we may assume that $\lim_{n\to\infty} u(t_n) = \infty$. By monotonicity, it then follows that $\phi_r = \lim_{n\to\infty}\phi(u(t_n)) = \sup\phi$, contradicting the assumption that $\phi_r \in \operatorname{int}(\operatorname{im}\phi)$. ∎

For $\alpha \in \mathbb{R}$, we define the exponentially weighted space $\mathcal{M}_\alpha$ as the set of all locally finite signed Borel measures μ on $\mathbb{R}_+$ (see, e.g., [6]) with the property that the weighted measure $E \mapsto \int_E e^{-\alpha t}\mu(dt)$ belongs to $\mathcal{M}$.

Theorem 3.8. *Let $(A,B,C,D) \in \mathcal{R}$, $\phi \in \mathcal{N}$, and $(x_0,u_0) \in X \times \mathbb{R}$. Assume that $\mathcal{L}^{-1}(\mathbf{G}) \in \mathcal{M}$. Let $k : \mathbb{R}_+ \to (0,\infty)$ be bounded, monotone, and such that $k(t) \downarrow 0$ and $\int_0^t k =: K(t) \to \infty$ as $t \to \infty$. Let $r \in \mathbb{R}$ be feasible, that is, $\phi_r := [\mathbf{G}(0)]^{-1}r \in$* $\operatorname{clos}(\operatorname{im}\phi)$.

The unique solution $(x(\cdot),u(\cdot)) : \mathbb{R}_+ \to X \times \mathbb{R}$ of (10) *satisfies*

(a) $\lim_{t\to\infty}\phi(u(t)) = \phi_r$,

(b) $\lim_{t\to\infty}\|x(t) + A^{-1}B\phi_r\| = 0$,

(c) $\lim_{t\to\infty}[r - y(t) + (\mathbf{\Psi}_\infty x_0)(t)] = 0$, *where* $y(t) = C_L x(t) + D\phi(u(t))$,

(d) *if* $\phi_r \in \operatorname{im}\phi$, *then* $\lim_{t\to\infty}\operatorname{dist}(u(t),\phi^{-1}(\phi_r)) = 0$,

(e) *if* $\phi_r \in \operatorname{int}(\operatorname{im}\phi)$, *then* $u(\cdot)$ *is bounded, and*

(f) *if* $\phi_r \in \operatorname{im}\phi$ *is not a critical value of* ϕ, *then the convergence in* (a) *and* (b) *is of order* $\exp(-\rho K(t))$ *for some* $\rho > 0$; *moreover, the convergence in* (c) *is of the same order, provided that* $\mathcal{L}^{-1}(\mathbf{G}) \in \mathcal{M}_\alpha$ *for some* $\alpha < 0$.

Remark 3.9. (i) *The assumption that $\mathcal{L}^{-1}(\mathbf{G}) \in \mathcal{M}$ (or that $\mathcal{L}^{-1}(\mathbf{G}) \in \mathcal{M}_\alpha$ for some $\alpha < 0$) is not very restrictive and seems to be satisfied in all practical examples of exponentially stable systems. In particular, this assumption is satisfied if B or C is bounded (see Lemma* 2.3*).*

(ii) *Since $(\mathbf{\Psi}_\infty x_0)(t)$ converges exponentially to* 0 *as $t \to \infty\ \forall\, x_0 \in X_1 = \operatorname{dom}(A)$, it follows from* (c) *that the error $e(t) = r - y(t)$ converges to* 0 $\forall\, x_0 \in \operatorname{dom}(A)$. *(Moreover, the convergence is of order* $\exp(-\rho K(t))$ *if the extra assumptions in* (f) *are satisfied.) If C is bounded, then this statement is true $\forall\, x_0 \in X$. If C is unbounded and $x_0 \notin \operatorname{dom}(A)$, then $e(t)$ does not necessarily converge to* 0 *as $t \to \infty$. However, $e(t)$ is small for large t in the sense that $e(t) = e_1(t) + e_2(t)$, where the function e_1 is bounded with $\lim_{t\to\infty} e_1(t) = 0$ and $e_2 \in L^2_\alpha(\mathbb{R}_+,\mathbb{R})$ for some $\alpha < 0$.*

(iii) *In particular,* (d) *asserts that $u(t)$ converges as $t \to \infty$ if the set $\phi^{-1}(\phi_r)$ is a singleton, which, in turn, will be true if $\phi_r \in \operatorname{im}\phi$ is not a critical value of ϕ.*

Proof. Assertions (a)–(e) follow immediately from Theorem 3.6 combined with Lemma 3.7. It remains only to establish assertion (f). By hypothesis, $\phi_r \in \operatorname{im}\phi$ is not a critical value of ϕ, and so, by monotonicity, its preimage $\phi^{-1}(\phi_r)$ is a singleton $\{u_r\}$ and

$$\phi^\diamond(u_r) \geq \phi^-(u_r) =: \phi_r^- > 0.$$

Therefore, by assertion (d), $u(t) \to u_r$ as $t \to \infty$. By lower semicontinuity of ϕ^-, there exists $\sigma_1 > 0$ such that

$$\phi^\diamond(u(t)) \geq \phi^-(u(t)) \geq \tfrac{1}{2}\phi_r^- > 0 \quad \forall\, t \geq \sigma_1. \tag{24}$$

Define $\rho := \frac{1}{4}\mathbf{G}(0)\phi_r^- > 0$, and introduce exponentially weighted variables given by

$$z_e(t) := \exp(\rho K(t))[x(t) + A^{-1}B\phi(u(t))], \tag{25a}$$

$$v_e(t) := \exp(\rho K(t))[\phi(u(t)) - \phi_r] \tag{25b}$$

$\forall\, t \in \mathbb{R}_+$. Since (z_e, v_e) is absolutely continuous as an $(X_{-1} \times \mathbb{R})$-valued function, and using Proposition 3.4, we obtain by direct calculation

$$\dot{z}_e = (A + \rho k I)z_e - k\phi^\diamond(u)A^{-1}B(C_L z_e + \mathbf{G}(0)v_e), \tag{26a}$$

$$z_e(0) = x_0 + A^{-1}B\phi(u_0),$$

$$\dot{v}_e = -k\phi^\diamond(u)(C_L z_e + \mathbf{G}(0)v_e) + \rho k v_e, \tag{26b}$$

$$v_e(0) = \phi(u_0) - \phi_r.$$

For each (t, s) with $0 \leq s \leq t$, define

$$\mathbf{U}(t, s) := \exp(\rho[K(t) - K(s)])\mathbf{T}_{t-s}. \tag{27}$$

We briefly digress to prove a technicality.

