IMA Journal of Numerical Analysis (2007) **27**, 45–73
doi:10.1093/imanum/drl024
Advance Access publication on August 9, 2006

The $\mathcal{Z}$-transform and linear multistep stability

JAMES J. COUGHLAN†, ADRIAN T. HILL‡ AND HARTMUT LOGEMANN§
Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK

[Received on 21 June 2005; revised on 8 June 2006]

This paper makes systematic use of control-theoretic methods such as the $\mathcal{Z}$-transform, small-gain theorems and frequency-domain stability criteria in the analysis of the stability behaviour of linear multistep methods. Some of the results in Nevanlinna's work are recovered and a number of new boundedness and asymptotic properties of solutions of numerical schemes are obtained. In particular, we give a careful and detailed analysis of the nonlinear stability properties of strictly zero-stable methods.

Keywords: behaviour at infinity; control theory; linear multistep methods; nonlinear stability; $\mathcal{Z}$-transform.

1. Introduction

In this paper, we consider the ordinary differential equation (ODE) problem

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) = f(t, y(t)), \quad t \in \mathbb{R}_+, \quad y(0) = y_0,$$

where $f\colon \mathbb{R}_+ \times H_0 \to H$ and $H_0 \subset H$ is a subspace of the (possibly finite-dimensional) complex Hilbert space H. This is to be approximated by a linear multistep method of the form

$$\sum_{j=0}^{q} \alpha_j U_{n+j} = h \sum_{j=0}^{q} \beta_j f((n+j)h, U_{n+j}), \quad n \in \mathbb{Z}_+, \tag{1.1}$$

with fixed time-step $h > 0$ and initial data $U_0, \ldots, U_{q-1}$. Our main concern here is to find conditions on hf and the method under which the difference of two numerical solutions, $U_n^1 - U_n^2$, is bounded uniformly with respect to n (with bound in terms of the initial data). Closely related questions are:

(i) When does the error grow at worst linearly as a function of n?
(ii) When is U_n itself bounded uniformly in n?

These issues are also connected with previous work, the aim of which was to find error bounds independent of the Lipschitz constant of the nonlinearity f.

For A-stable methods, Dahlquist (1978) established that $\|U_n^1 - U_n^2\|$ is uniformly bounded for all hf satisfying the monotonicity condition

$$\mathrm{Re}\langle f(t,\xi_1) - f(t,\xi_2), \xi_1 - \xi_2 \rangle_H \leqslant 0, \quad t \geqslant 0, \quad \xi_1, \xi_2 \in H_0. \tag{1.2}$$

†Email: mapjjc@maths.bath.ac.uk
‡Corresponding author. Email: ath@maths.bath.ac.uk
§Email: hl@maths.bath.ac.uk

However, for the more commonly used $A(\alpha)$-stable methods, the results of Dahlquist (1978) only established stability for disks of the form

$$\{z \in \mathbb{C}\colon |z - c| \leqslant R\} \quad (\text{where } c \in \mathbb{C})$$

contained in S, the linear stability domain of the method. More precisely, $\|U_n^1 - U_n^2\|$ was shown to be bounded uniformly in n for all hf satisfying the incremental circle condition,

$$\|hf(t,\xi_1) - hf(t,\xi_2) - c(\xi_1 - \xi_2)\|_H \leqslant R\|\xi_1 - \xi_2\|_H, \quad t \geqslant 0, \ \ \xi_1, \xi_2 \in H_0. \tag{1.3}$$

This is a less satisfactory condition for the integration of stiff systems as it implies an upper bound on h. Related results were given by Nevanlinna (1977b).

Expanded regions of nonlinear stability for $A(\alpha)$-stable methods were obtained by Nevanlinna & Odeh (1981), who applied the theory of Popov multipliers from control theory, see Popov (1962). However, this was at the cost of additional assumptions on hf.

In this paper, our objective is to use the $\mathcal{Z}$-transform method to investigate the nonlinear stability of $A(\alpha)$-stable methods for hf satisfying an incremental circle condition. Although transform methods have already been used by Nevanlinna (1977a,b), the proofs and assumptions in these papers are somewhat difficult to follow. The presentation in this paper is in our opinion more transparent due to a systematic use of the $\mathcal{Z}$-transform and other control-theoretic techniques, such as ‘small-gain theorems’ (Section 4, see Remark 4.4(b)) and frequency-domain stability criteria (Sections 5 and 6, see Remark 5.3). The methods of this paper will also form the basis of future work by the authors on Popov multipliers in multistep stability which, in particular, will exploit insights from control theory on the trade-off between the expansion of the stability region beyond a disk and further assumptions on the structure of hf.

One class of new results in this paper is bounds on $\|U_n^1 - U_n^2\|$ purely in terms of the initial data; i.e. bounds independent of hf. These are especially important in applications to parabolic partial differential equations. Additionally, we prove new results on the behaviour of $(U_n^1 - U_n^2)$ as $n \to \infty$ and bounds are obtained for some classes of methods not considered in Nevanlinna (1977a,b), see Remarks 4.4(a) and 6.7.

The objective of bounding a single solution (U_n) of (1.1) leads to a consideration of the (weak) dissipativity condition

$$\mathrm{Re}\langle f(t,\xi), \xi\rangle_H \leqslant 0, \quad t \geqslant 0, \ \ \xi \in H_0, \tag{1.4}$$

and the nonincremental circle condition

$$\|hf(t,\xi) - c\xi\|_H \leqslant R\|\xi\|_H, \quad t \geqslant 0, \ \ \xi \in H_0, \tag{1.5}$$

where $c \in \mathbb{C}$ and $R > 0$. While (1.4) is relevant for A-stable methods, (1.5) is important in the context of $A(\alpha)$-stable methods. Condition (1.5) bounds the deviation of $hf(\xi)$ from the linear function $c\xi$. Assuming (1.5), we show that (U_n) is bounded uniformly in n, provided that the closed disk $|z - c| \leqslant R$ is in S. Under mildly strengthened hypotheses, new results on the qualitative asymptotic behaviour of U_n are also shown. Stability results involving the conditions (1.4) and (1.5) can frequently be invoked to derive corresponding ‘incremental’ results in which (1.4) and (1.5) are replaced by (1.2) and (1.3), see Sections 4 and 6.

