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Absolute-stability results in infinite dimensions

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We derive absolute-stability results of Popov and circle-criterion type for infinite-dimensional systems in an input–output setting. Our results apply to feedback systems in which the linear part is the series interconnection of an input–output stable linear system and an integrator, and the nonlinearity satisfies a sector condition which, in particular, allows for saturation and deadzone effects. We use the input–output theory developed to derive state-space results on absolute stability applying to feedback systems in which the linear part is the series interconnection of an exponentially stable, well-posed infinite-dimensional system and an integrator.

Keywords: absolute stability; circle criterion; infinite-dimensional systems; integral equations; Popov criterion; positive-real

1. Introduction

Consider the feedback system shown in figure 1, where L is a linear right-shift-invariant operator and N is a (possibly time-varying) static nonlinearity. For simplicity of presentation, we assume in this paragraph that L and N are 'scalar' systems, that is, L and N have only one input and one output channel. A sector condition for N is a condition of the form

$$a_1 v^2 \leqslant N(t,v)v \leqslant a_2 v^2, \quad \forall (t,v) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1.1}$$

where $-\infty \leqslant a_1 \leqslant a_2 \leqslant \infty$ and at least one of the sector bounds a_1 and a_2 is finite. Standard examples of sector-bounded nonlinearities are given by deadzone and saturation, both of which arise naturally in control engineering. An absolute-stability result for the feedback system shown in figure 1 is a stability criterion in terms of the transfer function or the frequency response of the linear system L and the sector bounds a_1 and a_2 of the nonlinearity N. Note that, given a linear system L and sector data a_1 and a_2, an absolute-stability criterion guarantees closed-loop stability for all nonlinearities N satisfying the sector condition (1.1).

Absolute-stability problems and their relations to positive-real conditions have played a prominent role in systems and control theory and have led to a number of important stability criteria for unity feedback controls applied to linear dynamical systems subject to static input or output nonlinearities (see Aizerman & Gantmacher

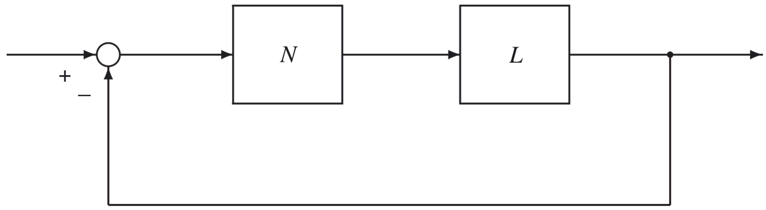

Figure 1. Basic feedback system

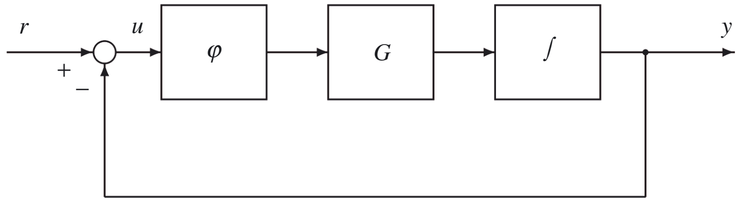

Figure 2. Feedback system with integrator

(1964), Hahn (1967), Khalil (1996), Lefschetz (1965), Leonov *et al.* (1996), Narendra & Taylor (1973), Popov (1962), Sastry (1999), Vidyasagar (1993) and Willems (1970) for the finite-dimensional case, and Bucci (1999), Corduneanu (1973), Curtain *et al.* (2003), Curtain & Oostveen (2001), Desoer & Vidyasagar (1975), Grabowski & Callier (2002), Leonov *et al.* (1996), Logemann & Curtain (2000), Mees (1981), Sastry (1999), Vidyasagar (1993) and Wexler (1979, 1980) for the infinite-dimensional case, to mention just a few references).

In this paper we study an absolute-stability problem for the feedback system shown in figure 2. The input–output operator G is linear, right-shift invariant and bounded from $L^p(\mathbb{R}_+, U)$ into itself for both $p = 2$ and $p = \infty$, where U is a real separable Hilbert space. The nonlinearity φ is static, possibly time-varying, and satisfies a certain sector condition (more details are given below). It is well known that G can be represented by a transfer function $\boldsymbol{G}$ which is analytic and bounded on the open right-half of the complex plane. For simplicity, we assume in the introduction that $\boldsymbol{G}$ admits an analytic extension to an open neighbourhood of 0 (this assumption will be weakened in §§ 2 and 3). In less general contexts (more restrictive assumptions on G, (finite-dimensional) state-space settings, $\dim U = 1$), the feedback scheme shown in figure 2 has been frequently considered in absolute-stability theory (see, for example, Bucci 1999; Corduneanu 1973, p. 91; Lefschetz 1965, p. 19; Wexler 1980). In particular, the nonlinear feedback systems considered originally by Lur'e (1957, p. 44) and Popov (1962) can be rewritten in the form given by figure 2 (see Lefschetz 1965, p. 18 and p. 87, respectively). We mention that, by right-shift invariance, the operator G commutes with integration; consequently, G and the integrator may be interchanged in figure 2. Due to the integrator (which arises naturally in many control systems), the feedback system shown in figure 2 is sometimes said to be of indirect type (see Lefschetz 1965, p. 18; Popov 1962).

In § 2 it is shown that if φ satisfies certain standard regularity conditions (including a Lipschitz-type condition) and $r : \mathbb{R}_+ \to U$ is continuous, then the feedback system in figure 2 has a unique continuous solution u which can be continued to the right as long as it remains bounded. In fact, the existence and uniqueness result in § 2 (proposition 2.1) is more general in the sense that it allows for unbounded G. The first theorem in § 3 (theorem 3.3), a result of Popov type, assumes that φ is time-independent. It shows, in particular, that if $\boldsymbol{G}(0)$ is invertible and if there exists a linear, bounded, self-adjoint operator $P : U \to U$, a linear, bounded, invertible operator $Q : U \to U$ with $Q\boldsymbol{G}(0) = [Q\boldsymbol{G}(0)]^* \geqslant 0$ and a number $q \geqslant 0$ such that

$$P + \frac{1}{2}\left(q\boldsymbol{G}(\mathrm{i}\omega) + \frac{1}{\mathrm{i}\omega}Q\boldsymbol{G}(\mathrm{i}\omega) + q\boldsymbol{G}^*(\mathrm{i}\omega) - \frac{1}{\mathrm{i}\omega}\boldsymbol{G}^*(\mathrm{i}\omega)Q^*\right) \geqslant 0, \quad \text{a.e. } \omega \in \mathbb{R}, \tag{1.2}$$

then for any $r : \mathbb{R}_+ \to U$ with $\dot{r}, \ddot{r} \in L^1(\mathbb{R}_+, U)$ and for any locally Lipschitz gradient field $\varphi : U \to U$ with a non-negative potential and such that

$$\langle \varphi(v), Qv \rangle \geqslant \langle \varphi(v), P\varphi(v) \rangle, \quad \forall v \in U,$$

the solution u of the feedback system shown in figure 2 exists on $\mathbb{R}_+$ (no finite escape time), $u, \dot{u}, y \in L^\infty(\mathbb{R}_+, U)$ and, under certain mild extra assumptions, $u(t)$ and $y(t)$ converge as $t \to \infty$ and $\lim_{t\to\infty} \varphi(u(t)) = 0$.

The second theorem in § 3 (theorem 3.10), a result of circle-criterion type, shows that if the positive-real condition (1.2) holds with $q = 0$, then we can allow the nonlinearity φ to be time-varying. More precisely, we show that if $\boldsymbol{G}(0)$ is invertible and if there exist a linear, bounded, self-adjoint operator $P : U \to U$ and a linear, bounded, invertible operator $Q : U \to U$ with $Q\boldsymbol{G}(0) = [Q\boldsymbol{G}(0)]^* \geqslant 0$ and such that (1.2) holds with $q = 0$, then for any $r : \mathbb{R}_+ \to U$ with $\dot{r} \in L^1(\mathbb{R}_+, U)$ and for any nonlinearity $\varphi : \mathbb{R}_+ \times U \to U$ satisfying certain standard regularity assumptions (including a Lipschitz-type condition) and such that

$$\langle \varphi(t, v), Qv \rangle \geqslant \langle \varphi(t, v), P\varphi(t, v) \rangle, \quad \forall (t, v) \in \mathbb{R}_+ \times U, \tag{1.3}$$

the solution u of the feedback system shown in figure 2 exists on $\mathbb{R}_+$ (no finite escape time) and $u, y \in L^\infty(\mathbb{R}_+, U)$.

If $Q = I$ and $P = (1/a)I$ for some $0 < a \leqslant \infty$, then (1.3) is equivalent to the standard sector condition

$$\langle \varphi(t, v), v \rangle \geqslant \frac{\|\varphi(t, v)\|^2}{a}, \quad \forall (t, v) \in \mathbb{R}_+ \times U,$$

or, equivalently,

$$\left\langle \varphi(t, v), \frac{\varphi(t, v)}{a} - v \right\rangle \leqslant 0, \quad \forall (t, v) \in \mathbb{R}_+ \times U.$$

Section 4 is devoted to applications of the input–output results in § 3 to the class of well-posed state-space systems which are documented, for example, in Curtain & Weiss (1989), Salamon (1987, 1989), Staffans (1997, 2001, 2004), Staffans & Weiss (2002) and Weiss (1989, 1994). We remark that the class of well-posed, linear, infinite-dimensional systems allows for considerable unboundedness in the control and observation operators and is therefore rather general: it includes most distributed parameter systems and time-delay systems (retarded and neutral) which are of interest in applications. Finally, we mention that in Curtain *et al.* (2003) the same absolute-stability problems were studied under the considerably stronger assumption that (1.2) holds with εI (for some $\varepsilon > 0$) replacing 0 on the right-hand side of (1.2).

However, in the present paper the assumptions on G are stronger than in Curtain *et al.* (2003) (where it was not required that G is L^∞-stable) and the assumptions on r are somewhat different to those imposed in Curtain *et al.* (2003). Moreover, the conclusions of theorems 3.3 and 3.10 also differ from those of the main results in Curtain *et al.* (2003). Therefore, the results in this paper and in Curtain *et al.* (2003) are difficult to compare: while the two papers are complementary, it would be inappropriate to interpret the current paper as a generalization of Curtain *et al.* (2003).

(*a*) *Notation and terminology*

Let X be a real or complex Banach space and let $1 \leqslant p \leqslant \infty$. For $\tau \geqslant 0$, let $\boldsymbol{R}_\tau$ denote the corresponding right-shift operator on $L^p_{\mathrm{loc}}(\mathbb{R}_+, X)$, where $\mathbb{R}_+ := [0, \infty)$. For $0 < \tau < \tau^* \leqslant \infty$, the truncation operator $\boldsymbol{P}_\tau : L^p_{\mathrm{loc}}([0, \tau^*), X) \to L^p(\mathbb{R}_+, X)$ is given by $(\boldsymbol{P}_\tau u)(t) = u(t)$ if $t \in [0, \tau]$ and $(\boldsymbol{P}_\tau u)(t) = 0$ if $t > \tau$. For $\alpha \in \mathbb{R}$, we define the exponentially weighted L^p-space

$$L^p_\alpha(\mathbb{R}_+, X) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}_+, X) \mid f(\cdot)\exp(-\alpha\cdot) \in L^p(\mathbb{R}_+, X)\}$$

and endow it with the norm

$$\|f\|_{p,\alpha} := \left(\int_0^\infty \|\mathrm{e}^{-\alpha\theta} f(\theta)\|^p \, \mathrm{d}\theta \right)^{1/p}.$$

For an arbitrary interval $J \subset \mathbb{R}_+$, $C(J, X)$ denotes the space of all continuous functions defined on J with values in X. For $a \in \mathbb{R}$, let $W^{1,1}_{\mathrm{loc}}([a, \infty), X)$ denote the space of all functions $f \in L^1_{\mathrm{loc}}([a, \infty), X)$ for which there exists $g \in L^1_{\mathrm{loc}}([a, \infty), X)$ such that

$$f(t) - f(a) = \int_a^t g(\theta) \, \mathrm{d}\theta \quad \forall t \geqslant a.$$

Moreover, $W^{2,1}_{\mathrm{loc}}([a, \infty), X)$ denotes the space of all functions $f \in L^1_{\mathrm{loc}}([a, \infty), X)$ for which there exists $g \in W^{1,1}_{\mathrm{loc}}([a, \infty), X)$ such that

$$f(t) - f(a) = \int_a^t g(\theta) \, \mathrm{d}\theta \quad \text{for all } t \geqslant a.$$

For $\alpha \in \mathbb{R}$, $H^\infty(\mathbb{C}_\alpha, X)$ denotes the space of bounded holomorphic functions defined on $\mathbb{C}_\alpha$ with values in X, where $\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\}$. We say that a function $f : \mathbb{R}_+ \to X$ approaches a non-empty subset $W \subset X$ as $t \to \infty$, if

$$\lim_{t \to \infty} (\inf\{\|f(t) - w\| \mid w \in W\}) = 0.$$

$\mathcal{B}(X, Y)$ denotes the space of bounded linear operators from X to Y (another Banach space); we write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$. Let $A : \mathrm{dom}(A) \subset X \to X$ be a densely defined linear operator, where $\mathrm{dom}(A)$ denotes the domain of A. The resolvent set of A is denoted by $\varrho(A)$. X_1 denotes the space $\mathrm{dom}(A)$ endowed with the graph norm of A, while X_{-1} denotes the completion of X with respect to the norm $\|x\|_{-1} = \|(\alpha I - A)^{-1} x\|$, where $\alpha \in \varrho(A)$ (different choices of α lead to equivalent norms) and $\|\cdot\|$ denotes the norm on X. The norm on X_1 is denoted by $\|x\|_1$. Clearly, $X_1 \subset X \subset X_{-1}$ and the canonical injections are bounded and dense. If A generates a strongly continuous semigroup $\boldsymbol{T} = (\boldsymbol{T}_t)_{t\geqslant 0}$ on X, then $\boldsymbol{T}$ restricts to a strongly continuous semigroup on X_1 and extends to a strongly continuous semigroup on X_{-1} with the exponential growth constant being the same on all three spaces. Correspondingly, A restricts to a generator on X_1 and extends to a generator on X_{-1}. We shall use the same symbol $\boldsymbol{T}$ (respectively, A) for the original semigroup (respectively, generator) and the associated restrictions and extensions: with this convention, we may write $A \in \mathcal{B}(X, X_{-1})$ (considered as a generator on X_{-1}, the domain of A is X). The Laplace transform is denoted by $\mathfrak{L}$.