LEMMA 3.10. *Let $s \in \mathbb{R}_+$, $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$, and, on $[s, \infty)$, define a function p by*

$$p(t) := \int_s^t \mathbf{U}(t, \xi)Bu(\xi)d\xi.$$

Then, $\forall\, t$, $p(t) \in X$ and, as an X_{-1}-valued function, p is absolutely continuous with

$$\dot{p}(t) = (A + \rho k(t)I)p(t) + Bu(t) \quad \textit{a.e.}$$

Proof. On $[s, \infty)$, define a function q by

$$q(t) := e^{-\rho K(t)}p(t) = \int_s^t \mathbf{T}_{t-\xi}Be^{-\rho K(\xi)}u(\xi)\, d\xi.$$

Clearly, $q(t) \in X\ \forall\, t$ and so $p(t) \in X\ \forall\, t$. Moreover, q is absolutely continuous as an X_{-1}-valued function and, by Pazy [18, Theorem 2.9, p. 109], for a.a. $t \geq s$,

$$\dot{q}(t) = Aq(t) + e^{-\rho K(t)}Bu(t).$$

Thus, as an X_{-1}-valued function, p is absolutely continuous with

$$\dot{p}(t) = \rho k(t)e^{\rho K(t)}q(t) + e^{\rho K(t)}\dot{q}(t) = (A + \rho k(t)I)p(t) + Bu(t)$$

for a.a. $t \geq s$. $\square$

Returning to the proof of the theorem, for notational convenience write

$$w_e := \phi^\diamond(u)\,[C_L z_e + \mathbf{G}(0)v_e]\,.$$

Let $s \in \mathbb{R}_+$ and, on $[s,\infty)$, define $f := f_1 - A^{-1}f_2$ with

$$f_1(t) := \mathbf{U}(t,s)z_e(s), \quad f_2(t) := \int_s^t \mathbf{U}(t,\xi)Bk(\xi)w_e(\xi)d\xi.$$

Clearly, $f_1(t) \in X\ \forall\ t$ and, as an X_{-1}-valued function, f_1 is absolutely continuous with

$$\dot f_1(t) = (A + \rho k(t)I)f_1(t) \quad \text{a.e.}$$

By Lemma 3.10, it now follows that $f(t) \in X\ \forall\ t$ and, as an X_{-1}-valued function, f is absolutely continuous with

$$\begin{aligned}\dot f(t) &= (A+\rho k(t)I)f_1(t) - A^{-1}\left((A+\rho k(t)I)f_2(t) + Bw_e(t)\right)\\ &= (A+\rho k(t)I)f(t) - A^{-1}Bw_e(t) \quad \text{a.e.}\end{aligned}$$

In view of (26a) (together with uniqueness of solutions), we may now conclude that

$$z_e(t) = \mathbf{U}(t,s)z_e(s) - A^{-1}\int_s^t \mathbf{U}(t,\xi)Bk(\xi)w_e(\xi)\,d\xi \quad \forall\ t,s \text{ with } 0 \le s \le t. \tag{28}$$

By exponential stability of the semigroup $\mathbf{T}$, there exist constants $N,\ \nu > 0$ such that $\|\mathbf{T}_t\| \le N\exp(-\nu t)\ \forall\ t \in \mathbb{R}_+$. Let $\varepsilon \in (0,\nu)$ be sufficiently small such that $(A+\varepsilon I, B, C, D) \in \mathcal{R}$ (recall Proposition 3.2). Fix $\sigma_2 > \sigma_1$ such that

$$k(\sigma_2) < \min\{\varepsilon/\rho, \nu/(\rho N)\}. \tag{29}$$

Again, we digress to prove a technicality.

LEMMA 3.11. *There exists constant $\gamma > 0$ such that, $\forall\ u \in L^2_{\text{loc}}(\mathbb{R}_+)$,*

$$\left(\int_s^t \left\|\int_s^\tau \mathbf{U}(\tau,\xi)Bu(\xi)d\xi\right\|^2 d\tau\right)^{\frac12} \le \gamma\left(\int_s^t u^2(\xi)d\xi\right)^{\frac12} \quad \forall\ s,t \ge \sigma_2 \text{ with } s \le t.$$

Proof. Let $s \ge \sigma_2$ and let $u \in L^2_{\text{loc}}(\mathbb{R}_+)$ be arbitrary. On $[s,\infty)$ define (as before) the function $p : t \mapsto \int_s^t \mathbf{U}(t,\xi)Bu(\xi)d\xi$. By Lemma 3.10, for a.a. $t \ge s$, we have $\dot p(t) = Ap(t) + \rho k(t)p(t) + Bu(t)$. Therefore,

$$p(t) = \int_s^t \mathbf{T}_{t-\xi}\rho k(\xi)p(\xi)\,d\xi + \int_s^t \mathbf{T}_{t-\xi}Bu(\xi)\,d\xi \quad \forall\ t \ge s.$$

Using exponential stability of the semigroup $\mathbf{T}$, monotonicity of k, and Lemma 2.2 (part (a)), we may conclude

$$\left(\int_s^t \|p(\xi)\|^2 d\xi\right)^{\frac12} \le \rho N\nu^{-1}k(\sigma_2)\left(\int_s^t \|p(\xi)\|^2 d\xi\right)^{\frac12} + \alpha_1\left(\int_s^t u^2(\xi)\,d\xi\right)^{\frac12} \quad \forall\ t \ge s.$$

By (29), $1 - \rho N\nu^{-1}k(\sigma_2) > 0$, and the result follows. $\square$

Once more, we return to the proof of the theorem. By monotonicity of k, $K(t) - K(s) \le k(s)(t-s)\,\forall\, t,s$ with $0 \le s \le t$. Since $k(\sigma_2) \le \varepsilon/\rho$, it follows that

$$\exp(\rho[K(t)-K(s)]) \le \exp(\varepsilon[t-s]) \quad \forall\ t,s \text{ with } \sigma_2 \le s \le t. \tag{30}$$

Observe that, $\forall\, t,s$ with $\sigma_2 \le s \le t$,

$$\begin{aligned} |C_L\mathbf{U}(t,s)z_e(s)| &= |C_L\mathbf{T}_{t-s}z_e(s)|\exp(\rho[K(t)-K(s)]) \\ &\le |C_L\mathbf{T}_{t-s}\exp(\varepsilon[t-s])z_e(s)|. \end{aligned}$$

Invoking (4), (5) (in the context of the regular system $(A+\varepsilon I, B, C, D)$), (28), and Lemma 3.11, and recalling that C_LA^{-1} maps X boundedly into $\mathbb{R}$, there exist constants $\alpha_2, \alpha_3 > 0$ such that

$$\begin{aligned} \left(\int_s^t |C_Lz_e|^2\right)^{1/2} &\le \alpha_2\|z_e(s)\| + k(s)\alpha_3\left(\int_s^t |\phi^\diamond(u)|^2|C_Lz_e|^2\right)^{1/2} \\ &\quad + \alpha_3\mathbf{G}(0)\left(\int_s^t |k\phi^\diamond(u)v_e|^2\right)^{1/2} \end{aligned} \tag{31}$$

$\forall\, t,s$ with $\sigma_2 \le s \le t$.