The methodology used in this paper requires us to first write (1.1) as a convolution identity. If (U_n) is the solution of (1.1), then one may define sequences u, r and s, such that

$$u(n) := U_n, \quad r(n) := \begin{cases} \alpha_{q-n}, & 0 \leqslant n \leqslant q, \\ 0, & n > q, \end{cases} \quad \text{and} \quad s(n) := \begin{cases} \beta_{q-n}, & 0 \leqslant n \leqslant q, \\ 0, & n > q. \end{cases} \tag{1.6}$$

Method (1.1) may then be rewritten as the convolution identity

$$r * u = s * (f_h \circ u) + v, \tag{1.7}$$

where

$$(f_h \circ u)(n) := hf(nh, U_n), \quad n \in \mathbb{Z}_+, \tag{1.8}$$

$$v(n) := \begin{cases} (r * u)(n) - (s * (f_h \circ u))(n), & 0 \leqslant n \leqslant q - 1, \\ 0, & n \geqslant q, \end{cases} \tag{1.9}$$

and $(a * b)(n) = \sum_{k=0}^{n} a(n-k)b(k)$, $n \in \mathbb{Z}_+$, denotes the convolution between two sequences.

Further analysis makes use of the $\mathcal{Z}$-transform

$$\hat{a}(z) := \sum_{n=0}^{\infty} a(n) z^{-n},$$

for $z \in \mathbb{C}$ with $|z|$ sufficiently large. While much of the $\mathcal{Z}$-transform methodology is similar to more familiar Fourier techniques, the $\mathcal{Z}$-transform, as is well-known in control theory, is a more natural tool in the context of stability analysis (see Section 2 for more details on the $\mathcal{Z}$-transform).

The structure of the paper is as follows. In Section 2, standard results on sequences, convolutions and $\mathcal{Z}$-transforms are presented. In Section 3, the convolution identity (1.7) is proved, together with related identities. In Section 4, we consider hf satisfying a ‘strict’ version of (1.3):

$$\sup_{t \geqslant 0, \xi_1, \xi_2 \in H_0, \xi_1 \neq \xi_2} \frac{\|hf(t,\xi_1) - hf(t,\xi_2) - c(\xi_1 - \xi_2)\|_H}{\|\xi_1 - \xi_2\|_H} < R.$$

A straightforward proof using the ‘small-gain’ idea shows that $(U_n^1 - U_n^2)$ is bounded uniformly in n. In Section 5, we derive a stability criterion of a control-theoretic nature which guarantees certain boundedness and asymptotic properties for the solutions of a nonlinear discrete-time Volterra equation. This criterion, which we expect to be of independent interest in discrete-time and sampled-data control theory, is used in Section 6 in the context of a careful and detailed stability analysis of strictly-zero stable methods (1.1) with hf satisfying the (nonstrict) circle condition (1.3). Some of the results of Nevanlinna (1977a,b) are recovered and a number of new boundedness and asymptotic properties of the solutions of (1.1) are obtained.

Notation. Let $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, $\mathbb{N} := \mathbb{Z}_+ \backslash \{0\}$ and $\mathbb{R}_+ := [0, \infty)$. For $c \in \mathbb{C}$ and $R > 0$ define

$$\mathbb{B}(c, R) := \{z \in \mathbb{C}\colon |z - c| < R\}.$$

For $\alpha > 0$ set

$$\mathbb{E}_\alpha := \{z \in \mathbb{C}\colon |z| > \alpha\} = \mathbb{C} \backslash \overline{\mathbb{B}}(0, \alpha).$$

Let H be a complex Hilbert space and H_0 a subspace of H. The vector space of all unilateral sequences (defined on $\mathbb{Z}_+$) with values in H is denoted by $\mathcal{S}(H)$. For $1 \leqslant p \leqslant \infty$, let $l^p(H)$ denote the l^p-space of unilateral H-valued sequences. In the special case $H = \mathbb{C}$, we write l^p for $l^p(\mathbb{C})$ and $\mathcal{S}$ for $\mathcal{S}(\mathbb{C})$. For $K \subset H$ a nonempty subset and $\xi_0 \in H$, we define

$$\operatorname{dist}(\xi_0, K) := \inf_{\xi \in K} \|\xi_0 - \xi\|_H.$$

For $\varphi\colon \mathbb{Z}_+ \times H_0 \to H$ and $x\colon \mathbb{Z}_+ \to H_0$, by slight abuse of notation, we denote the function $n \mapsto \varphi(n, x(n))$ by $\varphi \circ x$.

2. Preliminaries: convolutions and $\mathcal{Z}$-transforms

For $a \in \mathcal{S}$ and $b \in \mathcal{S}(H)$, we define the 'convolution' $a * b$ of a and b by

$$(a * b)(n) := \sum_{k=0}^{n} a(n-k)b(k).$$

The convolution product in the space $\mathcal{S}$ is commutative and the sequence δ defined by

$$\delta(n) := \begin{cases} 1, & n = 0, \\ 0, & n \in \mathbb{N}, \end{cases}$$

is the unit element. A sequence $a \in \mathcal{S}$ is invertible (i.e. there exists $a^{-1} \in \mathcal{S}$ such that $a * a^{-1} = a^{-1} * a = \delta$) if and only if $a(0) \neq 0$.

Defining $\theta \in \mathcal{S}$ by

$$\theta(n) = 1, \quad n \in \mathbb{Z}_+,$$

summation of $a \in \mathcal{S}(H)$, Σa, can be represented by convolution with θ:

$$(\Sigma a)(n) := \sum_{j=0}^{n} a(j) = (\theta * a)(n).$$

It is easily verifiable that

$$\theta^{-1}(n) := \begin{cases} 1, & n = 0, \\ -1, & n = 1, \\ 0, & n \geqslant 2. \end{cases}$$

Defining the backward difference operator $\nabla : \mathcal{S}(H) \to \mathcal{S}(H)$ by

$$(\nabla a)(n) := \begin{cases} a(0), & n = 0, \\ a(n) - a(n-1), & n \in \mathbb{N}, \end{cases}$$

it follows that $\nabla a = \theta^{-1} * a$ and

$$(\Sigma \nabla)(a) = (\nabla \Sigma)(a) = a, \quad a \in \mathcal{S}(H).$$

Let $a \in \mathcal{S}(H)$. The $\mathcal{Z}$-'transform' $\mathcal{Z}a$ of a is defined by

$$\hat{a}(z) := (\mathcal{Z}a)(z) := \sum_{j=0}^{\infty} a(j) z^{-j}, \tag{2.1}$$

where z is a complex variable. We say that a is $\mathcal{Z}$-'transformable' if the series in (2.1) converges for some $z = z_0 \in \mathbb{C}\backslash\{0\}$, in which case it converges absolutely for all $z \in \mathbb{C}$ with $|z| > |z_0|$. It is an elementary fact from the theory of power series, see e.g. Conway (1978, p. 31), that a is $\mathcal{Z}$-transformable if and only if