2. Existence and uniqueness of solutions to the feedback system

Throughout this section let U be a real Hilbert space. Let $G : L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ be a linear, continuous and causal operator, where we regard the space $L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ as a Fréchet space with its topology given by the family of seminorms $u \mapsto \|\boldsymbol{P}_n u\|_{L^2}$, $n \in \mathbb{N}$. Recall that G is called causal if $\boldsymbol{P}_\tau G \boldsymbol{P}_\tau = \boldsymbol{P}_\tau G$ for all $\tau > 0$ (equivalently, G is causal if, for every $\tau > 0$ and every $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ such that $u = 0$ on $[0, \tau]$, we have that $Gu = 0$ on $[0, \tau]$.) Note that a linear operator $G : L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ is continuous and causal if and only if for every $\tau \in \mathbb{R}$ there exists a constant $\gamma_\tau \geqslant 0$ such that

$$\|\boldsymbol{P}_\tau G u\|_{L^2} \leqslant \gamma_\tau \|\boldsymbol{P}_\tau u\|_{L^2}, \quad \forall u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U).$$

Consider the following Volterra equation

$$u(t) = r(t) - \int_0^t (G(\varphi \circ u))(\theta)\,\mathrm{d}\theta, \quad t \geqslant 0, \tag{2.1}$$

which describes the feedback system shown in figure 2. In (2.1), $r : \mathbb{R}_+ \to U$ is the input of the feedback system (or forcing function), $\varphi : \mathbb{R}_+ \times U \to U$ is a time-dependent nonlinearity and $\varphi \circ u$ denotes the function $t \mapsto \varphi(t, u(t))$. Let $0 < T \leqslant \infty$. In order to define the concept of a solution of (2.1) on $[0, T)$, we need to give a meaning to Gv for $v \in L^2_{\mathrm{loc}}([0, T), U)$ if T is finite (recall that G operates on L^2_{loc}-functions defined on the whole time-axis $\mathbb{R}_+$). This can be done as follows: we define an operator $G_T : L^2_{\mathrm{loc}}([0, T), U) \to L^2_{\mathrm{loc}}([0, T), U)$ by setting

$$(G_T v)(t) = (G\boldsymbol{P}_\tau v)(t), \quad 0 \leqslant t \leqslant \tau < T.$$

Since G is causal, this definition does not depend on the choice of τ and so G_T is well defined. Note that $G_T(L^2([0, T), U)) \subset L^2([0, T), U)$ for finite T. In the following we will not distinguish between G and G_T and we omit the subscript T.

A function $u : [0, T) \to U$ is called a *solution* of (2.1) on $[0, T)$, if the function $t \mapsto \varphi(t, u(t))$ is in $L^2_{\mathrm{loc}}([0, T), U)$ (so that $G(\varphi \circ u)$ is defined on the interval $[0, T)$) and (2.1) holds for all $t \in [0, T)$. If r is continuous, then so is the right-hand side of (2.1), and hence any solution u of (2.1) is then necessarily continuous.

The following lemma shows that if r is continuous and φ satisfies certain standard regularity conditions (including a Lipschitz-type condition), then (2.1) has a unique continuous solution which can be continued as long as it remains bounded.

Proposition 2.1. *Let $G : L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ be a linear, continuous and causal operator, let $r \in C(\mathbb{R}_+, U)$ and let $\varphi\colon \mathbb{R}_+ \times U \to U$ be such that $t \mapsto \varphi(t, v)$*

is measurable for every $v \in U$, $t \mapsto \varphi(t,0)$ *is in* $L^2_{\mathrm{loc}}(\mathbb{R}_+,U)$ *and, for every bounded set* $V \subset U$, *there exists* $\lambda_V \in L^2_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R})$ *such that*

$$\sup_{v,w\in V} \frac{\|\varphi(t,v)-\varphi(t,w)\|}{\|v-w\|} \leqslant \lambda_V(t), \quad \text{a.e. } t \geqslant 0. \tag{2.2}$$

Then the Volterra equation (2.1) has at most one solution on any given interval $[0,\tau)$, *where* $0<\tau\leqslant\infty$, *and it has a unique continuous solution* u *defined on a maximal interval of existence* $[0,T)$, *where* $0<T\leqslant\infty$. *If* $T<\infty$, *then* $\limsup_{t\to T}\|u(t)\|=\infty$. *Under the additional assumption that there exists a function* $\gamma \in L^2_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R})$ *such that*

$$\|\varphi(t,v)\| \leqslant \gamma(t)(1+\|v\|), \quad \forall (t,v)\in\mathbb{R}_+\times U, \tag{2.3}$$

we have $T=\infty$.

Proof. It has been proved in Curtain *et al.* (2003) that (2.1) has at most one solution on any given interval $[0,\tau)$ (where $0<\tau\leqslant\infty$) and that it has a unique continuous solution defined on a maximal interval of existence $[0,T)$. Moreover, it was shown in Curtain *et al.* (2003) that if $T<\infty$, then $\limsup_{t\to T}\|u(t)\|=\infty$. To prove the remaining claim, assume that there exists $\gamma \in L^2(\mathbb{R}_+,\mathbb{R})$ such that (2.3) holds. Let u be the unique continuous solution of (2.1) defined on the maximal interval of existence $[0,T)$. Seeking a contradiction, assume that $T<\infty$. Then $\limsup_{t\to T}\|u(t)\|=\infty$; in particular, u is unbounded on $[0,T)$. It follows from (2.1) that

$$\|u(t)\| \leqslant \|r(t)\| + \beta\sqrt{t}\biggl(\int_0^t \|(\varphi\circ u)(\theta)\|^2\,\mathrm{d}\theta\biggr)^{1/2} \quad \forall t\in[0,T),$$

where $\beta>0$ is a suitable constant which exists by the continuity of G on $L^2_{\mathrm{loc}}(\mathbb{R}_+,U)$. Consequently,

$$\|u(t)\|^2 \leqslant 2\|r(t)\|^2 + 2\beta^2 t\int_0^t \gamma^2(\theta)(1+\|u(\theta)\|)^2\,\mathrm{d}\theta, \quad \forall t\in[0,T).$$

Setting $\alpha := 2\max_{t\in[0,T]}\|r(t)\|^2 + 4T\beta^2\int_0^T \gamma^2(\theta)\,\mathrm{d}\theta$, we may conclude that

$$\|u(t)\|^2 \leqslant \alpha + 4T\beta^2\int_0^t \gamma^2(\theta)\|u(\theta)\|^2\,\mathrm{d}\theta, \quad \forall t\in[0,T).$$

An application of Gronwall's lemma (see, for example, Desoer & Vidyasagar 1975, p. 252; or Khalil 1996, p. 63) now shows that u is bounded on $[0,T)$, yielding a contradiction. ■

3. Absolute stability results in an input–output setting

Throughout this section let U be a real separable Hilbert space. Let $\alpha\in\mathbb{R}$ and let $G\in\mathcal{B}(L^2_\alpha(\mathbb{R}_+,U))$ be right-shift invariant, i.e. $\boldsymbol{R}_\tau G = G\boldsymbol{R}_\tau$ for all $\tau\geqslant 0$. As a consequence of right-shift invariance, G is causal, and so G can be extended to a linear, continuous, right-shift-invariant (and hence causal) operator mapping $L^2_{\mathrm{loc}}(\mathbb{R}_+,U)$ into itself. We shall use the same symbol G to denote the original operator on $L^2_\alpha(\mathbb{R}_+,U)$ and its right-shift-invariant extension to $L^2_{\mathrm{loc}}(\mathbb{R}_+,U)$. As is well-known, a right-shift-invariant operator $G\in\mathcal{B}(L^2_\alpha(\mathbb{R}_+,U))$ has a *transfer function* $\boldsymbol{G}\in H^\infty(\mathbb{C}_\alpha,\mathcal{B}(U_{\mathrm{c}}))$ in the sense that

$$(\mathfrak{L}(Gu))(s) = \boldsymbol{G}(s)\mathfrak{L}(u)(s), \quad \forall u\in L^2_\alpha(\mathbb{R}_+,U),\ \forall s\in\mathbb{C}_\alpha,$$

where U_c denotes the complexification of U. The operator G is called *regular* if $\boldsymbol{G}(s)$ converges in the strong operator topology as $s \to \infty$ in $\mathbb{R}_+$. Furthermore, we say that G is *weakly regular* if $\boldsymbol{G}(s)$ converges in the weak operator topology as $s \to \infty$ in $\mathbb{R}_+$. Of course, if $\dim U < \infty$, then the concepts of regularity and weak regularity coincide.

Under the additional assumption $G \in \mathcal{B}(L^\infty_\alpha(\mathbb{R}_+, U))$, the following proposition shows that G is weakly regular and that, in the case of finite-dimensional U, G is a convolution operator, the kernel of which is a matrix-valued Borel measure on $\mathbb{R}_+$.

Proposition 3.1. *Let $G \in \mathcal{B}(L^2_\alpha(\mathbb{R}_+, U))$ be a right-shift-invariant operator and assume that $G \in \mathcal{B}(L^\infty_\alpha(\mathbb{R}_+, U))$; then G is weakly regular. Moreover, under the additional assumption that U is finite-dimensional (that is, $U = \mathbb{R}^m$ for some positive integer m), G is a convolution operator, the kernel of which is a locally bounded $\mathbb{R}^{m\times m}$-valued Borel measure μ on $\mathbb{R}_+$ such that the exponentially weighted measure μ_α is bounded, where μ_α is defined by $\mathrm{d}\mu_\alpha = \mathrm{e}^{-\alpha\cdot}\,\mathrm{d}\mu$.*

Clearly, any locally bounded $\mathbb{R}^{m\times m}$-valued Borel measure μ on $\mathbb{R}_+$, with the property that the exponentially weighted measure μ_α is bounded, defines via $u \mapsto \mu \star u$ a linear right-shift-invariant operator in $\mathcal{B}(L^2_\alpha(\mathbb{R}_+, \mathbb{R}^m)) \cap \mathcal{B}(L^\infty_\alpha(\mathbb{R}_+, \mathbb{R}^m))$. Proposition 3.1 says that all right-shift-invariant operators in $\mathcal{B}(L^2_\alpha(\mathbb{R}_+, \mathbb{R}^m)) \cap \mathcal{B}(L^\infty_\alpha(\mathbb{R}_+, \mathbb{R}^m))$ are of this form. A proof of proposition 3.1 can be found in the appendix.

For the rest of this section we assume that $\alpha = 0$. Since U is separable, the transfer function $\boldsymbol{G}$ of a right-shift-invariant operator $G \in \mathcal{B}(L^2(\mathbb{R}_+, U))$ has strong non-tangential limits at almost every point $\mathrm{i}\omega$ on the imaginary axis (see Rosenblum & Rovnyak 1985, theorem B on p. 85) and this limit is denoted by $\boldsymbol{G}(\mathrm{i}\omega)$ (whenever it exists). We introduce the following assumption.

Assumption 3.2. The limit $\boldsymbol{G}(0) := \lim_{s\to 0,\ s\in\mathbb{C}_0} \boldsymbol{G}(s)$ exists and

$$\limsup_{s\to 0,\ s\in\mathbb{C}_0} \left\| \frac{1}{s}(\boldsymbol{G}(s) - \boldsymbol{G}(0)) \right\| < \infty.$$

We first consider the feedback system shown in figure 2 for a class of time-independent nonlinearities φ, so-called gradient fields, a concept which we now define. For a C^1 function $\Phi\colon U \to \mathbb{R}$, let $\Phi'\colon U \to U^*$ denote the derivative of Φ. Using the Riesz representation theorem for Hilbert spaces, we define the gradient $\nabla\Phi\colon U \to U$ of Φ by

$$\langle (\nabla\Phi)(v), w\rangle = [(\Phi')(v)](w), \quad \forall w \in U.$$

For all $v \in U$ we have that $\|(\nabla\Phi)(v)\| = \|\Phi'(v)\|$, and therefore, by the C^1 property of Φ, the gradient $\nabla\Phi$ is continuous. As usual, a function $\varphi\colon U \to U$ is called a gradient field if there exists a C^1 function $\Phi\colon U \to \mathbb{R}$ (sometimes called potential) such that $\varphi = \nabla\Phi$.

