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Absolute stability and integral control

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Absolute stability results of both circle criterion and Popov type are derived for finite-dimensional linear plants with non-linearity in the feedback loop. The linear plant contains an integrator (and so is not asymptotically stable). The (possibly time-varying) non-linearity satisfies a particular sector condition which allows for cases with zero lower gain (such as saturation and deadzone). The conjunction of stable, but not asymptotically stable, linear plants and non-linearities with possibly zero lower gain is a distinguishing feature of the paper. The absolute stability results are invoked in proving convergence and stability properties of low-gain integral feedback control for tracking of constant reference signals in the context of exponentially stable linear systems subject to input and output non-linearities.

1. Introduction

Absolute stability, and its relation to the concept of positive-real transfer functions, permeates much of the classical and modern control literature, see, for example, Aizerman and Gantmacher (1964), Lefschetz (1965), Meyer (1965), Hahn (1967), Willems (1970), Narendra and Taylor (1973), Molander and Willems (1980), Vidyasagar (1993), Leonev *et al*. (1996a,b), Rantzer (1996), Megretski and Rantzer (1997), Aeyels *et al*. (1998), Sastry (1999), Jönsson and Megretski (2000), Lozano *et al*. (2000), Arcak and Kokotović (2001), Arcak and Teel (2002), Johansson and Robertsson (2002), Khalil (2002), Arcak *et al*. (2003) and Curtain *et al*. (2003). As one of the more recent developments, we mention the integral quadratic constraint methodology (Megretski and Rantzer 1997, Jönsson and Megretski 2000). Of particular importance are absolute stability results of circle-criterion type and those of Popov type, each applicable in the context of a canonical feedback structure with a linear plant Λ in the forward path and a non-linearity f in the feedback path (see figure 1). Absolute stability criteria are not only useful for stability analysis, but they have also been used in the context of control synthesis, see, for example, Molander and Willems (1980), Arcak and Kokotović (2001), Johansson and Robertsson (2002) and Arcak *et al*. (2003).

In the present paper, we address absolute stability issues in the setting of finite-dimensional, single-input–single-output plants Λ which contain an integrator (and so are not asymptotically stable). We consider both time-varying and time-invariant non-linearities f satisfying a particular sector condition which, in the time-varying case, posits the existence of constants $t_0 \geq 0$ and $\beta > 0$ such that $\xi f(t,\xi) \geq \beta f^2(t,\xi)$ for all $t \geq t_0$ and all $\xi \in \mathbb{R}$ or, in the time-invariant case, posits the existence of $\beta \geq 0$ such that $\xi f(\xi) \geq \beta f^2(\xi)$ for all $\xi \in \mathbb{R}$. Moreover, the lower gain of f, that is, $\inf\{|f(t,\xi)/\xi|: t \geq t_0, \xi \neq 0\}$ (or $\inf\{|f(\xi)/\xi|: \xi \neq 0\}$), may be zero (as is the case for deadzone non-linearities and bounded non-linearities such as saturation). The conjunction of stable, but not asymptotically stable, linear plants Λ and non-linearities f with possibly zero

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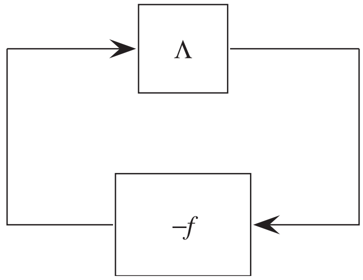

Figure 1. Feedback system with non-linearity.

lower gain (the so-called critical cases of the circle and Popov criteria) is a distinguishing feature of the paper.

In Theorems 1 and 2, we present a result of circle-criterion type in the context of time-varying non-linearities and a result of Popov type for time-invariant nonlinearities, respectively. Whilst these results are reminiscent of a number of well-known stability criteria pertaining to non-critical cases, we emphasize that (to our knowledge) Theorems 1 and 2, which apply to the critical cases, are unavailable in the literature. We elaborate on this in Remarks 1 and 3.

The main motivation for developing absolute stability criteria for feedback structures of the form of figure 1, wherein the linear plant Λ contains an integrator and the non-linearity has possibly zero lower gain, is their application in proving convergence and stability properties of low-gain integral feedback control for tracking of constant reference signals in the context of exponentially stable linear systems Σ subject to input/actuator φ (saturation, for example) and output/sensor ψ non-linearities.

Under an appropriate non-linear transformation, the closed-loop system in figure 2 has the structure of figure 1 to which our absolute stability results apply to deduce a threshold value of the integrator gain k under which performance of the closed-loop is assured (both, constant and time-varying gains k are considered). Whilst these applications have significant overlap with those of Fliegner *et al.* (2001, 2003), we stress that the absolute stability approach adopted in the present paper differs fundamentally from the arguments used in Fliegner *et al.* (2001, 2003) and provides a natural and unified framework for investigations on low-gain integral control. The terminology "low-gain feedback" is also used in other contexts, see, for example, Grognard *et al.* (1998) and Saberi *et al.* (2000). However, we remark that the low-gain designs therein are of a state feedback nature with considerable state-space system data requirements. This contrasts with the output feedback structure underlying the low-gain integral control design in § 3 which requires only limited

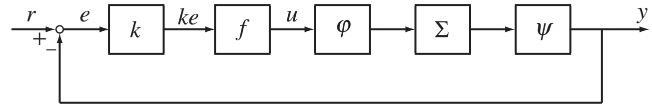

Figure 2. Integral control with input and output non-linearities.

knowledge of plant data in order to determine an appropriate scalar gain k (see figure 2).

The structure of the paper is as follows. In § 2, we present two absolute stability results: the first (Theorem 1) of circle-criterion type in a context of time-varying non-linearities and the second (Theorem 2) of Popov type in a context of time-invariant non-linearities. The novel features of these results are placed in perspective with the existing literature on absolute stability. In § 3, the problem of tracking constant reference signals r for systems with the structure of figure 2 is considered. In particular, systems with non-linearity in both the input and output channels are considered in § 3.1 and analysed in the "circle criterion" context of Theorem 1; systems with non-linearity in the input channel only are investigated in § 3.2 and analysed in the "Popov" context of Theorem 2.

Notation and terminology. For $h\colon J \subset \mathbb{R} \to \mathbb{R}$ and $W \subset \mathbb{R}$ we set $h^{-1}(W) := \{\xi \in J\colon h(\xi) \in W\}$. For $w \in \mathbb{R}$ we write $h^{-1}(w)$ in place of the more cumbersome $h^{-1}(\{w\})$. A function $h\colon \mathbb{R} \to \mathbb{R}$ is said to be piecewise continuously differentiable if h is continuous and there exists a strictly increasing bi-sequence $(a_j)_{j\in\mathbb{Z}}$ with $a_j \to \pm\infty$ as $j \to \pm\infty$ such that h is continuously differentiable on the closed interval $[a_j, a_{j+1}]$ for every $j \in \mathbb{Z}$. The left and right derivatives of h at $\xi \in \mathbb{R}$ are denoted by $h'_-(\xi)$ and $h'_+(\xi)$, respectively. Finally, $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ (the punctured real line).

2. Absolute stability

In this section, we present two absolute stability results: the first (Theorem 1) of circle-criterion type and the second (Theorem 2) of Popov type.

2.1. *Preliminaries*

With reference to figure 1, we consider real, linear, single-input–single-output systems Λ of the form

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{b}u(t), \quad z(0) = z^0, \quad y(t) = \mathbf{c}z(t) \tag{1}$$

with $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{b}, \mathbf{c}^T \in \mathbb{R}^n$. The following hypothesis remains in force throughout this section.

(H) $\mathbf{A}$ has a simple eigenvalue at zero and every other eigenvalue of $\mathbf{A}$ has negative real part.

Therefore, with zero input, Λ is stable but not asymptotically stable. By hypothesis (H), we may infer the existence of a real invertible matrix T such that

$$T\mathbf{A}T^{-1} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad T\mathbf{b} = \begin{bmatrix} b \\ \beta \end{bmatrix}, \quad \mathbf{c}T^{-1} = \begin{bmatrix} c & \gamma \end{bmatrix}, \tag{2}$$

where $A \in \mathbb{R}^{(n-1)\times(n-1)}$ is Hurwitz (that is, the spectrum of A is contained in the open left half complex plane $\{s \in \mathbb{C} : \operatorname{Re} s < 0\}$), $b, c^T \in \mathbb{R}^{n-1}$ and $\beta, \gamma \in \mathbb{R}$. Therefore, system (1) may be expressed in the equivalent form

$$\left.\begin{array}{l} \dot{z}_1(t) = Az_1(t) + bu(t), \quad z_1(0) = z_1^0, \\ \dot{z}_2(t) = \beta u(t), \quad z_2(0) = z_2^0, \\ y(t) = cz_1(t) + \gamma z_2(t), \end{array}\right\} \tag{3}$$

where

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := Tz(t), \quad \begin{bmatrix} z_1^0 \\ z_2^0 \end{bmatrix} := Tz^0.$$

The transfer function $\mathbf{G}$ of (1) (equivalently (3)) is given by

$$\mathbf{G}(s) := \mathbf{c}(sI - \mathbf{A})^{-1}\mathbf{b} = c(sI - A)^{-1}b + \frac{\gamma\beta}{s}$$

with

$$\lim_{s \to 0} s\mathbf{G}(s) = \gamma\beta \in \mathbb{R}. \tag{4}$$

Before developing our variants of the circle and Popov criteria, we first present a technicality which facilitates their proof.

Define the rational function H by $H(s) := c(sI - A)^{-1}b$. Since A is Hurwitz, the function $\omega \mapsto (qi\omega + 1)H(i\omega)$ is bounded for each $q \in \mathbb{R}$; moreover, $(qi\omega + 1)H(i\omega) \to qcb$ as $\omega \to \pm\infty$. Noting that, for all $q \in \mathbb{R}$,

$$\operatorname{Re}[(qi\omega + 1)\mathbf{G}(i\omega)] = q\gamma\beta + \operatorname{Re}[(qi\omega + 1)H(i\omega)], \quad \forall\, \omega \in \mathbb{R}^*,$$

we may conclude that

$$-\infty < \inf_{\omega \in \mathbb{R}^*} \operatorname{Re}[(qi\omega + 1)\mathbf{G}(i\omega)] \le q\mathbf{cb} = q(cb + \gamma\beta), \quad \forall\, q \in \mathbb{R}, \tag{5}$$

and so, for every $q \in \mathbb{R}$, there exists $\eta \ge 0$ such that

$$\eta + \inf_{\omega \in \mathbb{R}^*} \operatorname{Re}[(qi\omega + 1)\mathbf{G}(i\omega)] = \eta + q\gamma\beta + \inf_{\omega \in \mathbb{R}} \operatorname{Re}[(qi\omega + 1)H(i\omega)] > 0. \tag{6}$$

Lemma 1: *Let $q \in \mathbb{R}$ and let $\eta \ge 0$ be such that (6) holds. There exists $P \in \mathbb{R}^{(n-1)\times(n-1)}$ such that $P = P^T > 0$ and*

$$\begin{bmatrix} PA + A^T P & Pb - (qA^T + I)c^T \\ b^T P - c(qA + I) & -2\eta - 2q\mathbf{cb} \end{bmatrix} < 0. \tag{7}$$

Proof: By (6),

$$\eta + q\gamma\beta + \operatorname{Re}[(qi\omega + 1)H(i\omega)] > 0, \quad \forall \omega \in \mathbb{R} \cup \{\pm\infty\}.$$

Writing

$$M = \begin{bmatrix} 0 & -(qA^T + I)c^T \\ -c(qA + I) & -2\eta - 2q\mathbf{cb} \end{bmatrix},$$

we may conclude that, for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$,

$$\begin{bmatrix} (i\omega I - A)^{-1}b] \\ 1 \end{bmatrix}^* M \begin{bmatrix} (i\omega I - A)^{-1}b \\ 1 \end{bmatrix}$$
$$= -2\operatorname{Re}\big(c(qA + I)(i\omega - A)^{-1}b\big) - 2\eta - 2q\mathbf{cb}$$
$$= -2\big(\eta + q\gamma\beta + \operatorname{Re}[(qi\omega + 1)H(i\omega)]\big) < 0.$$

By an application of the variant of the Kalman–Yakubovich–Popov Lemma given in Rantzer (1996), it follows that there exists $P = P^T > 0$ such that (7) holds.