$$r_a := \limsup_{n \to \infty} \|a(n)\|_H^{1/n} < \infty,$$

in which case (2.1) converges absolutely if $|z| > r_a$ and diverges if $|z| < r_a$. For $\eta > r_a$ we have that

$$\hat{a}(\eta\, \mathrm{e}^{\mathrm{i}\omega}) = \sum_{k=0}^{\infty} (a(k)\eta^{-k})\mathrm{e}^{-\mathrm{i}k\omega}, \quad \omega \in [0, 2\pi),$$

showing that the function $\omega \mapsto \hat{a}(\eta\, \mathrm{e}^{\mathrm{i}\omega})$ is the discrete Fourier transform of the sequence $(a(k)\eta^{-k})_{k\in\mathbb{Z}_+}$. If $a \in \mathcal{S}(H)$ is $\mathcal{Z}$-transformable, then, for every $\eta > r_a$, the function $\hat{a}$ is holomorphic and bounded on $\mathbb{E}_\eta$. Conversely, if $\eta > 0$ and $A\colon \mathbb{E}_\eta \to H$ is holomorphic and bounded, then $a \in \mathcal{S}(H)$ defined by

$$a(n) := \frac{1}{2\pi\,\mathrm{i}} \int_{|z|=\nu} A(z) z^{n-1} \mathrm{d}z = \frac{\nu^n}{2\pi} \int_0^{2\pi} A(\nu\, \mathrm{e}^{\mathrm{i}\omega}) \mathrm{e}^{n\mathrm{i}\omega} \mathrm{d}\omega, \quad \text{where } \nu > \eta,$$

is the unique $\mathcal{Z}$-transformable sequence (with $r_a \leqslant \eta$) such that $\hat{a}(z) = A(z)$ for all $z \in \mathbb{E}_\eta$ and we write $a = \mathcal{Z}^{-1}(A)$.

Under the $\mathcal{Z}$-transform, convolutions become multiplications: if $a \in \mathcal{S}$ and $b \in \mathcal{S}(H)$ are $\mathcal{Z}$-transformable, then $a * b$ is $\mathcal{Z}$-transformable and

$$\widehat{(a * b)}(z) = \hat{a}(z)\hat{b}(z), \quad z \in \mathbb{C} \text{ s.t. } |z| > \max\{r_a, r_b\}.$$

For $a, b \in l^1(H) \subset l^2(H)$ the ‘Parseval–Bessel’ identity holds:

$$\sum_{k=0}^{\infty} \langle a(k), b(k)\rangle_H = \frac{1}{2\pi} \int_0^{2\pi} \langle \hat{a}(\mathrm{e}^{\mathrm{i}\omega}), \hat{b}(\mathrm{e}^{\mathrm{i}\omega})\rangle_H \, \mathrm{d}\omega.$$

A sequence $a \in \mathcal{S}(H)$ is said to be exponentially decaying if there exist $\eta \in (0, 1)$ and $M > 0$ such that

$$\|a(n)\|_H \leqslant M\eta^n, \quad n \in \mathbb{Z}_+.$$

Note that $a \in \mathcal{S}(H)$ is exponentially decaying if and only if $r_a < 1$.

The following three results contain some standard results on convolutions and $\mathcal{Z}$-transforms and will be used freely in the following sections.

LEMMA 2.1 Assume that $a \in l^1$ and let $1 \leqslant p \leqslant \infty$. The following statements hold:

(a) $\|a * b\|_{l^p(H)} \leqslant \|a\|_{l^1} \|b\|_{l^p(H)}$ for all $b \in l^p(H)$.

(b) The l^2-induced norm of the operator $l^2(H) \to l^2(H)$, $b \mapsto a * b$, is given by

$$\sup_{\omega\in[0,2\pi)} |\hat{a}(\mathrm{e}^{\mathrm{i}\omega})| = \sup_{|z|\geqslant 1} |\hat{a}(z)|.$$

(c) If $b \in \mathcal{S}(H)$, then

$$\lim_{n\to\infty} b(n) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} (a * b)(n) = 0.$$

Proof. We refer to Desoer & Vidyasagar (1975, p. 244) for the proof of part (a) and to Partington (1997, p. 85) for part (b). To prove part (c), assume that $b(n) \to 0$ as $n \to \infty$. For every $n \in \mathbb{Z}_+$, let m_n denote the largest integer less than or equal to $n/2$. Since

$$(a * b)(n) = \sum_{k=0}^{m_n} a(n-k)b(k) + \sum_{k=m_n+1}^{n} a(n-k)b(k),$$

we obtain

$$\|(a * b)(n)\|_H \leqslant \|b\|_{l^\infty(H)} \sum_{k \geqslant n/2} |a(k)| + \|a\|_{l^1} \sup_{k \geqslant n/2} \|b(k)\|_H .$$

By assumption $b(k) \to 0$ as $k \to \infty$ and, since $a \in l^1$,

$$\lim_{n\to\infty} \sum_{k \geqslant n/2} |a(k)| = 0.$$

Consequently, $(a * b)(n) \to 0$ as $n \to \infty$. □

For every $a \in \mathcal{S}$, we define

$$\|a\|_{\mathcal{L}(l^p, l^q)} := \sup_{\substack{b \in l^p(H) \\ b \neq 0}} \frac{\|a * b\|_{l^q(H)}}{\|b\|_{l^p(H)}}.$$

It is convenient to set $\|a\|_{\mathcal{L}(l^p)} := \|a\|_{\mathcal{L}(l^p, l^p)}$. If $\|a\|_{\mathcal{L}(l^p, l^q)} < \infty$, then the operator $b \mapsto a * b$ is bounded from $l^p(H)$ into $l^q(H)$ and the norm of this operator is given by $\|a\|_{\mathcal{L}(l^p, l^q)}$. Lemma 2.1 says that, for $a \in l^1$,

$$\|a\|_{\mathcal{L}(l^p)} \leqslant \|a\|_{l^1}, \quad 1 \leqslant p \leqslant \infty,$$

and

$$\|a\|_{\mathcal{L}(l^2)} = \sup_{\omega \in [0, 2\pi)} |\hat{a}(\mathrm{e}^{i\omega})| = \sup_{|z| \geqslant 1} |\hat{a}(z)|.$$

Furthermore, if $a \in l^1$, it is easy to see that $\|a\|_{\mathcal{L}(l^1)} = \|a\|_{\mathcal{L}(l^\infty)} = \|a\|_{l^1}$.