Inequality (31) is the exponentially weighted version of (16). Following the argument in the proof of Theorem 3.6, (31) may be used to derive an exponentially weighted version of (14), i.e., there exist positive constants $\gamma_1, \gamma_2 > 0$ and $\sigma_3 \ge \sigma_2$ such that

$$\int_s^t |C_Lz_e|\,|k\phi^\diamond(u)v_e| \le \gamma_1\|z_e(s)\|\left(\int_s^t k^2\phi^\diamond(u)v_e^2\right)^{1/2} + \gamma_2\int_s^t k^2\phi^\diamond(u)v_e^2 \tag{32}$$

$\forall\ t,s$ with $\sigma_3 \le s \le t$.

By (26b), for a.a. $t \ge 0$,

$$v_e(t)\dot v_e(t) = -k(t)\mathbf{G}(0)\phi^\diamond(u(t))v_e^2 + \rho k(t)v_e^2(t) - k(t)\phi^\diamond(u(t))v_e(t)C_Lz_e(t). \tag{33}$$

By (24), $\mathbf{G}(0)\phi^\diamond(u(t)) - \rho \ge \frac{1}{2}\mathbf{G}(0)\phi_r^- - \rho = \rho > 0\ \forall\ t \ge \sigma_3$. Hence, we have

$$v_e(t)\dot v_e(t) \le -\tfrac{1}{2}\rho k(t)v_e^2(t) + |C_Lz_e(t)|\,|k(t)\phi^\diamond(u(t))v_e(t)| \quad \text{for a.a. } t \ge \sigma_3.$$

Integrating this inequality and using (32) and monotonicity of k yields, $\forall\ t,s$ with $t \ge s \ge \sigma_3$,

$$v_e^2(t) \le v_e^2(s) + 2\gamma_1\sqrt{\lambda k(s)}\|z_e(s)\|\left(\int_s^t kv_e^2\right)^{1/2} - \int_s^t(\rho - 2k\gamma_2\lambda)kv_e^2. \tag{34}$$

Fix $\sigma \ge \sigma_3$ such that $\rho - 2k(t)\gamma_2\lambda > \frac{1}{2}\rho\ \forall\ t \ge \sigma$. From (34) and (32), we deduce

$$\int_\sigma^\infty kv_e^2 < \infty.$$

Hence, by (34), $v_e(\cdot) = \exp(\rho K(\cdot))[\phi(u(\cdot)) - \phi_r]$ is bounded and so the convergence in (a) is of order $\exp(-\rho K(t))$. We proceed to prove that the convergence in (b) is of the same order. Define $x_r := -A^{-1}B\phi_r$, and introduce a new variable given by

$$x_e(t) = \exp(\rho K(t))[x(t) - x_r] \quad \forall\ t \ge 0.$$

It suffices to show that $x_e(\cdot)$ is bounded. By (8) and (25), we have

$$\dot{x}_e = (A + \rho k I)x_e + Bv_e, \quad x_e(0) = x_0 - x_r,$$

and so, $\forall\ t \geq \sigma$,

$$x_e(t) = \mathbf{T}_{t-\sigma}x_e(\sigma) + \int_\sigma^t \mathbf{T}_{t-\xi}Bv_e(\xi)\,d\xi + \int_\sigma^t \mathbf{T}_{t-\xi}\rho k(\xi)x_e(\xi)\,d\xi.$$

Therefore, by boundedness of v_e together with Lemma 2.2 (part (b)) and exponential stability of $\mathbf{T}$, there exists a constant $\beta > 0$ such that

$$\sup_{s\in[\sigma,t]} \|x_e(s)\| \leq \beta + \rho N\nu^{-1}k(\sigma) \sup_{s\in[\sigma,t]} \|x_e(s)\| \quad \forall\ t \geq \sigma.$$

Since $\sigma \geq \sigma_2$, we have, by (29), $\rho N\nu^{-1}k(\sigma) < 1$, and hence we may conclude boundedness of x_e. Therefore, the convergence in part (b) is of order $\exp(-\rho K(t))$.

It remains only to prove that the convergence in (c) is also of order $\exp(-\rho K(t))$, provided that $\mu := \mathcal{L}^{-1}(\mathbf{G}) \in \mathcal{M}_\alpha$ for some $\alpha < 0$. Denoting the unit-step function by θ, we have $\forall\ t \geq 0$

$$|r - y(t) + (\mathbf{\Psi}_\infty x_0)(t)| \leq |[\mu \star (\phi(u) - \phi_r\theta)(t)]| + |\phi_r[(\mu\star\theta)(t) - \mathbf{G}(0)]|. \tag{35}$$

For convenience we set $w(t) = \exp(\rho K(t))\ \forall\ t \geq 0$. We have already shown that the function $t \mapsto w(t)|\phi(u(t)) - \phi_r|$ remains bounded as $t \to \infty$. If we extend w to a function defined on $\mathbb{R}$ by setting $w(t) = 1\ \forall\ t < 0$, then it is easy to show that w is a submultiplicative weight function in the sense of [7, p. 118]. Moreover, since $\mu \in \mathcal{M}_\alpha$, the measure $\mu_w : E \mapsto \int_E w(t)\mu(dt)$ belongs to $\mathcal{M}$. Hence, by [7, Theorem 3.5, part (i), p. 119], we may conclude that the function $t \mapsto w(t)[\mu\star(\phi(u)-\phi_r\theta)](t)$ is bounded on $\mathbb{R}_+$.

Since $\mu_w \in \mathcal{M}$ (a space of finite measures), $\int_0^\infty w(t)|\mu|(dt)$, where $|\mu|$ denotes the total variation of μ. Hence

$$|w(t)[(\mu\star\theta)(t) - \mathbf{G}(0)]| = w(t)\left|\int_t^\infty \mu(d\tau)\right| \leq \int_0^\infty w(\tau)|\mu|(d\tau) < \infty,$$

showing that the function $t \mapsto w(t)[(\mu\star\theta)(t) - \mathbf{G}(0)]$ is bounded on $\mathbb{R}_+$. Consequently, appealing to (35), we deduce that the function

$$\mathbb{R}_+ \to \mathbb{R}, \quad t \mapsto \exp(\rho K(t))|r - y(t) + (\mathbf{\Psi}_\infty x_0)(t)|$$

is bounded. $\square$

3.2. Adaptive gain. Whilst Theorem 3.8 identifies conditions under which the tracking objective is achieved through the use of a monotone gain function, the resulting control strategy is somewhat unsatisfactory insofar as the gain function is selected a priori: no use is made of the output information from the plant to update the gain. We now consider the possibility of exploiting this output information to generate, by feedback, an appropriate gain function. In particular, let the gain $k(\cdot)$ be generated by the law:

$$k(t) = \frac{1}{l(t)}, \quad \dot{l}(t) = |r - y(t)|, \quad l(0) = l_0 > 0, \tag{36}$$

which yields the feedback system

$$\dot{x}(t) = Ax(t) + B\phi(u(t)), \quad x(0) = x_0 \in X, \tag{37a}$$

$$\dot{u}(t) = (1/l(t))[r - C_L x(t) - D\phi(u(t))], \quad u(0) = u_0 \in \mathbb{R}, \tag{37b}$$

$$\dot{l}(t) = |r - C_L x(t) - D\phi(u(t))|, \quad l(0) = l_0 \in (0, \infty). \tag{37c}$$

The concept of a solution of this feedback system is the obvious modification of the solution concept defined in subsection 3.1. The proof of the following existence and uniqueness result can be found in the appendix.

