A circle criterion for strong integral input-to-state stability

Chris Guiver and Hartmut Logemann

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Abstract

We present sufficient conditions for integral input-to-state stability (iISS) and strong iISS of the zero equilibrium pair of continuous-time forced Lur'e systems, where by strong iISS we mean the conjunction of iIIS and small-signal ISS. Our main results are reminiscent of the complex Aizerman conjecture and the well-known circle criterion. We derive a number of corollaries, including a result on stabilisation by static feedback in the presence of input saturation. Whilst it is known that forced Lur'e systems with unstable linear component and bounded nonlinearity cannot be ISS, we show that they may well be strongly iISS.

Keywords. asymptotic stabilization, circle criterion, Lur'e systems, robust control of nonlinear systems, strong iISS

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1 Introduction

In this paper, we study (strong) integral input-to-state stability (iISS) properties of the following class of forced Lur'e systems:

$$\dot{x} = Ax + Bf(Cx) + v, \quad x(0) = x^0 \in \mathbb{R}^n\,. \tag{1.1}$$

Here A, B and C are real matrices, f is a (nonlinear) function, x denotes the state and v is a forcing function (also named, or interpreted, as a disturbance, control or input). Lur'e systems are a common and important class of nonlinear control system, and arise in a number of engineering scenarios, such as the stabilisation of linear systems by saturated static feedback. The name Lur'e is attributed to A.I. Lur'e, a Soviet scientist who made early contributions to the stability theory of continuous-time Lur'e systems. The study of the stability properties of Lur'e systems constitutes *absolute stability theory* which, loosely speaking, seeks to conclude stability of the feedback system (1.1) through the interplay of frequency domain properties of the linear system given by (A, B, C) and boundedness or sector properties of the nonlinearity f. Absolute stability theory is a mature subject see, for example, [14, 15, 16, 22, 23, 40], and has used state-space methods to analyze global asymptotic stability properties (of the unforced version of (1.1)) [14, 22] and input-output methods to establish L^2 and L^∞-stability [10, 38]. More recently, absolute stability ideas have been merged with ISS theory to obtain ISS criteria which resemble classical absolute stability results, see [3, 12, 18, 19, 29, 30, 31]. The ISS stability criteria obtained in [29] have been used in [6] to prove converging-input converging-state properties for Lur'e systems.

Strong iISS is a recent stability concept, introduced in [7], and is the conjunction of iISS and ISS with respect to small signals, or small-signal ISS. The concept of ISS was introduced in [32] and is the subject of numerous papers including, for example [1, 20, 33, 34, 35] and the tutorial papers [9, 36]. ISS is a stability notion with respect to a state/forcing trajectory, which, for simplicity, we assume to be the zero trajectory. Roughly speaking, ISS means that the state has "nice" boundedness properties with respect to initial states and potentially persistent forcing which are expressed in terms of suitable comparison functions. Over the last 30 years, an extensive ISS Lyapunov theory has been developed, successfully synthesizing state-space and input-output viewpoints and resulting in a comprehensive and powerful stability theory for nonlinear control systems.

Integral ISS was introduced in [35], further developed in [2] and ensures that the states of controlled systems are uniformly bounded by separate terms involving the initial conditions and the integral of the input, respectively. In particular, iISS implies boundedness of the state when subject to forcing which has "finite energy". ISS implies iISS, but the converse is false in general.

The small-signal ISS property guarantees that the boundedness of the state is robust with respect to small, but potentially persistent, forcing. Indeed, the somewhat pathological behaviour that the state may become unbounded when subject to forcing with arbitrarily small L^∞-norm is ruled out by small-signal ISS. The emphasis of the current paper is on strong iISS, the conjunction of iISS and small-signal ISS: strong iISS is an "intermediate property" which has the benefits of "the robustness strengths of ISS and the generality of iISS" [7].

The results obtained in this paper are reminiscent of the complex Aizerman conjecture [15, 16] and the circle criterion [14, 19, 22]: we show that, when suitably modified, these classical absolute stability results are sufficient for (strong) iISS. In particular, our main result, Theorem 3.1, shows that if every complex output feedback gain matrix in the open ball $\mathbb{B}_{\mathbb{C}}(K, r)$ (centred at K and of radius r) stabilises the underlying linear system (A, B, C), then the zero equilibrium pair of the forced Lur'e system (1.1) is strongly iISS for all f satisfying a related "nonlinear ball" condition, namely,

$$\frac{\|f(z) - Kz\|}{\|z\|} \leq r - \beta(\|z\|) \quad \forall\, z \neq 0,$$

where β is any continuous positive function defined on $(0, \infty)$ such that the function $s \mapsto s\beta(s)$ is strictly increasing.

Furthermore, Proposition 3.9 provides a similar result for iISS which holds under a "nonlinear ball" condition weaker than that imposed in Theorem 3.1. The contribution of this work very much resonates with that in the paper [29], where ISS stability conditions for Lur'e systems are derived which are inspired by the complex Aizerman conjecture and the circle criterion. In particular, "nonlinear ball" conditions play an important role in [29]. Not surprisingly, the sense in which the "nonlinear ball" condition holds is crucial for the stability property which may be inferred, see the discussion in Section 5, in particular Theorem 5.1. The consequences of Theorem 3.1 include a strong iISS version of the well-known circle criterion which provides a sufficient condition for strong iISS in terms of positive-real and sector properties of the transfer function and the nonlinearity, respectively, see Corollary 3.7.

One motivation for the present work is its relevance to the problem of stabilizing unstable linear systems by saturated feedback. It is known that the zero equilibrium pair of the resulting Lur'e system cannot be ISS, see [29, Proposition 3.4]. However, in Proposition 4.2, we provide sufficient conditions for the zero equilibrium pair of such a feedback interconnection to be strongly iISS for a class of saturating nonlinearities. We remark that both Proposition 4.2 and Theorem 3.1 overlap with a result in the recent paper [4]: in particular, [4, Theorem 2] may be seen as a special case of our Theorem 3.1, see Remark 4.5 for a detailed discussion.

The paper is organised as follows. Section 2 gathers notation and preliminaries. Our main results are presented in Section 3, and are applied to a class of Lur'e systems with saturating nonlinearities in Section 4. Section 5 contains a discussion which places the our work in the wider context provided by related papers in the literature.

2 Notation and preliminaries

The set of positive integers is denoted by $\mathbb{N}$, and $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. We set $\mathbb{R}_+ := \{r \in \mathbb{R} : r \geq 0\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the usual real and complex n-dimensional vector spaces, respectively, both equipped with the 2-norm denoted by $\|\cdot\|$ induced by the standard inner product $\langle \cdot, \cdot \rangle$.

For $m \in \mathbb{N}$, let $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ denote the normed linear spaces of $n \times m$ matrices with real and complex entries, respectively, both equipped with the operator norm induced by the 2-norm, also denoted by $\|\cdot\|$.

A matrix $M \in \mathbb{C}^{n\times n}$ is said to be *Hurwitz* if all its eigenvalues have negative real parts. Furthermore, for positive semi-definite symmetric $M \in \mathbb{R}^{n\times n}$, we define the semi-norm $|\cdot|_M$ via

$$|z|_M^2 := \langle z, Mz\rangle \quad \forall\, z \in \mathbb{R}^n\,.$$

We note that $|z|_M = 0$ if, and only if, $z \in \ker M$. Moreover, $|z|_M = \|M^{1/2}z\|$, where $M^{1/2}$ denote the unique positive semi-definite square root of M. Consequently, we obtain the Cauchy-Schwarz inequality $\langle z_1, Mz_2\rangle \le |z_1|_M|z_2|_M$ for all $z_1, z_2 \in \mathbb{R}^n$ which we shall make extensive use of. The semi-norm $|\cdot|_M$ is a norm if, and only if, M is positive-definite. Clearly, we have that $|\cdot|_I = \|\cdot\|$. The semi-norm $|\cdot|_M$ induces an operator (matrix) semi-norm, denoted by the same symbol. In particular, if $N \in \mathbb{R}^{n\times m}$, then $|N|_M = 0$ if, and only if, $\operatorname{im} N \subset \ker M$.

Setting $s_M(z) := |z|_M$, a straightforward calculation shows that

$$(\nabla s_M)(z) = \frac{1}{|z|_M} Mz \quad \forall\, z \in \mathbb{R}^n \quad \text{such that} \quad |z|_M \neq 0\,.$$

For $K \in \mathbb{R}^{m\times p}$ and $r > 0$, we set

$$\mathbb{B}_{\mathbb{F}}(K, r) := \big\{Z \in \mathbb{F}^{m\times p} \;:\; \|Z - K\| < r\big\},$$

the open ball in $\mathbb{F}^{m\times p}$ centred at K and of radius r, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

We recall terminology and definitions pertaining to so-called comparison functions. Let $\mathcal{K}$ denote the set of all continuous and strictly increasing functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi(0) = 0$. Note that if $\phi \in \mathcal{K}$, then $\phi(s) > 0$ for all $s > 0$. The subset of $\mathcal{K}$ consisting of all unbounded functions in $\mathcal{K}$ is denoted $\mathcal{K}_\infty$. Obviously, if $\phi \in \mathcal{K}_\infty$, then $\phi(s) \to \infty$ as $s \to \infty$. The set $\mathcal{KL}$ consists of all functions $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy $\phi(\cdot, t) \in \mathcal{K}$ for all $t \ge 0$ and, for all $s \ge 0$, $\phi(s, \cdot)$ is non-increasing with $\phi(s, t) \to 0$ as $t \to \infty$. The reader is referred to [21] for more information on comparison functions.

The linear space of (equivalence classes of) measurable, locally essentially bounded functions $f : \mathbb{R}_+ \to \mathbb{R}^n$ is denoted by $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$. If $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, then

$$\|f\|_{L^\infty(0,t)} := \operatorname*{ess\,sup}_{\tau\in[0,t]} \|f(\tau)\| < \infty \quad \forall\, t \ge 0\,.$$

As usual, $L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ denotes the space of all measurable, essentially bounded functions $\mathbb{R}_+ \to \mathbb{R}^n$. For $f \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$, we write

$$\|f\|_{L^\infty} := \|f\|_{L^\infty(0,\infty)} = \operatorname*{ess\,sup}_{\tau\in[0,\infty)} \|f(\tau)\|\,.$$

In the following, for $z \in \mathbb{R}^n$, the constant function $\mathbb{R}_+ \to \mathbb{R}^n$, $t \mapsto z$ will also be denoted by z.

Consider the initial value problem (1.1). Here, and throughout, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $f : \mathbb{R}^p \to \mathbb{R}^m$ in (2.1b) is locally Lipschitz. For given $x^0 \in \mathbb{R}^n$ and $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, we let $x = x(\cdot\,; x^0, v)$ denote the unique maximally defined forward solution of the initial value problem (1.1), which is continuous and differentiable almost everywhere it is defined. It is well-known that if f is affine linearly bounded, then the Lur'e system (1.1) is forward complete, meaning that for all $x^0 \in \mathbb{R}^n$ and all $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, the solution x is defined for all $t \ge 0$ (see, for example, [25, Proposition 4.12]). In this paper the nonlinearity f will always be affine linearly bounded.

We refer to (1.1) with $v = 0$, as the *unforced* Lur'e system. A constant solution of the unforced Lur'e system is called an equilibrium. If 0 is an equilibrium, then we abbreviate "global asymptotic stability of the zero equilibrium" to the familiar "0-GAS".