In the following, if $a \in \mathcal{S}$ and $k \in \mathbb{N}$, then

$$a^k := \underbrace{a * a * \cdots * a}_{k \text{ factors}} .$$

For $\xi \in \mathbb{C}$, define

$$\vartheta_\xi := (0, 1, \xi, \xi^2, \xi^3, \ldots).$$

Note that $\hat{\vartheta}_\xi(z) = 1/(z - \xi)$.

A rational function A is said to be 'proper' if $\lim_{|z|\to\infty} |A(z)| < \infty$.

LEMMA 2.2 Let A be a proper rational function. Then there exists a $\mathcal{Z}$-transformable sequence $a \in \mathcal{S}$ such that $a = \mathcal{Z}^{-1}(A)$ and a is of the form

$$a = \gamma\delta + \sum_{k=1}^{m} \sum_{j=1}^{m_k} \gamma_{kj} \vartheta_{z_k}^{j}, \tag{2.2}$$

where $\gamma, \gamma_{kj} \in \mathbb{C}$ are suitable coefficients, the z_k are the poles of A and m_k denotes the multiplicity of z_k.

Proof. If p_k is the principal part of the Laurent expansion of A at z_k, then

$$p_k(z) = \sum_{j=1}^{m_k} \frac{\gamma_{kj}}{(z - z_k)^j},$$

where the γ_{kj} are suitable constants. The function $B := A - \sum_{k=1}^{m} p_k$ is a rational function without any poles, and hence B must be a polynomial. Since A is proper, it follows that B is a constant polynomial equal to some $\gamma \in \mathbb{C}$. Therefore, $A = \gamma + \sum_{k=1}^{m} p_k$, and thus $a := \mathcal{Z}^{-1}(A)$ is of the form (2.2). □

The following corollary is an immediate consequence of Lemma 2.2.

COROLLARY 2.3 For a proper rational function A, the following statements hold:

(a) If A is holomorphic in $\overline{\mathbb{E}}_1$, then $\mathcal{Z}^{-1}(A)$ is exponentially decaying; in particular, $\mathcal{Z}^{-1}(A) \in l^1$.

(b) If A is holomorphic in $\mathbb{E}_1$ and has only simple poles $\{z_k\}_{k=1}^{m}$ on the complex unit circle, then

$$\mathcal{Z}^{-1}(A) = a_0 + \sum_{k=1}^{m} \gamma_k \vartheta_{z_k},$$

where $a_0 \in \mathcal{S}$ is exponentially decaying and γ_k denotes the residue of A at z_k.

3. The method as a convolution identity

We assume throughout the paper that the method (1.1) has coefficients $\alpha_0, \ldots, \alpha_q, \beta_0, \ldots, \beta_q \in \mathbb{R}$ with $\alpha_q > 0$ and that the polynomials

$$\rho(z) := \sum_{j=0}^{q} \alpha_j z^j, \quad \sigma(z) := \sum_{j=0}^{q} \beta_j z^j \tag{3.1}$$

are coprime. We observe that

$$\hat{r}(z) = z^{-q}\rho(z), \quad \hat{s}(z) = z^{-q}\sigma(z), \quad z \in \mathbb{C}, \tag{3.2}$$

where r and s are given by (1.6). The method is said to be ‘consistent’ if $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$. A polynomial $p(z)$ satisfies the ‘root condition’ if

$$p(z) = 0 \text{ implies either } |z| < 1, \text{ or } |z| = 1 \text{ and } p'(z) \neq 0.$$

The method (1.1) is said to be ‘zero-stable’ if ρ satisfies the root condition. The method (1.1) is ‘strictly zero-stable’ if it is zero-stable and $z = 1$ is the only root of ρ on the complex unit circle. The ‘linear stability domain’ S for method (1.1) is the set

$$S := \{\zeta \in \mathbb{C}\colon \rho(z) - \zeta\sigma(z) \text{ satisfies the root condition}\}.$$

We emphasise that the properties of consistency, zero-stability and strict zero-stability will only be assumed where indicated.

In applications, the boundedness of several related quantities is considered: the numerical solution, the numerical error and the difference between two numerical solutions. The following lemma is applicable in all these cases.

LEMMA 3.1 Suppose that $D_n \in H$ for $n \geqslant q$, and that $\varphi\colon \mathbb{Z}_+ \times H_0 \to H$. Suppose also that $X_n \in H_0$, $n \in \mathbb{Z}_+$, satisfies

$$\sum_{j=0}^{q} \alpha_j X_{n+j} = \sum_{j=0}^{q} \beta_j \varphi(n+j, X_{n+j}) + D_{n+q}, \quad n \in \mathbb{Z}_+.$$

Then, for $x \in \mathcal{S}(H_0)$ given by $x(n) = X_n$, $n \in \mathbb{Z}_+$, and $r, s \in \mathcal{S}$ as defined in (1.6),

$$r * x = s * (\varphi \circ x) + \upsilon, \tag{3.3}$$

where υ is given by

$$\upsilon(n) := \begin{cases} (r * x)(n) - (s * (\varphi \circ x))(n), & 0 \leqslant n \leqslant q-1, \\ D_n, & n \geqslant q. \end{cases}$$

Proof. Considering the left-hand side of (3.3), the finite support of r implies that

$$(r * x)(n+q) = \sum_{k=0}^{n+q} r(n+q-k)x(k) = \sum_{j=0}^{q} r(q-j)x(n+j) = \sum_{j=0}^{q} \alpha_j X_{n+j}, \quad n \in \mathbb{Z}_+.$$

Similarly, considering the right-hand side of (3.3),

$$(s * (\varphi \circ x))(n+q) + \upsilon(n+q) = \sum_{j=0}^{q} \beta_j \varphi(n+j, X_{n+j}) + D_{n+q}, \quad n \in \mathbb{Z}_+.$$

Hence, $(r * x)(n) = (s * (\varphi \circ x))(n) + \upsilon(n)$ for all $n \geqslant q$. By definition of υ it now follows that (3.3) holds. □

Consider the ODE problem,

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) = f(t, y(t)), \quad t \in \mathbb{R}_+, \quad y(0) = y_0, \tag{3.4}$$

where $f\colon \mathbb{R}_+ \times H_0 \to H$ satisfies suitable regularity conditions. We describe three situations in numerical analysis to which Lemma 3.1 applies.