LEMMA 3.12. *Let $(A, B, C, D) \in \mathcal{R}$, $\phi \in \mathcal{N}$, and $r \in \mathbb{R}$. For each $(x_0, u_0, l_0) \in X \times \mathbb{R} \times (0, \infty)$, the initial-value problem given by* (37) *has a unique solution defined on $\mathbb{R}_+$.*

We now arrive at the main adaptive control result.

THEOREM 3.13. *Let $(A, B, C, D) \in \mathcal{R}$, $\phi \in \mathcal{N}$, and let $r \in \mathbb{R}$ be such that $\phi_r := [\mathbf{G}(0)]^{-1} r \in \operatorname{clos}(\operatorname{im} \phi)$. Assume that $\mathcal{L}^{-1}(\mathbf{G}) \in \mathcal{M}$.*

$\forall\ (x_0, u_0, l_0) \in X \times \mathbb{R} \times (0, \infty)$, the unique solution of the initial-value problem given by (37) *is such that assertions* (a) *to* (e) *of Theorem* 3.8 *hold. Moreover, if $\phi_r \in \operatorname{im} \phi$ is not a critical value and $\mathcal{L}^{-1}(\mathbf{G}) \in \mathcal{M}_\alpha$ for some $\alpha < 0$, then the monotone gain k converges to a positive value.*

Proof. Set $k(t) = 1/l(t)$. Since $l(\cdot)$ is monotone increasing, either $l(t) \to \infty$ as $t \to \infty$ (Case 1), or $l(t) \to l^* \in (0, \infty)$ as $t \to \infty$ (Case 2). We consider these two cases separately.

Case 1. In this case, $k(t) \downarrow 0$ as $t \to \infty$ and the hypotheses of Theorem 3.6 are satisfied. Therefore, $(\phi(u))(\cdot)$ converges. It follows that $\lim_{t\to\infty} (\mathcal{L}^{-1}(\mathbf{G}) \star \phi(u))(t)$ converges (and so, in particular, is a bounded function). Moreover, by exponential stability, $\mathbf{\Psi}_\infty x_0 \in L^1(\mathbb{R}_+, \mathbb{R})$, and it follows from

$$\dot{l}(t) = |r - y(t)| \le |r - (\mathcal{L}^{-1}(\mathbf{G}) \star \phi(u))(t)| + |(\mathbf{\Psi}_\infty x_0)(t)|,$$

via integration that

$$k(t) = \frac{1}{l(t)} \ge \frac{1}{\alpha + \beta t} \quad \forall\ t \ge 0, \tag{38}$$

where

$$\alpha := l_0 + \int_0^\infty |\mathbf{\Psi}_\infty x_0(\tau)| \, d\tau, \quad \beta \ge \sup_{t \ge 0} |r - (\mathcal{L}^{-1}(\mathbf{G}) \star \phi(u))(t)|.$$

Therefore, assertions (a) to (e) of Theorem 3.8 hold.

Case 2. In this case, $k(t) \to k^* := 1/l^* > 0$ as $t \to \infty$. By boundedness of $l(\cdot)$ and (36), we may conclude that $e(\cdot) := r - C_L x(\cdot) - D\phi(u(\cdot)) \in L^1(\mathbb{R}_+)$ and so (by (10b)) $u(t)$ converges to a finite limit as $t \to \infty$. Consequently, $\phi(u(t))$ converges to a finite limit as $t \to \infty$, and hence, by Lemma 3.7, assertions (a) to (e) of Theorem 3.8 hold.

Finally, assume that $\phi_r \in \operatorname{im} \phi$ is not a critical value and that $\mathcal{L}^{-1}(\mathbf{G}) \in \mathcal{M}_\alpha$ for some $\alpha < 0$. We will show that the monotone gain k converges to a positive value. Seeking a contradiction, suppose that the monotone function l is unbounded (equivalently, $k(t) \downarrow 0$ as $t \to \infty$). Then the hypotheses of Theorem 3.6 are satisfied and so (38) holds. By Theorem 3.8, $\phi(u(\cdot))$ converges to ϕ_r, and $y(\cdot) - (\mathbf{\Psi}_\infty x_0)(\cdot)$

converges to r; moreover, the convergence is of order $\exp(-\rho K(t))$ for some $\rho > 0$; that is, there exists constant $L > 0$ such that

$$|r - y(t) + (\mathbf{\Psi}_\infty x_0)(t)| \le L \exp(-\rho K(t)) \quad \forall\, t \in \mathbb{R}_+. \tag{39}$$

Choose $\gamma \ge \beta$ such that $\rho/\gamma < 1$. By (38), $k(t) = 1/l(t) \ge (\alpha + \gamma t)^{-1}$ $\forall\, t \in \mathbb{R}_+$. Therefore,

$$K(t) = \int_0^t k \;\ge\; \ln[((\alpha + \gamma t)/\alpha)^{1/\gamma}] \quad \forall\, t \ge 0.$$

Consequently for a.a. $t \ge 0$,

$$\dot{l}(t) = |r - y(t)| \le L \exp(-\rho K(t)) + |(\mathbf{\Psi}_\infty x_0)(t)| \le M(\alpha + \gamma t)^{-\eta} + |(\mathbf{\Psi}_\infty x_0)(t)|,$$

where $\eta = \rho/\gamma \in (0, 1)$ and $M = L\alpha^\eta$. Since, by exponential stability, $\mathbf{\Psi}_\infty x_0 \in L^1(\mathbb{R}_+, \mathbb{R})$, integration gives

$$l(t) \le N(\alpha + \gamma t)^{1-\eta} \quad \forall\, t \ge 0,$$

for some suitable constant $N > 0$. It follows that

$$-K(t) = -\int_0^t k \le -(N\gamma\eta)^{-1}\,[(\alpha + \gamma t)^\eta - \alpha^\eta] \quad \forall\, t \ge 0.$$

Therefore, $\exp(-\rho K(\cdot))$ is of class $L^1(\mathbb{R}_+, \mathbb{R})$, and, by (39), it follows that $|r - y(\cdot) + (\mathbf{\Psi}_\infty x_0)(\cdot)|$ is also of class $L^1(\mathbb{R}_+, \mathbb{R})$. Since $\mathbf{\Psi}_\infty x_0 \in L^1(\mathbb{R}_+, \mathbb{R})$, we have $|r - y(\cdot)| \in L^1(\mathbb{R}_+, \mathbb{R})$. This contradicts the supposition of unboundedness of $l(\cdot)$. Therefore, $l(\cdot)$ is bounded. $\square$