Lur'e systems (1.1) may be seen as the closed-loop system arising from the feedback interconnection of the forced linear system

$$\dot{x} = Ax + Bu + v, \quad x(0) = x^0, \quad y = Cx\,, \tag{2.1a}$$

with state x, input u, output y and forcing v, and the static nonlinear output feedback

$$u = f(y)\,. \tag{2.1b}$$

We call $(x^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^n$ an equilibrium pair of (1.1) if $Ax^* + Bf(Cx^*) + v^* = 0$, that is, x^* is a constant solution of (1.1) with constant forcing $v(t) \equiv v^*$. Following [7], an equilibrium pair (x^*, v^*) is said to be strongly iISS if it is iISS and small-signal ISS. The iISS property was introduced in [35], see also [2], and means that there exist $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$ such that, for all $x^0 \in \mathbb{R}^n$ and all $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$,

$$\|x(t; x^0, v) - x^*\| \leq \beta(\|x^0 - x^*\|, t) + \gamma_1 \left(\int_0^t \gamma_2(\|v(s) - v^*\|) \, ds \right) \quad \forall\, t \geq 0\,. \tag{2.2}$$

Furthermore, we say that an equilibrium pair (x^*, v^*) is small-signal ISS if there exist $R > 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that: for all $x^0 \in \mathbb{R}^n$, all $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, and all $t \geq 0$, The constant R which appears in the definition of small-signal ISS is called an input threshold [7].

$$\|v - v^*\|_{L^\infty(0,t)} < R \quad \Rightarrow \quad \|x(t; x^0, v) - x^*\| \leq \beta(\|x^0 - x^*\|, t) + \gamma(\|v - v^*\|_{L^\infty(0,t)})\,. \tag{2.3}$$

Remark 2.1. For $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ and $\tau \geq 0$, define $v_\tau \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ by $v_\tau(t) := v(\tau + t)$ for all $t \geq 0$. Invoking the identity $x(t + \tau; x^0, v) = x(t; x(\tau; x^0, v), v_\tau)$ (which holds for all $t \geq 0$), the following consequences of the iISS and small-signal ISS properties are easy to prove.

(i) If (2.2) holds and $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ is such that the function $s \mapsto \gamma_2(\|v(s) - v^*\|)$ is integrable, then $x(t; x^0, v) \to x^*$ as $t \to \infty$.

(ii) If an equilibrium pair (x^*, v^*) is small-signal ISS and $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ is such that $v(t) \to v^*$ as $t \to \infty$, then $x(t; x^0, v) \to x^*$ as $t \to \infty$. ◇

The following lemma is easily established and hence its proof is omitted.

Lemma 2.2. *Let $(x^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^n$ be an equilibrium pair of* (1.1) *and define $\tilde{f}(z) := f(z + Cx^*) - f(Cx^*)$. For given $x^0 \in \mathbb{R}^n$ and $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, the function $x : \mathbb{R}_+ \to \mathbb{R}^n$ solves* (1.1) *if, and only if, $x - x^*$ is a solution of*

$$\dot{z} = Az + B\tilde{f}(Cz) + w, \quad z(0) = x^0 - x^*, \tag{2.4}$$

where $w = v - v^*$. Furthermore, the equilibrium pair (x^*, v^*) of (1.1) *is* (*strongly*) *iISS if, and only if, the equilibrium pair* $(0, 0)$ *of* (2.4) *is* (*strongly*) *iISS.*

Consequently, we will assume that $f(0) = 0$ and shall focus attention on the equilibrium pair $(0, 0)$, which we call the zero equilibrium pair.

Throughout the present work, we let $\mathbf{G}$, given by $\mathbf{G}(s) = C(sI - A)^{-1}B$, denote the transfer function of the linear system specified by (A, B, C) (that is, the transfer function of (2.1a) from input u to output y). Applying static output feedback to (A, B, C) with gain $K \in \mathbb{R}^{m \times p}$ leads to the linear system specified by $(A + BKC, B, C)$, the transfer function of which shall be denoted by $\mathbf{G}^K$. A straightforward calculation shows that

$$\mathbf{G}^K(s) = C(sI - A - BKC)^{-1}B = \mathbf{G}(s)(I - K\mathbf{G}(s))^{-1}\,.$$

As usual, $H^\infty(\mathbb{C}^{p \times m})$ denotes the space of holomorphic, bounded functions $\mathbb{C}_0 \to \mathbb{C}^{p \times m}$, which is a Banach space when equipped with the norm

$$\|\mathbf{H}\|_{H^\infty} := \sup_{\operatorname{Re} s \geq 0} \|\mathbf{H}(s)\| = \operatorname*{ess\,sup}_{\omega \in \mathbb{R}} \|\mathbf{H}(i\omega)\| \quad \forall\, \mathbf{H} \in H^\infty(\mathbb{C}^{p \times m})\,.$$

We set $H^\infty := H^\infty(\mathbb{C})$. Transfer functions of the form $\mathbf{G}(s) = C(sI - A)^{-1}B$ are strictly proper, rational, matrix-valued functions, which belong to $H^\infty(\mathbb{C}^{p \times m})$ if, and only if, every pole of $\mathbf{G}$ has negative real part.

For $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, let $\mathbb{S}_{\mathbb{F}}(\mathbf{G})$ denote the set of stabilising output feedback gains in $\mathbb{F}^{m \times p}$, that is,

$$\mathbb{S}_{\mathbb{F}}(\mathbf{G}) := \big\{ K \in \mathbb{F}^{m \times p} : \mathbf{G}^K \in H^\infty(\mathbb{C}^{p \times m}) \big\}.$$

We shall typically impose the assumption that the triple (A, B, C) is stabilisable and detectable, in which case,

$$\mathbb{S}_{\mathbb{F}}(\mathbf{G}) := \big\{ K \in \mathbb{F}^{m \times p} : A + KBC \text{ is Hurwitz} \big\}\,.$$

The following result provides a characterisation of balls of stabilising *complex* feedback gains in terms of a related H^∞-norm condition. A proof may be found in, for example, [11, Proposition 5.6] or [29, Lemma 2.1].

Lemma 2.3. *For $K \in \mathbb{R}^{m\times p}$ and $r > 0$, $\mathbb{B}_\mathbb{C}(K, r) \subseteq \mathbb{S}_\mathbb{C}(\mathbf{G})$ if, and only if, $\|\mathbf{G}^K\|_{H^\infty} \le 1/r$.*

We emphasize that the Lemma 2.3 does not hold if in the statement $\mathbb{C}$ is replaced by $\mathbb{R}$. More specifically, the inclusion $\mathbb{B}_\mathbb{R}(K, r) \subseteq \mathbb{S}_\mathbb{R}(\mathbf{G})$ does in general not imply that $\|\mathbf{G}^K\|_{H^\infty} \le 1/r$.

We shall make extensive use of the following result, the so-called complex Aizerman conjecture. For a proof we refer the reader to [15, Theorem 3.14 and Corollary 3.15] or [16, Theorem 5.6.22].

Theorem 2.4. *Assume that the triple (A, B, C) is stabilisable and detectable, and that $f : \mathbb{R}^p \to \mathbb{R}^m$ is locally Lipschitz. Assume further that $K \in \mathbb{R}^{m\times p}$ and $r > 0$ are such that $\mathbb{B}_\mathbb{C}(K, r) \subseteq \mathbb{S}_\mathbb{C}(\mathbf{G})$. If*

$$\|f(z) - Kz\| < r\|z\| \quad \forall\, z \in \mathbb{R}^p,\ z \neq 0\,, \tag{2.5}$$

then the zero equilibrium of the unforced Lur'e system (1.1) *is globally asymptotically stable.*

3 Strong iISS for forced Lur'e systems

Our main result is the following.

Theorem 3.1. *Assume that the triple (A, B, C) is stabilisable and detectable, and that $f : \mathbb{R}^p \to \mathbb{R}^m$ is locally Lipschitz. Assume further that $K \in \mathbb{R}^{m\times p}$ and $r > 0$ are such that $\mathbb{B}_\mathbb{C}(K, r) \subseteq \mathbb{S}_\mathbb{C}(\mathbf{G})$. If there exists $\alpha \in \mathcal{K}$ such that*

$$\|f(z) - Kz\| \le r\|z\| - \alpha(\|z\|) \quad \forall\, z \in \mathbb{R}^p\,, \tag{3.1}$$

then the zero equilibrium pair of (1.1) *is strongly iISS. Moreover, the iISS estimate* (2.2) *holds with $\gamma_2(z) = az$, for some $a > 0$.*

If, in (3.1), $\alpha \in \mathcal{K}_\infty$, then the zero equilibrium pair is ISS by [29, Theorem 3.1], and hence strongly iISS. This follows as ISS implies iISS by [35, Corollary 4]. Therefore, the novel situation here is when $\alpha \in \mathcal{K}\backslash\mathcal{K}_\infty$. The claim that the estimate (2.2) holds with γ_2 a linear function implies that solutions of (1.1) are bounded if $v \in L^1(\mathbb{R}_+, \mathbb{R}^n)$.

Invoking Lemma 2.3, the following nonlinear small-gain version of Theorem 3.1 is immediate.

Corollary 3.2. *Assume that the triple (A, B, C) is stabilisable and detectable, $K \in \mathbb{S}_\mathbb{R}(\mathbf{G})$ and that $f : \mathbb{R}^p \to \mathbb{R}^m$ is locally Lipschitz. If there exists $\alpha \in \mathcal{K}$ such that*

$$\|\mathbf{G}^K\|_{H^\infty} \frac{\|f(z) - Kz\|}{\|z\|} \le 1 - \frac{\alpha(\|z\|)}{\|z\|} \quad \forall\, z \in \mathbb{R}^p,\ z \neq 0, \tag{3.2}$$

then the zero equilibrium pair of (1.1) *is strongly iISS. Moreover, the iISS estimate* (2.2) *holds with $\gamma_2(z) = az$, for some $a > 0$.*

In the context of small-signal ISS, it is important and interesting to compute or estimate the largest possible value for the input threshold R appearing in (2.3) in terms of the data of the Lur'e system, including the comparison function α in (3.1) and (3.2). Whilst this is difficult in general (although see [7, Theorems 1 and 2]), we give a class of examples where the "optimal" value for R turns out to be equal to $\lim_{s\to\infty} \alpha(s)$.

Example 3.3. The forced scalar differential equation

$$\dot{x} = -x + f(x) + v, \quad x(0) = x^0\,, \tag{3.3}$$

where $f : \mathbb{R} \to \mathbb{R}$, $A = -1$ and $B = C = 1$ is a special case of (1.1). Here $\mathbf{G}(s) = 1/(s + 1)$ and so, trivially, $\|\mathbf{G}\|_{H^\infty} = \mathbf{G}(0) = 1$. Hence, $\mathbb{B}_\mathbb{C}(K, r) \subseteq \mathbb{S}_\mathbb{C}(\mathbf{G})$ with $K = 0$ and $r = 1$. We shall revisit this example throughout the paper in various contexts.