If $\lim_{t\to\infty} f(t)$ exists for a function $f : \mathbb{R}_+ \to U$, we denote this limit by f^∞, i.e. $f^\infty := \lim_{t\to\infty} f(t)$. We say that a (strongly) measurable function $f : \mathbb{R}_+ \to U$ has an *essential limit* at ∞ if there exists $l \in U$ such that $\operatorname{ess\,sup}_{\tau\geqslant t} \|f(\tau) - l\|$ tends to 0 as $t \to \infty$ and we write $\operatorname{ess\,lim}_{t\to\infty} f(t) = l$. It is a routine exercise to show that if two functions $f, g : \mathbb{R}_+ \to U$ are equal almost everywhere, then $\operatorname{ess\,lim}_{t\to\infty} f(t)$ exists if and only if $\operatorname{ess\,lim}_{t\to\infty} g(t)$ exists, in which case the two limits coincide. Moreover, $\operatorname{ess\,lim}_{t\to\infty} f(t) = l$ if and only if there exists a function $\tilde{f} : \mathbb{R}_+ \to U$ such that $\tilde{f}(t) = f(t)$ for almost all $t \in \mathbb{R}_+$ and $\lim_{t\to\infty}\tilde{f}(t) = l$. If f is continuous, then ess $\lim_{t\to\infty} f(t)$ exists if and only if $\lim_{t\to\infty} f(t)$ exists, in which case the two limits coincide. We are now in the position to formulate the first one of the two main results of this section, a stability criterion of Popov type.

Theorem 3.3. *Let $G \in \mathcal{B}(L^2(\mathbb{R}_+, U)) \cap \mathcal{B}(L^\infty(\mathbb{R}_+, U))$ be a right-shift-invariant operator with transfer function $\boldsymbol{G}$ satisfying assumption 3.2 with $\boldsymbol{G}(0)$ invertible, and let $\varphi\colon U \to U$ be a locally Lipschitz continuous gradient of a non-negative C^1 function Φ. Let $r \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}_+, U)$ with*

$$\dot{r} \in L^1(\mathbb{R}_+, U), \qquad \ddot{r} \in L^1(\mathbb{R}_+, U). \tag{3.1}$$

Moreover, assume that there exist $P, Q \in \mathcal{B}(U)$ with P self-adjoint, Q invertible and $Q\boldsymbol{G}(0) = [Q\boldsymbol{G}(0)]^* \geqslant 0$ and $q \geqslant 0$ such that

$$\langle \varphi(v), Qv\rangle \geqslant \langle \varphi(v), P\varphi(v)\rangle, \quad \forall v \in U, \tag{3.2}$$

$$P + \frac{1}{2}\left(q\boldsymbol{G}(\mathrm{i}\omega) + \frac{1}{\mathrm{i}\omega}Q\boldsymbol{G}(\mathrm{i}\omega) + q\boldsymbol{G}^*(\mathrm{i}\omega) - \frac{1}{\mathrm{i}\omega}\boldsymbol{G}^*(\mathrm{i}\omega)Q^*\right) \geqslant 0, \quad \text{a.e. } \omega \in \mathbb{R}. \tag{3.3}$$

Let u denote the unique continuous solution of (2.1) defined on a maximal interval of existence $[0, T)$. Then the following conclusions hold.

(i) *The solution u of (2.1) exists on $\mathbb{R}_+$ (that is, $T = \infty$) and there exists a constant $K > 0$ (which depends only on Q and $\boldsymbol{G}$ but not on r) such that*

$$\|u\|_{L^\infty} + \sup_{t\geqslant 0}\left\|\int_0^t (\varphi\circ u)(\theta)\,\mathrm{d}\theta\right\| + \left(\int_0^\infty \langle(\varphi\circ u)(\theta), Qu(\theta) - P(\varphi\circ u)(\theta)\rangle\,\mathrm{d}\theta\right)^{1/2} \leqslant K\eta(r), \tag{3.4}$$

where

$$\eta(r) := \sqrt{q\Phi(r(0))} + \|r(0)\| + \|\dot{r}\|_{L^1} + q\|\ddot{r}\|_{L^1}. \tag{3.5}$$

(ii) $\dot{u} \in L^\infty(\mathbb{R}_+, U)$, $\varphi\circ u \in L^\infty(\mathbb{R}_+, U)$, $G(\varphi\circ u) \in L^\infty(\mathbb{R}_+, U)$,

$$\int_0^{\cdot} (G(\varphi\circ u))(\theta)\,\mathrm{d}\theta \in L^\infty(\mathbb{R}_+, U)$$

and

$$\lim_{t\to\infty}\langle(\varphi\circ u)(t), Qu(t) - P(\varphi\circ u)(t)\rangle = 0. \tag{3.6}$$

(iii) *Under the following two additional assumptions, $u(t)$ approaches $\varphi^{-1}(\{0\})$ as $t \to \infty$ and $\lim_{t\to\infty}(\varphi\circ u)(t) = 0$.*

Assumption 3.4. $\varphi^{-1}(\{0\}) \cap V$ is compact for any bounded closed set $V \subset U$.

Assumption 3.5. $\inf_{v\in V}\langle\varphi(v), Qv - P\varphi(v)\rangle > 0$ for any bounded, closed and non-empty set $V \subset U$ which does not intersect $\varphi^{-1}(\{0\})$.

Under assumptions 3.4 and 3.5 and assumption 3.6 below, $u(t)$ converges to a limit $u^\infty \in \varphi^{-1}(\{0\})$ as $t \to \infty$.

Assumption 3.6. $\varphi^{-1}(\{0\}) \cap V$ is totally disconnected for any bounded closed set $V \subset U$.

(iv) *Assume that assumptions 3.4–3.6 hold and that the operator H defined by*

$$(Hw)(t) := \int_0^t (Gw)(\theta)\,\mathrm{d}\theta - \boldsymbol{G}(0)\int_0^t w(\theta)\,\mathrm{d}\theta, \quad \forall w \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U),\ \forall t \in \mathbb{R}_+, \tag{3.7}$$

has the property that $\lim_{t\to\infty}(Hw)(t) = 0$ for all bounded and continuous w with $\lim_{t\to\infty} w(t) = 0$. Then

$$\boldsymbol{G}(0)\lim_{t\to\infty}\int_0^t (\varphi \circ u)(\theta)\,\mathrm{d}\theta = \lim_{t\to\infty} r(t) - \lim_{t\to\infty} u(t). \tag{3.8}$$

In particular, $\lim_{t\to\infty}\int_0^t (\varphi\circ u)(\theta)\,\mathrm{d}\theta$ exists.

(v) *Assume that assumptions 3.4 and 3.5 hold and that G has the property that*

$$\operatorname*{ess\,lim}_{t\to\infty}(Gw)(t) = 0,$$

whenever w is bounded and continuous with $\lim_{t\to\infty} w(t) = 0$. Then

$$\operatorname*{ess\,lim}_{t\to\infty}\dot{u}(t) = 0.$$

(vi) *If we relax condition (3.1) and only require that, for all $\tau > 0$,*

$$r \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, U) \cap W^{2,1}_{\mathrm{loc}}([\tau,\infty), U)$$

and that

$$\dot{r} \in L^1(\mathbb{R}_+, U), \qquad \ddot{r} \in L^1([\tau,\infty), U), \tag{3.9}$$

then all the conclusions in statements (i)–(v) *remain valid with the following exceptions: in statement* (i), *the left-hand side of (3.4) is still finite, but it is no longer bounded in terms of $\eta(r)$, and, in statement* (ii), *the condition $\dot{u} \in L^\infty(\mathbb{R}_+, U)$ needs to be replaced by $\dot{u} \in L^\infty([\tau,\infty), U)$ (for every $\tau > 0$).*

(vii) *Assume that φ satisfies the additional condition*

$$\|\varphi(v)\| \leqslant \gamma(1 + \|v\|), \quad \forall v \in U,$$

for some $\gamma > 0$. If we relax condition (3.1) and only require that there exists $\tau > 0$ such that $r \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, U) \cap W^{2,1}_{\mathrm{loc}}([\tau,\infty), U)$ and (3.9) holds, then all the conclusions in statements (i)–(v) *remain valid with the following exceptions: in statement* (i), *the left-hand side of (3.4) is still finite, but it is no longer bounded in terms of $\eta(r)$, and, in statement* (ii), *the condition $\dot{u} \in L^\infty(\mathbb{R}_+, U)$ needs to be replaced by $\dot{u} \in L^\infty([\tau,\infty), U)$.*

Statement (iii) of theorem 3.3 is reminiscent of Corduneanu (1973, theorem 3.1, p. 91), where it is assumed that $\dim U = 1$, $P = 0$, $Q = 1$, $\varphi^{-1}(\{0\}) = \{0\}$ and that G has a convolution kernel of the form $g + g_0\delta$, where $g \in L^1(\mathbb{R}_+)$, $g_0 \in \mathbb{R}$ and δ denotes the unit mass at 0. Assumption 3.4 holds trivially if $\dim U < \infty$. It is clear that assumption 3.5 is always satisfied if $\dim U = 1$, $Q > 0$ and $P = 0$. If

$\dim U < \infty$, then it follows from a well-known theorem of Paley and Wiener that the assumption on the operator H in statement (iv) holds if $\boldsymbol{G} \in H^\infty(\mathbb{C}_\alpha, \mathcal{B}(U))$ for some $\alpha < 0$. Furthermore, if $\dim U < \infty$, proposition 3.1 guarantees that the convolution kernel of G is a bounded Borel measure on $\mathbb{R}_+$ and so, by Gripenberg *et al.* (1990, theorem 6.1 part (ii), p. 96), the extra assumption on G imposed in statement (v) is satisfied. We mention that statement (vi) is essential for interesting applications of theorem 3.3 to well-posed infinite-dimensional state-space systems (see § 4). Finally, theorem 3.3 shows that the signal y in figure 2 is bounded (by statement (ii)) and that, under the additional assumptions of statement (iv), $y(t)$ converges as $t \to \infty$.

Before we prove theorem 3.3, we state a slightly simplified version of this result (where P and Q are scalars and $P \geqslant 0$) in the form of a corollary. To this end, let $0 < a \leqslant \infty$ and let $\varphi\colon U \to U$ be a function satisfying the following sector condition

$$\langle \varphi(v), v \rangle \geqslant \frac{1}{a} \|\varphi(v)\|^2, \quad \forall v \in U, \tag{3.10}$$

where $1/\infty := 0$. It is not difficult to show that for any gradient field $\varphi : U \to U$ satisfying (3.10) there exists a C^1-potential $\Phi : U \to \mathbb{R}$ satisfying

$$\Phi(v) = \int_0^1 \langle \varphi(\theta v), v \rangle \, \mathrm{d}\theta \geqslant 0,$$

where the non-negativity follows from the sector condition (3.10). The following result is now an immediate consequence of theorem 3.3.

Corollary 3.7. *Let $G \in \mathcal{B}(L^2(\mathbb{R}_+, U)) \cap \mathcal{B}(L^\infty(\mathbb{R}_+, U))$ be a right-shift-invariant operator with transfer function $\boldsymbol{G}$ satisfying assumption 3.2 with $\boldsymbol{G}(0)$ invertible and $\boldsymbol{G}(0) = \boldsymbol{G}^*(0) \geqslant 0$, let $\varphi\colon U \to U$ be a locally Lipschitz continuous gradient field such that the sector condition (3.10) holds for some $0 < a \leqslant \infty$ and let $r \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}_+, U)$ with $\dot r, \ddot r \in L^1(\mathbb{R}_+, U)$. If there exists $q \geqslant 0$ such that*

$$\frac{1}{a} I + \frac{1}{2}\left[\left(q + \frac{1}{\mathrm{i}\omega}\right)\boldsymbol{G}(\mathrm{i}\omega) + \left(q - \frac{1}{\mathrm{i}\omega}\right)\boldsymbol{G}^*(\mathrm{i}\omega)\right] \geqslant 0, \quad \text{a.e. } \omega \in \mathbb{R}, \tag{3.11}$$

then the conclusions of theorem 3.3 hold with $P = (1/a)I$, $Q = I$ and $\Phi : U \to \mathbb{R}_+, v \mapsto \int_0^1 \langle \varphi(\theta v), v \rangle \, \mathrm{d}\theta$.