4. Example: Controlled diffusion process with output delay. Consider a diffusion process (with diffusion coefficient $a > 0$ and with Dirichlet boundary conditions), on the one-dimensional spatial domain $I = [0, 1]$, with scalar nonlinear pointwise control action (applied at point $x_b \in I$, via a nonlinearity ϕ with Lipschitz constant $\lambda > 0$) and delayed (delay $h \ge 0$) pointwise scalar observation (output at point $x_c \in I$). We formally write this single-input, single-output system (previously considered, in a nonadaptive control context, in the precursor [9] to the present paper) as

$$z_t(t, x) = a z_{xx}(t, x) + \delta(x - x_b)\phi(u(t)), \quad y(t) = z(t - h, x_c),$$

$$z(t, 0) = 0 = z(t, 1) \quad \forall\, t > 0.$$

For simplicity, we assume zero initial conditions:

$$z(t, x) = 0 \quad \forall\, (t, x) \in [-h, 0] \times [0, 1].$$

With input $\phi(u(\cdot))$ and output $y(\cdot)$, this example qualifies as a regular linear system with transfer function given by

$$\mathbf{G}(s) = \frac{e^{-sh/a} \sinh(x_b\sqrt{s}\,) \sinh((1 - x_c)\sqrt{s}\,)}{a\sqrt{s} \sinh\sqrt{s}}.$$

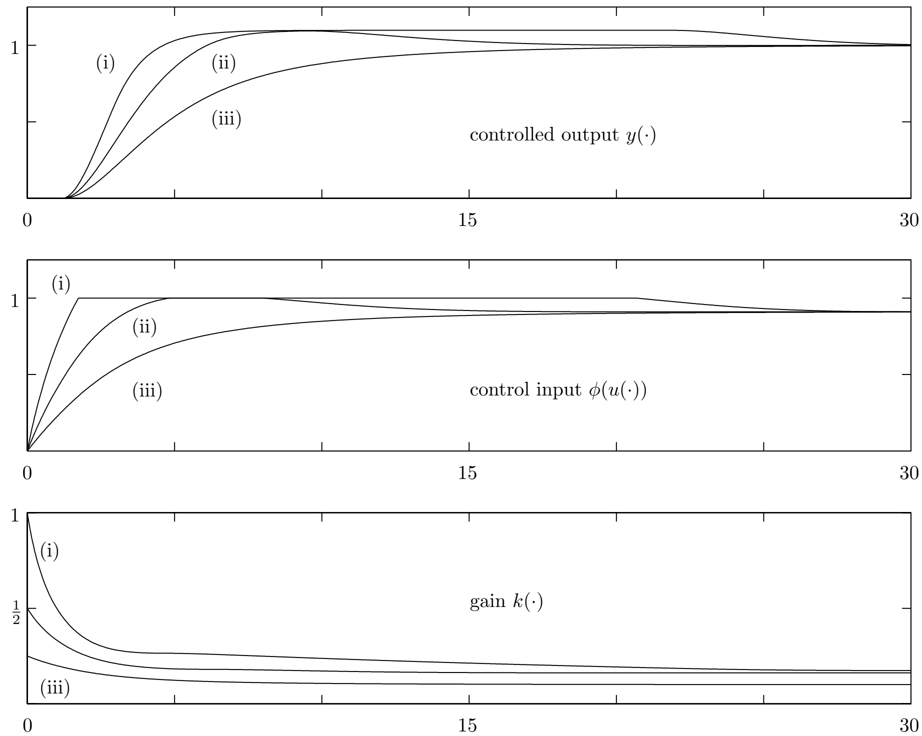

FIG. 3. *Controlled output, control input, and adapting gain.*

Therefore, by Theorem 3.8, the adaptive integral control

$$u(t) = \int_0^t k(t)[r - y(t)]\,dt, \qquad k(t) = \frac{1}{l(t)},$$

where the evolution of $l(t)$ is given by the adaptation law

$$\dot{l}(t) = |r - y(t)|, \quad l(0) = l_0 > 0,$$

guarantees asymptotic tracking of every constant reference signal r satisfying

$$\frac{r}{\mathbf{G}(0)} = \frac{ar}{x_b(1 - x_c)} \in \operatorname{clos}(\operatorname{im}\phi).$$

For purposes of illustration, we adopt the following values:

$$a = 0.1, \quad x_b = \frac{1}{3}, \quad x_c = \frac{2}{3}, \quad h = 0.1.$$

We consider a nonlinearity ϕ of saturation type, defined as follows

$$u \mapsto \phi(u) := \begin{cases} 1, & u \geq 1, \\ u, & u \in (0,1), \\ 0, & u \leq 0, \end{cases}$$

in which case $\lambda = 1$. For unit reference signal $r = 1$, we have

$$\frac{r}{\mathbf{G}(0)} = \frac{a}{x_b(1 - x_c)} = 0.9 \in \operatorname{int}(\operatorname{im}\phi).$$

Figure 3 depicts the behavior of the system (with reference $r = 1$) under adaptive integral control in each of the following three cases:

$$\text{(i) } l_0 = 1, \quad \text{(ii) } l_0 = 2, \quad \text{(iii) } l_0 = 4.$$

This figure was generated using SIMULINK simulation software within MATLAB wherein a truncated eigenfunction expansion, of order 10, was adopted to model the diffusion process.

5. Appendix: Proof of Lemmas 3.5 and 3.12. In proving Lemmas 3.5 and 3.12, we will first study an abstract Volterra integrodifferential equation. Let $\alpha \geq 0$, and let $w_\alpha \in C([0, \alpha], \mathbb{R}^n)$. Consider the initial-value problem

$$\dot{w}(t) = (Vw)(t), \quad t \geq \alpha, \tag{40a}$$

$$w(t) = w_\alpha(t), \quad t \in [0, \alpha], \tag{40b}$$

where the operator $V : C(\mathbb{R}_+, \mathbb{R}^n) \to L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ is causal and *weakly Lipschitz* in the following sense.

$\forall\ \alpha \geq 0,\ \delta > 0,\ \rho > 0$, and $\theta \in C([0, \alpha], \mathbb{R}^n)$, there exists a continuous function $f : [0, \delta] \to \mathbb{R}_+$, with $f(0) = 0$, such that

$$\int_\alpha^{\alpha+\varepsilon} \|(Vv)(t) - (Vw)(t)\|\, dt \leq f(\varepsilon) \sup_{\alpha \leq t \leq \alpha+\varepsilon} \|v(t) - w(t)\|$$

$\forall\ \varepsilon \in [0, \delta]$ and $\forall\ v, w \in \mathcal{C}(\alpha, \delta, \rho, \theta)$, where

$$\mathcal{C}(\alpha, \delta, \rho, \theta) := \{w \in C([0, \alpha + \delta], \mathbb{R}^n)|\ w(t) = \theta(t)\ \forall\ t \in [0, \alpha],$$
$$\|w(t) - \theta(\alpha)\| \leq \rho\ \forall\ t \in [\alpha, \alpha + \delta]\}.$$

A *solution* of the initial-value problem (40) on an interval $[0, \beta)$, where $\beta > \alpha$, is a function $w \in C([0, \beta), \mathbb{R}^n)$, with $w(t) = w_\alpha(t)\ \forall\ t \in [0, \alpha]$, such that w is absolutely continuous on $[\alpha, \beta)$ and (40a) is satisfied for a.a. $t \in [\alpha, \beta)$.