Let $\phi \in \mathcal{K}\backslash\mathcal{K}_\infty$ be such that $\phi(s) < s$ for all $s > 0$ and fix $f(x) = x - \mathrm{sign}(x)\phi(|x|)$ for all $x \in \mathbb{R}$. Then

$$|f(x)| = |x| - \phi(|x|), \quad \forall\, x \in \mathbb{R}\,,$$

and so, choosing $\alpha = \phi$, condition (3.1) and the small-gain inequality (3.2) are satisfied. By Theorem 3.1 (or Corollary 3.2), the zero equilibrium pair is strongly iISS, and so, in particular, has the small signal ISS property. The closed-loop feedback system may be written as

$$\dot{x} = -\mathrm{sign}(x)\phi(|x|) + v\,,$$

and, setting $R^* := \lim_{s\to\infty} \alpha(s) = \lim_{s\to\infty} \phi(s)$, it is clear that inputs v with $\|v\|_{L^\infty} < R^*$ lead to bounded state trajectories. Furthermore, we claim that, for every $R \in (0, R^*)$, there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that (2.3) holds (with $x^* = v^* = 0$). To prove this, it is sufficient to show that, for any $R \in (0, R^*)$, there exist ψ and λ in $\mathcal{K}_\infty$ such that, for every solution x generated by an input v with $\|v\|_{L^\infty} \le R$, we have

$$\frac{1}{2}\frac{d}{dt}x^2(t) \le -\psi(|x(t)|) + \lambda(\|v\|_{L^\infty}) \quad \text{a.e. } t \ge 0. \tag{3.4}$$

Choose $L > 0$ such that $\phi(s) > R$ for all $s \ge L$, set $\psi_1(s) := s(\phi(s) - R)$ for $s \ge L$, let $\psi_0 : [0, L] \to \mathbb{R}_+$ be strictly increasing such that $\psi_0(L) = \psi_1(L)$ and $\psi_0(s) \le s\phi(s)$ and define a $\mathcal{K}_\infty$-function ψ_2 by

$$\psi_2(s) := \begin{cases} \psi_0(s), & s \in [0, L) \\ \psi_1(s), & s \ge L\,. \end{cases}$$

Then, for all for v such that $\|v\|_{L^\infty} \le R$,

$$\frac{1}{2}\frac{d}{dt}x^2(t) = x(t)\dot{x}(t) \le -|x(t)|\phi(|x(t)|) + |x(t)||v(t)| \le -\psi_2(|x(t)|) + L|v(t)| \quad \text{a.e. } t \ge 0.$$

Consequently, with the choices $\psi = \psi_2$ and $\lambda(s) = Ls$, inequality (3.4) holds for all inputs v with $\|v\|_{L^\infty} \le R$.

Finally, we claim that the constant input $v(t) \equiv R^*$ generates divergent state trajectories: indeed, defining the function F by

$$F(z) := \int_0^z \frac{d\xi}{R^* - \mathrm{sign}(\xi)\phi(|\xi|)} \quad \forall\, z \in \mathbb{R},$$

and invoking separation of variables, we see that the solution x with initial value equal to 0 satisfies $F(x(t)) = t$ for all $t \ge 0$, implying that $x(t) \to \infty$ as $t \to \infty$.

To conclude the example, we note that any complex feedback gain k with $\mathrm{Re}\, k \ge 1 = r$ destabilizes $\mathbf{G}$ and that, with our choice $\alpha = \phi$, conditions (3.1) and (3.2) hold with equality. This means that both r and α are as large as they can be (for fixed $K = 0$), and thus, in this sense, the scenario considered in the example is "extreme". ◇

The proof of Theorem 3.1 is facilitated by the following technical lemma.

Lemma 3.4. *Given $\kappa \in \mathcal{K}\backslash\mathcal{K}_\infty$, there exist $R, d > 0$ such that*

$$2zw \le 2z\kappa(z) + dw \quad \forall\, z \ge 0,\ \forall\, w \in [0, R]\,. \tag{3.5}$$

Proof. Since $\kappa \in \mathcal{K}\backslash\mathcal{K}_\infty$, it follows that

$$R_0 := \sup_{s\ge 0} \kappa(s) = \lim_{s\to\infty} \kappa(s) > 0\,,$$

is finite. Furthermore, $\kappa : \mathbb{R}_+ \to [0, R_0)$ is bijective, with strictly increasing inverse $\kappa^{-1} : [0, R_0) \to \mathbb{R}_+$. Fix $R \in [0, R_0)$. For $z \ge 0$ and $w \in [0, R]$, we see that

$$w \le \kappa(z) \quad \Rightarrow \quad 2zw \le 2z\kappa(z)\,. \tag{3.6}$$

Similarly, for $z \geq 0$ and $w \in [0, R]$,

$$\kappa(z) < w \quad \Rightarrow \quad z \leq \kappa^{-1}(w)\,,$$

so that

$$\kappa(z) < w \quad \Rightarrow \quad 2zw \leq 2\kappa^{-1}(R)w\,. \tag{3.7}$$

Combining (3.6) and (3.7), we conclude that (3.5) holds with $d := 2\kappa^{-1}(R) > 0$. $\square$

Proof of Theorem 3.1. Trivially, if (3.1) is satisfied with $\alpha = \beta \in \mathcal{K}_\infty$, then it *a fortiori* holds for any $\alpha \in \mathcal{K}\backslash\mathcal{K}_\infty$ such that $\alpha(s) \leq \beta(s)$ for all $s \geq 0$. Therefore, without loss of generality, we may assume that $\alpha \in \mathcal{K}\backslash\mathcal{K}_\infty$ (see also the commentary following the statement of Theorem 3.1). In particular, α is bounded. As a matter of convenience, we define the function

$$F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \;\; (z, w) \mapsto Az + Bf(Cz) + w,$$

and so, the differential equation in (1.1) can be written in the form $\dot{x} = F(x, v)$.

There are two claims to prove: first, that the zero equilibrium pair of (1.1) is small-signal ISS and, second, that it is iISS.

Step I: small-signal ISS. To establish the small-signal ISS property, it suffices to find $R > 0$, a continuously differentiable $V : \mathbb{R}^n \to \mathbb{R}_+$, and $\alpha_1, \alpha_2, \gamma, \mu \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|z\|) \leq V(z) \leq \alpha_2(\|z\|) \quad \forall\, z \in \mathbb{R}^n\,, \tag{3.8}$$

and, furthermore,

$$\langle (\nabla V)(z), F(z, w)\rangle \leq -\gamma(\|z\|) + \mu(\|w\|) \quad \forall\, (z, w) \in \mathbb{R}^n \times \mathbb{R}^n \;\text{ with } \|w\| \leq R, \tag{3.9}$$

see [7, Section IV. A.]. A function V which satisfies (3.8) and (3.9) can be constructed in the same way as in the proof of [29, Theorem 3.1], provided that any references to [29, Lemma 2.4] are replaced by references to Lemma 3.4. Note that the properties (3.8) and (3.9) are reminiscent of those of an ISS Lyapunov function for (1.1), the only difference being that, for an ISS Lyapunov function, the inequality in (3.9) would have to be satisfied for all $(z, w) \in \mathbb{R}^n \times \mathbb{R}^n$.

Step II: iISS. By [2, Theorem 1, Remark II.3], the zero equilibrium pair of (1.1) is iISS if the unforced Lur'e system is 0-GAS and (1.1) is *zero-output dissipative*, meaning that there exist a continuously differentiable radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+$ with $V(0) = 0$ and $V(z) > 0$ for $z \neq 0$ and $\zeta \in \mathcal{K}$ such that

$$\langle (\nabla V)(z), F(z, w)\rangle \leq \zeta(\|w\|) \quad \forall\, (z, w) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{3.10}$$

In fact, we shall construct a V so that (3.10) is satisfied with ζ a linear function, which is important for establishing the claim that γ_2 may be chosen as a linear function.

The condition (3.1) implies that (2.5) holds, and so an application of Theorem 2.4 ensures that the unforced Lur'e system is 0-GAS. Therefore, it remains to establish both the zero-output dissipative property, and the claim regarding the functional form of γ_2 in (2.2). The inequality (3.1) implies that

$$-r^2\|z\|^2 + \|f(z) - Kz\|^2 \leq -r\|z\|\alpha(\|z\|) \quad \forall\, z \in \mathbb{R}^p\,. \tag{3.11}$$

Defining $f^K : \mathbb{R}^p \to \mathbb{R}^m$ via

$$f^K(z) := f(z) - Kz \quad \forall\, z \in \mathbb{R}^p\,, \tag{3.12}$$

we see immediately from (3.1) that

$$\|f^K(z)\|^2 \leq r^2\|z\|^2 \quad \forall\, z \in \mathbb{R}^p\,. \tag{3.13}$$

Since $\mathbf{G}^K \in H^\infty(\mathbb{C}^{p\times m})$, it follows from the stabilisability and detectability assumptions that $A^K := A + BKC$ is Hurwitz. Moreover, $\|\mathbf{G}^K\|_{H^\infty} \leq 1/r$ by Lemma 2.3. Now the triple (A^K, B, rC) is a stabilisable and detectable realization of the transfer function $r\mathbf{G}^K$ and an application of the bounded real lemma (see, for example, [16, Theorem 5.3.25, Remark 5.3.27, p. 604]) to (A^K, B, rC) shows that there exist a positive semi-definite $P = P^T \in \mathbb{R}^{n\times n}$ and $L \in \mathbb{R}^{m\times n}$ such that

$$(A^K)^T P + PA^K + r^2 C^T C = -L^T L \quad \text{and} \quad PB = -L^T \,. \tag{3.14}$$

Note that (3.14) implies that $\ker P \subseteq \ker C$. Obviously, $|\cdot|_P$ is a norm on $(\ker P)^\perp$ and so there exists $\ell > 0$ such that $\|z\| \le \ell |z|_P$ for all $z \in (\ker P)^\perp$. Hence, for all $z = z_1 + z_2 \in \mathbb{R}^n$, $z_1 \in (\ker P)^\perp$ and $z_2 \in \ker P$, we have that

$$\|Cz\| = \|Cz_1 + Cz_2\| = \|Cz_1\| \le \|C\|\|z_1\| \le \ell\|C\||z_1|_P = \ell\|C\||z_1+z_2|_P = d|z|_P \,, \tag{3.15}$$

where $d := \ell\|C\|$. Without loss of generality, we may assume that P is not the zero-matrix (because if $P = 0$, then $C = 0$, A is Hurwitz by detectability and the Lur'e system (1.1) reduces to the linear system $\dot{x} = Ax + v$ which is trivially iISS with linear γ_2 in the iISS estimate (2.2)).