Case 1 *(The numerical solution).* For the method given by (1.1), Lemma 3.1 may be applied with

$$X_n := U_n; \quad \varphi(n, \xi) := hf(nh, \xi), \quad \xi \in H_0; \quad D_{n+q} := 0, \quad n \in \mathbb{Z}_+.$$

Case 2 *(The numerical error).* Let y be the solution of (3.4). The 'truncation error' $T_{n+q} \in H$, $n \in \mathbb{Z}_+$, is defined by

$$\sum_{j=0}^{q} \alpha_j y((n+j)h) = h \sum_{j=0}^{q} \beta_j f((n+j)h, y((n+j)h)) + hT_{n+q}, \quad n \in \mathbb{Z}_+.$$

Lemma 3.1 may be applied with

$$X_n := y(nh) - U_n, \quad n \in \mathbb{Z}_+,$$
$$\varphi(n, \xi) := hf(nh, y(nh)) - hf(nh, y(nh) - \xi), \quad n \in \mathbb{Z}_+, \quad \xi \in H_0,$$
$$D_n := hT_n, \quad n \geqslant q,$$

where U_n is the solution of (1.1).

Case 3 *(Difference of two numerical solutions).* If $U_n^1, U_n^2 \in H_0, n \in \mathbb{Z}_+$, are two solutions of (1.1), then Lemma 3.1 may be applied with

$$\begin{aligned} X_n &:= U_n^1 - U_n^2, \quad n \in \mathbb{Z}_+, \\ \varphi(n,\xi) &:= hf(nh, U_n^1) - hf(nh, U_n^1 - \xi), \quad n \in \mathbb{Z}_+, \ \ \xi \in H_0, \\ D_n &:= 0, \quad n \in \mathbb{Z}_+. \end{aligned}$$

4. A simple circle criterion

In this section, the initial aim is to establish boundedness of (U_n), a solution of (1.1), under the assumption that f satisfies the circle condition (1.5):

$$\|hf(t,\xi) - c\xi\|_H \leqslant R\|\xi\|_H, \quad t \geqslant 0, \ \ \xi \in H_0.$$

This is related to the incremental circle condition (1.3) assumed by Dahlquist (1978, Section 5) in investigating the problem of bounding the difference between two solutions of (1.1). For the case of bounding a single solution (U_n), the results of Dahlquist (1978, Section 5), proved using G-norms and Möbius maps, imply that if $\overline{\mathbb{B}}(c, R) \subset S$ and f satisfies (1.5) then $\|U_n\|_H$ is bounded uniformly in n. In Sections 5 and 6, we give a different proof of this result, using the convolution representation of the method, $\mathcal{Z}$-transform theory and summation by parts.

Here, we give a simple proof of a somewhat weaker result, using the convolution representation derived in Case 1 of Section 3, by means of a technique known as the small-gain argument in control theory, see e.g. Desoer & Vidyasagar (1975) and Vidyasagar (1993). This essentially means that the nonlinear bound is obtained as a perturbation of a linear, constant coefficient, problem: in this case, the numerical solution of

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{c}{h}y.$$

Such an argument is feasible because $\|(r - cs)^{-1} * s\|_{\mathcal{L}(l^2)}$ can be 'exactly' expressed in terms of the polynomials $\rho(z)$ and $\sigma(z)$ (see (3.2)) and thus be related to the stability domain S, as in the following lemma.

LEMMA 4.1 For $c \in \mathbb{C}$ and $R > 0$, the following equivalences hold for method (1.1)*:*

$$\mathbb{B}(c,R) \subset S \quad \Longleftrightarrow \quad \inf_{|z|\geqslant 1} |\rho(z)/\sigma(z) - c| \geqslant R \quad \Longleftrightarrow \quad \sup_{|z|\geqslant 1} |\sigma(z)/(\rho(z) - c\sigma(z))| \leqslant 1/R.$$

Proof. Assuming $\mathbb{B}(c,R) \subset S$, we deduce that $\mathbb{B}(c,R) \subset \mathrm{int}(S)$. Hence, if $\zeta \in \mathbb{B}(c,R)$, then $\rho(z) - \zeta\sigma(z) = 0$ implies that $|z| < 1$. Thus,

$$\inf_{|z|\geqslant 1} |\rho(z)/\sigma(z) - c| \geqslant R, \tag{4.1}$$

or, equivalently,

$$\sup_{|z|\geqslant 1} |\sigma(z)/(\rho(z) - c\sigma(z))| \leqslant 1/R. \tag{4.2}$$

Conversely, assume (4.2) or, equivalently, (4.1) holds. For $\zeta \in \mathbb{C}\backslash S$, there exists $z_0 \in \mathbb{C}$ with $|z_0| \geqslant 1$ and such that $\rho(z_0) - \zeta\sigma(z_0) = 0$. Thus, by (4.1), $|\zeta - c| \geqslant R$, which implies $\zeta \in \mathbb{C}\backslash\mathbb{B}(c,R)$. We deduce that $\mathbb{B}(c,R) \subset S$. □

For $m \in \mathbb{Z}_+$, let $P^{(m)}: \mathcal{S}(H) \to \mathcal{S}(H)$ be the projection such that

$$(P^{(m)}a)(n) = \begin{cases} a(n), & 0 \leqslant n \leqslant m, \\ 0, & n > m, \end{cases} \qquad n \in \mathbb{Z}_+, \quad a \in \mathcal{S}(H).$$

The following properties of $P^{(m)}$ are readily verified.

LEMMA 4.2 For $a \in \mathcal{S}$, $b \in l^2(H)$, $\lim_{m\to\infty} \|P^{(m)}b - b\|_{l^2(H)} = 0$ and

$$P^{(m)}(a * b) = P^{(m)}a * P^{(m)}b, \quad \|P^{(m)}b\|_{l^2(H)} \leqslant \|b\|_{l^2(H)}, \quad m \in \mathbb{Z}_+.$$

The following theorem is the main result of this section.