Strictly speaking, to make sense of (40), we have to give a meaning to $(Vw)(t)$, $t \in [0, \beta)$, when w is a continuous function defined on a *finite* interval $[0, \beta)$ (recall that V operates on the space of continuous functions defined on the *infinite* interval $\mathbb{R}_+$). This can be done easily using causality of V: $\forall\ t \in [0, \beta)$, $(Vw)(t) := (Vw^*)(t)$, where $w^* : \mathbb{R}_+ \to \mathbb{R}^n$ is any continuous function with $w^*(s) = w(s)\ \forall\ s \in [0, t]$.

PROPOSITION 5.1. *For every $\alpha \geq 0$ and every $w_\alpha \in C([0, \alpha], \mathbb{R}^n)$, there exists a unique solution $w(\cdot)$ of (40) defined on a maximal interval $[0, t_{\max})$, with $t_{\max} > \alpha$. Moreover, if $t_{\max} < \infty$, then*

$$\limsup_{t \to t_{\max}} |w(t)| = \infty. \tag{41}$$

Proof. Fix $\alpha \geq 0$, $w_\alpha \in C([0, \alpha], \mathbb{R}^n)$ arbitrarily. Define a continuous extension $w^*_\alpha : \mathbb{R}_+ \to \mathbb{R}^n$ of w_α by setting $w^*_\alpha(t) = w_\alpha(\alpha)\ \forall\ t > \alpha$. For later convenience, we introduce the continuous function

$$\varepsilon \mapsto g(\varepsilon) := \int_\alpha^{\alpha+\varepsilon} \|(Vw^*_\alpha)(t)\|\, dt.$$

We proceed in three steps.

Step 1. Existence and uniqueness on a small interval.

For each $\varepsilon \in (0,1)$, define

$$\mathcal{C}_\varepsilon := \mathcal{C}(\alpha, \varepsilon, 1, w_\alpha),$$

which, endowed with the metric

$$(v, w) \mapsto \sup_{\alpha \le t \le \alpha+\varepsilon} \|v(t) - w(t)\|,$$

is a complete metric space.

Existence and uniqueness of a solution on a small interval is proved by showing that

$$(\Gamma w)(t) := \begin{cases} w_\alpha(t), & 0 \le t \le \alpha, \\ w_\alpha(\alpha) + \displaystyle\int_\alpha^t (Vw)(\tau)\, d\tau, & \alpha \le t \le \alpha + \varepsilon \end{cases}$$

defines a contraction on $\mathcal{C}_\varepsilon$ for sufficiently small $\varepsilon > 0$.

By the weak Lipschitz property of V, there exists a continuous function $f : [0,1] \to \mathbb{R}_+$ with $f(0) = 0$, such that, $\forall\, \varepsilon \in (0,1)$, $v, w \in \mathcal{C}_\varepsilon$, and $t \in [\alpha, \alpha+\varepsilon]$,

$$\begin{aligned} \|(\Gamma w)(t) - w_\alpha(\alpha)\| &\le \int_\alpha^{\alpha+\varepsilon} \|(Vw)(\tau)\|\, d\tau \\ &\le g(\varepsilon) + \int_\alpha^{\alpha+\varepsilon} \|(Vw)(\tau) - (Vw_\alpha^*)(\tau)\|\, d\tau \\ &\le g(\varepsilon) + f(\varepsilon) \\ &\le 1 \qquad \text{for all sufficiently small } \varepsilon > 0 \end{aligned} \tag{42}$$

and

$$\begin{aligned} \|(\Gamma v)(t) - (\Gamma w)(t)\| &\le \int_\alpha^{\alpha+\varepsilon} \|(Vv)(\tau) - (Vw)(\tau)\|\, d\tau \\ &\le f(\varepsilon) \sup_{\alpha \le \tau \le \alpha+\varepsilon} \|v(\tau) - w(\tau)\| \\ &\le \tfrac{1}{2} \sup_{\alpha \le \tau \le \alpha+\varepsilon} \|v(\tau) - w(\tau)\| \quad \text{for all sufficiently small } \varepsilon > 0. \end{aligned} \tag{43}$$

By (42), $\Gamma(\mathcal{C}_\varepsilon) \subset \mathcal{C}_\varepsilon$ for all sufficiently small $\varepsilon > 0$. Consequently, we obtain from (43) that Γ is a contraction on $\mathcal{C}_\varepsilon$ for all sufficiently small $\varepsilon > 0$.

Step 2. Extended uniqueness.

Let $v : [0, \beta_1) \to \mathbb{R}^n$ and $w : [0, \beta_2) \to \mathbb{R}^n$, $\beta_1, \beta_2 > \alpha$, be solutions of (40) (existence of v and w is ensured by Step 1).

We claim that $v(t) = w(t)\ \forall\ t \in [0, \beta)$, where $\beta = \min\{\beta_1, \beta_2\}$. Seeking a contradiction, suppose that there exists $t \in (0, \beta)$ such that $v(t) \ne w(t)$. Defining

$$t^* = \inf\{t \in (0, \beta) \,|\, v(t) \ne w(t)\},$$

it follows that $t^* > \alpha$ (by Step 1), $t^* < \beta$ (by supposition), and $v(t^*) = w(t^*)$ (by continuity of v and w). Clearly, the initial-value problem

$$\dot z(t) = (Vz)(t), \;\; t \ge t^*\,; \quad z(t) = v(t), \;\; t \in [0, t^*]$$

is solved by v and w. This implies (by the argument in Step 1) that there exists an $\varepsilon > 0$ such that $v(t) = w(t)\ \forall\, t \in [0, t^* + \varepsilon)$, which contradicts the definition of t^*.

Step 3. Continuation of solutions.

Let $\alpha \geq 0$ and $w_\alpha \in C([0,\alpha], \mathbb{R}^n)$ be arbitrary and, as before, let w_α^* be the continuous extension of w_α with $w_\alpha^*(t) = w_\alpha(\alpha)\ \forall\, t > \alpha$.