Next, since A^K is Hurwitz, there exists positive definite $Q_0 = Q_0^T \in \mathbb{R}^{n\times n}$ such that

$$(A^K)^T Q_0 + Q_0 A^K = -I \,.$$

Then, it follows that

$$\begin{aligned}\langle A^K z + Bf^K(Cz), Q_0 z\rangle + \langle z, Q_0(A^K z + Bf^K(Cz))\rangle &= \langle((A^K)^T Q_0 + Q_0 A^K)z, z\rangle + 2\langle z, Q_0 B f^K(Cz)\rangle \\ &\le -\|z\|^2 + 2\|Q_0 B\|\|z\|\|f^K(Cz)\| \;\; \forall\, z \in \mathbb{R}^n.\end{aligned} \tag{3.16}$$

By using the inequality

$$2z_1 z_2 \le \rho z_1^2 + z_2^2/\rho \quad \forall\, z_1, z_2 \in \mathbb{R},\ \forall\, \rho > 0\,,$$

on the second term on the right hand side of (3.16) and defining Q as a suitable positive multiple of Q_0, we conclude that there exists $\delta > 0$ such that

$$\langle A^K z + Bf^K(Cz), Qz\rangle + \langle z, Q(A^K z + Bf^K(Cz))\rangle \le -\delta\|z\|^2 + \|f^K(Cz)\|^2 \quad \forall\, z \in \mathbb{R}^n \,. \tag{3.17}$$

By equivalence of the norms $\|\cdot\|$ and $|\cdot|_Q$, there exist $q_1, q_2 > 0$ such that

$$q_1\|z\| \le |z|_Q \le q_2\|z\| \quad \forall\, z \in \mathbb{R}^n \,. \tag{3.18}$$

Furthermore, there exists $\mu > 0$ such that

$$|z|_P \le \mu |z|_Q \quad \forall\, z \in \mathbb{R}^n \,.$$

We require some further notation and preliminary estimates. To that end, set

$$j_0 := \delta/(2q_2^2) > 0, \quad j_1 := \min\{1, \sqrt{j_0}/r\} > 0, \quad j_2 := \min\{1, \sqrt{j_0}/(dr)\} > 0\,, \tag{3.19}$$

and define $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\theta(s) = \begin{cases} 0 & s = 0 \\ \dfrac{1}{s}\displaystyle\int_0^s \alpha(j_1\tau)\,d\tau & s > 0\,. \end{cases}$$

The function θ is positive and continuously differentiable on $(0,\infty)$ and satisfies

$$0 \le \theta(s) \le \alpha(j_1 s) \le \alpha(s) \quad \forall\, s \ge 0\,, \tag{3.20}$$

and

$$\theta(s) + s\theta'(s) = \frac{d}{ds}(s\theta(s)) = \alpha(j_1 s) \quad \forall\, s > 0\,. \tag{3.21}$$

The combination of (3.20) and (3.21) yields that $\theta'(s) \ge 0$ for all $s > 0$, and so θ is non-decreasing. Choose $\nu > 0$ sufficiently small so that

$$\min\left\{s^2, \frac{1}{1+s}\right\} = s^2 \quad \forall\, s \in \left[0, \frac{r^3\nu^2}{j_0}\right]. \tag{3.22}$$

Since the function $s \mapsto s\theta(j_2 s)$ is in $\mathcal{K}_\infty$ it is clear that there exists $\varepsilon_1 > 0$ such that

$$\frac{\varepsilon_1 r|C|_Q\mu\|\alpha\|_{L^\infty}}{q_1}s^3 \leq 1 \quad \text{for all } s \geq 0 \text{ such that } s\theta(j_2 s) \leq \tfrac{r^3\nu^2}{j_0}. \tag{3.23}$$

Choose $\varepsilon_2 > 0$ sufficiently small so that

$$\varepsilon_2 \min\Big\{s^2\theta^2(j_2 s), \frac{1}{1+s\theta(j_2 s)}\Big\}s\alpha(j_1 j_2 s) \leq \frac{q_1\alpha(\nu)\alpha(j_1\nu/d)}{r|C|_Q\mu} \quad \forall\, s \geq 0. \tag{3.24}$$

Such a choice is possible owing to the boundedness of α and the fact that θ is non-decreasing. Set $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ and define $k, h : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$k(s) = \varepsilon \min\big\{s^2, \frac{1}{1+s}\big\} \quad \text{and} \quad h(s) = \int_0^s k(\tau)\, d\tau \quad \text{for all } s \geq 0.$$

Define the function $V : \mathbb{R}^n \to \mathbb{R}_+$ by

$$V(z) := 2|z|_P\theta(|z|_P) + 2h\big(|z|_Q\theta(j_2|z|_Q)\big) \quad \forall\, z \in \mathbb{R}^n.$$

This function is continuously differentiable and satisfies $V(0) = 0$ and $V(z) > 0$ for $z \neq 0$. Moreover, by setting

$$V_P(\xi) := 2|\xi|_P\theta(|\xi|_P) \quad \text{and} \quad V_Q(\xi) := 2h\big(|\xi|_Q\theta(j_2|\xi|_Q)\big)\,,$$

it is clear that

$$V(\xi) \geq V_Q(\xi) = 2h\big(|\xi|_Q\theta(j_2|\xi|_Q)\big)\,,$$

and so V is radially unbounded as

$$h(s) = \int_0^s k(\tau)\, d\tau \geq \varepsilon \int_1^s \frac{1}{1+\tau}\, d\tau \quad \forall\, s \geq 1$$

diverges as $s \to \infty$.

We will show that there exists $c > 0$ such that

$$\langle(\nabla V)(z), F(z,w)\rangle \leq c\|w\| \quad \forall\,(z,w) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{3.25}$$

A straightforward calculation using (3.21) shows that

$$\begin{aligned}\langle(\nabla V_P)(z), F(z,w)\rangle &= \frac{2\,\langle Pz, F(z,w)\rangle}{|z|_P}\alpha(j_1|z|_P)\\ &= 2\,\langle A^K z + Bf^K(Cz) + w, Pz\rangle\frac{\alpha(j_1|z|_P)}{|z|_P} \quad \forall\,(z,w) \in \mathbb{R}^n \times \mathbb{R}^n \text{ with } z \notin \ker P,\end{aligned} \tag{3.26}$$

where we have used that $F(z,w) = A^K z + Bf^K(Cz) + w$. Using the Cauchy-Schwarz inequality, we majorize the terms in (3.26) that contain w by

$$2\,\langle w, Pz\rangle\frac{\alpha(j_1|z|_P)}{|z|_P} \leq 2\alpha(j_1|z|_P)|w|_P \leq 2\sqrt{\|P\|}\|\alpha\|_{L^\infty}\|w\| = c_1\|w\|\,, \tag{3.27}$$

where $c_1 := 2\sqrt{\|P\|}\|\alpha\|_{L^\infty}$. Next, we estimate the terms in (3.26) which do not contain w. To this end, using (3.14), we compute

$$\begin{aligned}2\langle A^K z + Bf^K(Cz), Pz\rangle &= \langle(((A^K)^T P + PA^K)z, z\rangle + 2\langle PBf^K(Cz), z\rangle\\ &= -\langle(r^2C^TC + L^TL)z, z\rangle - 2\langle f^K(Cz), Lz\rangle\\ &= -r^2\|Cz\|^2 - \|Lz + f^K(Cz)\|^2 + \|f^K(Cz)\|^2\\ &\leq -r^2\|Cz\|^2 + \|f^K(Cz)\|^2\\ &\leq -r\|Cz\|\alpha(\|Cz\|) \quad \forall\, z \in \mathbb{R}^n,\end{aligned} \tag{3.28}$$

where the last inequality follows from (3.11). Since α is non-negative, we obtain from (3.28) that

$$2\langle A^K z + Bf^K(Cz), Pz\rangle \frac{\alpha(j_1|z|_P)}{|z|_P} \leq -r\|Cz\|\alpha(\|Cz\|)\frac{\alpha(j_1|z|_P)}{|z|_P} \quad \forall\, z \in \mathbb{R}^n,\ z \notin \ker P. \tag{3.29}$$

Therefore, the conjunction of (3.26), (3.27) and (3.29) gives,

$$\langle(\nabla V_P)(z), F(z,w)\rangle \leq -r\|Cz\|\alpha(\|Cz\|)\frac{\alpha(j_1|z|_P)}{|z|_P} + c_1\|w\| \quad \forall\,(z,w) \in \mathbb{R}^n \times \mathbb{R}^n,\ z \notin \ker P. \tag{3.30}$$

Next we estimate the inner product $\langle(\nabla V_Q)(z), F(z,w)\rangle$. To this end, set $k_z := k\big(|z|_Q\theta(j_2|z|_Q)\big)$ for $z \in \mathbb{R}^n$. We claim that

$$k_z r^2\|Cz\|^2\frac{\alpha(j_1j_2|z|_Q)}{|z|_Q} \leq r\|Cz\|\alpha(\|Cz\|)\frac{\alpha(j_1|z|_P)}{|z|_P} + j_0k_z|z|_Q\alpha(j_1j_2|z|_Q) \quad \forall\, z \in \mathbb{R}^n,\ z \notin \ker P\,, \tag{3.31}$$

where j_0, j_1 and j_2 are defined in (3.19). So as to avoid disrupting the flow of the present argument, we postpone the proof of (3.31) until later.

A calculation similar to that leading to (3.26) yields

$$\begin{aligned}\langle(\nabla V_Q)(z), F(z,w)\rangle &= 2h'(|z|_Q\theta(j_2|z|_Q))\langle F(z,w), Qz\rangle\frac{\alpha(j_1j_2|z|_Q)}{|z|_Q}\\ &= 2k\big(|z|_Q\theta(j_2|z|_Q)\big)\langle A^Kz + Bf^K(Cz) + w, Qz\rangle\frac{\alpha(j_1j_2|z|_Q)}{|z|_Q}\\ &\qquad \forall\,(z,w) \in \mathbb{R}^n \times \mathbb{R}^n,\ z \neq 0,\end{aligned} \tag{3.32}$$

where we have used that $h' = k$ and $F(z,w) = A^Kz + Bf^K(Cz) + w$. We estimate the terms on the right hand side of (3.32) with and without v separately. For this purpose, invoking the Cauchy-Schwarz inequality again, we see that

$$2k\big(|z|_Q\theta(j_2|z|_Q)\big)\,\langle w, Qz\rangle\frac{\alpha(j_1j_2|z|_Q)}{|z|_Q} \leq 2\|k\|_{L^\infty}\|\alpha\|_{L^\infty}|w|_Q = c_2\|w\| \quad \forall\,(z,w) \in \mathbb{R}^n \times \mathbb{R}^n,\ z \neq 0,$$

where $c_2 := 2q_2\|k\|_{L^\infty}\|\alpha\|_{L^\infty}$ and q_2 is as in (3.18). Returning to (3.32), we now use (3.13) and (3.17) to estimate that

$$\begin{aligned}2k_z\,\langle A^Kz + Bf^K(Cz), Qz\rangle\frac{\alpha(j_1j_2|z|_Q)}{|z|_Q} &\leq k_z\Big(-\delta\|z\|^2 + \|f^K(Cx)\|^2\Big)\frac{\alpha(j_1j_2|z|_Q)}{|z|_Q}\\ &\leq k_z\Big(-\delta\|z\|^2 + r^2\|Cz\|^2\Big)\frac{\alpha(j_1j_2|z|_Q)}{|z|_Q} \quad \forall\, z \in \mathbb{R}^n,\ z \neq 0,\end{aligned}$$

and so,

$$\langle(\nabla V_Q)(z), F(z,w)\rangle \leq k_z\Big(-\delta\|z\|^2 + r^2\|Cz\|^2\Big)\frac{\alpha(j_1j_2|z|_Q)}{|z|_Q} + c_2\|w\| \quad \forall\,(z,w) \in \mathbb{R}^n \times \mathbb{R}^n,\ z \neq 0. \tag{3.33}$$

By choice of the constant $j_0 > 0$ we have that

$$j_0k_z|z|_Q\alpha(j_1j_2|z|_Q) \leq \delta\|z\|^2k_z\frac{\alpha(j_1j_2|z|_Q)}{|z|_Q} \quad \forall\, z \in \mathbb{R}^n,\ z \neq 0. \tag{3.34}$$

Summing (3.30) and (3.33), and using (3.31) and (3.34), we arrive at the estimate

$$\begin{aligned}\langle(\nabla V)(z), F(z,w)\rangle &= \langle(\nabla V_P(z), F(z,w)\rangle + \langle(\nabla V_Q)(z), F(z,w)\rangle\\ &\leq (c_1 + c_2)\|w\| \quad \forall\,(z,w) \in \mathbb{R}^n \times \mathbb{R}^n,\ z \notin \ker P.\end{aligned} \tag{3.35}$$

Let now $z \in \ker P$. Choosing $z^\perp \in (\ker P)^\perp$, $z^\perp \neq 0$ (which is possible since $P \neq 0$), we have that $z_\lambda := z + \lambda z^\perp \notin \ker P$ for every non-zero $\lambda \in \mathbb{R}$. For $w \in \mathbb{R}^n$, it follows from (3.35) that

$$\langle(\nabla V)(z_\lambda), F(z_\lambda, w)\rangle \leq (c_1 + c_2)\|w\| \quad \forall\, \lambda \in \mathbb{R},\ \lambda \neq 0.$$

Letting $\lambda \to 0$ and invoking the continuity of ∇V and F, it follows that $\langle (\nabla V)(z), F(z,w)\rangle \le (c_1+c_2)\|w\|$. Consequently, (3.35) extends to all $(z,w) \in \mathbb{R}^n \times \mathbb{R}^n$, and we see that (3.25) holds with $c = c_1 + c_2$.