In the single-input–single-output case (i.e. $\dim U = 1$), (3.11) simplifies to

$$\frac{1}{a} + \mathrm{Re}\left[\left(q + \frac{1}{\mathrm{i}\omega}\right)\boldsymbol{G}(\mathrm{i}\omega)\right] \geqslant 0, \quad \text{a.e. } \omega \in \mathbb{R}.$$

Proof of theorem 3.3. Let us begin by observing that

$$\|r\|_{L^\infty} \leqslant \|r(0)\| + \|\dot r\|_{L^1}. \tag{3.12}$$

Moreover, since $\dot r \in L^1(\mathbb{R}_+, U)$ and $\ddot r \in L^1(\mathbb{R}_+, U)$, we have that

$$\lim_{t \to \infty} \dot r(t) = 0, \tag{3.13}$$

and so, $\dot r(0) = -\int_0^\infty \ddot r(\theta) \, \mathrm{d}\theta$. Consequently,

$$\dot r(t) = \dot r(0) + \int_0^t \ddot r(\theta) \, \mathrm{d}\theta = -\int_t^\infty \ddot r(\theta) \, \mathrm{d}\theta, \quad \forall t \in \mathbb{R}_+,$$

showing that

$$\|\dot{r}\|_{L^\infty} \leqslant \|\ddot{r}\|_{L^1}. \tag{3.14}$$

By lemma 2.1, the Volterra equation (2.1) has a unique solution u defined on a maximal interval of existence $[0, T)$, where $0 < T \leqslant \infty$. As in the proof of theorem 3.1 in Curtain *et al.* (2003), it can be shown that

$$\begin{aligned} q(\Phi \circ u)(t) + \tfrac{1}{2}\langle \varphi_u(t), Q\boldsymbol{G}(0)\varphi_u(t)\rangle + \int_0^t \langle (\varphi \circ u)(\theta), Qu(\theta) - P(\varphi \circ u)(\theta)\rangle \,\mathrm{d}\theta \\ \leqslant q\Phi(r(0)) + \int_0^t \langle (\varphi \circ u)(\theta), q\dot{r}(\theta) + Qr(\theta)\rangle \,\mathrm{d}\theta, \quad \forall t \in [0, T), \end{aligned} \tag{3.15}$$

where we have introduced the abbreviation

$$\varphi_u(t) := \int_0^t (\varphi \circ u)(\theta) \,\mathrm{d}\theta. \tag{3.16}$$

Note that (3.15) is identical to equation (3.19) in Curtain *et al.* (2003) with $\varepsilon = 0$.

Step 1. Proof of statement (i). It follows from the assumption that the operator $Q\boldsymbol{G}(0)$ is coercive. Consequently, there exists $\delta > 0$ such that

$$\tfrac{1}{2}\langle \varphi_u(t), Q\boldsymbol{G}(0)\varphi_u(t)\rangle \geqslant \delta \|\varphi_u(t)\|^2.$$

Integration by parts yields

$$\begin{aligned} \int_0^t \langle (\varphi \circ u)(\theta), q\dot{r}(\theta) + Qr(\theta)\rangle \,\mathrm{d}\theta = \langle \varphi_u(t), q\dot{r}(t) + Qr(t)\rangle \\ - \int_0^t \langle \varphi_u(\theta), q\ddot{r}(\theta) + Q\dot{r}(\theta)\rangle \,\mathrm{d}\theta. \end{aligned} \tag{3.17}$$

Hence

$$\left| \int_0^t \langle (\varphi \circ u)(\theta), q\dot{r}(\theta) + Qr(\theta)\rangle \,\mathrm{d}\theta \right| \leqslant (\|q\dot{r} + Qr\|_{L^\infty} + \|q\ddot{r} + Q\dot{r}\|_{L^1}) \sup_{0 \leqslant \theta \leqslant t} \|\varphi_u(\theta)\|.$$

Substituting these estimates into (3.15), we obtain

$$\begin{aligned} q(\Phi \circ u)(t) + \delta\|\varphi_u(t)\|^2 + \int_0^t \langle (\varphi \circ u)(\theta), Qu(\theta) - P(\varphi \circ u)(\theta)\rangle \,\mathrm{d}\theta \\ \leqslant q\Phi(r(0)) + (\|q\dot{r} + Qr\|_{L^\infty} + \|q\ddot{r} + Q\dot{r}\|_{L^1}) \sup_{0 \leqslant \theta \leqslant t} \|\varphi_u(\theta)\|, \quad \forall t \in [0, T). \end{aligned} \tag{3.18}$$

In particular,

$$\delta\|\varphi_u(t)\|^2 \leqslant q\Phi(r(0)) + (\|q\dot{r} + Qr\|_{L^\infty} + \|q\ddot{r} + Q\dot{r}\|_{L^1}) \sup_{0 \leqslant \theta \leqslant t} \|\varphi_u(\theta)\|, \quad \forall t \in [0, T). \tag{3.19}$$

For every $t \in [0, T)$, choose $\theta_t \in [0, t]$ such that

$$\|\varphi_u(\theta_t)\| = \sup_{0 \leqslant \theta \leqslant t} \|\varphi_u(\theta)\|.$$

Using (3.12) and (3.14), it follows from (3.19) that there exist constants $K_1, K_2 > 0$ such that

$$\|\varphi_u(\theta_t)\|^2 - K_1(\|r(0)\| + \|\dot{r}\|_{L^1} + q\|\ddot{r}\|_{L^1})\|\varphi_u(\theta_t)\| \leqslant K_2 q\Phi(r(0)), \quad \forall t \in [0, T).$$

Completing the square shows that there exists a constant $K_3 > 0$ such that

$$\|\varphi_u(t)\| \leqslant \|\varphi_u(\theta_t)\| \leqslant K_3\eta(r), \quad \forall t \in [0, T),$$

where $\eta(r)$ is given by (3.5). Substituting this back into (3.18) we obtain most of (3.4) (restricted to the interval $[0, T)$); the only missing part is the bound on $\|u\|_{L^\infty}$, which we derive as follows. Since G is right-shift invariant, it commutes with the integration operator, and hence from (2.1),

$$u(t) = r(t) - \int_0^t (G(\varphi \circ u))(\theta)\, \mathrm{d}\theta = r(t) - (G\varphi_u)(t), \quad 0 \leqslant t < T.$$

This, combined with the L^∞-bounds for r (see (3.12)) and for φ_u and the assumption that $G \in \mathcal{B}(L^\infty(\mathbb{R}_+, U))$, implies that $\|u(t)\|$ is bounded on $[0, T)$ by $K\eta(r)$ for some constant $K > 0$, and so, by lemma 2.1, $T = \infty$ and (3.4) holds.

Step 2. Proof of statement (ii). The boundedness of u and the local Lipschitz continuity of φ implies that $\varphi \circ u$ is bounded. Since $G \in \mathcal{B}(L^\infty(\mathbb{R}_+, U))$, $G(\varphi \circ u)$ is bounded. Therefore, differentiating (2.1) and invoking (3.14) shows that $\dot{u}$ is bounded. Furthermore, by (3.4), $\int_0^\cdot (\varphi \circ u)(\theta)\, \mathrm{d}\theta \in L^\infty(\mathbb{R}_+, U)$, and so, since $G \in \mathcal{B}(L^\infty(\mathbb{R}_+, U))$,

$$\int_0^\cdot (G(\varphi \circ u))(\theta)\, \mathrm{d}\theta = G\left(\int_0^\cdot (\varphi \circ u)(\theta)\, \mathrm{d}\theta\right) \in L^\infty(\mathbb{R}_+, U),$$

where we have used that, by right-shift invariance, G commutes with the integral operator $\int_0^\cdot$. Finally, since u and $\dot{u}$ are bounded and φ is locally Lipschitz continuous, u and $\varphi \circ u$ are uniformly continuous, and so is $\langle \varphi \circ u, Qu - P(\varphi \circ u) \rangle$, which also belongs to $L^1(\mathbb{R}_+, U)$. Thus, by Barbălat's lemma (see, for example, Corduneanu 1973, p. 89; or Khalil 1996, p. 192), $\langle (\varphi \circ u)(t), Qu(t) - P(\varphi \circ u)(t) \rangle \to 0$ as $t \to \infty$.

Step 3. Proof of statement (iii). Since u is bounded, there exists a closed bounded ball $W \subset U$ such that $u(t) \in W$ for all $t \geqslant 0$. By assumption 3.4, $\varphi^{-1}(\{0\}) \cap W$ is compact. Hence, for arbitrary $\delta > 0$, $\varphi^{-1}(\{0\}) \cap W$ is contained in a finite union of open balls with radius δ, each ball centred at some point in $\varphi^{-1}(\{0\}) \cap W$. Call this union W_δ. We claim that $u(t) \in W_\delta$ for all sufficiently large t. This is trivially true if $W \subset W_\delta$. If $W \not\subset W_\delta$, then the set $V := W \backslash W_\delta$ is non-empty. Moreover, V is bounded and closed with $\varphi^{-1}(\{0\}) \cap V = \emptyset$, and so, by assumption 3.5, $\inf_{v \in V} \langle \varphi(v), Qv - P\varphi(v) \rangle > 0$. We know from (3.6) that $\lim_{t\to\infty} \langle (\varphi \circ u)(t), Qu(t) - P(\varphi \circ u)(t) \rangle = 0$, and so, also in this case, $u(t) \in W_\delta$ for all sufficiently large $t \geqslant 0$. Thus, as $t \to \infty$, $u(t)$ approaches $\varphi^{-1}(\{0\}) \cap W$ (and so, *a fortiori*, $u(t)$ approaches $\varphi^{-1}(\{0\})$). Using the compactness of $\varphi^{-1}(\{0\}) \cap W$ and the continuity of u, a routine argument shows that the trajectory $\{u(t) \mid t \in \mathbb{R}_+\}$ of u is precompact. It follows that $\lim_{t\to\infty} (\varphi \circ u)(t) = 0$. Moreover, by a standard result, the ω-limit set of u

$$\Omega_u := \{l \in U \mid u(t_k) \to l \text{ for some sequence } t_k \to \infty\}$$

is non-empty, compact, connected and is approached by $u(t)$ as $t \to \infty$. Consequently, $\Omega_u \subset \varphi^{-1}(\{0\}) \cap W$. Assuming that assumption 3.6 holds, $\varphi^{-1}(\{0\}) \cap W$ is totally disconnected, and so we may conclude that Ω_u is a singleton. Thus, $u(t)$ converges to some $u^\infty \in \varphi^{-1}(\{0\})$ as $t \to \infty$.

Step 4. Proof of statements (iv) and (v). Clearly, $\varphi \circ u$ is continuous and, by statements (ii) and (iii), $\varphi \circ u$ is bounded and $\lim_{t\to\infty}(\varphi \circ u)(t) = 0$. Moreover, by (3.7), $H(\varphi \circ u)$ is continuous. Consequently, by the assumed property of H,

$$\lim_{t\to\infty}(H(\varphi \circ u))(t) = \operatorname*{ess\,lim}_{t\to\infty}(H(\varphi \circ u))(t) = 0.$$

Since $\dot{r} \in L^1(\mathbb{R}_+, U)$, $\lim_{t\to\infty} r(t)$ exists, and so statement (iv) follows from statement (iii), (2.1) and (3.7). Similarly, statement (v) is a consequence of statement (ii), (3.13) and (2.1).

Step 5. Proof of statement (vi). Only step 1 in the preceding proof needs changing, the other parts of the proof still apply in this case too. By lemma 2.1, there exists a unique solution of (2.1) defined on a maximal interval of existence $[0, T)$, where $0 < T \leqslant \infty$. Let $\tau \in (0, T)$ and consider the arguments in step 1 for $t \in [\tau, T)$. The only significant change is that this time we write the integral on the right-hand side of (3.15) in the form

$$\begin{aligned}\int_0^t \langle(\varphi \circ u)(\theta), q\dot{r}(\theta) + Qr(\theta)\rangle\,\mathrm{d}\theta &= \int_0^\tau \langle(\varphi \circ u)(\theta), q\dot{r}(\theta) + Qr(\theta)\rangle\,\mathrm{d}\theta + \langle\varphi_u(t), q\dot{r}(t) + Qr(t)\rangle \\ &\quad - \langle\varphi_u(\tau), q\dot{r}(\tau) + Qr(\tau)\rangle - \int_\tau^t \langle\varphi_u(\theta), q\ddot{r}(\theta) + Q\dot{r}(\theta)\rangle\,\mathrm{d}\theta,\end{aligned}$$

which can be estimated by

$$\begin{aligned}\left|\int_0^t \langle(\varphi \circ u)(\theta), q\dot{r}(\theta) + Qr(\theta)\rangle\,\mathrm{d}\theta\right| &\leqslant \left|\int_0^\tau \langle(\varphi \circ u)(\theta), q\dot{r}(\theta) + Qr(\theta)\rangle\,\mathrm{d}\theta\right| \\ &\quad + (2\|q\dot{r} + Qr\|_{L^\infty} + \|q\ddot{r}+Q\dot{r}\|_{L^1}) \sup_{0\leqslant\theta\leqslant t}\|\varphi_u(\theta)\|.\end{aligned}$$

The argument now proceeds in the same way as in step 1 with an extra term

$$\left|\int_0^\tau \langle(\varphi \circ u)(\theta), q\dot{r}(\theta) + Qr(\theta)\rangle\,\mathrm{d}\theta\right|$$

on the right-hand side of (3.18) and the constant multiplying $\sup_{0\leqslant\theta\leqslant t}\|\varphi_u(\theta)\|$ replaced by the larger constant $2\|q\dot{r} + Qr\|_{L^\infty} + \|q\ddot{r} + Q\dot{r}\|_{L^1}$.