THEOREM 4.3 Assume that for some $c \in \mathbb{C}$ and $R > 0$, $\mathbb{B}(c, R) \subset S$ and that

$$\|hf(nh, \xi) - c\xi\|_H \leqslant R_1 \|\xi\|_H, \quad (n, \xi) \in \mathbb{Z}_+ \times H_0, \tag{4.3}$$

for some $R_1 < R$. Then the solution of (1.1) satisfies

$$\sup_{n\geqslant 0} \|U_n\|_H \leqslant \left(\sum_{n=0}^{\infty} \|U_n\|_H^2\right)^{1/2} \leqslant \gamma \left(\sum_{j=0}^{q-1} \|U_j\|_H^2\right)^{1/2},$$

where

$$\gamma := \frac{R}{R - R_1} \frac{\max_{\omega\in[0,2\pi)} |\rho(\mathrm{e}^{\mathrm{i}\omega})| + (R_1 + |c|) \max_{\omega\in[0,2\pi)} |\sigma(\mathrm{e}^{\mathrm{i}\omega})|}{\min_{\omega\in[0,2\pi)} |(\rho - c\sigma)(\mathrm{e}^{\mathrm{i}\omega})|}. \tag{4.4}$$

Proof. As $c \in \operatorname{int}(S)$, $\rho(z) - c\sigma(z) = 0$ implies $|z| < 1$. Also, by Lemma 4.1, $\frac{\sigma}{\rho - c\sigma}(z)$ and $\frac{z^q}{(\rho - c\sigma)(z)}$ are proper rational functions. Hence, by Corollary 2.3 part (a) and Lemma 2.1,

$$\|(r - cs)^{-1} * s\|_{\mathcal{L}(l^2)} = \max_{\omega\in[0,2\pi)} \left|\frac{\sigma}{\rho - c\sigma}(\mathrm{e}^{\mathrm{i}\omega})\right| = \frac{1}{\min_{\omega\in[0,2\pi)} \left|\frac{\rho}{\sigma}(\mathrm{e}^{\mathrm{i}\omega}) - c\right|} \leqslant \frac{1}{R},$$

$$\|(r - cs)^{-1}\|_{\mathcal{L}(l^2)} = \frac{1}{\min_{\omega\in[0,2\pi)} |(\rho - c\sigma)(\mathrm{e}^{\mathrm{i}\omega})|} =: \mu,$$

where, of course, $(r - cs)^{-1}$ denotes the inverse of $r - cs$ with respect to convolution. By (1.9),

$$r * u = s * (f_h \circ u) + v, \tag{4.5}$$

where $f_h: \mathbb{Z}_+ \times H_0 \to H$ is given by

$$f_h(n, \xi) := hf(nh, \xi), \quad (n, \xi) \in \mathbb{Z}_+ \times H_0. \tag{4.6}$$

Hence, for $m \geqslant q$,

$$u^{(m)} = P^{(m)}((r - cs)^{-1} * s * (f_h \circ u^{(m)} - cu^{(m)})) + P^{(m)}((r - cs)^{-1} * v), \tag{4.7}$$

where $u^{(m)} := P^{(m)}u$. From Lemma 4.2,

$$\|u^{(m)}\|_{l^2(H)} \leqslant R^{-1}\| f_h \circ u^{(m)} - cu^{(m)}\|_{l^2(H)} + \mu\|v\|_{l^2(H)}.$$

Now, (4.3) implies that

$$\| f_h \circ u^{(m)} - cu^{(m)}\|_{l^2(H)} = \left(\sum_{n=0}^{m} \|hf(nh, u(n)) - cu(n)\|_H^2\right)^{1/2} \leqslant R_1\|u^{(m)}\|_{l^2(H)}.$$

Hence, using $R_1 < R$,

$$\|u^{(m)}\|_{l^2(H)} \leqslant \frac{R_1}{R}\|u^{(m)}\|_{l^2(H)} + \mu\|v\|_{l^2(H)} \leqslant \mu R\frac{\|v\|_{l^2(H)}}{R - R_1}. \tag{4.8}$$

To bound $\|v\|_{l^2(H)}$, we deduce from (1.9) and $m \geqslant q$ that

$$v(n) := \begin{cases} (r * u^{(m)})(n) - (s * (f_h \circ u^{(m)}))(n), & 0 \leqslant n \leqslant q - 1, \\ 0, & n \geqslant q. \end{cases}$$

Now,

$$\| f_h \circ u^{(m)}\|_{l^2(H)} \leqslant |c|\|u^{(m)}\|_{l^2(H)} + \| f_h \circ u^{(m)} - cu^{(m)}\|_{l^2(H)} \leqslant (|c| + R_1)\|u^{(m)}\|_{l^2(H)}.$$

Using $\|r\|_{\mathcal{L}(l^2)} = \max_{\omega\in[0,2\pi)} |\rho(\mathrm{e}^{\mathrm{i}\omega})|$ and $\|s\|_{\mathcal{L}(l^2)} = \max_{\omega\in[0,2\pi)} |\sigma(\mathrm{e}^{\mathrm{i}\omega})|$ to bound $\|v\|_{l^2(H)}$ in (4.8), we deduce that

$$\left(\sum_{n=0}^{m} \|u(n)\|_H^2\right)^{1/2} \leqslant \gamma \left(\sum_{j=0}^{q-1} \|u(j)\|_H^2\right)^{1/2}, \quad m \geqslant q.$$

The proof is completed by letting m tend to ∞. □

REMARK 4.4

(a) While Theorem 4.3 has some overlap with a result by Nevanlinna (1977b) (see Theorem 3.1 in Nevanlinna, 1977b, with $\theta < 1$ in the notation of Nevanlinna, 1977b), there are considerable differences:

 (a1) The approach adopted in Nevanlinna (1977b) considers f that are independent of t and requires that the map $I - ahf$ is bijective and its inverse is globally Lipschitz,[1] where a is a constant which appears in a frequency-domain condition involving ρ and σ (in particular $1/a \in \mathrm{int}(S)$), making the application of the main result in Nevanlinna (1977b) potentially awkward if $a \neq 0$ (which, e.g. is the case if S is bounded).

 (a2) The condition in Nevanlinna (1977b) on the linear stability domain involves disks with centre on the real axis (see p. 60 in Nevanlinna, 1977b), while Theorem 4.3 allows for disks with centre anywhere in the complex plane.

[1] An inspection of the proof of Theorem 3.1 in Nevanlinna (1977b) shows that, if it is the aim to obtain bounds on the numerical solution rather than the error, then the global Lipschitz assumption on $(I - ahf)^{-1}$ can be replaced by a global linear boundedness condition, i.e. $\sup_{\xi\in H}(\|(I - ahf)^{-1}\xi\|_H/\|\xi\|_H) < \infty$.

(b) The proof of Theorem 4.3 is essentially a combined application of ‘loop-transformation’ and small-gain ideas familiar in control theory (see Desoer & Vidyasagar, 1975; Vidyasagar, 1993): loop-transformation applied to (4.5) gives (4.7) which satisfies the small-gain condition, as follows from the assumption that $R_1 < R$.

(c) Note that Theorem 4.3 cannot be applied if f is bounded and there exists a $z \in \overline{\mathbb{E}}_1$ such that $\rho(z) = 0$ (the latter is true if (1.1) is consistent), as follows from Lemma 4.1 and (4.3).