We proceed to show that γ_2 in (2.2) is a linear function. This follows from [39, Theorem 2.4], the fact that (3.10) holds with linear ζ and the following estimate which is trivially valid for every compact set $\Gamma \subset \mathbb{R}^n$

$$\|F(z,w)\| = \|Az + Bf(Cz) + w\| \le \mu(1 + \|w\|) \quad \forall\, (z,w) \in \Gamma \times \mathbb{R}^n,$$

where we have used (3.13) and $\mu > 0$ is a constant which depends on Γ, (A,B,C) and f.

It remains to establish (3.31), which we rewrite as

$$k_z r^2 \|Cz\|^2 \frac{\alpha(j_1 j_2 |z|_Q)}{|z|_Q} \le r\|Cz\|\alpha(\|Cz\|)\frac{\alpha(j_1|z|_P)}{|z|_P} + j_0 k_z |z|_Q^2 \frac{\alpha(j_1 j_2|z|_Q)}{|z|_Q} \quad \forall\, z \in \mathbb{R}^n,\ z \notin \ker P. \tag{3.36}$$

Recall the notation $k_z = k\big(|z|_Q\theta(j_2|z|_Q)\big)$. Since (3.36) clearly holds if $Cz = 0$, we assume that $Cz \neq 0$ and consider two exhaustive cases.

Case 1: $\|Cz\| \ge \nu$, where $\nu > 0$ is as in (3.22). In this case it follows from (3.15) that

$$\frac{\nu}{d} \le |z|_P, \tag{3.37}$$

and so, we may estimate

$$\begin{aligned} k\big(|z|_Q\theta(j_2|z|_Q)\big) r\|Cz\| \frac{\alpha(j_1 j_2|z|_Q)}{|z|_Q}|z|_P &\le \frac{r|C|_Q\mu}{q_1} k\big(|z|_Q\theta(j_2|z|_Q)\big)|z|_Q \alpha(j_1 j_2|z|_Q) \\ &\le \alpha(\nu)\alpha(j_1\nu/d) \quad \text{by (3.24)}, \\ &\le \alpha(\|Cz\|)\alpha(j_1|z|_P)\,, \end{aligned} \tag{3.38}$$

where we have used (3.37) and that α is increasing. Dividing both sides of (3.38) by $|z|_P > 0$ and multiplying by $r\|Cz\|$ gives (3.36), as required.

Case 2: $\|Cz\| < \nu$. We notice that the first and third terms in (3.36) have nonzero coefficients in common, and so if

$$r^2\|Cz\|^2 \le j_0|z|_Q^2\,,$$

then (3.36) holds. We, therefore, assume that $j_0|z|_Q^2 \le r^2\|Cz\|^2$. Hence, $j_0|z|_Q^2 \le r^2\|Cz\|^2 \le r^2\nu^2$, and so,

$$|z|_Q\theta(j_2|z|_Q) \le |z|_Q\alpha(j_1 j_2|z|_Q) \le r|z|_Q^2 \le \frac{r^3\nu^2}{j_0}\,, \tag{3.39}$$

where we have used (3.20), that α is increasing with $\alpha(s) \le rs$, and that $j_1, j_2 \le 1$. Consequently, in light of (3.22), for such $|z|_Q$,

$$k\big(|z|_Q\theta(j_2|z|_Q)\big) = \varepsilon|z|_Q^2\theta^2(j_2|z|_Q) \le \varepsilon_1|z|_Q^2\theta^2(j_2|z|_Q)\,. \tag{3.40}$$

We also require the following estimates: first,

$$\theta(j_2|z|_Q) \le \alpha(j_2 j_1|z|_Q) \le \alpha(j_1|z|_Q) \le \alpha(j_1(r/\sqrt{j_0})\|Cz\|) \le \alpha(\|Cz\|)\,, \tag{3.41}$$

by (3.19) and (3.20) and as α increasing. Second,

$$\theta(j_2|z|_Q) \le \alpha(j_1 j_2|z|_Q) \le \alpha(j_1 j_2(r/\sqrt{j_0})\|Cz\|) \le \alpha(j_1(j_2 rd/\sqrt{j_0})|z|_P) \le \alpha(j_1|z|_P)\,, \tag{3.42}$$

by (3.15), (3.19) and (3.20).

Appealing to (3.39)–(3.42), we now have the following upper bounds

$$\begin{aligned} k\big(|z|_Q\theta(j_2\|z\|_Q)\big) r\|Cz\| \frac{\alpha(j_1 j_2|z|_Q)}{|z|_Q}|z|_P &\le \varepsilon_1|z|_Q^2\theta^2(j_2\|z\|_Q) r\|Cz\| \frac{\alpha(j_1 j_2|z|_Q)}{|z|_Q}|z|_P \\ &\le \varepsilon_1 r|z|_Q^2\|Cz\|\alpha(\|Cz\|)\alpha(j_1|z|_P)\frac{\alpha(j_1 j_2|z|_Q)}{|z|_Q}|z|_P \\ &\le \frac{\varepsilon_1 r|C|_Q\mu\|\alpha\|_{L^\infty}}{q_1}|z|_Q^3\alpha(\|Cz\|)\alpha(j_1|z|_P) \\ &\le \alpha(\|Cz\|)\alpha(j_1|z|_P)\,, \end{aligned} \tag{3.43}$$

where the final inequality above follows from the definition of ε_1 in (3.23) and crucially uses (3.39). Dividing both sides of (3.43) by $|z|_P > 0$ and multiplying by $r\|Cz\|$ gives (3.36), completing the proof. $\square$

We note that although the model data A, B, C, K and f in Theorem 3.1 are assumed to be real, a key hypothesis is that every feedback gain in the complex ball $\mathbb{B}_{\mathbb{C}}(K, r)$ is stabilising for $\mathbf{G}$. Our next result shows that the complex ball condition may be weakened to a real ball condition, provided a suitable additional assumption is satisfied. We say that a proper rational matrix $\mathbf{H} \in H^\infty(\mathbb{C}^{p\times m})$ has the *real supremum value property* if there exists $s_* \in \{s \in \mathbb{C} : \operatorname{Re} s \geq 0\} \cup \{\infty\}$ such that

$$\|\mathbf{H}\|_{H^\infty} = \|\mathbf{H}(s_*)\| \quad \text{and} \quad \mathbf{H}(s_*) \in \mathbb{R}^{p\times m},$$

where $\mathbf{H}(\infty) := \lim_{|s|\to\infty} \mathbf{H}(s)$. We are now in position to state the following corollary to Theorem 3.1.

Corollary 3.5. *Assume that the triple (A, B, C) is stabilisable and detectable, and that $f : \mathbb{R}^p \to \mathbb{R}^m$ is locally Lipschitz. Assume further that $r > 0$ and $K \in \mathbb{R}^{m\times p}$ are such that $\mathbb{B}_{\mathbb{R}}(K, r) \subseteq \mathbb{S}_{\mathbb{R}}(\mathbf{G})$ and $\mathbf{G}^K$ has the real supremum value property. If there exists $\alpha \in \mathcal{K}$ such that (3.1) holds, then the zero equilibrium pair of (1.1) is strongly iISS and the iISS estimate (2.2) holds with $\gamma_2(z) = az$, for some $a > 0$.*

Proof. We claim that

$$\|\mathbf{G}^K\|_{H^\infty} \leq 1/r\,. \tag{3.44}$$

If (3.44) holds, then Lemma 2.3 yields that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, and the corollary now follows from Theorem 3.1.

It remains to prove (3.44). Seeking a contradiction, suppose that $\|\mathbf{G}^K\|_{H^\infty} > 1/r$. By the real supremum value property, there exists $s_* \in \{s \in \mathbb{C} : \operatorname{Re} s \geq 0\} \cup \{\infty\}$ such that $\mathbf{G}^K(s_*) \in \mathbb{R}^{p\times m}$ and

$$\|\mathbf{G}^K(s_*)\| = \|\mathbf{G}^K\|_{H^\infty} > \frac{1}{r}\,.$$

Note that here $\|\mathbf{G}^K(s_*)\|$ is the 2-norm induced norm of $\mathbf{G}^K(s_*)$ as an operator from $\mathbb{C}^m$ to $\mathbb{C}^p$. Since, for $R \in \mathbb{R}^{p\times m}$, the real and complex operator norms induced by the 2-norm coincide [37], that is,

$$\sup_{x\in\mathbb{R}^m,\, x\neq 0} \frac{\|Rx\|}{\|x\|} = \sup_{z\in\mathbb{C}^m,\, z\neq 0} \frac{\|Rz\|}{\|z\|},$$

there exists $u \in \mathbb{R}^m$ such that $\|u\| = 1$ and

$$\|\mathbf{G}^K(s_*)u\| = \|\mathbf{G}^K(s_*)\| = \|\mathbf{G}^K\|_{H^\infty}.$$

Set $\gamma := \|\mathbf{G}^K(s_*)u\| = \|\mathbf{G}^K\|_{H^\infty} > 1/r$, and define $w := (1/\gamma)\mathbf{G}^K(s_*)u \in \mathbb{R}^p$ and $L \in \mathbb{R}^{m\times p}$ by

$$Ly := \frac{\langle y, w\rangle}{\gamma} u \quad \forall\, y \in \mathbb{R}^p\,.$$

We see that

$$(I - L\mathbf{G}^K(s_*))u = u - L\mathbf{G}^K(s_*)u = 0\,,$$

so $L \notin \mathbb{S}_{\mathbb{R}}(\mathbf{G}^K)$, and, moreover,

$$\|Ly\| = \frac{|\langle y, w\rangle|}{\gamma}\|u\| \leq \frac{\|y\|}{\gamma} < r\|y\| \quad \forall\, y \in \mathbb{R}^p\,,$$

from which we deduce that $L \in \mathbb{B}_{\mathbb{R}}(0, r)$. We conclude that $\mathbb{B}_{\mathbb{R}}(0, r) \not\subseteq \mathbb{S}_{\mathbb{R}}(\mathbf{G}^K)$, showing that $\mathbb{B}_{\mathbb{R}}(K, r) \not\subseteq \mathbb{S}_{\mathbb{R}}(\mathbf{G})$, and thus yielding a contradiction. Consequently, (3.44) holds. $\square$

Below we provide some classes of systems for which the real supremum value assumption of Corollary 3.5 is satisfied.