Step 6. Proof of statement (vii). By lemma 2.1, there exists a unique solution of (2.1) defined on $[0, \infty)$. Statement (vii) can now be proved by setting $T = \infty$ and applying the arguments used in the proof of statement (vi) (see step 5). ∎

Remark 3.8. The above proof shows that if the positive-real condition (3.3) holds for $q = 0$, then statements (i)–(v) of theorem 3.3 are true for a larger class of input (or forcing) functions r, namely for all $r \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, U)$ with $\dot{r} \in L^1(\mathbb{R}_+, \mathbb{R})$.

Next we show that if the positive-real condition (3.3) holds with $q = 0$, then we can allow for time-dependent nonlinearities φ. To this end, we state and prove the following lemma.

Lemma 3.9. *Assume that* $\varphi\colon \mathbb{R}_+ \times U \to U$ *has the property that for every bounded set* $V \subset U$ *there exists a function* $\lambda_V : \mathbb{R}_+ \to \mathbb{R}$ *such that* (2.2) *holds. Moreover, assume that there exist* $P, Q \in \mathcal{B}(U)$ *with* $P = P^* \geqslant 0$ *and* Q *invertible and such that*

$$\langle \varphi(t,v), Qv\rangle \geqslant \langle \varphi(t,v), P\varphi(t,v)\rangle, \quad \forall (t,v) \in \mathbb{R}_+ \times U.$$

Then φ *is unbiased, that is,* $\varphi(t,0) = 0$ *for almost all* $t \in \mathbb{R}_+$.

Proof. Let $V \subset U$ be a bounded open set with $0 \in V$. Then there exists a set $E \subset \mathbb{R}_+$ of measure zero such that

$$\|\varphi(t,v) - \varphi(t,0)\| \leqslant \lambda_V(t)\|v\|, \quad \forall t \in \mathbb{R}_+ \setminus E,\ \forall v \in V. \tag{3.20}$$

We claim that $\varphi(t,0) = 0$ for all $t \in \mathbb{R}_+ \setminus E$. Seeking a contradiction, assume that there exists $\tau \in \mathbb{R}_+ \setminus E$ such that $w := \varphi(\tau, 0) \neq 0$. Defining $v := -Q^{-1}w$, we have that

$$\langle w, Qv\rangle = -\|w\|^2 < 0.$$

Let $h > 0$. Since $P \geqslant 0$, it follows from the sector condition on φ that

$$0 \leqslant \langle \varphi(\tau, hv), Qhv\rangle = h\langle \varphi(\tau, hv), Qv\rangle. \tag{3.21}$$

By (3.20), the function $v \mapsto \varphi(\tau, v)$ is continuous at 0, and so, as $h \downarrow 0$,

$$\langle \varphi(\tau, hv), Qv\rangle \to \langle w, Qv\rangle = -\|w\|^2 < 0.$$

Therefore, the right-hand side of (3.21) is negative for all sufficiently small $h > 0$, yielding a contradiction. ■

We are now in the position to formulate and prove our main absolute-stability result, a stability criterion of circle-criterion type, for the case that the nonlinearity φ in the feedback system shown in figure 2 is time dependent. We use the notation $\varphi \circ u$ for the function $t \mapsto \varphi(t, u(t))$.

Theorem 3.10. *Let* $G \in \mathcal{B}(L^2(\mathbb{R}_+, U)) \cap \mathcal{B}(L^\infty(\mathbb{R}_+, U))$ *be a right-shift-invariant operator with transfer function* $\boldsymbol{G}$ *satisfying assumption 3.2 and* $\boldsymbol{G}(0)$ *invertible. Assume that* $\varphi\colon \mathbb{R}_+ \times U \to U$ *is such that* $t \mapsto \varphi(t,v)$ *is measurable for every* $v \in U$ *and, for every bounded set* $V \subset U$, *there exists* $\lambda_V \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ *such that* (2.2) *holds. Let* $r \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, U)$ *with* $\dot{r} \in L^1(\mathbb{R}_+, U)$. *Moreover, assume that there exist* $P, Q \in \mathcal{B}(U)$ *with* P *self-adjoint,* Q *invertible and* $Q\boldsymbol{G}(0) = [Q\boldsymbol{G}(0)]^* \geqslant 0$ *and such that*

$$\langle \varphi(t,v), Qv\rangle \geqslant \langle \varphi(t,v), P\varphi(t,v)\rangle, \quad \forall (t,v) \in \mathbb{R}_+ \times U, \tag{3.22}$$

$$P + \frac{1}{2\mathrm{i}\omega}(Q\boldsymbol{G}(\mathrm{i}\omega) - \boldsymbol{G}^*(\mathrm{i}\omega)Q^*) \geqslant 0, \quad \text{a.e. } \omega \in \mathbb{R}. \tag{3.23}$$

Then (2.1) *has a unique continuous solution* u *defined on* $\mathbb{R}_+$ *(no finite escape time) and there exists a constant* $K > 0$ *(which depends only on* Q *and* $\boldsymbol{G}$ *but not on* r*) such that*

$$\|u\|_{L^\infty} + \sup_{t \geqslant 0}\left\|\int_0^t (\varphi \circ u)(\theta)\,\mathrm{d}\theta\right\| + \left(\int_0^\infty \langle (\varphi \circ u)(\theta), Qu(\theta) - P(\varphi \circ u)(\theta)\rangle\,\mathrm{d}\theta\right)^{1/2} \leqslant K\eta(r), \tag{3.24}$$

where $\eta(r) := \|r(0)\| + \|\dot{r}\|_{L^1}$.

Proof. Letting $\omega \to \infty$, it follows from (3.23) that $P \geqslant 0$, and so, by lemma 3.9, φ is unbiased. An application of lemma 2.1 shows that the Volterra equation (2.1) has a unique solution u defined on a maximal interval of existence $[0, T)$, where $0 < T \leqslant \infty$. Note that (3.18) remains valid with $q = 0$, that is,

$$\delta\|\varphi_u(t)\|^2 + \int_0^t \langle(\varphi \circ u)(\theta), Qu(\theta) - P(\varphi \circ u)(\theta)\rangle \,\mathrm{d}\theta \\ \leqslant (\|Qr\|_{L^\infty} + \|Q\dot{r}\|_{L^1}) \sup_{0 \leqslant \theta \leqslant t} \|\varphi_u(\theta)\|, \quad \forall t \in [0, T). \quad (3.25)$$

■

The same arguments as in step 1 in the proof of theorem 3.3 can now be used to prove the claim.

4. Absolute-stability results for well-posed state-space systems

In this section we use the results of § 3 to derive absolute-stability results for well-posed state-space systems. There are a number of equivalent definitions of well-posed systems (see Curtain & Weiss 1989; Salamon 1987, 1989; Staffans 1997, 2001, 2004; Staffans & Weiss 2002; Weiss 1989, 1994). We will be brief in the following and refer the reader to the above references for more details. Throughout this section, we shall be considering a well-posed system Σ with state-space X, input space U and output space $Y = U$, generating operators (A, B, C), input–output operator G and transfer function $\boldsymbol{G}$. Here X and U are real separable Hilbert spaces, A is the generator of a strongly continuous semigroup $\boldsymbol{T} = (\boldsymbol{T}_t)_{t \geqslant 0}$ on X, $B \in \mathcal{B}(U, X_{-1})$ and $C \in \mathcal{B}(X_1, U)$, where X_{-1} and X_1 are the usual extrapolation and interpolation spaces of X (see § 1 *a*). The norms on X, X_{-1} and X_1 are denoted by $\|\cdot\|$, $\|\cdot\|_{-1}$ and $\|\cdot\|_1$, respectively. Moreover, the operator B is an *admissible control operator* for $\boldsymbol{T}$, i.e. for each $t \in \mathbb{R}_+$ there exists $\beta_t \geqslant 0$ such that

$$\left\| \int_0^t \boldsymbol{T}_{t-\theta} Bv(\theta) \,\mathrm{d}\theta \right\| \leqslant \beta_t \|v\|_{L^2([0,t],U)}, \quad \forall v \in L^2([0, t], U);$$

the operator C is an *admissible observation operator* for $\boldsymbol{T}$, i.e. for each $t \in \mathbb{R}_+$ there exists $\gamma_t \geqslant 0$ such that

$$\left(\int_0^t \|C\boldsymbol{T}_\theta z\|^2 \,\mathrm{d}\theta \right)^{1/2} \leqslant \gamma_t \|z\|, \quad \forall z \in X_1.$$

The control operator B is said to be *bounded* if it is bounded as a map from the input space U to the state space X; otherwise it is said to be *unbounded*. The observation operator C is said to be *bounded* if it can be extended continuously to X; otherwise, C is said to be *unbounded*.

The so-called *Λ-extension* C_Λ of C is defined by

$$C_\Lambda z = \lim_{s \to \infty,\ s \in \mathbb{R}_+} Cs(sI - A)^{-1} z,$$

with $\mathrm{dom}(C_\Lambda)$ consisting of all $z \in X$ for which the above limit exists. Furthermore, we define the *weak Λ-extension* $C_{\Lambda\mathrm{w}}$ of C by

$$\langle w, C_{\Lambda\mathrm{w}} z\rangle = \lim_{s \to \infty,\ s \in \mathbb{R}_+} \langle w, Cs(sI - A)^{-1} z\rangle, \quad \forall w \in U,$$

with $\mathrm{dom}(C_{\Lambda\mathrm{w}})$ consisting of all $z \in X$ for which the above weak limit exists. Obviously, $C_{\Lambda\mathrm{w}}$ is an extension of C_Λ. For every $z \in X$, $\boldsymbol{T}_t z \in \mathrm{dom}(C_\Lambda)$ for a.e. $t \in \mathbb{R}_+$ so that the function $C_\Lambda \boldsymbol{T} z : t \mapsto C_\Lambda \boldsymbol{T}_t z$ is defined almost everywhere. If $\alpha > \omega(\boldsymbol{T})$, then, for every $z \in X$, $C_\Lambda \boldsymbol{T} z \in L^2_\alpha(\mathbb{R}_+, U)$, where

$$\omega(\boldsymbol{T}) := \lim_{t\to\infty} \frac{1}{t} \ln \|\boldsymbol{T}_t\|$$

denotes the exponential growth constant of $\boldsymbol{T}$. The transfer function $\boldsymbol{G}$ satisfies

$$\frac{1}{s - s_0}(\boldsymbol{G}(s) - \boldsymbol{G}(s_0)) = -C(sI-A)^{-1}(s_0 I - A)^{-1}B, \quad \forall s,\ s_0 \in \mathbb{C}_{\omega(\boldsymbol{T})},\ s \neq s_0, \tag{4.1}$$

and $\boldsymbol{G} \in H^\infty(\mathbb{C}_\alpha, \mathcal{B}(U_\mathrm{c}))$ for every $\alpha > \omega(\boldsymbol{T})$. Moreover, the input–output operator $G : L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ is continuous and right-shift invariant; for every $\alpha > \omega(\boldsymbol{T})$, $G \in \mathcal{B}(L^2_\alpha(\mathbb{R}_+, U))$ and

$$(\mathfrak{L}(Gv))(s) = \boldsymbol{G}(s)(\mathfrak{L}(v))(s), \quad \forall s \in \mathbb{C}_\alpha,\ \forall v \in L^2_\alpha(\mathbb{R}_+, U).$$

If B or C is bounded and $\dim U < \infty$, then

$$G \in \mathcal{B}(L^\infty_\alpha(\mathbb{R}_+, U)), \quad \forall \alpha > \omega(\boldsymbol{T}) \tag{4.2}$$

(see Logemann & Ryan 2000, lemma 2.3). While in general (4.2) does not hold, there are many examples of systems with control and observation operators both unbounded, for which (4.2) is satisfied (this includes retarded systems with input and output delays).

In the following let $s_0 \in \mathbb{C}_{\omega(\boldsymbol{T})}$ be fixed, but arbitrary. For $x^0 \in X$ and $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, let x and y denote the state and output functions of Σ, respectively, corresponding to the initial condition $x(0) = x^0 \in X$ and the input function v. Then

$$x(t) = \boldsymbol{T}_t x^0 + \int_0^t \boldsymbol{T}_{t-\theta} B v(\theta)\,\mathrm{d}\theta \quad \forall t \in \mathbb{R}_+,$$

$x(t) - (s_0 I - A)^{-1} B v(t) \in \mathrm{dom}(C_\Lambda)$ for a.e. $t \in \mathbb{R}_+$, and

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bv(t), \quad x(0) = x^0, \quad \text{a.e. } t \in \mathbb{R}_+, \\ y(t) &= C_\Lambda(x(t) - (s_0 I - A)^{-1} B v(t)) + \boldsymbol{G}(s_0) v(t), \quad \text{a.e. } t \in \mathbb{R}_+. \end{aligned}\right\} \tag{4.3}$$

Of course, the differential equation in (4.3) has to be interpreted in X_{-1}. Note that the second equation in (4.3) yields the following formula for the input–output operator G:

$$(Gv)(t) = C_\Lambda\left[\int_0^t \boldsymbol{T}_{t-\theta} B v(\theta)\,\mathrm{d}\theta - (s_0 I - A)^{-1} B v(t)\right] + \boldsymbol{G}(s_0) v(t),$$
$$\forall v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U),\ \text{a.e. } t \in \mathbb{R}_+. \tag{4.4}$$

In the following we identify Σ and (4.3) and refer to (4.3) as a well-posed system. If $\omega(\boldsymbol{T}) < 0$, then the well-posed system (4.3) is said to be *exponentially stable*.