(d) Using a routine argument involving exponential weighting, the conclusions of Theorem 4.3 can be strengthened (without changing the assumptions). In fact, it may be shown that there exist $\Gamma > 0$ and $\eta \in (0, 1)$, depending only on c, R, R_1 and the method (ρ, σ), such that the solution of (1.1) satisfies

$$\|U_n\|_H \leqslant \Gamma \eta^n \left(\sum_{j=0}^{q-1} \|U_j\|_H^2 \right)^{1/2}, \quad n \in \mathbb{Z}_+.$$

We now return to the problem of bounding the difference between two solutions of (1.1). Before stating our result, we make an observation on the relationship between the two circle conditions (1.3) and (1.5), the proof of which is self-evident.

LEMMA 4.5 Suppose that $(U_n^1)_{n\in\mathbb{Z}_+}$ and $(U_n^2)_{n\in\mathbb{Z}_+}$ are two solutions of (1.1) and that Condition (1.3),

$$\|hf(t, \xi_1) - hf(t, \xi_2) - c(\xi_1 - \xi_2)\|_H \leqslant R\|\xi_1 - \xi_2\|_H, \quad t \geqslant 0, \ \ \xi_1, \xi_2 \in H_0,$$

is satisfied for some $c \in \mathbb{C}$ and $R > 0$. Then,

$$\sum_{j=0}^{q} \alpha_j (U_{n+j}^1 - U_{n+j}^2) = \sum_{j=0}^{q} \beta_j \psi(n + j, U_{n+j}^1 - U_{n+j}^2), \quad n \in \mathbb{Z}_+,$$

where $\psi\colon \mathbb{Z}_+ \times H_0 \to H$, defined by

$$\psi(n, \xi) := hf(nh, \xi + U_n^2) - hf(nh, U_n^2), \quad n \in \mathbb{Z}_+, \ \ \xi \in H_0,$$

satisfies the (nonincremental) circle condition

$$\|\psi(n, \xi) - c\xi\| \leqslant R\|\xi\|, \quad n \in \mathbb{Z}_+, \ \ \xi \in H_0.$$

The following result, bounding $\|U_n^1 - U_n^2\|$, follows from a combination of Theorem 4.3 and Lemma 4.5 (or by analogy with the proof of Theorem 4.3).

COROLLARY 4.6 Assume that for some $c \in \mathbb{C}$ and $R > 0$, $\mathbb{B}(c, R) \subset S$ and that

$$\|hf(nh, \xi_1) - hf(nh, \xi_2) - c(\xi_1 - \xi_2)\|_H \leqslant R_1\|\xi_1 - \xi_2\|_H, \quad n \in \mathbb{Z}_+, \ \ \xi_1, \xi_2 \in H_0, \tag{4.9}$$

for some $R_1 < R$. Let (U_n^1) and (U_n^2) be two solutions of (1.1). Then

$$\sup_{n\geqslant 0} \|U_n^1 - U_n^2\|_H \leqslant \left(\sum_{n=0}^{\infty} \|U_n^1 - U_n^2\|_H^2 \right)^{1/2} \leqslant \gamma \left(\sum_{j=0}^{q-1} \|U_j^1 - U_j^2\|_H^2 \right)^{1/2},$$

where γ is given by (4.4).

A conclusion similar to Corollary 4.6 is obtained, under weaker assumptions, in Corollary 6.8. Both these results are directly comparable to the work of Dahlquist (1978, Section 5).

5. A stability result for a class of nonlinear discrete-time Volterra equations

Consider the nonlinear discrete-time Volterra equation

$$x(n) = \sum_{j=0}^{n} g(n-j)\varphi(j, x(j)) + w(n),$$

where the convolution kernel $g\colon \mathbb{Z}_+ \to \mathbb{C}$, the (time-dependent) nonlinearity $\varphi\colon \mathbb{Z}_+ \times H_0 \to H$ and the forcing function $w\colon \mathbb{Z}_+ \to H$ are given. The above equation can be written in the more compact form

$$x = g * (\varphi \circ x) + w. \tag{5.1}$$

A solution of (5.1) is a H_0-valued function x defined on $\mathbb{Z}_+$ satisfying (5.1). Trivially, there exists at least one solution (a unique solution, respectively) of (5.1) if, for every $n \in \mathbb{Z}_+$, the map

$$H_0 \to H, \quad \xi \mapsto \xi - g(0)\varphi(n, \xi)$$

is surjective (bijective, respectively).

The subspace of all functions $w \in \mathcal{S}(H)$ which admits a decomposition of the form $w = w_0\theta + w_1$, where $w_0 \in H$ and $w_1 \in l^p(H)$, is denoted by $H + l^p(H) =: m^p(H)$. Endowed with the norm

$$\|w\|_{m^p(H)} := \|w_0\|_H + \|w_1\|_{l^p(H)},$$

the space $m^p(H)$ is complete. We say that a subset of H is 'precompact' if its closure is compact. Occasionally, we shall impose the following assumption on φ.

$$\left.\begin{array}{l}\text{The function } \varphi \text{ does not depend on } t \text{ and } \varphi^{-1}(0) \cap B\\ \text{is precompact for every bounded set } B \subset H.\end{array}\right\} \tag{5.2}$$

It will be explicitly stated when (5.2) is assumed to hold.

The following theorem is the main result of this section. We use the convention $a/\infty := 0$ for $a \in \mathbb{R}$.

THEOREM 5.1 Let $g = g_0\theta + g_1$, where $g_0 \in (0, \infty)$ and $g_1 \in l^1$, let φ be sector-bounded in the sense that there exists $d \in (0, \infty]$ such that

$$\operatorname{Re}\langle \varphi(n, \xi), \xi\rangle_H \leqslant -\|\varphi(n, \xi)\|_H^2/d, \quad (n, \xi) \in \mathbb{Z}_+ \times H_0, \tag{5.3}$$

and assume that there exists $\varepsilon \geqslant 0$ such that

$$1/d + \operatorname{Re}\hat{g}(\mathrm{e}^{\mathrm{i}\omega}) \geqslant \varepsilon, \quad \omega \in (0, 2\pi). \tag{5.4}$$

(A) If $\varepsilon > 0$ and $w \in m^2(H)$, then every solution x of (5.1) has the following properties:

(A1) There exists a constant $K > 0$ (depending only on ε, d and g, but not on w) such that

$$\begin{aligned}&\|x\|_{l^\infty(H)} + \|\nabla x\|_{l^2(H)} + \|\varphi \circ x\|_{l^2(H)} + (\|\operatorname{Re}\langle \varphi \circ x, x\rangle_H\|_{l^1})^{1/2}\\ &\quad + \|\Sigma(\varphi \circ x)\|_{l^\infty(H)} \leqslant K\|w\|_{m^2(H)}.\end{aligned} \tag{5.5}$$

(A2) The limit $\lim_{n\to\infty}\|x(n)\|_H$ exists and is finite; in particular, if $\dim H = 1$, then $\lim_{n\to\infty} x(n)$ exists.