Example 3.6. The following examples have the property that

$$\|\mathbf{G}^K\|_{H^\infty} = \|\mathbf{G}^K(0)\|\,. \tag{3.45}$$

Since $\mathbf{G}^K(0)$ is real, it follows that $\mathbf{G}^K$ satisfies the real supremum value property.

(1) Recall that a square matrix is called Metzler if every off-diagonal entry is nonnegative (see, for example, [5, Ch. 6]). If (A, B, C) and K are such that $A + BKC$ is Hurwitz and Metzler and B and C are nonnegative, that is $B \in \mathbb{R}_+^{n\times m}$ and $C \in \mathbb{R}_+^{p\times n}$, then $\mathbf{G}^K$ is the transfer function of a stable positive system, and so satisfies (3.45) by, for instance, [17, Theorem 5].

(2) If K and (A, B, C) are such that $(A + BKC, B, C)$ is a so-called symmetric system, meaning $A + BKC = (A + BKC)^T$ and $C = B^T$, then (3.45) holds, see [24, Remark 4.1 2.].

(3) Let $(\tilde{A}, \tilde{b}, \tilde{c}^T) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^{1\times n}$ with Hurwitz $\tilde{A}$, transfer function $\mathbf{H}$, and let g be a real parameter. Consider the integral control feedback system

$$\dot{x} = \tilde{A}x + \tilde{b}u, \quad y = \tilde{c}^T x, \quad \dot{u} = v - gy\,,$$

where v is an external input, which may be described by the triple (A_g, b, c^T) (with input v and output y) given by

$$A_g := \begin{pmatrix} \tilde{A} & \tilde{b} \\ -g\tilde{c}^T & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c^T := \begin{pmatrix} \tilde{c}^T & 0 \end{pmatrix}.$$

A routine calculation shows that the transfer function $\mathbf{G}_g$ of the triple (A_g, b, c^T) is given by

$$\mathbf{G}_g(s) = \frac{\mathbf{H}(s)}{s + g\mathbf{H}(s)}\,.$$

It follows from [26, Proposition 3.9] that if $\mathbf{H}(0) \neq 0$, then there exists $g^* > 0$ such that, for all g with $g\mathbf{H}(0) > 0$ and $0 < |g| < g^*$,

$$\|\mathbf{G}_g\|_{H^\infty} = \frac{1}{|g|} = |\mathbf{G}_g(0)|\,.$$

In particular, (3.45) holds with $\mathbf{G} = \mathbf{G}_g$ and $K = 0$. ◇

The next result is a circle criterion for strong iISS. Recall that a square proper rational matrix-valued function $s \mapsto \mathbf{H}(s)$ of a complex variable s is said to be *positive real* if for every $s \in \mathbb{C}_0$, which is not a pole of $\mathbf{H}$, the matrix $[\mathbf{H}(s)]^* + \mathbf{H}(s)$ is positive semi-definite. Here the superscript $*$ denotes the Hermitian transpose. It follows from [11, Proposition 3.3] that if $\mathbf{H}$ is positive real, then $\mathbf{H}$ is holomorphic on $\mathbb{C}_0$.

Corollary 3.7. *Assume that the triple (A, B, C) is stabilisable and detectable, and that $f : \mathbb{R}^p \to \mathbb{R}^m$ is locally Lipschitz. Assume further that $K_1, K_2 \in \mathbb{R}^{m\times p}$ are such that $(I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ is positive real. If f additionally satisfies*

$$\langle f(z) - K_1 z, f(z) - K_2 z\rangle \leq -\|z\|\alpha(\|z\|) \quad \forall\, z \in \mathbb{R}^p\,, \tag{3.46}$$

for some $\alpha \in \mathcal{K}$, then the zero equilibrium pair of (1.1) *is strongly iISS and the iISS estimate* (2.2) *holds with $\gamma_2(z) = az$, for some $a > 0$.*

We note that if, in (3.46), the function α is in $\mathcal{K}_\infty$, then the zero equilibrium pair is ISS, see [29, Corollary 3.10].

Proof of Corollary 3.7: The proof is essentially the same as that of [29, Corollary 3.10]: the only difference is that Theorem 3.1 takes over the role of [29, Theorem 3.2]. Note that, in the current context, the function β defined in [29, equation (3.36)] belongs to $\mathcal{K}\backslash\mathcal{K}_\infty$ and not to $\mathcal{K}_\infty$ (as is the case in [29]). □

Example 3.8. An electrical circuit is considered in [19, p.34], consisting of a current source with current ν, an inductor with inductance $\ell > 0$, a capacitor with capacitance $c > 0$, and an element, such as a filament lamp, which has a nonlinear current-voltage characteristic given by the continuous function

$h : \mathbb{R} \to \mathbb{R}$. Letting x_1 and x_2 denote the current through the inductor and voltage across the capacitor, respectively, we obtain from Kirchoff's laws the Lur'e system (1.1) with

$$v = \begin{pmatrix} 0 \\ \nu/c \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1/\ell \\ -1/c & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = C^T, \tag{3.47}$$

and $f : \mathbb{R} \to \mathbb{R}$ given by $f(y) := -h(y)/c$. We assume that h satisfies

(h.1) $zh(z) \geq 0$ for all $z \in \mathbb{R}$;

(h.2) there exist $\mu > 0$ and $\beta \in \mathcal{K}$ such that $\beta(|z|) \leq |h(z)| \leq \mu|z|$ for all $z \in \mathbb{R}$.

The transfer function $\mathbf{G}(s) = C(sI - A)^{-1}B$ is given by

$$\mathbf{G}(s) = \frac{s}{s^2 + \omega} = \frac{1}{2(s - i\omega)} + \frac{1}{2(s + i\omega)}, \quad \text{where } \omega := 1/(c\ell) > 0.$$

which is positive real. Hence, we have that $(I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ is positive real when $K_1 = 0$ and for all $K_2 < 0$. Moreover, for $K_2 = -k_2$, with $k_2 > 0$ we see that

$$\begin{aligned}
\langle f(z) - K_1 z, f(z) - K_2 z \rangle &= -\frac{h(z)}{c}\left(-\frac{h(z)}{c} + k_2 z\right) \\
&= \frac{1}{c^2}h^2(z) - \frac{k_2}{c}zh(z) \\
&\leq \frac{\mu}{c^2}zh(z) - \frac{k_2}{c}zh(z) = -\frac{1}{c^2}\left(-\mu + ck_2\right)zh(z) \quad \forall\, z \in \mathbb{R},
\end{aligned}$$

by (h.1) and (h.2). Fix $k_2 > \mu/c$ and set $\varepsilon := (-\mu + ck_2)/c^2 > 0$. By (h.2), we have that

$$\langle f(z) - K_1 z, f(z) - K_2 z \rangle \leq -\varepsilon|z|\beta(|z|) \quad \forall\, z \in \mathbb{R},$$

and therefore, by Corollary 3.7 with $\alpha := \varepsilon\beta$, it follows that the zero equilibrium pair of (1.1) specified by (3.47) is strongly iISS. If, in fact, β in (h.2) satisfies the stronger condition $\beta \in \mathcal{K}_\infty$ (and so necessarily h must be unbounded), then the zero equilibrium pair of (1.1) specified by (3.47) is ISS, as follows from [29, Corollary 3.10]. ◇

Our final result of this section demonstrates that, under an additional assumption on the linear system, a well-known condition which is sufficient for 0-GAS of the unforced system (1.1) guarantees iISS of the zero equilibrium pair of (1.1).

Proposition 3.9. *Imposing the assumptions of* Theorem 2.4, *assume further that* (A, B, C) *is controllable or observable. Then the zero equilibrium pair of* (1.1) *is iISS. Moreover, the iISS estimate* (2.2) *holds with* $\gamma_2(z) = az$, *for some* $a > 0$.

By way of comparing the "nonlinear ball" conditions (3.1) and (2.5), note that the latter is equivalent to the existence of a continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\alpha(0) = 0$, $\alpha(s) > 0$ for $s > 0$ and

$$\|f(z) - Kz\| \leq r\|z\| - \alpha(\|z\|) \quad \forall\, z \in \mathbb{R}^p.$$

Proof of Proposition 3.9. By [2, Theorem 1, Remark II.3], it suffices to prove 0-GAS of the unforced system (1.1) and the zero-output dissipativity property. The 0-GAS property follows from Theorem 2.4. It remains to establish the zero-output dissipativity property. To which end, let r, K and f be as in the statement of the claim, set $A^K := A + BKC$ and let P be as in (3.14). Note that the controllability/observability assumption implies that P is positive definite (if the observability assumption holds, then this follows from [16, Remark 5.6.24] and the controllable case can be dealt with by a duality argument). Set

$$k(s) := \min\left\{s, \frac{1}{1+s}\right\} \quad \text{and} \quad h(s) := \int_0^s k(\tau)\, d\tau \quad \forall\, s \geq 0,$$

and define $V : \mathbb{R}^n \to \mathbb{R}_+$ by

$$V(z) := h\big(\langle z, Pz\rangle\big) \quad \forall\, z \in \mathbb{R}^n\,.$$

It is easily seen that V is continuously differentiable, radially unbounded, $V(0) = 0$ and $V(z) > 0$ for all $z \neq 0$ (where we have used that P is positive definite). Defining $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(z, w) = Az + Bf(Cz) + w = A^K z + Bf^K(Cz) + w \quad \forall\,(z, w) \in \mathbb{R}^n \times \mathbb{R}^n,$$

where f^K is given by (3.12), a routine calculation similar to that leading to (3.28) shows that

$$\begin{aligned}\langle(\nabla V)(z), F(z, w)\rangle &= 2h'(\langle z, Pz\rangle)\langle F(z, w), Pz\rangle \\ &\leq k(\langle z, Pz\rangle)\big[-r^2\|Cz\|^2 + \|f^K(Cz)\|^2\big] + 2k(\langle z, Pz\rangle)\langle Pz, w\rangle \quad \forall\,(z, w) \in \mathbb{R}^n \times \mathbb{R}^n.\end{aligned}$$

Invoking (3.13), we obtain

$$\langle(\nabla V)(z), F(z, w)\rangle \leq 2k(|z|_P^2)|z|_P|w|_P \leq c\|w\| \quad \forall\,(z, w) \in \mathbb{R}^n \times \mathbb{R}^n,$$

for some suitable constant $c > 0$, where we have used that the function $s \mapsto k(s^2)s$ is bounded. We have now established that the zero-output dissipativity property holds.

The argument that γ_2 in (2.2) may be chosen as a linear function is the same as that in the proof of Theorem 3.1. ☐

We do not know whether or not the controllability/observability assumption in Proposition 3.9 is necessary. What is clear, as the following example demonstrates, is that the "nonlinear ball" condition (2.5), which, by Proposition 3.9, is sufficient for iISS, is *not* sufficient for strong iISS.

Example 3.10. Consider the scalar Lur'e system (3.3) from Example 3.3. The triple $(A, B, C) = (-1, 1, 1)$ is trivially controllable and observable. Let f in (3.3) be given by

$$f : \mathbb{R} \to \mathbb{R}, \quad f(z) = z - z\mathrm{e}^{-z^2} \quad \forall\, z \in \mathbb{R}\,.$$

Obviously, with the choices $K = 0$ and $r = 1$, there does not exist $\alpha \in \mathcal{K}$ such that (3.1) is satisfied, but (2.5) does hold. In particular, by Proposition 3.9, the zero equilibrium pair is iISS.