2.2. *A result of circle-criterion type*

We now focus on stability properties of the linear system (1) under output feedback with time-varying non-linearity in the feedback loop (recall figure 1). First, we make precise the class $\mathcal{N}$ of allowable non-linearities. A function

$$f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \quad (t, \xi) \mapsto f(t, \xi)$$

is deemed to be of class $\mathcal{N}$ if $f(\cdot, \xi)$ is measurable for all ξ and $f(t, \cdot)$ is locally Lipschitz, uniformly with respect to t on bounded intervals, and there exists a non-negative function $c_f \in L^1_{\text{loc}}(\mathbb{R}_+)$ such that

$$|f(t, \xi)| \le c_f(t)\big[1 + |\xi|\big], \quad \forall\, t \in \mathbb{R}_+, \quad \forall\, \xi \in \mathbb{R}. \tag{8}$$

A function $f \in \mathcal{N}$ is said to be asymptotically autonomous with limit f_a if $f_a: \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and, for all $R > 0$ and $\varepsilon > 0$, there exists $\tau > 0$ such that,

$$\text{ess-sup}_{t \geq \tau} |f(t, \xi) - f_a(\xi)| < \varepsilon, \quad \forall \xi \in [-R, R].$$

It follows from standard results in ordinary differential equations (see, for example, pp. 121/122 in Walter (1998)) combined with Gronwall's lemma that, for $f \in \mathcal{N}$, the initial-value problem for the feedback system

$$\dot{z}(t) = \mathbf{A}z(t) - \mathbf{b}f(t, \mathbf{c}z(t)), \quad z(0) = z^0, \tag{9}$$

has a unique absolutely continuous solution defined on $\mathbb{R}_+$ (no finite escape time) which satisfies the differential equation in (9) for almost all $t \in \mathbb{R}_+$. We denote this solution by $z(\cdot, z^0)$.

The next theorem is a stability result of circle-criterion type. For completeness, we have included therein (*viz.* statement 1(b)) a well-known classical result on exponential stability, see Vidyasagar (1993) and Lozano *et al.* (2000). However, we emphasize the novelty of all other assertions of the theorem which pertain to feedback systems for which (i) the linear part contains an integrator (i.e., we are considering a so-called particular or critical case in the terminology of Aizerman and Gantmacher (1964), Narendra and Taylor (1973) and Leonov *et al.* (1996b), implying that the linear system is not asymptotically stable) and (ii) the "lower gain" $\inf\{|f(t, \xi)/\xi| : t \geq t_0, \xi \in \mathbb{R}^*\}$ of the non-linearity f may be zero (which, for example, is the case for bounded non-linearities such as saturation and deadzone). In fact, one of the motivations for studying this situation is its importance in the application to the low-gain integral control problem in the presence of input non-linearities of saturation type (see § 3).

Theorem 1: *Assume that* $\lim_{s \to 0} s\mathbf{G}(s) > 0$. *Let* $0 < \alpha < \infty$ *be such that*

$$\frac{1}{\alpha} + \inf_{\omega \in \mathbb{R}^*} \operatorname{Re} \mathbf{G}(i\omega) > 0. \tag{10}$$

1. *If* $f \in \mathcal{N}$ *is such that, for some* $t_0 \geq 0$,

$$\xi f(t, \xi) \geq \frac{1}{\alpha} f^2(t, \xi), \quad \forall\, t \geq t_0, \quad \forall\, \xi \in \mathbb{R}, \tag{11}$$

then the following statements hold.

(a) *There exists* $M \geq 1$ *such that*

$$\|z(t + t_0, z^0)\| \leq M \|z(t_0, z^0)\|, \quad \forall\, (t, z^0) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

(b) *If there exists* $\alpha_0 > 0$ *such that* $\alpha_0 \xi^2 \leq \xi f(t, \xi)$ *for all* $t \geq t_0$ *and all* $\xi \in \mathbb{R}$, *then there exist* $M \geq 1$ *and* $\rho > 0$ *such that*

$$\|z(t + t_0, z^0)\| \leq M e^{-\rho t} \|z(t_0, z^0)\|, \quad \forall\, (t, z^0) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

(c) *For each* $z^0 \in \mathbb{R}^n$, *the function* $t \mapsto y(t) = \mathbf{c}z(t, z^0)$ *is such that* $y(\cdot) f(\cdot, y(\cdot)) \in L^1(\mathbb{R}_+)$ *and* $f(\cdot, y(\cdot)) \in L^2(\mathbb{R}_+)$.

(d) *For each* $z^0 \in \mathbb{R}^n$, $z(t, z^0)$ *converges as* $t \to \infty$ *to a limit in* $\ker \mathbf{A}$, *that is,*

$$z^\infty := \lim_{t \to \infty} z(t, z^0) \in \ker \mathbf{A},$$

and, moreover, $\dot{z}(\cdot, z^0) \in L^2(\mathbb{R}_+)$.

(e) *If* f *is asymptotically autonomous with limit* f_a, *then, for each* $z^0 \in \mathbb{R}^n$, $\lim_{t \to \infty} z(t, z^0) = z^\infty \in \ker \mathbf{A}$ *is such that* $\mathbf{c}z^\infty \in f_a^{-1}(0)$.

(f) *If* f *is asymptotically autonomous with limit* f_a *and* $f_a^{-1}(0) = \{0\}$, *then* 0 *is a globally attractive equilibrium of the feedback system* (9).

2. *If* $f \in \mathcal{N}$ *is of the form* $f(t, \xi) = k(t) g(\xi)$, *where* k *is measurable, bounded and non-negative with* $\limsup_{t \to \infty} k(t) \leq 1$, g *is locally Lipschitz with* $\liminf_{\xi \to 0} g(\xi)/\xi > 0$ *and*

$$0 < \xi g(\xi) \leq \alpha \xi^2, \quad \forall\, \xi \in \mathbb{R}^*,$$

then there exists $N \geq 1$ *such that*

$$\|z(t, z^0)\| \leq N \|z^0\|, \quad \forall\, (t, z^0) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Furthermore, for each $r > 0$, *there exist* $L \geq 1$ *and* $\rho > 0$ *such that, for all* $z^0 \in \mathbb{R}^n$ *with* $\|z^0\| \leq r$,

$$\|z(t, z^0)\| \leq L e^{-\rho \int_0^t k(\tau) d\tau} \|z^0\|, \quad \forall\, t \in \mathbb{R}_+.$$

In particular, if $k \notin L^1(\mathbb{R}_+)$, *then* $\lim_{t \to \infty} z(t, z^0) = 0$ *for all* $z^0 \in \mathbb{R}^n$; *if* $\liminf_{t \to \infty} k(t) > 0$, *then* 0 *is a semi-globally exponentially stable equilibrium of system* (9).

Remark 1: In contrast to Theorem 1, in most of the circle-criterion type results available in the literature (such as Narendra and Taylor (1973), Vidyasagar (1993), Sastry (1999), Lozano *et al.* (2000) and Khalil (2002)), the lower gain of the non-linearity is either assumed to be positive, or, if the lower gain is allowed to be zero, the linear part is assumed to be asymptotically stable (and so does not contain an integrator): an exception is Willems (1970), where stability of the origin and Lagrange stability is proved for (9) without assuming that the lower gain of f is positive. However, none of the statements of Theorem 1 can be found in Willems (1970). Also note that, in contrast to most related results in the literature, Theorem 1 does not impose controllability or observability on the linear system. Finally, in Aeyels *et al.* (1998), a "relaxed" circle criterion is given for the linear time-varying feedback system obtained from (9) by considering functions f of the form $f(t,\xi) = k(t)\xi$ (this is a special case of the setting in assertion 2 of Theorem 1). Assuming that the positive real condition (10) holds and $0 \le k(t) \le \alpha$, Theorem 12 in Aeyels *et al.* (1998) guarantees asymptotic stability of the origin, provided that k satisfies a certain additional condition: the point of interest in the present context is that the latter condition is satisfied by a large class of gain functions k with $\liminf_{t\to\infty} k(t) = 0$. (We mention that the relaxed circle criterion (Aeyels *et al.* 1998, Theorem 12) is not correct in the generality stated in Aeyels *et al.* (1998) wherein the only regularity assumption explicitly imposed on k is (Lebesgue) measurability; it is not difficult to construct counterexamples with continuous k. However, uniform continuity of k is sufficient for the assertions of Theorem 12 in Aeyels *et al.* (1998) to hold.)

Proof of Theorem 1. Let T be an invertible real matrix such that (2) holds. Let $z^0 \in \mathbb{R}^n$ be arbitrary and define $z_1\colon \mathbb{R}_+ \to \mathbb{R}^{n-1}$, $z_2\colon \mathbb{R}_+ \to \mathbb{R}$, $z_1^0 \in \mathbb{R}^{n-1}$ and $z_2^0 \in \mathbb{R}$ by

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := Tz(t,z^0), \qquad \begin{bmatrix} z_1^0 \\ z_2^0 \end{bmatrix} := Tz^0. \tag{12}$$

In view of the equivalence of (1) and (3) we have that

$$\left.\begin{aligned} \dot z_1(t) &= Az_1(t) - bf(t,y(t)), \quad z_1(0) = z_1^0 \\ \dot z_2(t) &= -\beta f(t,y(t)), \quad z_2(0) = z_2^0 \\ y(t) &= cz_1(t) + \gamma z_2(t). \end{aligned}\right\} \tag{13}$$

Invoking the positive-real condition (10), Lemma 1 guarantees the existence of a matrix $P = P^T > 0$ such that

$$Q := \begin{bmatrix} PA + A^TP & Pb - c^T \\ b^TP - c & -2/\alpha \end{bmatrix} < 0.$$

Define the quadratic form $V\colon \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ by

$$V(\xi_1,\xi_2) := \langle \xi_1, P\xi_1\rangle + (\gamma/\beta)\xi_2^2. \tag{14}$$