The above formulae for the output, the input–output operator and the transfer function reduce to a more recognizable form for the subclasses of regular and weakly regular systems. Recall that the well-posed system (4.3) is called *regular* (*weakly* *regular*) if the input–output operator G is regular (weakly regular) in the sense of § 3. It is clear that if (4.3) is regular (weakly regular), then there exists $D \in \mathcal{B}(U)$ such that $\boldsymbol{G}(s) \to D$ in the strong (weak) operator topology as $s \to \infty$ in $\mathbb{R}_+$. The operator D is called the *feedthrough operator* of (4.3). If (4.3) is weakly regular, then $x(t) \in \mathrm{dom}(C_{\Lambda\mathrm{w}})$ for a.e. $t \in \mathbb{R}_+$ and the output equation in (4.3) and the formula (4.4) for the input–output operator simplify to

$$y(t) = C_{\Lambda\mathrm{w}} x(t) + Dv(t), \quad \text{a.e. } t \in \mathbb{R}_+, \tag{4.5}$$

and

$$(Gv)(t) = C_{\Lambda\mathrm{w}} \int_0^t \boldsymbol{T}_{t-\theta} Bv(\theta)\,\mathrm{d}\theta + Dv(t), \quad \forall v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U), \text{ a.e. } t \in \mathbb{R}_+, \tag{4.6}$$

respectively; moreover, $(sI - A)^{-1}BU \subset \mathrm{dom}(C_{\Lambda\mathrm{w}})$ for all $s \in \varrho(A)$ and we have that

$$\boldsymbol{G}(s) = C_{\Lambda\mathrm{w}}(sI - A)^{-1}B + D, \quad \forall s \in \mathbb{C}_{\omega(\boldsymbol{T})}. \tag{4.7}$$

If (4.3) is regular, then we may replace $C_{\Lambda\mathrm{w}}$ by C_Λ in formulae (4.5)–(4.7). It can be shown that if B is a bounded control operator or if C is a bounded observation operator, then (4.3) is regular.

Remark 4.1. Under the additional condition that $G \in \mathcal{B}(L^\infty_\alpha(\mathbb{R}_+, U))$ for some $\alpha \in \mathbb{R}$ (which will be assumed in theorems 4.4 and 4.5, the main results of this section), it follows from proposition 3.1 that (4.3) is weakly regular (regular, if $\dim U < \infty$) and so, the output equation in (4.3) and the formula (4.4) can be replaced by the more familiar-looking equations (4.5) and (4.6), respectively.

The following lemma will be needed later.

Lemma 4.2. *Assume that $\boldsymbol{T}$ is exponentially stable* (*i.e.* $\omega(\boldsymbol{T}) < 0$) *and let* $2 \leqslant p \leqslant \infty$. *Then there exist* $\alpha > 0$ *such that*

$$\left\| \int_0^t \boldsymbol{T}_{t-\theta} Bv(\theta)\,\mathrm{d}\theta \right\| \leqslant \alpha \|v\|_{L^p(\mathbb{R}_+, U)}, \quad \forall t \in \mathbb{R}_+,\ \forall v \in L^p(\mathbb{R}_+, U).$$

Proof. For $p = 2$ the lemma is well known and follows from the exponential stability of $\boldsymbol{T}$ and the admissibility of B. Hence we assume that $2 < p \leqslant \infty$. Let $v \in L^p(\mathbb{R}_+, U)$. It is convenient to introduce the continuous function $z : \mathbb{R}_+ \to X$ defined by

$$z(t) := \int_0^t \boldsymbol{T}_{t-\theta} Bv(\theta)\,\mathrm{d}\theta, \quad \forall t \in \mathbb{R}_+.$$

Let $\delta \in (0, 1)$ be fixed, but arbitrary. It follows from the exponential stability of $\boldsymbol{T}$ that there exists $\tau > 0$ such that $\|\boldsymbol{T}_\tau\| < \delta$. We define $t_n := n\tau$, where $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The admissibility of $\boldsymbol{T}$ guarantees that there exists $\beta_1 > 0$ (independent of v) such that

$$\left\| \int_{t_n}^t \boldsymbol{T}_{t-\theta} Bv(\theta)\,\mathrm{d}\theta \right\| \leqslant \beta_1 \|v\|_{L^2(t_n, t)} \leqslant \beta_1 \tau^{(p-2)/(2p)} \|v\|_{L^p(t_n, t_{n+1})}, \quad \forall t \in [t_n, t_{n+1}],\ \forall n \in \mathbb{N}_0, \tag{4.8}$$

where we define $(p-2)/(2p) = 1/2$ if $p = \infty$. Applying the variation-of-parameter formula

$$z(t) = \boldsymbol{T}_{t-t_n} z(t_n) + \int_{t_n}^{t} \boldsymbol{T}_{t-\theta} B v(\theta)\, \mathrm{d}\theta, \quad \forall t \in [t_n, t_{n+1}],\ \forall n \in \mathbb{N}_0 \tag{4.9}$$

for $t = t_{n+1}$ and using the estimate (4.8) shows that

$$\|z(t_{n+1})\| \leqslant \delta \|z(t_n)\| + \beta_1 \tau^{(p-2)/(2p)} \|v\|_{L^p(t_n, t_{n+1})}, \quad \forall n \in \mathbb{N}_0.$$

Since $z(t_0) = z(0) = 0$, $\delta \in (0,1)$ and $v \in L^p(\mathbb{R}_+, U)$, we may conclude that there exists $\beta_2 > 0$ (independent of v) such that

$$\|z(t_n)\| \leqslant \beta_2 \|v\|_{L^p(\mathbb{R}_+, U)}, \quad \forall n \in \mathbb{N}_0. \tag{4.10}$$

Setting $\beta_3 := \sup_{0 \leqslant \theta \leqslant \tau} \|\boldsymbol{T}_\theta\|$ and invoking (4.8) and (4.9), we obtain

$$\|z(t)\| \leqslant \beta_3 \|z(t_n)\| + \beta_1 \tau^{(p-2)/(2p)} \|v\|_{L^p(\mathbb{R}_+, U)}, \quad \forall t \in [t_n t_{n+1}],\ \forall n \in \mathbb{N}_0.$$

Combining this with (4.10) yields

$$\|z(t)\| \leqslant \beta \|v\|_{L^p(\mathbb{R}_+, U)}, \quad \forall t \in \mathbb{R}_+,$$

where $\beta := \beta_1 \tau^{(p-2)/(2p)} + \beta_2 \beta_3$. ■

Let $\varphi : \mathbb{R}_+ \times U \to U$ be a (time-dependent) static nonlinearity, and consider the well-posed system (4.3), with input nonlinearity $v = \varphi \circ u$, in feedback interconnection with the integrator $\dot{u} = -y$, i.e.

$$\left.\begin{aligned} \dot{x} &= Ax + B(\varphi \circ u), && x(0) = x^0, \\ \dot{u} &= -[C_\Lambda(x - (s_0 I - A)^{-1} B(\varphi \circ u)) + \boldsymbol{G}(s_0)(\varphi \circ u)], && u(0) = u^0 \in U, \end{aligned}\right\} \tag{4.11}$$

where $\varphi \circ u$ denotes the function $t \mapsto \varphi(t, u(t))$. A *solution* of (4.11) on the interval $[0, T)$ (where $0 < T \leqslant \infty$) is a continuous function $[0, T) \to X \times U, t \mapsto (x(t), u(t))$ such that $\varphi \circ u \in L^2_{\mathrm{loc}}([0, T), U)$, $x(t) - (s_0 I - A)^{-1} B(\varphi \circ u)(t) \in \mathrm{dom}(C_\Lambda)$ for a.e. $t \in [0, T)$, $C_\Lambda[x - (s_0 I - A)^{-1} B(\varphi \circ u)] \in L^1_{\mathrm{loc}}([0, T), U)$ and for all $t \in [0, T)$

$$\begin{aligned} x(t) &= x^0 + \int_0^t (Ax(\theta) + B(\varphi \circ u)(\theta))\, \mathrm{d}\theta, \\ u(t) &= u^0 - \int_0^t [C_\Lambda(x(\theta) - (s_0 I - A)^{-1} B(\varphi \circ u)(\theta)) + \boldsymbol{G}(s_0)(\varphi \circ u)(\theta)]\, \mathrm{d}\theta. \end{aligned}$$

Under the assumption that φ satisfies (3.2) and the positive-real condition (3.3) holds with εI (for some $\varepsilon > 0$) replacing 0 on the right-hand side of (3.3), aspects of the asymptotic and stability behaviour of (4.11) have been studied in Curtain *et al.* (2003) and Logemann & Curtain (2000). In less general form (B and/or C bounded, $\dim U = 1$), the feedback system (4.11) has been studied in Bucci (1999) and Wexler (1979, 1980).

The proof of the following existence and uniqueness result can be found in Curtain *et al.* (2003).

Proposition 4.3. *Let $\varphi\colon \mathbb{R}_+ \times U \to U$ be such that $t \mapsto \varphi(t,v)$ is measurable for every $v \in U$, $t \mapsto \varphi(t,0)$ is in $L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ and, for every bounded set $V \subset U$, there exists $\lambda_V \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ such that (2.2) holds. Then the initial-value problem (4.11) has a unique solution (x,u) defined on a maximal interval of existence $[0,T)$, where $0 < T \leqslant \infty$. If $T < \infty$, then $\limsup_{t\to T}\|u(t)\| = \infty$.*

As usual, the semigroup $\boldsymbol{T}$ is called *eventually differentiable* if there exists $\tau > 0$, such that for every $z \in X$ the function $t \mapsto \boldsymbol{T}_t z$ is differentiable on (τ, ∞). If τ can be chosen $\tau = 0$, then $\boldsymbol{T}$ is called *immediately differentiable*. The following theorem gives an absolute-stability result for the system (4.11) with time-independent φ.

Theorem 4.4. *Assume that the well-posed system (4.3) is exponentially stable, the semigroup $\boldsymbol{T}$ is immediately differentiable, $G \in \mathcal{B}(L^\infty(\mathbb{R}_+, U))$, $\boldsymbol{G}(0)$ is invertible and $\varphi\colon U \to U$ is a locally Lipschitz continuous gradient of a non-negative C^1-function $\Phi : U \to \mathbb{R}$. Furthermore, assume that there exist self-adjoint $P \in \mathcal{B}(U)$, invertible $Q \in \mathcal{B}(U)$ with $Q\boldsymbol{G}(0) = [Q\boldsymbol{G}(0)]^* \geqslant 0$ and a number $q \geqslant 0$ such that (3.2) and (3.3) hold. Let $(x^0, u^0) \in X \times U$ and let (x,u) be the unique solution of (4.11) defined on a maximal interval of existence $[0,T)$. Then the following statements hold.*

(i) *The solution (x,u) exists on $\mathbb{R}_+$ (that is, $T = \infty$), $x \in L^\infty(\mathbb{R}_+, X)$ and $u \in L^\infty(\mathbb{R}_+, U)$.*

(ii) *If φ satisfies assumptions 3.4–3.6 in theorem 3.3, then*

$$\lim_{t\to\infty} u(t) \in \varphi^{-1}(\{0\}) \quad \textit{and} \quad \lim_{t\to\infty} \|x(t)\| = 0.$$

(iii) *If $\varphi^{-1}(\{0\}) = \{0\}$ and $\inf_{v\in V}\langle \varphi(v), Qv - P\varphi(v)\rangle > 0$ for any bounded, closed and non-empty set $V \subset U$ which does not contain 0, then the zero solution of (4.11) is globally attractive, that is*

$$\lim_{t\to\infty} u(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} \|x(t)\| = 0$$

for all $(x^0, u^0) \in X \times U$.

(iv) *If we relax the immediate differentiability assumption on $\boldsymbol{T}$ to eventual differentiability, then statements* (i)–(iii) *remain true, provided that φ satisfies the additional assumption*

$$\|\varphi(v)\| \leqslant \gamma(1 + \|v\|), \quad \forall v \in U,$$

for some $\gamma > 0$.

(v) *If (3.3) holds for $q = 0$, then, without any differentiability assumption on $\boldsymbol{T}$, statements* (i)–(iii) *remain true. Moreover, for each $R > 0$, there exists a constant $K_R > 0$ such that*

$$\|x\|_{L^\infty(\mathbb{R}_+, X)} + \|u\|_{L^\infty(\mathbb{R}_+, U)} \leqslant K_R(\|x^0\| + \|u^0\|) \tag{4.12}$$

for all $(x^0, u^0) \in X \times U$ with $\|x^0\| + \|u^0\| \leqslant R$.

Assuming that $\dim U = 1$, $P = 0$ and $Q = 1$ and imposing stronger boundedness assumptions on B and on C, it has been shown in Bucci (1999) and Wexler (1979) that under the conditions of statement (iii) the zero solution of (4.11) is globally asymptotic stable. In statement (v), the inequality (4.12) shows that the zero solution of (4.11) is stable and that the solutions of (4.11) are equibounded.