Let $\varepsilon > 0$ be arbitrary and let $x^0 > 1/\sqrt{2}$ be sufficiently large so that $x^0\mathrm{e}^{-(x^0)^2} \leq \varepsilon/2$. Since $z \mapsto z\mathrm{e}^{-z^2}$ is decreasing for $z \geq 1/\sqrt{2}$, we have that the solution $x(t) := x(t; x^0, \varepsilon)$ satisfies

$$\dot{x}(t) = -x(t)\mathrm{e}^{-x(t)^2} + \varepsilon \geq \varepsilon/2 \quad \forall\, t \geq 0.$$

Hence $x(t) = x(t; x^0, \varepsilon) \to \infty$ as $t \to \infty$. As $\varepsilon > 0$ was arbitrary, we conclude that the small-signal ISS property does not hold, and so the zero equilibrium pair is not strongly iISS. ◇

4 Strong iISS for Lur'e systems with saturating nonlinearities

The present section is motivated by [29, Proposition 3.4] which, under mild assumptions, states that if, in the Lur'e system (1.1) the matrix A is unstable and f is bounded, then the $(0, 0)$ equilibrium *cannot* be ISS. The situation wherein A is unstable and f is bounded arises, for instance, in the stabilisation of linear systems (2.1a) by saturated, static state- or output-feedback. The main result in this section provides sufficient conditions for strong iISS of the zero equilibrium pair of forced Lur'e systems for a given class of nonlinearities which seeks to capture the notion of saturation.

To be specific, and based in part on the saturation functions considered in [8], we consider the class $\mathcal{F}$ of locally Lipschitz functions $f : \mathbb{R}^m \to \mathbb{R}^m$ which have the following two properties:

($\mathcal{F}$.1) $\|f(w)\|^2 \leq \langle f(w), w\rangle \quad \forall\, w \in \mathbb{R}^m$;

($\mathcal{F}$.2) there exist $\delta > 0$ and $\gamma > 0$ such that

$$\|f(w)\| = \|w\| \;\; \forall\, w \in \mathbb{R}^m,\; \|w\| \leq \delta \quad \text{and} \quad \langle f(w), w\rangle \geq \gamma\|w\| \;\; \forall\, w \in \mathbb{R}^m,\; \|w\| > \delta.$$

Note that if $(\mathcal{F}.1)$ holds, then $f(0) = 0$.

Before stating our main result of the section we provide two examples of functions which belong to $\mathcal{F}$.

Example 4.1. (1) For $\delta > 0$, we define $\theta : \mathbb{R}^m \to \mathbb{R}^m$ by

$$\theta(w) := \begin{cases} w & \|w\| \leq \delta \\ \delta \dfrac{w}{\|w\|} & \|w\| > \delta \,. \end{cases}$$

We claim that $\theta \in \mathcal{F}$ for every $\delta > 0$. Since $\theta(w) = a(\|w\|)w$ with $a(s) = 1$ if $s \leq \delta$ and $a(s) \leq 1$ if $s > \delta$, it is clear that $(\mathcal{F}.1)$ and the first condition in $(\mathcal{F}.2)$ hold. To verify the second condition in $(\mathcal{F}.2)$, we check that

$$\langle \theta(w), w \rangle = \left\langle \delta \frac{w}{\|w\|}, w \right\rangle = \delta \|w\| \quad \forall\, w \in \mathbb{R}^m,\ \|w\| > \delta \,,$$

and so the desired inequality holds for any $\gamma \in (0, \delta]$.

(2) For $a, b > 0$, we define $\zeta_{a,b} : \mathbb{R} \to \mathbb{R}$ by

$$\zeta_{a,b}(y) = \begin{cases} -a & y < -a \\ \ \ y & -a \leq y \leq b \\ \ \ b & y > b \,, \end{cases} \tag{4.1}$$

and define the diagonal saturation function $\phi : \mathbb{R}^m \to \mathbb{R}^m$ by

$$(\phi(w))_i = \zeta_{a_i, b_i}(w_i) \quad \forall\, w \in \mathbb{R}^m,\ \forall\, i \in \{1, 2, \ldots, m\} \,,$$

for some $a_i, b_i > 0$. We claim that $\phi \in \mathcal{F}$. Applying the readily established scalar inequalities

$$\zeta_{a,b}(z)^2 \leq \zeta_{a,b}(z) z \quad \forall\, z \in \mathbb{R} \,,$$

componentwise gives $(\mathcal{F}.1)$. Setting $\delta_i := \min\big\{a_i, b_i\big\}$, simple considerations yield that there exists positive ε_i such that

$$\zeta_{a_i, b_i}(z) z \geq \varepsilon_i |z| \quad \forall\, z \in \mathbb{R},\ |z| \geq \delta_i, \text{ for each } i \in \{1, 2, \ldots, m\}. \tag{4.2}$$

The δ_i are such that $\zeta_{a_i, b_i}(z) = z$ for $z \in [-\delta_i, \delta_i]$ for every $i \in \{1, 2, \ldots, m\}$. Defining $\delta := \min_i \delta_i > 0$, we have that

$$\phi(w) = w \quad \forall\, w \in \mathbb{R}^m,\ \|w\|_\infty \leq \delta \,,$$

where $\|w\|_\infty$ denotes the maximum-norm of w. Therefore, if $\|w\| \leq \delta$, then $\|w\|_\infty \leq \|w\| \leq \delta$, and for such w we have that $\phi(w) = w$ which gives the first equality in $(\mathcal{F}.2)$. Setting $\varepsilon := \min_i \varepsilon_i > 0$, suppose that $\|w\| \geq \delta$, and so $\|w\|_\infty = |w_j| \geq \delta/\sqrt{m}$, for some $j \in \{1, 2, \ldots, m\}$. Thus, invoking (4.2) that, we see that

$$\begin{aligned} \langle \phi(w), w \rangle &= \sum_{i=1}^m \zeta_{a_i, b_i}(w_i) w_i \geq \zeta_{a_j, b_j}(w_j) w_j \geq \min\big\{w_j^2, \varepsilon_j |w_j|\big\} \\ &\geq \min\left\{\frac{\delta}{\sqrt{m}}, \varepsilon\right\} |w_j| = \min\left\{\frac{\delta}{\sqrt{m}}, \varepsilon\right\} \|w\|_\infty \\ &\geq \gamma \|w\| \,, \end{aligned}$$

where $\gamma > 0$ is a suitable constant. We deduce that the inequality in $(\mathcal{F}.2)$ holds.

The functions in (1) and (2) coincide in the special case wherein $m = p = 1$ (single-input single-output) and $a = b = \delta$. $\diamond$

The main result of this section is stated and proven next. It provides conditions under which the zero equilibrium pair of the feedback interconnection of the linear system $\dot{x} = Ax + Bu + v$, $y = Cx$ and the saturated static output feedback $u = g(Ky) = g(KCx)$ with gain $K \in \mathbb{R}^{m \times p}$ and $g \in \mathcal{F}$ is strongly iISS.

Proposition 4.2. *Assume that the triple (A, B, C) is stabilisable and detectable. If there exists $K \in \mathbb{R}^{m\times p}$ such that $\mathbb{B}_{\mathbb{C}}(K, \|K\|) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, then, for every $g \in \mathcal{F}$ the zero equilibrium pair of the forced Lur'e system*

$$\dot{x} = Ax + Bg(KCx) + v, \quad x(0) = x^0\,, \tag{4.3}$$

is strongly iISS and the iISS estimate (2.2) *holds with $\gamma_2(z) = az$, for some $a > 0$.*

Before proving Proposition 4.2, we give two examples of classes of systems where the ball condition $\mathbb{B}_{\mathbb{C}}(K, \|K\|) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ is satisfied.

Example 4.3. (1) If $\mathbf{G}$ is positive real, then for every $k > 0$, it follows from [11, Lemma 2.4] and [11, Theorem 6.4] that $\mathbb{B}_{\mathbb{C}}(-kI, k) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, and so the ball condition in Proposition 4.2 holds with $K = -kI$.

(2) Suppose that $m = p$ and the transfer function $\mathbf{G}$ of (A, B, C) has the form $\mathbf{G}(s) = \mathbf{H}(s)/s$, for $\mathbf{H} \in H^\infty(\mathbb{C}^{m\times m})$ with $\mathbf{H}(0) = \mathbf{H}(0)^*$ positive definite. Then, by [26, Proposition 3.9], there exists $k^* > 0$ such that

$$\mathbf{G}^k = \mathbf{G}(I - k\mathbf{G})^{-1} \in H^\infty \quad \forall\, k \in (-k^*, 0)\,,$$

and, moreover,

$$\|\mathbf{G}^k\|_{H^\infty} = \frac{1}{|k|} \quad \forall\, k \in (-k^*, 0)\,.$$

Consequently, by Lemma 2.3, the ball condition in Proposition 4.2 is satisfied with $K = kI$ for all $k \in (-k^*, 0)$. ◇

Proof of Proposition 4.2: We seek to apply Theorem 3.1 with $r := \|K\|$ and $f : \mathbb{R}^p \to \mathbb{R}^m$ given by $f(z) := g(Kz)$. For which purpose, it only remains to check that there exists $\alpha \in \mathcal{K}$ such that (3.1) holds. Since $g \in \mathcal{F}$, there exist $\gamma, \delta > 0$ such that properties ($\mathcal{F}$.1) and ($\mathcal{F}$.2) hold. Let $z \in \mathbb{R}^p$. We distinguish two cases.

Case 1: $\|Kz\| \le \delta$. In this case,

$$\begin{aligned} \|f(z) - Kz\|^2 = \|g(Kz) - Kz\|^2 &= \|g(Kz)\|^2 - 2\langle g(Kz), Kz\rangle + \|Kz\|^2 \\ &\le \|Kz\|^2 - \|g(Kz)\|^2 = 0\,. \end{aligned} \tag{4.4}$$

Case 2: $\|Kz\| > \delta$. We now have that

$$\begin{aligned} \|f(z) - Kz\|^2 = \|g(Kz) - Kz\|^2 &= \|g(Kz)\|^2 - 2\langle g(Kz), Kz\rangle + \|Kz\|^2 \le \|Kz\|^2 - \langle g(Kz), Kz\rangle \\ &\le \|Kz\|^2 - \gamma\|Kz\| \le (\|Kz\| - \gamma/2)^2\,. \end{aligned}$$

Without loss of generality, we may assume that $2\gamma < \delta$ and, therefore, as $\|Kz\| > \delta \ge \gamma/2$, it follows that

$$\|f(z) - Kz\| \le \big|\|Kz\| - \gamma/2\big| = \|Kz\| - \gamma/2 \le \|K\|\|z\| - \gamma/2\,. \tag{4.5}$$

Hence, in light of (4.4) and (4.5), any $\alpha \in \mathcal{K}$ with the property that $\alpha(s) \le \min\{\|K\|s, \gamma/2\}$ for all $s \ge 0$ will ensure that f satisfies (3.1), completing the proof. □

Example 4.4. We consider (4.3) with model data

$$A = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 1/3 \\ -1 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{4.6}$$

The linear data in (4.6) is a special case of a circuit example from [13]. The triple (A, B, C) is controllable and observable, and A is so-called neutrally stable, as A has simple imaginary axis eigenvalues, namely 0 and $\pm i\sqrt{5/6}$. Moreover, the associated transfer function $\mathbf{G}$ is positive real by the positive real lemma (see, for example [41, Problem 12.3]), since the positive definite matrix $P^T = P := \operatorname{diag}\begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3}$ satisfies the positive real equations

$$A^T P + PA = 0 \quad \text{and} \quad B^T P - C = 0 .$$

We consider stabilisation by saturated static output feedback with $K = -I$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ the usual diagonal saturation function

$$g(y) = \begin{pmatrix} \zeta_{2,2}(y_1) \\ \zeta_{3,3}(y_2) \end{pmatrix} \quad \forall\, y \in \mathbb{R}^2 , \tag{4.7}$$

with $\zeta_{a,b}$ given by (4.1). By Proposition 4.2, the zero equilibrium pair of (4.3) with model data (4.6) and (4.7) is strongly iISS.