[1] By assumption, $\lim_{s\to 0} s\mathbf{G}(s) > 0$ and so, by (4), $\gamma/\beta > 0$, showing that V is positive definite. Noting that

$$-\gamma z_2(t)f(t,y(t)) = cz_1(t)f(t,y(t)) - y(t)f(t,y(t)), \quad \text{for a.a. } t \ge t_0,$$

we obtain

$$\begin{aligned} \frac{d}{dt}V(z_1(t),z_2(t)) &= \langle [PA + A^TP]z_1(t), z_1(t)\rangle \\ &\quad - 2\langle Pz_1(t), bf(t,y(t))\rangle - 2\gamma z_2(t)f(t,y(t)) \\ &= \left\langle \begin{bmatrix} z_1(t) \\ -f(t,y(t)) \end{bmatrix}, Q\begin{bmatrix} z_1(t) \\ -f(t,y(t)) \end{bmatrix}\right\rangle \\ &\quad - 2y(t)f(t,y(t)) + \frac{2}{\alpha}f^2(t,y(t)) \\ &\le -M_0\left[\|z_1(t)\|^2 + f^2(t,y(t))\right] \\ &\quad - 2y(t)f(t,y(t)) \\ &\quad + \frac{2}{\alpha}f^2(t,y(t)), \quad \text{for a.a. } t \ge t_0, \end{aligned} \tag{15}$$

where $M_0 > 0$ is such that $\langle \xi, Q\xi\rangle \le -M_0\|\xi\|^2$ for all $\xi \in \mathbb{R}^n$. Consequently, the sector condition (11) yields

$$\frac{d}{dt}V(z_1(t),z_2(t)) \le -M_0\left[\|z_1(t)\|^2 + f^2(t,y(t))\right] \le 0, \quad \text{for a.a. } t \ge t_0. \tag{16}$$

Statement (a) follows immediately. Moreover, we may conclude that $z_1 \in L^2(\mathbb{R}_+,\mathbb{R}^{n-1})$ and $f(\cdot,y(\cdot)) \in L^2(\mathbb{R}_+)$. By (15),

$$\int_{t_0}^t |y(s)f(s,y(s))|ds \le \frac{1}{2}V(z_1(t_0),z_2(t_0)) + \frac{1}{\alpha}\int_0^\infty f^2(s,y(s))ds < \infty, \quad \forall\, t \ge t_0,$$

which, together with (8), implies that the function $t \mapsto y(t)f(t,y(t))$ is in $L^1(\mathbb{R}_+)$, establishing statement (c). Since $f(\cdot,y(\cdot)) \in L^2(\mathbb{R}_+)$ and A is Hurwitz, it follows from (13) that $z_1, \dot z_1, \dot z_2 \in L^2(\mathbb{R}_+)$. Hence, $\dot z(\cdot,z^0) \in L^2(\mathbb{R}_+)$ and $z_1(t) \to 0$ as $t \to \infty$. By (16), the function $t \mapsto V(z_1(t),z_2(t))$ is non-increasing for $t \ge t_0$. Since V is non-negative, we obtain that $V(z_1(t),z_2(t))$ converges to a finite limit as $t \to \infty$. Combining this with the fact that $z_1(t)$ converges as $t \to \infty$, shows that $z_2^2(t)$ (and hence $z_2(t)$) converges to a finite limit as $t \to \infty$. We have now shown that, for some $z_2^\infty \in \mathbb{R}$, $(z_1(t),z_2(t)) \to (0,z_2^\infty)$ as $t \to \infty$, whence statement (d). Clearly, $y(t) = cz_1(t) + \gamma z_2(t) \to y^\infty := \gamma z_2^\infty$ as $t \to \infty$. Statement (e) follows if we can show that $f_a(y^\infty) = 0$. Suppose otherwise, that is, suppose $|f_a(y^\infty)| = 3\varepsilon$ for some $\varepsilon > 0$. Then there exists $\tau > 0$ such that

$|f_a(y(t))| \geq 2\varepsilon$ for all $t \geq \tau$. By boundedness of $y(\cdot)$ and asymptotic autonomy of f, there exists $\tau_1 \geq \tau$ such that $\text{ess-sup}_{t\geq\tau_1} |f(t, y(t)) - f_a(y(t))| < \varepsilon$. Therefore, $|f(t, y(t))| \geq \varepsilon$ for almost all $t \geq \tau_1$ which contradicts the fact that $f(\cdot, y(\cdot))$ is square integrable. This establishes statement (e). Statement (f) is a direct consequence of statements (d) and (e). It remains to prove statement (b). Assume the existence of $\alpha_0 > 0$ such that $\alpha_0\xi^2 \leq \xi f(t,\xi)$ for all $t \geq t_0$ and all $\xi \in \mathbb{R}$. Then $f^2(t, y(t)) \geq \alpha_0^2 y^2(t)$ for all $t \geq t_0$. By hypothesis, $\lim_{s\to 0} s\mathbf{G}(s) > 0$ and so, by (4), we have $\gamma \neq 0$. Therefore, the matrix

$$Z = \begin{bmatrix} 1 & 0 \\ \alpha_0 c & \alpha_0\gamma \end{bmatrix}$$

is invertible and so Z^TZ is positive definite. Therefore, there exists a constant $N_1 > 0$ such that

$$\begin{aligned}\|z_1(t)\|^2 + f^2(t, y(t)) &\geq \|z_1(t)\|^2 + \alpha_0^2 y^2(t) \\ &= \begin{bmatrix} z_1^T(t) & z_2^T(t) \end{bmatrix} Z^TZ \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \\ &\geq N_1 V(z_1(t), z_2(t)), \quad \forall\, t \geq t_0, \quad (17)\end{aligned}$$

which, together with (16), establishes statement (b).

[2] Let $t_0 \geq 0$ be such that $0 \leq k(t) \leq 1$ for all $t \geq t_0$. Then $0 \leq \xi f(t,\xi) \leq \alpha\xi^2$ for all $t \geq t_0$ and (11) holds. Define

$$\theta(t, z^0) := \begin{cases} \dfrac{g(\mathbf{c}z(t,z^0))}{\mathbf{c}z(t,z^0)}, & \mathbf{c}z(t,z^0) \neq 0 \\ \alpha, & \mathbf{c}z(t,z^0) = 0 \end{cases}, \quad \forall\, t \geq t_0, \quad \forall\, z^0 \in \mathbb{R}^n.$$

A routine application of Gronwall's lemma shows that there exists $N_2 \geq 1$ such that

$$\|z(t, z^0)\| \leq N_2\|z^0\|, \quad \forall\, t \in [0, t_0], \quad \forall\, z^0 \in \mathbb{R}^n. \quad (18)$$

Consequently, by statement (a) of part 1, there exists $N \geq 1$ such that

$$\|z(t, z^0)\| \leq N\|z^0\|, \quad \forall\, (t, z^0) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Let $r > 0$ be given, set $B := \{z \in \mathbb{R}^n : \|z\| \leq r\}$ and assume that $z^0 \in B$. By the assumptions on g, there exists $\varepsilon > 0$ (in general depending on the ball B, but not on z_0) such that

$$\frac{g(\mathbf{c}z(t,z^0))}{\mathbf{c}z(t,z^0)} \geq \varepsilon, \quad \text{for all } t \geq 0 \text{ with } \mathbf{c}z(t,z^0) \neq 0.$$

Thus,

$$\theta(t, z_0) \in [\varepsilon, \alpha], \quad \forall\, t \geq 0. \quad (19)$$

Moreover,

$$f(t, y(t)) = k(t)\theta(t, z^0)y(t), \quad \forall\, t \geq 0. \quad (20)$$

Set $K(t) := \int_0^t k(\tau)d\tau$ and introduce functions (parameterized by $\rho > 0$) v_ρ, w_ρ and y_ρ defined by

$$\begin{aligned} v_\rho(t) &:= \exp(\rho K(t))z_1(t), \quad w_\rho(t) := \exp(\rho K(t))z_2(t), \\ y_\rho(t) &:= \exp(\rho K(t))y(t); \quad \forall\, t \geq 0. \end{aligned}$$

Invoking (13), a straightforward calculation yields

$$\left.\begin{aligned} \dot{v}_\rho(t) &= Av_\rho(t) + \rho k(t)v_\rho(t) - be^{\rho K(t)}f(t, y(t)), \quad v_\rho(0) = z_1^0, \\ \dot{w}_\rho(t) &= \rho k(t)w_\rho(t) - \beta e^{\rho K(t)}f(t, y(t)), \quad w_\rho(0) = z_2^0, \\ y_\rho(t) &= cv_\rho(t) + \gamma w_\rho(t). \end{aligned}\right\} \quad (21)$$

The positive-real condition (10) yields the existence of a number $\tilde{\alpha} > \alpha$ so that

$$\frac{1}{\tilde{\alpha}} + \inf_{\omega\in\mathbb{R}^*} \operatorname{Re}\mathbf{G}(i\omega) > 0.$$

Hence, by Lemma 1, there exists $P = P^T > 0$ such that

$$\tilde{Q} := \begin{bmatrix} PA + A^TP & Pb - c^T \\ b^TP - c & -2/\tilde{\alpha} \end{bmatrix} < 0.$$

Associated with this matrix P, let V be the quadratic form given by (14). Then a routine calculation yields

$$\begin{aligned} \frac{d}{dt}V(v_\rho(t), w_\rho(t)) &= \langle [PA + A^TP]v_\rho(t), v_\rho(t)\rangle \\ &\quad - 2\langle Pv_\rho(t), be^{\rho K(t)}f(t, y(t))\rangle \\ &\quad + 2\rho k(t)\left(\langle v_\rho(t), Pv_\rho(t)\rangle + \frac{\gamma}{\beta}w_\rho^2(t)\right) \\ &\quad - 2\gamma e^{\rho K(t)}f(t, y(t)) \\ &\quad \times w_\rho(t), \quad \text{for a.a. } t \geq 0. \quad (22) \end{aligned}$$

Write $\delta := 1 - (\alpha/\tilde{\alpha}) > 0$. Invoking (19), (20) and the last of equations (21), together with (11), we have

$$\begin{aligned} \gamma e^{\rho K(t)}f(t, y(t))w_\rho(t) &= (\delta + (\alpha/\tilde{\alpha}))e^{2\rho K(t)}f(t, y(t))y(t) \\ &\quad - e^{\rho K(t)}f(t, y(t))cv_\rho(t) \\ &\geq \delta\varepsilon k(t)y_\rho^2(t) + \frac{1}{\tilde{\alpha}}\big(k(t)\theta(t, z^0)y_\rho(t)\big)^2 \\ &\quad - k(t)\theta(t, z^0)y_\rho(t)cv_\rho(t), \quad \forall\, t \geq t_0, \end{aligned}$$

which, when combined with (22), yields

$$\begin{aligned}\frac{d}{dt}&V(v_\rho(t), w_\rho(t))\\ &\le \langle [PA + A^T P]v_\rho(t), v_\rho(t)\rangle\\ &- 2\big(b^T P - c\big)v_\rho(t)k(t)\theta(t, z^0)y_\rho(t)\\ &- \frac{2}{\tilde\alpha}\big(k(t)\theta(t, z^0)y_\rho(t)\big)^2 - 2\delta\varepsilon k(t)y_\rho^2(t)\\ &+ 2\rho k(t)\big(\langle v_\rho(t), Pv_\rho(t)\rangle + \frac{\gamma}{\beta}w_\rho^2(t)\big)\\ =&\left\langle \begin{bmatrix} v_\rho(t) \\ -k(t)\theta(t, z^0)y_\rho(t)\end{bmatrix}, \tilde Q \begin{bmatrix} v_\rho(t) \\ -k(t)\theta(t, z^0)y_\rho(t)\end{bmatrix}\right\rangle\\ &- 2\delta\varepsilon k(t)y_\rho^2(t) + 2\rho k(t)\big(\langle v_\rho(t), Pv_\rho(t)\rangle\\ &+ \frac{\gamma}{\beta}w_\rho^2(t)\big), \quad \text{for a.a. } t \ge t_0.\end{aligned}$$