Let w be a solution of (40) on the interval $[0, \beta)$, $\alpha < \beta < \infty$. In order to prove that w can be extended to a maximal solution (which satisfies (41) if $t_{\max} < \infty$), it is sufficient to show that w can be continued to the right (beyond β) if w is bounded on $[0, \beta)$. Suppose that w is bounded. Set $\delta := \beta - \alpha$ and $\rho =: \sup\{\|w(\tau) - w_\alpha(\alpha)\| \mid \alpha < \tau < \beta\}$. By the weak Lipschitz property of V, there exists continuous $f : [0, \delta] \to \mathbb{R}_+$, with $f(0) = 0$, such that $\forall\, \varepsilon \in (0, \delta)$

$$\int_\alpha^{\alpha+\varepsilon} \|(Vw)(\tau)\|\, d\tau \leq g(\varepsilon) + \rho f(\varepsilon),$$

implying, by boundedness of g and f on $[0, \delta]$, that $Vw \in L^1([0, \beta], \mathbb{R}^n)$ and so the following limit exists:

$$\lim_{t \uparrow \beta} \int_\alpha^t (Vw)(\tau)\, d\tau =: L \in \mathbb{R}^n,$$

whence

$$L + w_\alpha(\alpha) = \lim_{t \uparrow \beta} (\Gamma w)(t).$$

Now $w(t) = (\Gamma w)(t)\ \forall\, t \in [0, \beta)$. Therefore, defining $w(\beta) = L + w_\alpha(\alpha)$ we can extend w into a continuous function on $[0, \beta]$. Finally, by the argument in Step 1, the initial-value problem

$$\dot{z}(t) = (Vz)(t),\ \ t \geq \beta\,; \quad z(t) = w(t),\ \ t \in [0, \beta]$$

has a unique solution w^* on $[0, \beta + \varepsilon)$ for some $\varepsilon > 0$. By causality of V, the function w^* is a solution of (40) on $[0, \beta + \varepsilon)$, and so w^* is a proper right continuation of w. $\square$

REMARK 5.2. *In what follows, we shall invoke Proposition* 5.1 *only in the special case* $\alpha = 0$. *Note, however, that Steps* 2 *and* 3 *in the above proof of the proposition required the local existence and uniqueness result in the more general context of* $\alpha \geq 0$.

In the following, Proposition 5.1 will be used to prove Lemmas 3.5 and 3.12. First note that, by setting $k(t) = 1/l(t)$, the adaptive feedback system (37) (with $(A, B, C, D) \in \mathcal{R}$) can be written in the following form which will be more convenient for our purposes:

$$\dot{x}(t) = Ax(t) + B\phi(u(t)), \quad x(0) = x_0 \in X, \tag{44a}$$

$$\dot{u}(t) = k(t)[r - C_L x(t) - D\phi(u(t))], \quad u(0) = u_0 \in \mathbb{R}, \tag{44b}$$

$$\dot{k}(t) = -k^2(t)|r - C_L x(t) - D\phi(u(t))|, \quad k(0) = k_0 \in (0, \infty). \tag{44c}$$

The feedback systems (10) and (44) are both of the form

$$\dot{x}(t) = Ax(t) + B\phi(u(t)), \quad x(0) = x_0 \in X \tag{45a}$$

$$\dot{u}(t) = \kappa(t)\theta(t)[r - C_L x(t) - D\phi(u(t))], \quad u(0) = u_0 \in \mathbb{R}, \tag{45b}$$

$$\dot{\theta}(t) = h(\theta(t))|r - C_L x(t) - D\phi(u(t))|, \quad \theta(0) = \theta_0 \in \mathbb{R}, \tag{45c}$$

where $\kappa \in L^\infty(\mathbb{R}_+, \mathbb{R})$ and $h : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz. To recover (10) from (45), set $h(\theta) \equiv 0$ and $\theta_0 = 1$ (in this case $\kappa(\cdot)$ plays the role of the gain function $k(\cdot)$). Considering the special case $\kappa(t) \equiv 1$ and $h(\theta) = -\theta^2$ gives the adaptive feedback equations (44) (with $k(\cdot) = \theta(\cdot)$).

For $a \in (0, \infty]$, a continuous function

$$[0, a) \to X \times \mathbb{R} \times \mathbb{R}, \quad t \mapsto (x(t), u(t), \theta(t))$$

is a *solution* of (45) if $(x(\cdot), u(\cdot), \theta(\cdot))$ is absolutely continuous as a $(X_{-1} \times \mathbb{R} \times \mathbb{R})$-valued function, $x(t) \in \mathrm{dom}(\mathrm{C_L})$ for a.a. $t \in [0, a)$, $(x(0), u(0), \theta(0)) = (x_0, u_0, \theta_0)$, and the differential equations in (45) are satisfied almost everywhere on $[0, a)$, where the derivative in (45a) should be interpreted in the space X_{-1}.

On noting that $C_L x(t) + D\phi(u(t)) = (\mathbf{\Psi}_\infty x_0)(t) + (\mathbf{F}_\infty \phi(u))(t)$ (with $\mathbf{\Psi}_\infty$ and $\mathbf{F}_\infty$ defined by (2)), the variable $x(t)$ can be eliminated from (45b) and (45c) to obtain

$$\dot{u}(t) = \kappa(t)\theta(t)[r - (\mathbf{\Psi}_\infty x_0)(t) - (\mathbf{F}_\infty \phi(u))(t)], \quad u(0) = u_0, \tag{46a}$$

$$\dot{\theta}(t) = h(\theta(t))|r - (\mathbf{\Psi}_\infty x_0)(t) - (\mathbf{F}_\infty \phi(u))(t)|, \quad \theta(0) = \theta_0. \tag{46b}$$

In order to proceed we need the following lemma.

LEMMA 5.3. *$\forall\ \alpha \geq 0$, $v \in C([0, \alpha], \mathbb{R})$, $\delta > 0$, and $\rho > 0$, there exist $\gamma_1, \gamma_2 > 0$ such that $\forall\ \varepsilon \in [0, \delta]$ and $u, w \in \mathcal{C}(\alpha, \delta, \rho, v)$*

$$\int_\alpha^{\alpha+\varepsilon} |(\mathbf{F}_\infty \phi(u))(\tau) - (\mathbf{F}_\infty \phi(w))(\tau)|\, d\tau \leq \varepsilon\gamma_1 \sup_{\alpha \leq \tau \leq \alpha+\varepsilon} |u(\tau) - w(\tau)|, \tag{47}$$

$$\int_\alpha^{\alpha+\varepsilon} |(\mathbf{F}_\infty \phi(u))(\tau)|\, d\tau \leq \varepsilon\gamma_1\rho + \sqrt{\varepsilon}\gamma_2. \tag{48}$$

Proof. Let $\alpha \geq 0$, $v \in C([0, \alpha], \mathbb{R})$, $\delta > 0$, $\rho > 0$, and $u, w \in \mathcal{C}(\alpha, \delta, \rho, v)$. Let λ be a Lipschitz constant for $\phi \in \mathcal{N}$. Then, using the Cauchy–Schwarz inequality and the boundedness of $\mathbf{F}_\infty$ as an operator from $L^2(\mathbb{R}_+, \mathbb{R})$ into $L^2(\mathbb{R}_+, \mathbb{R})$, we obtain $\forall$ $\varepsilon \in [0, \delta]$,

$$\begin{aligned}
\int_\alpha^{\alpha+\varepsilon} |\mathbf{F}_\infty \phi(u) - \mathbf{F}_\infty \phi(w)| &\leq \sqrt{\varepsilon} \left(\int_\alpha^{\alpha+\varepsilon} |\mathbf{F}_\infty \phi(u) - \mathbf{F}_\infty \phi(w)|^2 \right)^{1/2} \\
&\leq \sqrt{\varepsilon}\lambda \|\mathbf{F}_\infty\| \left(\int_\alpha^{\alpha+\varepsilon} |u - w|^2 \right)^{1/2} \\
&\leq \varepsilon\lambda \|\mathbf{F}_\infty\| \sup_{\alpha \leq \tau \leq \alpha+\varepsilon} |u(\tau) - w(\tau)|,
\end{aligned}$$

which is (47) with $\gamma_1 = \lambda\|\mathbf{F}_\infty\|$.