For a numerical simulation, we fix $x(0) = 0$ and consider two forcing terms $v, w : \mathbb{R}_+ \to \mathbb{R}^3$ given by

$$v(t) = \nu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad w(t) = \nu B \begin{pmatrix} (t+1)^{-1} \\ 0 \end{pmatrix} \quad \forall\, t \geq 0 , \tag{4.8}$$

where $\nu > 0$ will be used as a magnitude-tuning parameter. The numerical simulations were created using MATLAB [27], using the `ode45` command to solve the differential equations numerically. Figure 4.1 plots $\|x(t)\|$ against t when subject to the forcing v for different values of $\nu > 0$. We note that, by [29, Proposition 3.4], the zero equilibrium pair in this example is not ISS. ISS with respect to small signals ensures that the L^∞-norm of x is bounded for sufficiently small values of ν, and that the L^∞-norm of x decreases as ν does. In this example, we see (at least over the time interval computed) divergence when $\nu \geq 9$.

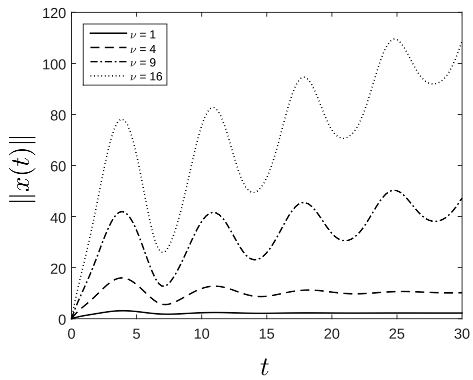

Figure 4.1: Small-signal ISS. Plot of $\|x(t)\|$ against t for different values of the tuning parameter ν when subject to the forcing v in (4.8).

Figure 4.2 plots $\|x(t)\|$ against t when subject to the forcing w for different values of $\nu > 0$. Again we see that L^∞-norm of x decreases as ν decreases. Moreover, in each simulation $\|x(t)\| \to 0$ as $t \to \infty$, as was to be expected, see Remark 2.1. $\diamond$

Remark 4.5. We claim that the conclusions of [4, Theorem 2] also follow from Theorem 3.1, which imposes weaker assumptions than [4, Theorem 2]. To see this, we note that [4, Theorem 2] assumes (in the notation of the current paper) that $A^* = -A$ and $C = B^*$. Thus, the triple (A, B, C) is positive real by, for example, [11, Corollary 7.4]. Invoking part (1) of Example 4.3, it follows that $\mathbb{B}_{\mathbb{C}}(-kI, k) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ for all $k > 0$. Moreover, [4, Theorem 2] assumes that (A, B) is controllable, and hence $(C, A) = (B^*, -A^*)$ is observable, which implies that (A, B, C) is stabilisable and detectable. Finally, it is straightforward to show that with $\sigma \in \mathscr{S}^m$ (the interested reader is referred to [4, Definition 4] for the definition of the class of saturation functions $\mathscr{S}^m$), there exist $g > 0$ and $\alpha \in \mathcal{K}\backslash\mathcal{K}_\infty$ such that

$$\| - \sigma(y) + gy\| \leq g\|y\| - \alpha(\|y\|) \quad \forall\, y \in \mathbb{R}^m ,$$

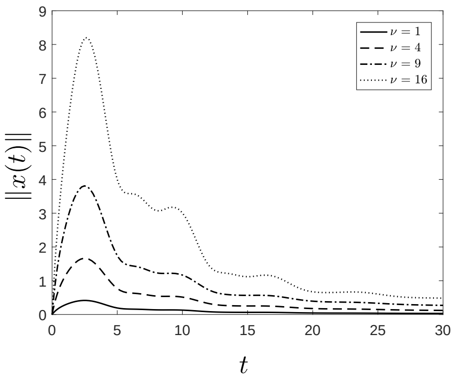

Figure 4.2: Integral ISS. Plot of $\|x(t)\|$ against t for different values of the tuning parameter ν when subject to the forcing w in (4.8).

and so strong iISS of the zero equilibrium pair of the Lur'e system with nonlinearity $f = -\sigma$ follows from Theorem 3.1 (where $K = -gI$ and $r = g$). We comment that a direct comparison of [4, Theorem 2] and Proposition 4.2 is difficult, however, because different assumptions are made about the saturation functions — indeed, neither $\mathcal{F} \subseteq \mathscr{S}^m$ or $\mathscr{S}^m \subseteq \mathcal{F}$. We note that $\mathcal{F}$ contains functions which are not "diagonal", which is not the case for $\mathscr{S}^m$. $\diamond$

5 Discussion

We conclude by placing the findings of this paper into the context given by related results in the literature, thereby providing a wider perspective. As usual, we assume that the nonlinearity f in (1.1) satisfies $f(0) = 0$. Recall that the zero equilibrium of the unforced Lur'e system (1.1) is said to be stable in the large if there exists $M > 0$ such that, for every initial condition x^0,

$$\|x(t; x^0, 0)\| \leq M\|x^0\| \quad \forall\, t \geq 0\,.$$

Furthermore, the zero equilibrium pair of (1.1) is said to be exponentially ISS if there exist $M, \gamma > 0$ such that, for all $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ and $x^0 \in \mathbb{R}^n$,

$$\|x(t; x^0, v)\| \leq M\big(\mathrm{e}^{-\gamma t}\|x^0\| + \|v\|_{L^\infty(0,t)}\big) \quad \forall\, t \geq 0\,.$$

Note that exponential ISS of the zero equilibrium pair implies that the zero equilibrium of the associated unforced system is globally exponentially stable.

The following result can be seen as variations on the complex Aizerman conjecture.

Theorem 5.1. *Given the forced Lur'e system* (1.1)*, assume that* (A, B, C) *is stabilisable and detectable, and let* $K \in \mathbb{R}^{m\times p}$*,* $r > 0$ *be such that* $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$*. Consider the following assumptions.*

(i) $\|f(z) - Kz\| \leq r\|z\| \quad \forall\, z \in \mathbb{R}^p$.

(ii) $\|f(z) - Kz\| < r\|z\| \quad \forall\, z \in \mathbb{R}^p,\ z \neq 0$.

(iii) *There exists* $\alpha \in \mathcal{K}$ *such that* $\|f(z) - Kz\| \leq r\|z\| - \alpha(\|z\|) \quad \forall\, z \in \mathbb{R}^p$.

(iv) *There exists* $\alpha \in \mathcal{K}_\infty$ *such that* $\|f(z) - Kz\| \leq r\|z\| - \alpha(\|z\|) \quad \forall\, z \in \mathbb{R}^p$.

(v) *There exists* $\rho \in (0, r)$ *such that* $\|f(z) - Kz\| \leq \rho\|z\| \quad \forall\, z \in \mathbb{R}^p$.

If (i) *or* (ii) *hold, then the zero equilibrium of the unforced system* (1.1) *is stable in the large or globally asymptotically stable, respectively. If* (ii) *holds and the triple is* (A, B, C) *is controllable or observable, then the zero equilibrium pair of* (1.1) *is iISS. If* (iii), (iv) *or* (v) *hold, then the zero equilibrium pair of* (1.1) *is strongly iISS, ISS or exponentially ISS, respectively.*

Proof. For a proof that assumption (i) implies stability in the large see; for example, [28, Proposition 8.2.1]. Theorem 2.4 shows that (ii) guarantees global asymptotic stability. The claim pertaining to iISS follows from Proposition 3.9. That assumption (iii) is sufficient for strong iISS follows from Theorem 3.1. The sufficiency of (iv) and (v) for ISS and exponential ISS, respectively, follows from [29, Theorem 3.2] (see also the commentary in [29, p.451]). □

Observe that the left hand sides of the inequalities in (i)–(v) are all equal to $\|f(z) - Kz\|$. However, from (i) to (v) we are demanding that increasingly strong inequalities hold by "shrinking" the right hand side. The following example demonstrates that any of the hypotheses (i)–(v) in Theorem 5.1 do not, in general, imply the stability notion guaranteed by the subsequent hypotheses.

Example 5.2. We consider again the scalar Lur'e system (3.3) from Example 3.3. As described there, (3.3) satisfies $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ with $K = 0$ and $r = 1$, where $\mathbf{G}(s) = 1/(1 + s)$.

(1) For $v = 0$, the linear function $z \mapsto f(z) = z$ satisfies condition (i) with equality, but not (ii). Trivially, in this case the solution of (3.3) is constant, $x(t, x^0, 0) \equiv x^0$, and so the zero equilibrium is stable in the large, but not globally asymptotically stable.

(2) Example 3.10 shows that the sufficient condition (ii) for global asymptotic stability of unforced Lur'e systems is not sufficient for strong iISS of the forced Lur'e system (1.1).

(3) It is straightforward to construct examples where (iii) holds, but (iv) does not, and the resulting zero equilibrium pair is not ISS, see [29].

(4) Finally, fixing $\kappa \in (0, 1)$, consider (3.3) with f given by

$$f : \mathbb{R} \to \mathbb{R}, \quad f(z) = \begin{cases} z - z^3 & |z| \in [0, \kappa) \\ z - \kappa^3 \ln\Big(\dfrac{|z|}{\kappa}\mathrm{e}\Big) & |z| \geq \kappa\,. \end{cases} \tag{5.1}$$

The function f in (5.1) satisfies (iv), but not (v). The zero equilibrium of the unforced system is globally asymptotically stable, but, for $x^0 \in (0, \kappa)$, the solution

$$x(t; x^0, 0) = \sqrt{\frac{(x^0)^2}{2t(x^0)^2 + 1}} \quad \forall\, t \geq 0\,,$$

does not converge to zero exponentially. Consequently, the zero equilibrium pair of the forced Lur'e system cannot be exponentially ISS. ◇

To summarise, we have derived sufficient conditions for iIIS and strong iISS of the zero equilibrium pair of finite-dimensional, continuous-time, forced Lur'e systems. Strong iISS is the combination of iISS and small-signal ISS, and was introduced in [7]. Our main result, Theorem 3.1 says that if a complex ball of linear static output-feedback gains is stabilising for a given linear system, then the zero equilibrium pair of the related forced Lur'e system (1.1) is strongly iISS for all nonlinearities f in (1.1) which satisfy a related nonlinear "ball" condition. Under a weaker assumption on the nonlinearity, Proposition 3.9 provides a similar sufficient condition for iISS. We note that these results are reminiscent of the complex Aizerman conjecture (a well-known result in absolute stability theory [15, 16]) and resonate with recent work on ISS of Lur'e systems [29]. Consequences of Theorem 3.1 include Corollary 3.7 and Proposition 4.2: the former extends the classical circle criterion to a strong iISS setting, the latter provides a sufficient condition for stabilisation of linear systems by saturated feedback and is related to the recent work [4].

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