We may now infer the following counterpart of (16)

$$\begin{aligned}\frac{d}{dt}V(v_\rho(t), w_\rho(t)) \le& -M_1\big[\|v_\rho(t)\|^2 + (k(t)\theta(t, z^0)y_\rho(t))^2\big]\\ &- 2\delta\varepsilon k(t)y_\rho^2(t) + 2\rho k(t)\big[\langle v_\rho(t), Pv_\rho(t)\rangle\\ &+ (\gamma/\beta)w_\rho^2(t)\big], \quad \text{for a.a. } t \ge t_0,\end{aligned}$$

where $M_1 > 0$ is such that $\langle \xi, \tilde Q\xi\rangle \le -M_1\|\xi\|^2$ for all $\xi \in \mathbb{R}^n$. Observing that, for some constant $M_2 > 0$ (not depending on z^0),

$$\begin{aligned}k(t)&\big[\langle v_\rho(t), Pv_\rho(t)\rangle\\ &+ (\gamma/\beta)w_\rho^2(t)\big] \le M_2\big[\|v_\rho(t)\|^2 + k(t)y_\rho^2(t)\big], \quad \forall\, t \ge t_0,\end{aligned}$$

we may conclude that

$$\begin{aligned}\frac{d}{dt}V(v_\rho(t), w_\rho(t)) \le& -(M_1 - 2\rho M_2)\|v_\rho(t)\|^2\\ &- 2(\delta\varepsilon - \rho M_2)k(t)y_\rho^2(t), \quad \text{for a.a. } t \ge t_0\end{aligned}$$

Therefore, choosing ρ such that $0 < \rho M_2 \le \min\{M_1/2, \delta\varepsilon\}$, it follows that, for all $t \ge t_0$, $V(v_\rho(t), w_\rho(t)) \le V(v_\rho(t_0), w_\rho(t_0))$. Hence there exits $M_3 \ge 1$ such that

$$\begin{aligned}\|z(t, z^0)\|&\\ \le M_3 \exp(-\rho K(t))&\|z(t_0, z^0)\|, \quad \forall\, t \ge t_0, \quad \forall\, z^0 \in B.\end{aligned}$$

By (18), it follows that for sufficiently large $L > 0$,

$$\|z(t, z^0)\| \le L\exp(-\rho K(t))\|z^0\|, \quad \forall\, t \ge 0, \quad \forall\, z^0 \in B.$$

□

Remark 2: We briefly consider the relation of the first part of the proof of Theorem 1 to passivity concepts (see, for example, Chapter 6 in Khalil (2002)). To this end, set

$$\hat A := \begin{bmatrix} A & 0 \\ 0 & 0\end{bmatrix}, \quad \hat b := \begin{bmatrix} b \\ \beta\end{bmatrix}, \quad \hat c := (c, \gamma),$$

and consider the controlled and observed system

$$\dot x = \hat A x + \hat b u, \quad w = \hat c x + (1/\alpha)u, \quad x(0) = x^0. \tag{23}$$

Using a calculation analogous to that leading to (16), shows that the function V defined in (14) satisfies

$$\begin{aligned}\langle (\nabla V)(\xi), \hat A\xi + \hat b v\rangle =& \left\langle \begin{bmatrix}\xi_1 \\ v\end{bmatrix}, Q\begin{bmatrix}\xi_1 \\ v\end{bmatrix}\right\rangle\\ &+ 2v\Big(\hat c\xi + \frac{v}{\alpha}\Big);\ \forall\, \xi = \begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix} \in \mathbb{R}^{n-1}\times\mathbb{R}, \quad \forall\, v \in \mathbb{R}.\end{aligned} \tag{24}$$

Since Q is negative definite, (24) shows in particular that (23) is input strictly passive. Equation (24) can be used to derive the conclusions of Theorem 1: simply consider (23) with

$$x^0 = \begin{bmatrix} z_1^0 \\ z_2^0\end{bmatrix} \quad \text{and} \quad u = -f(\cdot, cz_1 + \gamma z_2),$$

where z_1, z_2, z_1^0 and z_2^0 are given by (12). However, this point of view does not lead to any simplifications: equation (24) contains more information than just input strict passivity and the fine structure of (24) plays an important role in the analysis. The latter applies in particular to the proof of assertion 2 of Theorem 1.

2.3. *A result of Popov type*

In this section, we focus on stability properties of system (1) under output feedback with a time-invariant locally Lipschitz non-linearity $f: \mathbb{R} \to \mathbb{R}$ in the feedback path. In this case, the feedback system takes the form

$$\dot z(t) = \mathbf{A}z(t) - \mathbf{b}f(\mathbf{c}z(t)), \quad z(0) = z^0. \tag{25}$$

By the standard theory of differential equations, for each $z^0 \in \mathbb{R}^n$, the initial-value problem (25) has a unique continuously differentiable solution $t \mapsto z(t, z^0)$ defined on a maximal interval of existence $[0, t^*)$; moreover, if $z(\cdot, z^0)$ is bounded, then $t^* = \infty$ (see, for example, pp. 121/122 in Walter (1998)).

The next theorem is a stability result of Popov type which is reminiscent of a number of results which can be found in the literature, see Aizerman and Gantmacher (1964), Lefschetz (1965), Hahn (1967), Narendra and Taylor (1973), Willems (1970), Vidyasagar (1993) and Lozano *et al.* (2000), wherein, in contrast to the present paper, it is usually assumed that $f^{-1}(0) = \{0\}$. For completeness, we include the latter as a special case and so the corresponding result on global asymptotic stability in statement (f) of the theorem is well known.

As usual, we adopt the convention $1/\infty := 0$.

Theorem 2: *Assume that* $\lim_{s\to 0} s\mathbf{G}(s) > 0$. *Let* $0 < \alpha \le \infty$ *and* $q \ge 0$ *be such that*

$$\frac{1}{\alpha} + \inf_{\omega\in\mathbb{R}^*} \mathrm{Re}[(qi\omega + 1)\mathbf{G}(i\omega)] > 0. \tag{26}$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ *be locally Lipschitz with*

$$\xi f(\xi) \ge \frac{1}{\alpha} f^2(\xi), \quad \forall \xi \in \mathbb{R}. \tag{27}$$

For each $z^0 \in \mathbb{R}^n$, *the unique continuously differentiable solution* $t \mapsto z(t, z^0)$ *of the feedback system* (25) *exists on* $\mathbb{R}_+$ (*no finite escape time*) *and the following statements hold.*

(a) (*Stability in the large.*) *There exists* $M \ge 1$ *such that*

$$\|z(t, z^0)\| \le M\|z^0\|, \quad \forall (t, z^0) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

(b) *For each* $z^0 \in \mathbb{R}^n$, *the function* $t \mapsto y(t) = \mathbf{c}z(t, z^0)$ *is such that* $yf(y) \in L^1(\mathbb{R}_+)$ *and* $f(y) \in L^2(\mathbb{R}_+)$.

(c) *For each* $z^0 \in \mathbb{R}^n$, $z(t, z^0)$ *converges as* $t \to \infty$ *to a limit* $z^\infty \in \ker \mathbf{A}$ *with* $\mathbf{c}z^\infty \in f^{-1}(0)$; *moreover,* $\dot z(\cdot, z^0) \in L^2(\mathbb{R}_+)$.

(d) *If* f *is piecewise continuously differentiable and* $f'_-(\xi)f'_+(\xi) \ne 0$ *for all* $\xi \in f^{-1}(0)$, *then, for all* $z^0 \in \mathbb{R}^n$, $z(\cdot, z^0) - z^\infty \in L^2(\mathbb{R}_+)$, *where* z^∞ *is the limit of* $z(t, z^0)$ *as* $t \to \infty$, *the existence of which is guaranteed by* (c).

(e) *If* 0 *is an isolated point of* $f^{-1}(0)$, *then* 0 *is an asymptotically stable equilibrium.*

(f) *If* $f^{-1}(0) = \{0\}$, *then* 0 *is a globally asymptotically stable equilibrium.*

(g) *If* $\liminf_{\xi\to 0} f(\xi)/\xi > 0$, *then* 0 *is an exponentially stable equilibrium.*

(h) *If there exists* $\alpha_0 > 0$ *such that* $\alpha_0\xi^2 \le \xi f(\xi)$ *for all* $\xi \in \mathbb{R}$, *then* 0 *is a semi-globally exponentially stable equilibrium.*

(i) *If* $\alpha < \infty$ *and there exists* $\alpha_0 > 0$ *such that* $\alpha_0\xi^2 \le \xi f(\xi)$ for all $\xi \in \mathbb{R}$, *then* 0 *is a globally exponentially stable equilibrium.*

Remark 3: To the best of our knowledge, Theorem 2 (with the exception of statement (f)) is not available in the literature. However, we point out that statements (a) and (b) are implicit in the infinite-dimensional results in Curtain *et al.* (2003, Theorems 3.1 and 5.3). In contrast to the familiar Lyapunov techniques used in the proof of Theorem 2 of the present paper, the proofs of the results in Curtain *et al.* (2003), hinge on possibly less familiar integral equation techniques applied in the context of an input–output approach and involve many technical intricacies engendered by infinite dimensionality. We mention that, in common with statements (a)–(e), the assumption that $f^{-1}(0) = \{0\}$ is not imposed in Jönsson and Megretski (2000, Proposition 1): however, therein, the positive-real condition differs substantially from (26), the non-linearity is restricted to be an ideal deadzone and the conclusions are weaker than those in Theorem 2 (for example, Theorem 2 guarantees both stability in the large and convergence of the state whereas, in Jönsson and Megretski (2000, Proposition 1), it is guaranteed only that the state approaches a certain set which is not a singleton). Finally, we emphasize the importance of statement (c) in the context of the application of Theorem 2 to low-gain integral control (see § 3.2, proof of Theorem 4). Also note that in contrast to most related results in the literature (with Meyer (1965) being one of the few exceptions), Theorem 2 does not impose controllability or observability on the linear system.

Proof of Theorem 2. Let $z^0 \in \mathbb{R}^n$ and denote the maximal interval of existence of the unique solution $t \mapsto z(t, z^0)$ of (25) by $[0, t^*)$, where $0 < t^* \le \infty$. Let T be an invertible real matrix such that (12) holds. Define $z_1\colon [0, t^*) \to \mathbb{R}^{n-1}$, $z_2\colon [0, t^*) \to \mathbb{R}$, $z_1^0 \in \mathbb{R}^{n-1}$ and $z_2^0 \in \mathbb{R}$ by (2). In view of the equivalence of (1) and (3), we have that, for $t \in [0, t^*)$,

$$\left.\begin{aligned} \dot z_1(t) &= Az_1(t) - bf(y(t)), \quad z_1(0) = z_1^0,\\ \dot z_2(t) &= -\beta f(y(t)), \quad z_2(0) = z_2^0,\\ y(t) &= cz_1(t) + \gamma z_2(t). \end{aligned}\right\} \tag{28}$$

By Lemma 1, there exists $P = P^T > 0$ such that

$$Q := \begin{bmatrix} PA + A^TP & Pb - (qA^T + I)c^T \\ b^TP - c(qA + I) & -(2/\alpha) - 2q\mathbf{cb} \end{bmatrix} < 0$$

with the convention that $2/\alpha := 0$ for $\alpha = \infty$. Define the function $V\colon \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ by

$$V(\xi_1, \xi_2) := \langle \xi_1, P\xi_1\rangle + (\gamma/\beta)\xi_2^2 + 2q\int_0^{(c\xi_1+\gamma\xi_2)} f(\zeta)d\zeta. \tag{29}$$

By assumption, $\lim_{s\to 0} s\mathbf{G}(s) > 0$ and so, by (4), $\gamma/\beta > 0$. Moreover, the fact that $\xi f(\xi) \geq 0$ for all $\xi \in \mathbb{R}$ implies the non-negativity of the integral term on the right-hand side of (29). Hence, we may conclude that V is positive definite.