Proof of theorem 4.4. By proposition 4.3 there exists a unique solution (x, u) of (4.11) defined on the maximal interval of existence $[0, T)$, where $0 < T \leqslant \infty$. Obviously, u satisfies

$$u(t) = r(t) - \int_0^t (G(\varphi \circ u))(\theta)\, \mathrm{d}\theta, \quad \forall t \in [0, T), \tag{4.13}$$

where the function $r : \mathbb{R}_+ \to U$ is defined by

$$r(t) := u^0 - \int_0^t C_\Lambda \boldsymbol{T}_\theta x^0 \, \mathrm{d}\theta, \quad \forall t \in \mathbb{R}_+. \tag{4.14}$$

Our plan is to apply statement (vi) of theorem 3.3 to (4.13). To this end we need to verify the relevant assumptions. Clearly, by exponential stability, $G \in \mathcal{B}(L^2(\mathbb{R}_+, U))$. Furthermore, again using exponential stability, we see that $\boldsymbol{G}$ is analytic in a neighbourhood of 0 and hence, $\boldsymbol{G}$ satisfies assumptions 3.2. It follows from the definition of r that $r \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, U)$. By exponential stability, there exists $\alpha < 0$ such that $\dot{r} = -C_\Lambda \boldsymbol{T} x^0 \in L^2_\alpha(\mathbb{R}_+, U)$, and consequently $\dot{r} \in L^1(\mathbb{R}_+, U)$. Next we show that, for every $\tau > 0$, $r \in W^{2,1}_{\mathrm{loc}}([\tau, \infty), U)$ and $\ddot{r} \in L^1([\tau, \infty), U)$. Let $\tau > 0$. Since $\boldsymbol{T}$ is immediately differentiable, $\boldsymbol{T}_\tau x^0 \in \mathrm{dom}(A) = X_1$. Therefore,

$$\dot{r}(t) = -C_\Lambda \boldsymbol{T}_{t-\tau} \boldsymbol{T}_\tau x^0 = -C \boldsymbol{T}_{t-\tau} \boldsymbol{T}_\tau x^0, \quad \forall t \geqslant \tau. \tag{4.15}$$

Choose $z_n \in \mathrm{dom}(A^2)$ such that $\|z_n - \boldsymbol{T}_\tau x^0\|_1 \to 0$ as $n \to \infty$. Since

$$C \boldsymbol{T}_{t-\tau} z_n = C z_n + \int_\tau^t C \boldsymbol{T}_{\theta-\tau} A z_n \, \mathrm{d}\theta, \quad \forall t \geqslant \tau, \tag{4.16}$$

and $\lim_{n\to\infty} \|C_\Lambda \boldsymbol{T} A \boldsymbol{T}_\tau x^0 - C \boldsymbol{T} A z_n\|_{L^2(\mathbb{R}_+, U)} = 0$, by letting $n \to \infty$ in (4.16), we obtain

$$C \boldsymbol{T}_{t-\tau} \boldsymbol{T}_\tau x^0 = C \boldsymbol{T}_\tau x^0 + \int_\tau^t C_\Lambda \boldsymbol{T}_{\theta-\tau} A \boldsymbol{T}_\tau x^0 \, \mathrm{d}\theta, \quad \forall t \geqslant \tau.$$

Combining this with (4.15) shows that $r \in W^{2,1}_{\mathrm{loc}}([\tau, \infty), U)$ and that

$$\ddot{r}(t) = -C_\Lambda \boldsymbol{T}_{t-\tau} A \boldsymbol{T}_\tau x^0, \quad \forall t \geqslant \tau.$$

Since $C_\Lambda \boldsymbol{T} A \boldsymbol{T}_\tau x^0 \in L^2_\alpha(\mathbb{R}_+, U)$ for some $\alpha < 0$, it follows that $\ddot{r} \in L^1([\tau, \infty), U)$. We are now in the position to apply statement (vi) of theorem 3.3. Since (4.13) has a unique continuous solution with maximal interval of existence equal to $\mathbb{R}_+$ (by statement (vi) of theorem 3.3) and since this solution coincides with u on $[0, T)$ (by proposition 2.1), an application of proposition 4.3 shows that $T = \infty$. Statement (vi) of theorem 3.3 now yields that $u \in L^\infty(\mathbb{R}_+, U)$. As a consequence $\varphi \circ u \in L^\infty(\mathbb{R}_+, U)$ and so, using again exponential stability, $x \in L^\infty(\mathbb{R}_+, X)$ (see Logemann & Ryan 2000, Lemma 2.2*b*), completing the proof of statement (i). If φ satisfies assumptions 3.4–3.6 in theorem 3.3, then it follows immediately from statement (vi) of the same theorem that $\lim_{t\to\infty} u(t) \in \varphi^{-1}(\{0\})$. Therefore, $\lim_{t\to\infty}(\varphi \circ u)(t) = 0$ and so $\lim_{t\to\infty} \|x(t)\| = 0$ (see Logemann *et al.* 1998, Lemma 2.1 part (2)), yielding statement (ii). Statement (iii) follows by a very similar argument. Replacing in the preceding arguments statement (vi) of theorem 3.3 by statement (vii) of the same theorem, we obtain statement (iv). To prove statement (v), assume that (3.3) holds for $q = 0$. Using remark 3.8, it is clear that statements (i)–(iii) remain true. The remaining claim in statement (v) is a special case of theorem 4.5 below. ■

Finally, we apply theorem 3.10 to obtain an absolute-stability result for (4.11) with time-dependent φ.

Theorem 4.5. *Assume that the well-posed system (4.3) is exponentially stable, the operator G is in $\mathcal{B}(L^\infty(\mathbb{R}_+, U))$ with $\boldsymbol{G}(0)$ invertible and $\varphi\colon \mathbb{R}_+ \times U \to U$ is such that $t \mapsto \varphi(t, v)$ is measurable for every $v \in U$ and, for every bounded set $V \subset U$, there exists $2 \leqslant p \leqslant \infty$ and $\lambda_V \in L^p(\mathbb{R}_+, \mathbb{R})$ such that (2.2) holds. Furthermore, assume that there exist self-adjoint $P \in \mathcal{B}(U)$ and invertible $Q \in \mathcal{B}(U)$ with $Q\boldsymbol{G}(0) = [Q\boldsymbol{G}(0)]^* \geqslant 0$ such that (3.22) and (3.23) hold. Then, for every $(x^0, u^0) \in X \times U$, (4.11) has a unique solution (x, u) on $\mathbb{R}_+$ (no finite escape time) and, for each $R > 0$, there exists a constant $K_R > 0$ such that*

$$\|x\|_{L^\infty(\mathbb{R}_+,X)} + \|u\|_{L^\infty(\mathbb{R}_+,U)} \leqslant K_R(\|x^0\| + \|u^0\|). \tag{4.17}$$

for all $(x^0, u^0) \in X \times U$ with $\|x^0\| + \|u^0\| \leqslant R$.

Proof. Letting $\omega \to \infty$, it follows from the positive-real condition (3.23) that $P \geqslant 0$. Hence, by lemma 3.9, $\varphi(t, 0) = 0$ for almost all $t \in \mathbb{R}$. Therefore, proposition 4.3 shows that there exists a unique solution (x, u) of (4.11) defined on the maximal interval of existence $[0, T)$. As in the proof of theorem 4.4, we note that u satisfies (4.13) with r defined by (4.14). Moreover, as has been shown in the proof of theorem 4.4, $\dot{r} \in L^1(\mathbb{R}_+, U)$. Therefore, we may apply theorem 3.10 to (4.13). Since (4.13) has a unique continuous solution with maximal interval of existence equal to $\mathbb{R}_+$ (by theorem 3.10) and since this solution coincides with u on $[0, T)$ (by proposition 2.1), an application of proposition 4.3 shows that $T = \infty$. In order to establish the L^∞-bound (4.17) for the solution (x, u), we proceed in two steps.

Step 1. L^∞ bound for u. Defining r as in (4.14), it follows from theorem 3.10 that there exists $L_1 > 0$ such that, for all $(x^0, u^0) \in X \times U$,

$$\|u\|_{L^\infty(\mathbb{R}_+,U)} \leqslant L_1(\|r(0)\|+\|\dot{r}\|_{L^1}) \leqslant L_1\left[\|u^0\|+\frac{1}{\sqrt{2|\nu|}}\left(\int_0^\infty \|C_\Lambda \mathrm{e}^{-\nu\theta}\boldsymbol{T}_\theta x^0\|^2 \,\mathrm{d}\theta\right)^{1/2}\right], \tag{4.18}$$

where $\nu \in (\omega(\boldsymbol{T}), 0)$ is fixed, but arbitrary. By the admissibility of C and the exponential stability of the weighted semigroup $t \mapsto \mathrm{e}^{-\nu t}\boldsymbol{T}_t$, there exists $\beta > 0$ such that

$$\left(\int_0^\infty \|C_\Lambda \mathrm{e}^{-\nu\theta}\boldsymbol{T}_\theta x^0\|^2 \,\mathrm{d}\theta\right)^{1/2} \leqslant \beta\|x^0\|, \quad \forall x^0 \in X.$$

Combining this with (4.18) shows that there exists $L_2 > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R}_+,U)} \leqslant L_2(\|x^0\| + \|u^0\|), \quad \forall (x^0, u^0) \in X \times U. \tag{4.19}$$

Step 2. L^∞ bound for x. Let $R > 0$ and assume that $\|x^0\| + \|u^0\| \leqslant R$. From (4.19) we obtain $\|u\|_{L^\infty} \leqslant L_2 R$. By assumption there exist $2 \leqslant p \leqslant \infty$ and $\lambda \in L^p(\mathbb{R}_+, \mathbb{R})$ such that

$$\sup_{\|v\|,\|w\| \leqslant L_2 R} \frac{\|\varphi(t, v) - \varphi(t, w)\|}{\|v - w\|} \leqslant \lambda(t), \quad \text{a.e. } t \geqslant 0.$$

Using the unbiasedness of φ, we may conclude that

$$\|(\varphi \circ u)(t)\| = \|\varphi(t, u(t))\| \leqslant \lambda(t)\|u(t)\|, \quad \text{a.e. } t \geqslant 0.$$

Consequently, by (4.19),

$$\|(\varphi \circ u)(t)\| \leqslant L_2 \lambda(t)(\|x^0\| + \|u^0\|), \quad \text{a.e. } t \geqslant 0,$$

which in turn implies that $\varphi \circ u \in L^p(\mathbb{R}_+, U)$ and

$$\|\varphi \circ u\|_{L^p} \leqslant L_2 \|\lambda\|_{L^p}(\|x^0\| + \|u^0\|).$$

We use this inequality in an application of lemma 4.2 to obtain that there exists $L_3 > 0$ such that, for all (x^0, u^0) with $\|x^0\| + \|u^0\| \leqslant R$,

$$\|x\|_{L^\infty(\mathbb{R}_+, X)} \leqslant L_3(\|x^0\| + \|u^0\|). \tag{4.20}$$

The claim follows now from (4.19) and (4.20). ■

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Appendix A. Proof of proposition 3.1

Let $\mathbb{S} = \mathbb{R}, \mathbb{R}_+$ or $\mathbb{R}_-$, where $\mathbb{R}_- := (-\infty, 0]$. The space of all bounded U-valued continuous functions defined on $\mathbb{S}$ is denoted by $\mathrm{BC}(\mathbb{S}, U)$. Endowed with the norm

$$\|u\|_{\mathrm{BC}} := \sup_{t \in \mathbb{S}} \|u(t)\|,$$

$\mathrm{BC}(\mathbb{S}, U)$ is complete and hence a Banach space. Moreover, let $C_{\mathrm{c}}(\mathbb{S}, U)$ denote the space of U-valued functions on $\mathbb{S}$ with compact support. It is clear that $C_{\mathrm{c}}(\mathbb{S}, U)$ is contained in $\mathrm{BC}(\mathbb{S}, U)$. For $\tau \in \mathbb{R}$, let $\boldsymbol{S}_\tau$ define the (bilateral) shift operator (also called translation operator) on $L^1_{\mathrm{loc}}(\mathbb{R}, U)$ defined by $(\boldsymbol{S}_\tau u)(t) = u(t - \tau)$ for all $t \in \mathbb{R}$, so that $\tau > 0$ corresponds to a right shift and $\tau < 0$ to a left shift. It is convenient to introduce the canonical injection $\iota : L^1_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^1_{\mathrm{loc}}(\mathbb{R}, U)$ given by

$$(\iota u)(t) := \begin{cases} 0 & \text{if } t < 0, \\ u(t) & \text{if } t \geqslant 0. \end{cases}$$

The following lemma will be useful for the proof of proposition 3.1.

Lemma A 1. *Let $F \in \mathcal{B}(L^2(\mathbb{R}_+, U))$ be right-shift invariant. Assume that $F(C_{\mathrm{c}}(\mathbb{R}_+, U)) \subset L^\infty(\mathbb{R}_+, U)$ and that there exists $L \geqslant 0$ such that*

$$\|Fu\|_{L^\infty} \leqslant L \|u\|_{\mathrm{BC}}, \quad \forall u \in C_{\mathrm{c}}(\mathbb{R}_+). \tag{A 1}$$

Then there exists a unique operator $\tilde{F} \in \mathcal{B}(L^2(\mathbb{R}, U))$ such that the following statements hold.