To establish (48), define a continuous extension $v^* : \mathbb{R}_+ \to \mathbb{R}$ of v by setting $v^*(t) = v(\alpha)\ \forall\ t > \alpha$. Applying (47), it follows $\forall\ \varepsilon \in [0, \delta]$ that

$$\begin{aligned}
\int_\alpha^{\alpha+\varepsilon} |(\mathbf{F}_\infty \phi(u))(\tau)|\, d\tau &\leq \int_\alpha^{\alpha+\varepsilon} |(\mathbf{F}_\infty \phi(v^*))(\tau)|\, d\tau + \varepsilon\gamma_1 \sup_{\alpha \leq \tau \leq \alpha+\varepsilon} |u(\tau) - v^*(\tau)| \\
&\leq \sqrt{\varepsilon} \left(\int_\alpha^{\alpha+\delta} |(\mathbf{F}_\infty \phi(v^*))(\tau)|^2 \right)^{1/2} d\tau + \varepsilon\gamma_1\rho,
\end{aligned}$$

which yields (48) with $\gamma_2 := (\int_\alpha^{\alpha+\delta} |(\mathbf{F}_\infty \phi(v^*))(\tau)|^2 \, d\tau)^{1/2}$. □

Lemmas 3.5 and 3.12 are special cases of the following corollary.

COROLLARY 5.4. *Let* $(A,B,C,D) \in \mathcal{R}$, $\phi \in \mathcal{N}$, $r \in \mathbb{R}$, $\kappa \in L^\infty(\mathbb{R}_+,\mathbb{R})$, *and let* $h : \mathbb{R} \to \mathbb{R}$ *be locally Lipschitz. If* $h(\theta) \le 0 \; \forall \; \theta \in \mathbb{R}$ *and* $h(0) = 0$, *then* $\forall$ $(x_0,u_0,\theta_0) \in X \times \mathbb{R} \times (0,\infty)$, *the initial-value problem given by* (45) *has a unique solution defined on* $\mathbb{R}_+$.

Proof. Let $(x_0,u_0,\theta_0) \in X \times \mathbb{R} \times (0,\infty)$. It is clear that the map $V : C(\mathbb{R}_+,\mathbb{R}^2) \to L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^2)$ given by

$$V\begin{pmatrix} u \\ \theta \end{pmatrix}(t) = \begin{pmatrix} \kappa(t)\theta(t)[r - (\mathbf{\Psi}_\infty x_0)(t) - (\mathbf{F}_\infty \phi(u))(t)] \\ h(\theta(t))|r - (\mathbf{\Psi}_\infty x_0)(t) - (\mathbf{F}_\infty \phi(u))(t)| \end{pmatrix}$$

is causal, and it follows from Lemma 5.3 via a routine argument that it is also weakly Lipschitz. Hence it follows from Proposition 5.1 that the initial-value problem (46) has a unique solution (u,θ) on a maximal interval of existence $[0,t_{\max})$. To prove that $t_{\max} = \infty$, we first show that θ is bounded on $[0,t_{\max})$. Note that since $h \le 0$, $\theta(\cdot)$ is nonincreasing, and combining this with the assumption that $\theta_0 > 0$, we see that boundeness of $\theta(\cdot)$ follows if we can show that $\theta(t) > 0 \; \forall \; t \in [0,t_{\max})$. Seeking a contradiction, suppose that there exists a $t^* \in (0,t_{\max})$ such that $\theta(t^*) = 0$. Consider the following initial-value problem on $[0,t_{\max})$:

$$\dot{\zeta}(t) = h(\zeta(t))|e(t)|, \quad \zeta(t^*) = 0, \tag{49}$$

where $e(t) = r - (\mathbf{\Psi}_\infty x_0)(t) - (\mathbf{F}_\infty \phi(u))(t)$. Then $\theta(\cdot)$ is a solution of (49). Since $h(0) = 0$, the function $\zeta \equiv 0$ is also a solution of (49). By uniqueness it follows that $\theta \equiv 0$, which is in contradiction to $\theta_0 > 0$. Therefore, the function $\theta(\cdot)$ is bounded on $[0,t_{\max})$ and hence there exists a constant $\gamma > 0$ such that

$$|\kappa(t)\theta(t)| \le \gamma \quad \forall \, t \in [0,t_{\max}).$$

Let $[0,T)$ be an arbitrary interval with $[0,T) \subset [0,t_{\max})$ and $T < \infty$. Multiplying (46a) by u and estimating we obtain that, $\forall \; \tau \in [0,T)$,

$$u(\tau)\dot{u}(\tau) \le \gamma[r^2 + (\mathbf{\Psi}_\infty x_0)^2(\tau) + u^2(\tau) + |(\mathbf{F}_\infty \phi(u))(\tau)u(\tau)|\,]. \tag{50}$$

Integrating (50) from 0 to t and combining the estimate

$$\int_0^t |(\mathbf{F}_\infty \phi(u))u| \le \int_0^t |\mathbf{F}_\infty(\phi(u) - \phi(0))|\,|u| + \frac{1}{2}\left(\int_0^t (\mathbf{F}_\infty \phi(0))^2 + \int_0^t u^2\right),$$

the Cauchy–Schwarz inequality, and the global Lipschitz property of ϕ, we can show readily that there exist positive constants α and β such that, $\forall \; t \in [0,T)$,

$$u^2(t) \le \alpha + \beta \int_0^t u^2(\tau)\, d\tau.$$

An application of Gronwall's lemma then shows that $u^2(t) \le \alpha e^{\beta t} \; \forall \; t \in [0,T)$. Hence u is bounded on $[0,T)$. Since this holds $\forall \; T$ with $T \le t_{\max}$ and $T < \infty$, it follows by Proposition 5.1 that $t_{\max} = \infty$.

Finally, to obtain a solution of (45), *define*

$$x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B\phi(u(\tau))\, d\tau. \tag{51}$$

By well-posedness, x is a continuous X-valued function, and moreover, since A, considered as a generator on X_{-1}, is in $\mathcal{B}(X, X_{-1})$, the function $t \mapsto Ax(t)$ is a continuous X_{-1}-valued function. Consequently, by Pazy [18, Theorem 2.4, p. 107], we have that in X_{-1}

$$\dot{x}(t) = Ax(t) + B\phi(u(t)) \quad \forall\, t \in \mathbb{R}_+.$$

It follows that (x, u, θ) is the unique solution of (45) defined on $\mathbb{R}_+$. $\square$

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