Since

$$-\gamma z_2(t) f(y(t)) = c z_1(t) f(y(t)) - y(t) f(y(t)), \quad \forall\, t \in [0, t^*), \tag{30}$$

we arrive at a counterpart (16) in the proof of Theorem 1:

$$\begin{aligned}
&\frac{d}{dt} V(z_1(t), z_2(t)) \\
&\quad = \langle [PA + A^T P] z_1(t), z_1(t)\rangle - 2\langle P z_1(t), b f(y(t))\rangle \\
&\qquad - 2\gamma z_2(t) f(y(t)) + 2q f(y(t))\big(cAz_1(t) - \mathbf{cb} f(y(t))\big) \\
&\quad \leq \left\langle \begin{bmatrix} z_1(t) \\ -f(y(t)) \end{bmatrix}, Q \begin{bmatrix} z_1(t) \\ -f(y(t)) \end{bmatrix} \right\rangle - 2y(t) f(y(t)) + \frac{2}{\alpha} f^2(y(t)) \\
&\quad \leq -M_1\big[\|z_1(t)\|^2 + f^2(y(t))\big] \leq 0, \quad \text{for a.a. } t \in [0, t^*).
\end{aligned} \tag{31}$$

Consequently, the function $t \mapsto V(z_1(t), z_2(t))$ is bounded on $[0, t^*)$. Observing that there exists a constant $M_2 > 0$ such that $\|(\xi_1, \xi_2)\|^2 \leq M_2 V(\xi_1, \xi_2)$ for all $(\xi_1, \xi_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we may conclude that the function $t \mapsto (z_1(t), z_2(t))$ is bounded on $[0, t^*)$, which in turn implies that $t^* = \infty$. Statements (a), (b) and (f) of the theorem now follow by the arguments (*mutatis mutandis*) previously adopted in the proof of Theorem 1.

To prove statement (c) first note that, since $f(y) \in L^2(\mathbb{R}_+)$ and A is Hurwitz, $z_1, \dot z_1, \dot z_2 \in L^2(\mathbb{R}_+)$. Hence, $\dot z(\cdot, z^0) \in L^2(\mathbb{R}_+)$ and $\lim_{t\to\infty} z_1(t) = 0$. Next observe that it follows from (28) that $\gamma z_2 \dot z_2 = \beta f(y)(cz_1 - y)$ which, on integration, gives

$$\gamma z_2^2(t) = \gamma z_2^2(0) + 2\beta\left(\int_0^t f(y(\tau))[cz_1(\tau) - y(\tau)]d\tau\right), \quad \forall\, t \in \mathbb{R}_+. \tag{32}$$

By (31), $z_1 \in L^2(\mathbb{R}_+, \mathbb{R}^{n-1})$ and, since $f(y) \in L^2(\mathbb{R}_+)$, we have that $f(y)cz_1 \in L^1(\mathbb{R}_+)$. Furthermore, by statement (b), $yf(y) \in L^1(\mathbb{R}_+)$, showing that the integrand in the right-hand side of (32) is in $L^1(\mathbb{R}_+)$. Since $\gamma \neq 0$, (32) shows that $z_2^2(t)$, and hence $z_2(t)$, converges to a finite limit as $t \to \infty$. Therefore, $z^\infty := \lim_{t\to\infty} z(t, z^0) \in \ker \mathbf{A}$; furthermore, since $f(y) \in L^2(\mathbb{R}_+)$, we have $\mathbf{c}z^\infty = \lim_{t\to\infty} y(t) \in f^{-1}(0)$.

To prove statement (d), assume that f is piecewise continuously differentiable with $f'_-(\xi) f'_+(\xi) \neq 0$ for all $\xi \in f^{-1}(0)$. Since $z_1 \in L^2(\mathbb{R}_+)$, $\lim_{t\to\infty} z_1(t) = 0$ and $\gamma \neq 0$, it follows from the third equation in (28) that it is sufficient to show that $y - y^\infty \in L^2(\mathbb{R}_+)$, where $y^\infty := \mathbf{c}z^\infty$. By statement (c), $y^\infty \in f^{-1}(0)$ and so, by hypothesis, $f'_-(y^\infty) f'_+(y^\infty) \neq 0$. Hence there exist $\varepsilon > 0$ and $\delta > 0$ such that the restrictions f_1 and f_2 of f to the intervals $J_1 := [y^\infty - \varepsilon, y^\infty]$ and $J_2 := [y^\infty, y^\infty + \varepsilon]$, respectively, are continuously differentiable and

$$|f'_i(\xi)| \geq \delta > 0, \quad \forall\, \xi \in J_i, \ i = 1, 2.$$

As a consequence, for $i = 1, 2$, the inverse function $g_i := f_i^{-1} : f_i(J_i) \to J_i$ of f_i exists, is continuously differentiable and satisfies a Lipschitz condition on $f_i(J_i)$ with Lipschitz constant $l = 1/\delta > 0$. Since $y(t)$ converges to y^∞ as $t \to \infty$, there exists $\tau \geq 0$ such that $y(t) \in J_1 \cup J_2$ for all $t \geq \tau$. Using that $f_i(y^\infty) = f(y^\infty) = 0$, we obtain

$$\begin{aligned}
|y(t) - y^\infty| &= |g_i(f_i(y(t))) - g_i(f_i(y^\infty))| \\
&\leq l|f_i(y(t))|, \quad \forall\, t \in y^{-1}(J_i) \cap [\tau, \infty), \quad i = 1, 2.
\end{aligned}$$

Therefore,

$$|y(t) - y^\infty| \leq l|f(y(t))|, \quad \forall\, t \in [\tau, \infty),$$

and so, since $f(y) \in L^2(\mathbb{R}_+)$ (by statement (b)), we may conclude that $y - y^\infty \in L^2(\mathbb{R}_+)$.

We proceed to prove statement (e). Assume that 0 is an isolated point of $f^{-1}(0)$ and so there exists $\varepsilon > 0$ such that $|f(\xi)| > 0$ for all ξ with $0 < |\xi| \leq \varepsilon$. Define $\theta := \varepsilon/(M\|\mathbf{c}\|)$, with $M \geq 1$ as in statement (a). For each z^0 with $\|z^0\| \leq \theta$, we have $\lim_{t\to\infty} \mathbf{c}z(t, z^0) = y^\infty \in f^{-1}(0)$ (by statement (c)) and $|\mathbf{c}z(t, z^0)| \leq \|\mathbf{c}\|\|z(t, z^0)\| \leq M\|\mathbf{c}\|\|z^0\| \leq \varepsilon$ for all $t \geq 0$: therefore, $y^\infty = 0$ and so $z(t, z^0) \to 0$ as $t \to \infty$. This establishes local attractivity of 0 which, together with statement (a), yields statement (e).

To prove statements (g)–(i), note that for any $R > 0$ there exists $M_3 > 0$ such that

$$\begin{aligned}
&\|\xi_1\|^2 + \|\xi_2\|^2 \geq M_3 V(\xi_1, \xi_2), \\
&\quad \text{for all } (\xi_1, \xi_2) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ with } \|(\xi_1, \xi_2)\| \leq R,
\end{aligned} \tag{33}$$

which can easily be shown using the local Lipschitz condition on f. Consider statement (g) and assume that $\liminf_{\xi\to 0} f(\xi)/\xi > 0$. Then there exist $\alpha_0 > 0$ and $\varepsilon > 0$ such that $\alpha_0 \xi^2 \leq \xi f(\xi)$ for all $\xi \in \mathbb{R}$ with $|\xi| \leq \varepsilon$. As above in the proof of statement (e), define $\theta := \varepsilon/(M\|\mathbf{c}\|)$. Then, for all z^0 with $\|z^0\| \leq \theta$, $y(\cdot) = \mathbf{c}z(\cdot, z^0)$ is such that $|y(t)| \leq \varepsilon$ for all $t \geq 0$ and so $f^2(y(t)) \geq \alpha_0^2 y^2(t)$ for all $t \geq 0$. Invoking statement (a) and (33), we may conclude as in the proof of Theorem 1 (see (17)) that there exists a constant $M_4 > 0$ such that for every z^0 with $\|z^0\| \leq \theta$

$$\|z_1(t)\|^2 + f^2(t, y(t)) \geq M_4 V(z_1(t), z_2(t)), \quad \forall\, t \in \mathbb{R}_+. \tag{34}$$

Combining this with (31), we may infer the existence of a constant $M_5 > 0$ such that, for all (z_1^0, z_2^0) with $\|(z_1^0, z_2^0)\|$ sufficiently small,

$$\frac{d}{dt} V(z_1(t), z_2(t)) \leq -M_5 V(z_1(t), z_2(t)), \quad \forall\, t \in \mathbb{R}_+.$$

Statement (g) now follows. Next consider statement (h) and assume that $\alpha_0 > 0$ is such that $\alpha_0 \xi^2 \leq \xi f(\xi)$ for all $\xi \in \mathbb{R}$. The argument used to prove statement (g) now applies to any bounded set of initial conditions and hence yields semi-global exponential stability. Finally, the same argument can be used to prove statement (i): we only need to realize that if α is finite, then there exists $M_3 > 0$ such that the inequality in (33) holds for all $(\xi_1, \xi_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$. □

3. Low-gain integral control

In this section we apply Theorems 1 and 2 to obtain results on low-gain integral control. In particular, the problem of tracking constant reference signals $r \in \mathbb{R}$ for linear systems with nonlinearity in both the input and output channels will first be analysed (in § 3.1) in the "circle criterion" context of Theorem 1; linear systems with non-linearity in only the input channel are then considered (in § 3.2) and analysed in the "Popov" context of Theorem 2.

3.1. *Integral control in the presence of input and output non-linearities*

With reference to figure 2, the problem of tracking constant reference signals $r \in \mathbb{R}$ will be addressed in the context of a class of finite-dimensional, single-input–single-output, continuous-time, real, linear systems $\Sigma = (A, b, c, d)$ having a piecewise continuously differentiable non-linearity in both the input and output channels:

$$\begin{aligned} \dot{x} &= Ax + b\varphi(u), \quad x(0) = x^0 \in \mathbb{R}^n; \\ y &= \psi(cx + d\varphi(u)). \end{aligned} \tag{35}$$

In (35), A is assumed to be Hurwitz. Furthermore, the transfer function G, given by $G(s) = c(sI - A)^{-1}b + d$, is assumed to satisfy $G(0) > 0$. To achieve the objective of tracking a constant reference signal $r \in \mathbb{R}$, we consider integral control action

$$\begin{aligned} u(t) &= u^0 + \int_0^t k(\tau)[r - \psi(cx(\tau) + d\varphi(u(\tau))]d\tau \\ &= u^0 + \int_0^t k(\tau)[r - y(\tau)]\, d\tau, \end{aligned}$$

with control gain function k (possibly constant), yielding the following non-linear feedback system (illustrated in figure 2)

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + b\varphi(u(t)), & x(0) &= x^0 \in \mathbb{R}^n, \\ \dot{u}(t) &= k(t)[r - \psi(cx(t) + d\varphi(u(t)))], & u(0) &= u^0 \in \mathbb{R}. \end{aligned}\right\} \tag{36}$$

The objective in this subsection is to determine a gain function k such that the tracking error

$$e(t) := r - y(t) = r - \psi(cx(t) + d\varphi(u(t))$$

converges to 0 as $t \to \infty$. We introduce the set of feasible reference values

$$\mathcal{R} := \{\psi(G(0)v) \colon v \in \overline{\operatorname{im} \varphi}\}.$$

By continuity of φ and ψ, it is clear that $\mathcal{R}$ is an interval. The motivation for the introduction of $\mathcal{R}$ is as follows. If asymptotic tracking occurs for a given $r \in \mathbb{R}$, we would expect that $\varphi^r := \lim_{t\to\infty} \varphi(u(t))$ exists and is finite. Then, using the final value theorem, we may conclude that $\lim_{t\to\infty} y(t) = \psi(G(0)\varphi^r)$, and so, $r = \psi(G(0)\varphi^r)$, which in turn implies that $r \in \mathcal{R}$. In fact, it can be shown that if ψ is continuous and monotone, then $r \in \mathcal{R}$ is close to being a necessary condition for asymptotic tracking insofar as, if asymptotic tracking of r is achievable, whilst maintaining boundedness of $\varphi \circ u$, then $r \in \mathcal{R}$.