(i) *$\tilde{F}(\iota u) = \iota(Fu)$ for all $u \in L^2(\mathbb{R}_+, U)$.*

(ii) *$\tilde{F}$ is shift invariant, that is, $\boldsymbol{S}_\tau \tilde{F} = \tilde{F} \boldsymbol{S}_\tau$ for all $\tau \in \mathbb{R}$.*

(iii) *For every $u \in C_{\mathrm{c}}(\mathbb{R}, U)$, $\tilde{F}u$ is equal almost everywhere to a uniformly continuous function and satisfies $\|\tilde{F}u\|_{L^\infty} \leqslant L \|u\|_{\mathrm{BC}}$.*

It follows easily from statements (i) and (ii) that $\tilde{F}$ is causal, that is, if $\tau \in \mathbb{R}$ and $u \in L^2(\mathbb{R}, U)$ such that $u = 0$ on $(-\infty, \tau]$, then $\tilde{F}u = 0$ on $(-\infty, \tau]$.

Proof of lemma A 1. Let $\boldsymbol{F} \in H^\infty(\mathbb{C}_0, \mathcal{B}(U_{\rm c}))$ be the transfer function of F. The strong non-tangential limits of $\boldsymbol{F}(s)$ exists at almost every point $\mathrm{i}\omega$ on the imaginary axis and these limits are denoted by $\boldsymbol{F}(\mathrm{i}\omega)$. The function $\tilde{\boldsymbol{F}} : \mathbb{R} \to \mathcal{B}(U_{\rm c})$ defined by $\tilde{\boldsymbol{F}}(\omega) = \boldsymbol{F}(\mathrm{i}\omega)$ is in $L^\infty(\mathbb{R}, \mathcal{B}(U_{\rm c}))$. Let $\mathfrak{F}$ denote the Fourier transform on $L^2(\mathbb{R}, U)$. Defining $\tilde{F} : L^2(\mathbb{R}, U) \to L^2(\mathbb{R}, U)$ by

$$\tilde{F}u = \mathfrak{F}^{-1}(\tilde{\boldsymbol{F}}\mathfrak{F}(u)), \quad \forall u \in L^2(\mathbb{R}, U),$$

it is clear that $\tilde{F} \in \mathcal{B}(L^2(\mathbb{R}, U))$ and that statements (i) and (ii) hold. To prove statement (iii), let $u \in C_{\rm c}(\mathbb{R}, U)$. Choose $\sigma > 0$ such that $\operatorname{supp} u \subset [-\sigma, \sigma]$. Then $\operatorname{supp} \boldsymbol{S}_\sigma u \subset [0, 2\sigma]$ and, defining $u_\sigma := (\boldsymbol{S}_\sigma u)|_{\mathbb{R}_+}$, we obtain from statements (i) and (ii) that

$$\tilde{F}u = \boldsymbol{S}_{-\sigma}\tilde{F}\boldsymbol{S}_\sigma u = \boldsymbol{S}_{-\sigma}\tilde{F}(\iota u_\sigma) = \boldsymbol{S}_{-\sigma}\iota(Fu_\sigma). \tag{A 2}$$

Consequently, by (A 1),

$$\|\tilde{F}u\|_{L^\infty} = \|Fu_\sigma\|_{L^\infty} \leqslant L\|u_\sigma\|_{L^\infty} = L\|u\|_{\rm BC}, \quad \forall u \in C_{\rm c}(\mathbb{R}, U).$$

Combining this with statement (ii) shows that for every $\tau \in \mathbb{R}$

$$\|\boldsymbol{S}_\tau\tilde{F}u - \tilde{F}u\|_{L^\infty} = \|\tilde{F}(\boldsymbol{S}_\tau u - u)\|_{L^\infty} \leqslant L\|\boldsymbol{S}_\tau u - u\|_{\rm BC}, \quad \forall u \in C_{\rm c}(\mathbb{R}, U).$$

Therefore,

$$\lim_{\tau\to 0} \|\boldsymbol{S}_\tau\tilde{F}u - \tilde{F}u\|_{L^\infty} = 0, \quad \forall u \in C_{\rm c}(\mathbb{R}, U),$$

showing that, for $u \in C_{\rm c}(\mathbb{R}, U)$, the function $\tilde{F}u$ is equal almost everywhere to a uniformly continuous function (see, for example, Folland 1999, exercise 4, p. 239).

Finally, to prove uniqueness of $\tilde{F}$, let $\hat{F} \in \mathcal{B}(L^2(\mathbb{R}, U))$ be another shift-invariant operator with $\hat{F}(\iota u) = \iota(Fu)$ for all $u \in L^2(\mathbb{R}_+)$. Then, for $u \in C_{\rm c}(\mathbb{R}, U)$, equation (A 2) remains true with $\tilde{F}$ replaced by $\hat{F}$. Therefore,

$$\hat{F}u = \boldsymbol{S}_{-\sigma}\iota(Fu_\sigma) = \tilde{F}u, \quad \forall u \in C_{\rm c}(\mathbb{R}, U),$$

which, combined with the denseness of $C_{\rm c}(\mathbb{R}, U)$ in $L^2(\mathbb{R}, U)$, shows that $\hat{F} = \tilde{F}$. ■

Proof of proposition 3.1. We consider two cases.

Case 1. Finite-dimensional input space: $\dim U = m < \infty$. Let us first assume that $\alpha = 0$. By lemma A 1 there exists a shift-invariant operator $\tilde{G} \in \mathcal{B}(L^2(\mathbb{R}, \mathbb{R}^m))$ such that $\tilde{G}(\iota u) = \iota(Gu)$ for all $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. We define a linear injection $j : C_{\rm c}(\mathbb{R}_-, \mathbb{R}^m) \to C_{\rm c}(\mathbb{R}, \mathbb{R}^m)$ by

$$(ju)(t) := \begin{cases} u(t) & \text{if } t \leqslant 0, \\ u(0)(1-t) & \text{if } 0 < t < 1, \\ 0 & \text{if } t \geqslant 1. \end{cases}$$

Note that $\|ju\|_{\rm BC} = \|u\|_{\rm BC}$. By statement (iii) of lemma A 1, we may assume that $\tilde{G}u$ is continuous for all $u \in C_{\rm c}(\mathbb{R}, \mathbb{R}^m)$. Therefore,

$$\varGamma : C_{\rm c}(\mathbb{R}_-, \mathbb{R}^m) \to \mathbb{R}^m, \quad u \mapsto (\tilde{G}ju)(0)$$

is a well-defined bounded linear operator from $C_{\rm c}(\mathbb{R}_-, \mathbb{R}^m)$ to $\mathbb{R}^m$. The closure of $C_{\rm c}(\mathbb{R}_-, \mathbb{R}^m)$ with respect to the norm $\|\cdot\|_{\rm BC}$ is the space $C_0(\mathbb{R}_-, \mathbb{R}^m)$ of all continuous

$\mathbb{R}^m$-valued functions on $\mathbb{R}_-$ vanishing at $-\infty$. Consequently, Γ extends to a bounded linear operator on $C_0(\mathbb{R}_-,\mathbb{R}^m)$. It follows from the Riesz representation theorem (see Folland 1999, p. 223) that there exists a bounded $\mathbb{R}^{m\times m}$-valued Borel measure ν on $\mathbb{R}_-$ such that $\Gamma u = \int_{-\infty}^0 u(\theta)\,\mathrm{d}\nu(\theta)$ for all $u \in C_0(\mathbb{R}_-,\mathbb{R}^m)$.

Let $u \in C_\mathrm{c}(\mathbb{R}_+,\mathbb{R}^m)$ with $u(0)=0$, so that $\iota u \in C_\mathrm{c}(\mathbb{R},\mathbb{R}^m)$. Then, for all $t \in \mathbb{R}_+$,

$$(Gu)(t) = (\tilde{G}\iota u)(t) = (\boldsymbol{S}_{-t}\tilde{G}\iota u)(0) = (\tilde{G}\boldsymbol{S}_{-t}\iota u)(0).$$

For $t \in \mathbb{R}_+$ we define $u_{-t} := (\boldsymbol{S}_{-t}\iota u)|_{\mathbb{R}_-} \in C_\mathrm{c}(\mathbb{R}_-,\mathbb{R}^m)$. Trivially, $\boldsymbol{S}_{-t}\iota u$ and ju_{-t} coincide on $\mathbb{R}_-$, and therefore, by the causality of $\tilde{G}$, we obtain, for all $t \in \mathbb{R}_+$,

$$(Gu)(t) = (\tilde{G}ju_{-t})(0) = \Gamma u_{-t} = \int_{-\infty}^0 u_{-t}(\theta)\,\mathrm{d}\nu(\theta) = \int_{-\infty}^0 (\iota u)(\theta+t)\,\mathrm{d}\nu(\theta).$$

Consequently, for all $t \in \mathbb{R}_+$ and all $u \in C_\mathrm{c}(\mathbb{R}_+,\mathbb{R}^m)$ with $u(0)=0$,

$$(Gu)(t) = \int_{-\infty}^0 (\iota u)(\theta+t)\,\mathrm{d}\nu(\theta) = \int_{-t}^0 u(\theta+t)\,\mathrm{d}\nu(\theta).$$

Defining a bounded $\mathbb{R}^{m\times m}$-valued Borel measure μ on $\mathbb{R}_+$ by setting $\mu(E) := \nu(-E)$ for all Borel sets $E \subset \mathbb{R}_+$, it follows that for all $t \in \mathbb{R}_+$ and all $u \in C_\mathrm{c}(\mathbb{R}_+,\mathbb{R})$ with $u(0)=0$,

$$(Gu)(t) = \int_0^t u(t-\theta)\,\mathrm{d}\mu(\theta) = (\mu \star u)(t).$$

Since the set of all $u \in C_\mathrm{c}(\mathbb{R}_+,\mathbb{R}^m)$ with $u(0)=0$ is dense in $L^2(\mathbb{R}_+,\mathbb{R}^m)$, we may conclude that $Gu = \mu \star u$ for all $u \in L^2(\mathbb{R}_+,\mathbb{R}^m)$.

Let us now assume that $\alpha \in \mathbb{R}$ is arbitrary. Since G is a right-shift-invariant operator on $L^2_\alpha(\mathbb{R}_+,\mathbb{R}^m) \cap L^\infty_\alpha(\mathbb{R}_+,\mathbb{R}^m)$, it is not difficult to see that the operator G_α defined by $G_\alpha(u) := \mathrm{e}^{-\alpha\cdot}G(\mathrm{e}^{\alpha\cdot}u)$ is a right-shift-invariant operator on $L^2(\mathbb{R}_+,\mathbb{R}^m) \cap L^\infty(\mathbb{R}_+,\mathbb{R}^m)$. Therefore, by what we have already proved, there exists a bounded $\mathbb{R}^{m\times m}$-valued Borel measure ν on $\mathbb{R}_+$ such that $G_\alpha u = \nu \star u$ for all $u \in L^2(\mathbb{R}_+,\mathbb{R}^m)$. It follows that $Gu = \mu \star u$ for all $u \in L^2_\alpha(\mathbb{R}_+,\mathbb{R}^m)$, where the measure μ is given by $\mathrm{d}\mu = \mathrm{e}^{\alpha\cdot}\,\mathrm{d}\nu$. It is clear that μ is a locally bounded measure with μ_α bounded, since $\mu_\alpha = \nu$. Hence, the Laplace transform $\mathfrak{L}(\mu)$ of μ is holomorphic and bounded on $\mathbb{C}_\alpha$ and is equal to the transfer function $\boldsymbol{G}$ of G. Consequently, $\boldsymbol{G}(s)$ converges as $s \to \infty$ in $\mathbb{R}_+$, showing that G is regular.

Case 2. Infinite-dimensional input space: $\dim U = \infty$. It remains to show that in this case G is weakly regular. To this end, let $v, w \in U$ and consider

$$G_{v,w} : L^2_\alpha(\mathbb{R}_+,\mathbb{R}) \to L^2_\alpha(\mathbb{R}_+,\mathbb{R}), \quad u \mapsto \langle G(vu), w\rangle.$$

Our hypotheses on G imply that $G_{v,w}$ is a right-shift-invariant operator in

$$\mathcal{B}(L^2_\alpha(\mathbb{R}_+,\mathbb{R})) \cap \mathcal{B}(L^\infty_\alpha(\mathbb{R}_+,\mathbb{R})).$$

Hence, by case 1, $G_{v,w}$ is regular, that is, the limit of the transfer function of $G_{v,w}$ given by $s \mapsto \langle \boldsymbol{G}(s)v, w\rangle$ converges as $s \to \infty$ in $\mathbb{R}_+$, where $\boldsymbol{G}$ denotes the transfer function of G. This is true for all $v, w \in U$, and so, we may conclude that $\boldsymbol{G}(s)$ converges in the weak operator topology as $s \to \infty$ in $\mathbb{R}_+$, showing that G is weakly regular. ∎

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