The following lemma will be used later. The proof is straightforward and is therefore omitted.

Lemma 2: *Let $g : \mathbb{R} \to \mathbb{R}$ be piecewise continuously differentiable and non-decreasing. Define the function $g^\triangledown \colon \mathbb{R} \to \mathbb{R}$ by $g^\triangledown(\xi) := \min\{g'_-(\xi), g'_+(\xi)\}$. Then $g^\triangledown$ is Borel measurable, non-negative and $g^\triangledown \in L^\infty_{\mathrm{loc}}(\mathbb{R})$. If $v \colon \mathbb{R}_+ \to \mathbb{R}$ is absolutely continuous, then $g \circ v$ is absolutely continuous and*

$$\frac{d}{dt}(g \circ v)(t) = g^\triangledown(v(t))\dot{v}(t), \quad \text{for a.a. } t \in \mathbb{R}_+.$$

Under the extra assumption that g is globally Lipschitz continuous with Lipschitz constant $\lambda \geq 0$ we have $0 \leq g^\triangledown(\xi) \leq \lambda$ for all $\xi \in \mathbb{R}$.

If $g : \mathbb{R} \to \mathbb{R}$ is a piecewise continuously differentiable and non-decreasing function, then a point $w \in \mathbb{R}$ is said to be a critical value of g if there exists $\xi \in \mathbb{R}$ such that $g(\xi) = w$ and $g^{\nabla}(\xi) = 0$: a critical value w is said to be strongly critical if $g^{-1}(w)$ is not a singleton.

We are now in the position to formulate the main result of this subsection.

Theorem 3: *Consider the feedback system* (36). *Assume that A is Hurwitz, $G(0) = d - cA^{-1}b > 0$, $r \in \mathcal{R}$, $k \in L^{\infty}_{\text{loc}}(\mathbb{R}_+)$, φ and ψ are piecewise continuously differentiable and non-decreasing, φ is globally Lipschitz and there exists $\sigma > 0$ such that*

$$0 \le \xi(\psi(\xi) - \psi(0)) \le \sigma\xi^2, \quad \forall\, \xi \in \mathbb{R}. \tag{37}$$

Then, for each $(x^0, u^0, k) \in \mathbb{R}^n \times \mathbb{R} \times L^{\infty}_{\text{loc}}(\mathbb{R}_+)$, there exists a unique absolutely continuous solution $t \mapsto (x(t), u(t))$ of (36) *on $\mathbb{R}_+$ (no finite escape time). Moreover, there exists $\kappa_0 > 0$ such that, for each $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}$ and every non-negative-valued function $k \in L^{\infty}_{\text{loc}}(\mathbb{R}_+)$ with*

$$\limsup_{t\to\infty} k(t) < \kappa_0,$$

the unique solution $t \mapsto (x(t), u(t))$ of (32) *on $\mathbb{R}_+$ has the following properties*:

(a) *the limit $\varphi^r := \lim_{t\to\infty} \varphi(u(t))$ exists and is finite*;
(b) $\lim_{t\to\infty} x(t) = -A^{-1}b\varphi^r$;

furthermore, if $k \notin L^1(\mathbb{R}_+)$, then

(c) $\psi(G(0)\varphi^r) = r$;
(d) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \psi(cx(t) + d\varphi(u(t))) = r$;
(e) *if r is an interior point of $\mathcal{R}$, then u is bounded*;
(f) *if r is an interior point of $\mathcal{R}$ and $\psi(G(0)w) \neq r$ for all strongly critical values w of φ, then $u^r := \lim_{t\to\infty} u(t)$ exists, is finite and $\psi(G(0)\varphi(u^r)) = r$*;
(g) *if r is an interior point of $\mathcal{R}$ and is not a critical value of ψ, and $\psi(G(0)w) \neq r$ for all critical values w of φ, then the convergence in* (a), (b), (d) *and* (f) *is of order $\exp(-\rho\int_0^t k(\tau)d\tau)$ for some $\rho > 0$.*

Proof: Let $(x^0, u^0, k) \in \mathbb{R}^n \times \mathbb{R} \times L^{\infty}_{\text{loc}}(\mathbb{R}_+)$. It follows from standard results in ordinary differential equations (see, for example, pp. 121/122 in Walter (1998)) combined with Gronwall's lemma that (36) has a unique absolutely continuous solution $t \mapsto (x(t), u(t))$ defined on $\mathbb{R}_+$.

Since $r \in \mathcal{R}$, there exists $\varphi^* \in \overline{\text{im}\,\varphi}$ such that $\psi(G(0)\varphi^*) = r$. Defining

$$\bar{\psi}: \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \psi(\xi + (0)\varphi^*) - r, \tag{38}$$

it is clear that $\bar{\psi}(0) = 0$. Using the sector condition imposed on the function $\xi \mapsto \psi(\xi) - \psi(0)$, it is straightforward to show that there exists $\bar{\sigma} > 0$ such that

$$0 \le \xi\bar{\psi}(\xi) \le \bar{\sigma}\xi^2, \quad \forall\, \xi \in \mathbb{R}. \tag{39}$$

Therefore, defining $\theta: \mathbb{R} \to \mathbb{R}$ by $\theta(0) := 0$ and $\theta(\xi) := \bar{\psi}(\xi)/\xi$ for $\xi \neq 0$, it is obvious that

$$0 \le \theta(\xi) \le \bar{\sigma} \quad \text{and} \quad \bar{\psi}(\xi) = \theta(\xi)\xi\,; \quad \forall\, \xi \in \mathbb{R}.$$

Introduce new variables

$$z_1(t) := x(t) + A^{-1}b\varphi(u(t)),$$
$$z_2(t) := \varphi(u(t)) - \varphi^*\,; \quad \forall\, t \in \mathbb{R}_+,$$

and notice that

$$cx + d\varphi(u) = cz_1 + G(0)z_2 + G(0)\varphi^*.$$

By Lemma 2, $\dot{z}_2(t) = \varphi^{\nabla}(u(t))\dot{u}(t)$. Therefore,

$$\left.\begin{aligned} \dot{z}_1(t) &= Az_1(t) - k(t)\varphi^{\nabla}(u(t))A^{-1}b\bar{\psi}(w(t)),\\ \dot{z}_2(t) &= -k(t)\varphi^{\nabla}(u(t))\,\bar{\psi}(w(t)); \quad \text{for a.a. } t \ge 0, \end{aligned}\right\} \tag{40}$$

where $w := cz_1 + G(0)z_2$. Setting

$$f(t, \xi) := k(t)\varphi^{\nabla}(u(t))\theta(w(t))\xi, \quad \forall\, (t, \xi) \in \mathbb{R}_+ \times \mathbb{R},$$

(40) can be written in the form

$$\dot{z}(t) = \mathbf{A}z(t) - \mathbf{b}f(t, \mathbf{c}z(t)), \tag{41}$$

where

$$z(t) := \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad \mathbf{A} := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$
$$\mathbf{b} := \begin{bmatrix} A^{-1}b \\ 1 \end{bmatrix}, \quad \mathbf{c} := (c,\, G(0)).$$

Clearly $\mathbf{A}$ is such that Hypothesis (H) holds. Let $\mathbf{G}$ be the system transfer function given by $\mathbf{G}(s) := \mathbf{c}(sI - \mathbf{A})^{-1}\mathbf{b}$. Noting that $\mathbf{G}(s) = G(s)/s$ and invoking (5), we have

$$-\infty < \nu_0 := \inf_{\omega\in\mathbb{R}^*} \text{Re}\,\mathbf{G}(i\omega) = \inf_{\omega\in\mathbb{R}^*} \text{Re}(G(i\omega)/i\omega) \le 0. \tag{42}$$

Moreover, with a view to applying Theorem 1 to (41), observe that

$$\lim_{s\to 0} s\mathbf{G}(s) = G(0) > 0.$$

Let $\lambda_1 > 0$ be a Lipschitz constant for φ and define

$$\kappa_0 := \begin{cases} \dfrac{1}{(\lambda_1 \bar{\sigma} |\nu_0|)}, & \nu_0 \neq 0 \\ \infty, & \nu_0 = 0. \end{cases} \tag{43}$$

Assume that $k \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ is non-negative-valued with $\limsup_{t\to\infty} k(t) < \kappa_0$. Then there exist $t_0 \geq 0$ and $\alpha \in (0, \kappa_0 \lambda_1 \bar{\sigma})$ (in which case, $|\nu_0| < 1/\alpha$) such that

$$\bar{k}(t) := \frac{k(t)\varphi^\nabla(u(t))\theta(w(t))}{\alpha} \in [0, 1], \quad \forall\, t \geq t_0. \tag{44}$$

Note that f can be written in the form $f(t, \xi) = \bar{k}(t)\alpha\xi$ (and so (11) holds). Also, since $|\nu_0| < 1/\alpha$, we have

$$\frac{1}{\alpha} + \inf_{\omega \in \mathbb{R}^*} \operatorname{Re} \mathbf{G}(i\omega) > 0.$$

Proof of (a) and (b): An application of Theorem 1 to (41) now shows that $z^\infty := \lim_{t\to\infty} z(t) \in \ker \mathbf{A}$. Therefore, $\lim_{t\to\infty} z_1(t) = 0$ and $\lim_{t\to\infty} \varphi(u(t)) =: \varphi^r$ exists and is finite, whence statements (a) and (b).

Now assume that $k \notin L^1(\mathbb{R}_+)$.

Proof of (c) and (d): It suffices to show that $\psi(G(0)\varphi^r) = r$. Seeking a contradiction, suppose that $\psi(G(0)\varphi^r) \neq r$. This implies that $\delta := (r - \psi(G(0)\varphi^r))/2 \neq 0$. Using continuity of ψ, we obtain for sufficiently large $\tau > 0$

$$|y(t) - \psi(G(0)\varphi^r)| \leq |\delta|, \quad \forall\, t \geq \tau.$$

As a consequence, and noticing that $\dot{u}(t) = k(t)[r - y(t)] = k(t)[2\delta - y(t) + \psi(G(0)\varphi^r)]$, we have

$$-|\delta| k(t) \leq \dot{u}(t) - 2\delta k(t) \leq |\delta| k(t), \quad \forall\, t \geq \tau.$$

Since $\delta \neq 0$, either $\delta > 0$ or $\delta < 0$. Assume $\delta > 0$. Then $\dot{u}(t) \geq \delta k(t)$ for all $t \geq \tau$ which, on integration, yields $u(t) - u(\tau) \geq \delta \int_\tau^t k(s)ds$ for all $t \geq \tau$. Since $\int_\tau^t k(s)ds \to \infty$ as $t \to \infty$, we conclude that $u(t) \to \infty$ as $t \to \infty$, and hence, since φ is non-decreasing,

$$\sup \overline{\operatorname{im}}\, \varphi = \lim_{t\to\infty} \varphi(u(t)) = \varphi^r. \tag{45}$$

Let $\xi \in \overline{\operatorname{im}}\, \varphi$ with $\psi(G(0)\xi) = r$. Since ψ is non-decreasing, $G(0) > 0$ and $\delta > 0$, it follows that $\xi > \varphi^r$, contradicting (45). A similar argument shows that the assumption $\delta < 0$ also leads to a contradiction.

Proof of (e) and (f): To prove statements (e) and (f), assume that r is an interior point of $\mathcal{R}$. Using monotonicity of ψ and statement (a), we obtain that $\varphi^r = \lim_{t\to\infty} \varphi(u(t))$ is an interior point of $\operatorname{im} \varphi$. The monotonicity of φ then implies that u is bounded, yielding statement (e). Under the additional assumption that $\psi(G(0)w) \neq r$ for all strongly critical values w of φ, it follows that φ^r is not a strongly critical value of φ (since $\psi(G(0)\varphi^r) = r$ by statement (c)). Combined with the monotonicity of φ, this shows that $\varphi^{-1}(\varphi^r)$ is a singleton, that is $\varphi^{-1}(\varphi^r) = \{u^r\}$ for some $u^r \in \mathbb{R}$. By statement (a), $\lim_{t\to\infty} \varphi(u(t)) = \varphi^r$, and so, $\lim_{t\to\infty} u(t) = u^r$.

Proof of (g): It follows from an application of assertion 2 of Theorem 1 to (41) that there exist $L \geq 1$ and $\bar{\rho} > 0$ such that

$$\|z(t)\| \leq L \exp\left(-\bar{\rho} \int_0^t \bar{k}(\tau) d\tau\right), \quad \forall\, t \in \mathbb{R}_+,$$

where $\bar{k}$ is given by (44). Consequently, in order to show that the convergence of $\varphi(u(t))$, $x(t)$ and $y(t)$ is of order $\exp(-\rho \int_0^t k(\tau)d\tau)$ for some $\rho > 0$ as $t \to \infty$, it is sufficient to prove that $\liminf_{t\to\infty} \varphi^\nabla(u(t)) > 0$ and $\liminf_{t\to\infty} \theta(w(t)) > 0$. By statement (f), $\psi(G(0)\varphi(u^r)) = r$, and hence, by hypothesis, $\varphi(u^r)$ is not a critical value of φ, showing that $\varphi^\nabla(u^r) > 0$. Since φ is piecewise continuously differentiable it follows that

$$\liminf_{t\to\infty} \varphi^\nabla(u(t)) \geq \varphi^\nabla(u^r) > 0.$$

Furthermore, using that $\psi(G(0)\varphi^*) = \psi(G(0)\varphi^r) = r$ combined with the monotonicity of ψ and the assumption that r is not a critical value of ψ, we may conclude that $\varphi^* = \varphi^r$. It therefore follows that $w(t) = cz_1(t) + G(0)z_2(t) \to 0$ as $t \to \infty$. Consequently,

$$\liminf_{t\to\infty} \theta(w(t)) \geq \liminf_{\xi\to 0} \frac{\bar{\psi}(\xi)}{\xi} = \bar{\psi}^\nabla(0) = \psi^\nabla(G(0)\varphi^r) > 0,$$

where the last inequality follows from the fact that $\psi(G(0)\varphi^r) = r$ combined with the hypothesis that r is not a critical value of ψ.

To prove that the convergence of $u(t)$ is of order $\exp(-\rho \int_0^t k(\tau)d\tau)$ as $t \to \infty$, choose $\varepsilon > 0$ such that $\varphi^\nabla(\xi) > 0$ for all $\xi \in [u^r - \varepsilon, u^r + \varepsilon]$ (this is possible since $\varphi^\nabla(u^r) > 0$). Denoting the restriction of φ to $[u^r - \varepsilon, u^r + \varepsilon]$ by φ_ε, it is clear that the inverse function of φ_ε satisfies a Lipschitz condition, that is there exists $l > 0$ such that

$$|\varphi_\varepsilon^{-1}(\xi_1) - \varphi_\varepsilon^{-1}(\xi_2)| \leq l|\xi_1 - \xi_2|, \quad \forall\, \xi_1, \xi_2 \in \operatorname{im} \varphi_\varepsilon.$$

Choosing $\tau \geq 0$ sufficiently large, so that $u(t) \in [u^r - \varepsilon, u^r + \varepsilon]$ for all $t \geq \tau$, we obtain

$$|u(t) - u^r| = |\varphi_\varepsilon^{-1}(\varphi_\varepsilon(u(t))) - \varphi_\varepsilon^{-1}(\varphi_\varepsilon(u^r))| \\ \leq l|\varphi(u(t)) - \varphi^r|, \quad \forall\, t \geq \tau.$$

Since we have already proved that the convergence of $\varphi(u(t))$ is of order $\exp(-\rho\int_0^t k(\tau)d\tau)$ as $t \to \infty$, the above inequality shows that the same is true for the convergence of $u(t)$. □

Corollary 1: *Let the assumptions of Theorem* 3 *hold. Let* $\lambda_1 > 0$ *be a Lipschitz constant for the globally Lipschitz function* φ. *Additionally assume that* ψ *is globally Lipschitz*, *with Lipschitz constant* $\lambda_2 > 0$. *Then*

$$-\infty < \nu_0 = \inf_{\omega \in \mathbb{R}^*} \mathrm{Re}[G(i\omega)/i\omega] \leq 0$$

and, for each $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}$ *and every non-negative-valued function* $k \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ *satisfying*

$$\limsup_{t\to\infty} k(t) < \kappa_0 \\ := 1/(\lambda_1\lambda_2|\nu_0|) \quad \text{(with the convention } 1/0 := \infty), \quad (46)$$

the unique solution $t \mapsto (x(t), u(t))$ *of* (36) *on* $\mathbb{R}_+$ *has the properties* (a)–(g) *in Theorem* 3.

Proof: By (42), $-\infty < \nu_0 \leq 0$ and so, $0 < \kappa_0 \leq \infty$. Assume that the assumptions of Theorem 3 hold and that ψ is globally Lipschitz, with Lipschitz constant $\lambda_2 > 0$. Then (39) holds with $\bar{\sigma}$ replaced by λ_2. Formulae (43) and (46) for κ_0 then coincide and hence the claim follows from Theorem 3. □

Remark 4: If, under the assumptions of Theorem 3, sufficient plant information is available *a priori* to compute a suitable $\kappa_0 > 0$, then the tracking objective is achievable by a constant gain integral control

$$u(t) = u^0 + k\int_0^t [r - y(\tau)]d\tau, \quad u^0 \in \mathbb{R}, \quad k \in (0, \kappa_0).$$

In the case of globally Lipschitz output non-linearity ψ, Corollary 1 identifies the following as sufficient *a priori* information for the computation of κ_0: knowledge of Lipschitz constants λ_1 and λ_2 for the non-linearities φ and ψ, together with computability of the quantity $\nu_0 = \inf_{\omega\in\mathbb{R}^*} \mathrm{Re}[G(i\omega)/i\omega]$. In Logemann *et al.* (1999), it has been shown how $|\nu_0|$ (or upper bounds for $|\nu_0|$) can be obtained from frequency/step-response experiments. In the presence of time-invariant input non-linearities only, the "Popov" context of Theorem 2 can yield enhanced threshold gain values. This potential enhancement motivates the investigations in § 3.2 below.

Finally, we remark that if insufficient plant information is available a priori to compute a suitable value κ_0, then Theorem 3 ensures that the tracking objective is achievable by integral control with time-varying gain

$$u(t) = u^0 + \int_0^t k(\tau)[r - y(\tau)]d\tau, \quad u^0 \in \mathbb{R},$$

provided that $k \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ is non-negative-valued with $k \notin L^1(\mathbb{R}_+)$ and $k(t) \to 0$ as $t \to \infty$ (a canonical choice being k: $t \mapsto 1/(1+t)$): this control ensures the requisite performance for every system of the form (36) under the minimal hypotheses that A is Hurwitz, $G(0) > 0$, φ and ψ are piecewise continuously differentiable and non-decreasing, φ is globally Lipschitz, and (37) holds for some $\sigma > 0$.

3.2. *Integral control in the presence of input non-linearities only*

We now specialize to the case wherein the system under consideration has input non-linearity only. It will be shown that, in this situation, an application of the Popov-type result stated in Theorem 2 leads to a threshold gain value which in many cases is considerably larger than the constant κ_0 given by (46) (see § 3.3 for examples). With reference to figure 3, the problem of tracking constant reference signals $r \in \mathbb{R}$ will be addressed in the context of a class of finite-dimensional single-input single-output continuous-time real linear systems $\Sigma = (A, b, c, d)$ having a globally Lipschitz nonlinearity φ in the input channel:

$$\dot{x} = Ax + b\varphi(u), \quad x(0) = x^0 \in \mathbb{R}^n, \quad y = cx + d\varphi(u). \quad (47)$$

In (47), A is assumed to be Hurwitz. Furthermore, the transfer function G, given by $G(s) = c(sI - A)^{-1}b + d$, is assumed to satisfy $G(0) > 0$. To achieve the objective of tracking a constant reference signal $r \in \mathbb{R}$, we consider integral control action

$$u(t) = u^0 + k\int_0^t [r - y(\tau)]d\tau,$$

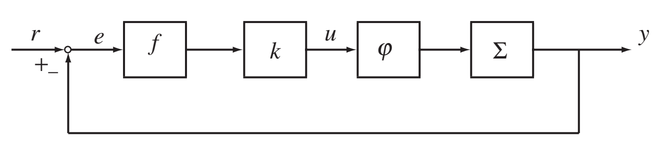

Figure 3. Constant low-gain integral control with input non-linearity.

with constant control gain $k > 0$, yielding the following non-linear feedback system (illustrated in figure 3)

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + b\varphi(u(t)), & x(0) &= x^0 \in \mathbb{R}^n, \\ \dot{u}(t) &= k[r - (cx(t) + d\varphi(u(t))], & u(0) &= u^0 \in \mathbb{R}. \end{aligned}\right\} \quad (48)$$

Theorem 4: *Consider the feedback system* (48). *Assume that A is Hurwitz, $G(0) = d - cA^{-1}b > 0$, φ is non-decreasing and globally Lipschitz, with Lipschitz constant $\lambda > 0$, and $r \in \mathbb{R}$ is such that $r/G(0) =: \varphi^r \in \operatorname{im}\varphi$. Define*

$$\nu_q := \inf_{\omega \in \mathbb{R}^*} \operatorname{Re}[(qi\omega + 1)G(i\omega)/i\omega], \quad \nu := \sup_{q \geq 0} \nu_q \quad \text{and}$$

$$\kappa := \begin{cases} \dfrac{1}{|\nu|\lambda}, & \nu < 0 \\ \infty, & \text{otherwise.} \end{cases} \quad (49)$$

Then $-\infty < \nu \leq \infty$ and $0 < \kappa \leq \infty$. For each $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}$ and $k \in (0, \kappa)$ there exists a unique continuously differentiable solution $t \mapsto (x(t), u(t))$ of (48) *defined on $\mathbb{R}_+$ (no finite escape time). The solution has the following properties.*

(a) $\lim_{t\to\infty} x(t) = -A^{-1}b\varphi^r$, $x + A^{-1}b\varphi^r \in L^2(\mathbb{R}_+)$ *and* $\dot{x} \in L^2(\mathbb{R}_+)$.
(b) $u^r := \lim_{t\to\infty} u(t)$ *exists, is finite,* $\varphi(u^r) = \varphi^r$, $\varphi(u) - \varphi^r \in L^2(\mathbb{R}_+)$ *and* $\dot{u} \in L^2(\mathbb{R}_+)$.
(c) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty}(cx(t) + d\varphi(u(t))) = r$ *and* $y - r \in L^2(\mathbb{R}_+)$.
(d) *If φ is piecewise continuously differentiable and φ^r is not a critical value of φ, then $u - u^r \in L^2(\mathbb{R}_+)$, where u^r is the limit of $u(t)$, the existence of which is guaranteed by* (b).

Proof: That $-\infty < \nu \leq \infty$ (and hence $0 < \kappa \leq \infty$) is an immediate consequence of (5). Let $(x^0, u^0, k) \in \mathbb{R}^n \times \mathbb{R} \times (0, \kappa)$. It follows from standard results in ordinary differential equations (see, for example, pp. 121/122 in Walter (1998)) combined with Gronwall's lemma that (48) has a unique continuously differentiable solution $t \mapsto (x(t), u(t))$ defined on $\mathbb{R}_+$.

Since $r/G(0) \in \operatorname{im}\varphi$, there exists $\varphi^r \in \operatorname{im}\varphi$ such that $G(0)\varphi^r = r$. Let $u^* \in \varphi^{-1}(\varphi^r)$ and define

$$f\colon \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \varphi(\xi + u^*) - \varphi^r.$$

Then $f(0) = 0$ and, since φ is non-decreasing and globally Lipschitz (with Lipschitz constant $\lambda > 0$), we have

$$0 \leq \xi f(\xi) \leq \lambda\xi^2, \quad \forall\, \xi \in \mathbb{R}.$$

Introduce new variables

$$z_1(t) := x(t) + A^{-1}b\varphi^r, \qquad z_2(t) := u(t) - u^*, \quad \forall\, t \in \mathbb{R}_+.$$

By direct calculation, we have

$$\begin{aligned} \dot{z}_1(t) &= Az_1(t) + bf(z_2(t)), \\ \dot{z}_2(t) &= -kcz_1(t) - kdf(z_2(t)) \end{aligned} \quad (50)$$

which, on writing

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad \mathbf{A} := \begin{bmatrix} A & 0 \\ -kc & 0 \end{bmatrix},$$
$$\mathbf{b} := \begin{bmatrix} -b \\ kd \end{bmatrix}, \quad \mathbf{c} := \begin{bmatrix} 0 & 1 \end{bmatrix},$$

takes the form of system (22), where $\mathbf{A}$ satisfies hypothesis (H). Moreover, setting $\mathbf{G}(s) = \mathbf{c}(sI - \mathbf{A})^{-1}\mathbf{b}$, we have

$$s\mathbf{G}(s) = k\big[c(sI - A)^{-1}b + d\big] = kG(s),$$

and so $\lim_{s\to 0} s\mathbf{G}(s) = kG(0) > 0$. Define $\varepsilon := (\kappa - k)/(2\lambda\kappa) > 0$ ($\varepsilon = 1/(2\lambda)$ if $\kappa = \infty$) and let $q \geq 0$ be such that $\inf_{\omega\in\mathbb{R}^*} \operatorname{Re}\big[(qi\omega + 1)G(i\omega)/i\omega\big] \geq \nu - (\varepsilon/k)$. Then,

$$\begin{aligned} &\inf_{\omega\in\mathbb{R}^*} \operatorname{Re}\big[(qi\omega + 1)\mathbf{G}(i\omega)\big] \\ &\quad = k \inf_{\omega\in\mathbb{R}^*} \operatorname{Re}\Big[(qi\omega + 1)G(i\omega)/i\omega\Big] \geq k\nu - \varepsilon \geq \varepsilon - \frac{1}{\lambda}. \end{aligned}$$

Therefore, the hypotheses of Theorem 2 hold with $\alpha = \lambda$ and so, by statement (c) of that theorem, $z(t) \to z^\infty \in \ker\mathbf{A}$ as $t \to \infty$ and $\dot{z} = (\dot{x}, \dot{u}) \in L^2(\mathbb{R}_+)$. It follows that $\lim_{t\to\infty} z_1(t) = 0$ and $\lim_{t\to\infty} z_2(t) = z_2^\infty \in f^{-1}(0)$. Moreover, by statement (b) of Theorem 2, $f(z_2) \in L^2(\mathbb{R}_+)$ and so, since A is Hurwitz, the first equation in (50) yields that $z_1 \in L^2(\mathbb{R}_+)$. Hence, statement (a) of the present theorem follows. From the convergence of z_2 we obtain that the limit $\lim_{t\to\infty} u(t) =: u^r$ exists and is finite. Since $f(z_2^\infty) = 0$, we may conclude that

$$\varphi(u^r) - \varphi^r = f(u^r - u^*) = f(z_2^\infty) = 0.$$

Furthermore, since $f(z_2) \in L^2(\mathbb{R}_+)$ and $\varphi(u) - \varphi^r = f(z_2)$, we have that $\varphi(u) - \varphi^r \in L^2(\mathbb{R}+)$, whence statement (b). To establish statement (c), simply observe that

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{t\to\infty} \big(cx(t) + d\varphi(u(t))\big) \\ &= (-cA^{-1}b + d)\varphi^r = G(0)\varphi^r = r \end{aligned}$$

and $r - y = \dot{u}/k \in L^2(\mathbb{R}_+)$. Finally, to prove statement (d), assume that φ is piecewise continuously differentiable and that φ^r is not a critical value of φ. Then f is piecewise continuously differentiable and $f'_-(\xi)f'_+(\xi) \neq 0$ for all $\xi \in f^{-1}(0)$. Therefore, by statement (d) of Theorem 2, $z - z^\infty \in L^2(\mathbb{R}_+)$, where $z^\infty := (0, z_2^\infty) = (0, u^r - u^*)$. Consequently, $u - u^r \in L^2(\mathbb{R}_+)$, completing the proof of statement (d) of the present theorem. □

3.3. *Discussion of the constant-gain case*

In the specific case of constant-gain integral control in the presence of time-invariant non-decreasing globally Lipschitz input nonlinearities only, Corollary 1 and Theorem 4 are both applicable: by (46) (with $\lambda_1 = \lambda$ and $\lambda_2 = 1$), the former yields the gain threshold value $\kappa_0 = 1/(\lambda|\nu_0|)$, whilst the latter provides, by (49), the threshold $\kappa = 1/(\lambda|\nu|)$. However, since $\kappa \geq \kappa_0$, Theorem 4 gives a potentially larger range of gains for which asymptotic tracking is guaranteed. Consequently, in the context of constant-gain integral control in the presence of input non-linearities only, it is advantageous to use Theorem 4 rather than Corollary 1. To illustrate this point, we present two classes of examples where κ is strictly greater than κ_0.

Example 1: Consider the transfer function G given by $G(s) = g_1/(s + g_2)$, where $g_1, g_2 > 0$. A routine calculation shows that $\nu_0 = -g_1/g_2^2$ and $\nu = 0$, so that $\kappa/\kappa_0 = |\nu_0|/|\nu| = \infty$.

Example 2: Consider the transfer function G given by $G(s) = g_1/(s^2 + g_2 s + g_3)$, where $g_1, g_2, g_3 > 0$. Defining

$$F(q,\omega) := \mathrm{Re}\,\frac{(qi\omega + 1)G(i\omega)}{i\omega} = g_1 \frac{q(g_3 - \omega^2) - g_2}{(g_3 - \omega^2)^2 + g_2^2\omega^2},$$

we have that, for all $q \geq 0$, $F(q, \pm\sqrt{g_3}) = -g_1/(g_2 g_3)$. Hence,

$$\nu = \sup_{q \geq 0} \nu_q = \sup_{q \geq 0}\left(\inf_{\omega \in \mathbb{R}} F(q,\omega)\right) \leq -\frac{g_1}{g_2 g_3}.$$

A routine argument shows that $\inf_{\omega\in\mathbb{R}} F(g_2/g_3, \omega) = -g_1/(g_2 g_3)$ and therefore we may conclude that

$$\nu = -\frac{g_1}{g_2 g_3}.$$

Next we compute ν_0. An elementary calculation shows that $(\partial F/\partial\omega)(0,\omega) = 0$ if and only if $\omega(2\omega^2 - 2g_3 + g_2^2) = 0$. We consider two cases.

Case 1: $2g_3 > g_2^2$.
Write $F(0,\omega)$ in the form

$$F(0,\omega) = -\frac{g_1 g_2}{g_3^2 + (g_2^2 - 2g_3 + \omega^2)\omega^2}.$$

Since the term $(g_2^2 - 2g_3 + \omega^2)\omega^2$ is negative for all sufficiently small $|\omega|$, $\omega \neq 0$, it follows that the function $\omega \mapsto F(0,\omega)$ has a local maximum at $\omega = 0$ and a global minimum at $\omega = \pm\sqrt{(2g_3 - g_2^2)/2}$. Thus,

$$\nu_0 = F\left(0, \pm\sqrt{\frac{(2g_3 - g_2^2)}{2}}\right) = -\frac{g_1}{g_2(g_3 - g_2^2/4)},$$

and consequently,

$$\frac{\kappa}{\kappa_0} = \frac{|\nu_0|}{|\nu|} = \frac{\nu_0}{\nu} = \frac{g_3}{g_3 - g_2^2/4} \in (1,2).$$

Case 2: $2g_3 \leq g_2^2$.
In this case, the function $\omega \mapsto F(0,\omega)$ has a global minimum at $\omega = 0$ and $\nu_0 = F(0,0) = -g_1 g_2/g_3^2$. Consequently,

$$\frac{\kappa}{\kappa_0} = \frac{|\nu_0|}{|\nu|} = \frac{\nu_0}{\nu} = \frac{g_2^2}{g_3} \in [2,\infty).$$

Finally, let $\lambda > 0$ and let $(A, b, c, 0)$ be an asymptotically stable realization of $G(s)$. Then the conclusions of Theorem 4 are valid for all non-decreasing globally Lipschitz non-linearities φ with Lipschitz constant λ, all reference values $r \in \mathbb{R}$ with $rg_3/g_1 \in \mathrm{im}\,\varphi$ and all initial values, provided that the gain k is in the interval $(0,\kappa)$, where $\kappa = 1/(|\nu|\lambda) = g_2 g_3/(g_1\lambda)$. We claim that κ is the largest number with this property. To this end consider the (linear) function φ given by $\varphi(\xi) = \lambda\xi$ and the reference value $r = 0$. An application of the Routh-Hurwitz criterion then shows that the corresponding (linear) closed-loop system is asymptotically stable if and only if $k \in (0,\kappa)$, implying in particular that for every $k \geq \kappa$, there exists an initial condition for which statement (a) of Theorem 4 is not valid. In this context it is interesting to note that Theorem 2, together with the Routh-Hurwitz criterion, yields that the Aizerman conjecture holds for any stabilizable and detectable realization of the transfer function $\mathbf{G}(s) = G(s)/s = g_1/(s^3 + g_2 s^2 + g_3 s)$.

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