(A3) If (5.2) holds and if

$$\left.\begin{array}{l}\inf_{\xi\in B\cap H_0}\|\varphi(\xi)\| > 0 \text{ for every bounded closed}\\ \text{set } B\subset H \text{ such that } \varphi^{-1}(0)\cap B=\emptyset,\end{array}\right\} \tag{5.6}$$

then $\lim_{n\to\infty}\operatorname{dist}(x(n),\varphi^{-1}(0)) = 0$.

(B) If $\varepsilon = 0$ and $w \in m^1(H)$, then every solution x of (5.1) has the following properties:

(B1) There exists a constant $K > 0$ (depending only on d and g, but not on w) such that

$$\|x\|_{l^\infty(H)} + \left(\left\|\operatorname{Re}\left\langle \varphi\circ x, x+\frac{1}{d}(\varphi\circ x)\right\rangle_H\right\|_{l^1}\right)^{1/2} + \|\Sigma(\varphi\circ x)\|_{l^\infty(H)} \leqslant K\|w\|_{m^1(H)}. \tag{5.7}$$

(B2) If (5.2) holds and if

$$\left.\begin{array}{l}\sup_{\xi\in B\cap H_0}\operatorname{Re}\langle\varphi(\xi),\xi+\varphi(\xi)/d\rangle < 0 \text{ for every bounded}\\ \text{closed set } B\subset H \text{ such that } \varphi^{-1}(0)\cap B=\emptyset,\end{array}\right\} \tag{5.8}$$

then $\lim_{n\to\infty}\operatorname{dist}(x(n),\varphi^{-1}(0)) = 0$.

REMARK 5.2

(a) Under the assumptions of statement (A1) (respectively, statement (B1)) of Theorem 5.1 and the additional assumptions that φ does not depend on time, φ is continuous and H_0 is closed, it follows from (5.5) (respectively, (5.7)) that, if the limit $\lim_{n\to\infty} x(n) =: x^\infty$ exists, then $x^\infty \in \varphi^{-1}(0)$.

(b) Assume that φ does not depend on t. If $\dim H < \infty$ and φ is continuous, then assumption (5.2) is satisfied. If additionally H_0 is closed, then (5.6) also holds.

Proof of Theorem 5.1. We have

$$x - g*(\varphi\circ x) = w,$$

or equivalently,

$$x - g_1*(\varphi\circ x) - g_0(\theta*(\varphi\circ x)) = w. \tag{5.9}$$

Forming the inner product with $(\varphi\circ x)(j)$ and summing from 0 to n in (5.9) yields,

$$\sum_{j=0}^{n}\langle(\varphi\circ x)(j), w(j)\rangle_H = \sum_{j=0}^{n}\langle(\varphi\circ x)(j), x(j)\rangle_H - \sum_{j=0}^{n}\langle(\varphi\circ x)(j), (g_1*(\varphi\circ x))(j)\rangle_H$$
$$- g_0\sum_{j=0}^{n}\left\langle(\varphi\circ x)(j), \sum_{l=0}^{j}(\varphi\circ x)(l)\right\rangle_H. \tag{5.10}$$

A simple proof by induction on n yields,

$$2\mathrm{Re}\sum_{j=0}^{n}\left\langle(\varphi\circ x)(j),\sum_{l=0}^{j}(\varphi\circ x)(l)\right\rangle_H=\left\|\sum_{j=0}^{n}(\varphi\circ x)(j)\right\|_H^2+\sum_{j=0}^{n}\|(\varphi\circ x)(j)\|_H^2. \tag{5.11}$$

Taking real parts in (5.10) and using (5.11) gives,

$$\begin{aligned}\mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ x)(j),w(j)\rangle_H=&\ \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ x)(j),x(j)\rangle_H-\frac{g_0}{2}\left\|\sum_{j=0}^{n}(\varphi\circ x)(j)\right\|_H^2\\&-\mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ x)(j),(g_1*(\varphi\circ x))(j)\rangle_H\\&-\frac{g_0}{2}\sum_{j=0}^{n}\|(\varphi\circ x)(j)\|_H^2.\end{aligned} \tag{5.12}$$

Noting that $\hat{g}_1(z)=\hat{g}(z)-g_0z/(z-1)$ and

$$\mathrm{Re}\frac{e^{i\omega}}{e^{i\omega}-1}=\frac{1}{2},\quad \omega\in(0,2\pi),$$

we obtain using (5.4)

$$\mathrm{Re}\,\hat{g}_1(e^{i\omega})=\mathrm{Re}\,\hat{g}(e^{i\omega})-\frac{g_0}{2}\geqslant\varepsilon-\frac{1}{d}-\frac{g_0}{2},\quad \omega\in(0,2\pi). \tag{5.13}$$

Define $v\colon\mathbb{Z}_+\to H$ by

$$v(j)=\begin{cases}(\varphi\circ x)(j), & \text{if } 0\leqslant j\leqslant n,\\ 0, & \text{otherwise.}\end{cases}$$

Invoking the Parseval–Bessel identity and using (5.13), we derive that

$$\begin{aligned}\mathrm{Re}\sum_{j=0}^{\infty}\langle v(j),(g_1*v)(j)\rangle_H&=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\langle\hat{v}(e^{i\omega}),(\widehat{g_1*v})(e^{i\omega})\rangle_H\,d\omega\\&\geqslant\frac{1}{2\pi}\left(\varepsilon-\frac{1}{d}-\frac{g_0}{2}\right)\int_0^{2\pi}\|\hat{v}(e^{i\omega})\|_H^2\,d\omega\\&=\left(\varepsilon-\frac{1}{d}-\frac{g_0}{2}\right)\sum_{j=0}^{\infty}\|v(j)\|_H^2.\end{aligned}$$

Hence,

$$\mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ x)(j),(g_1*(\varphi\circ x))(j)\rangle_H\geqslant\left(\varepsilon-\frac{1}{d}-\frac{g_0}{2}\right)\sum_{j=0}^{n}\|(\varphi\circ x)(j)\|_H^2. \tag{5.14}$$

Combining (5.12) and (5.14) gives

$$\operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ x)(j),x(j)\rangle_H+\frac{1}{d}\sum_{j=0}^{n}\|(\varphi\circ x)(j)\|_H^2-\varepsilon\sum_{j=0}^{n}\|(\varphi\circ x)(j)\|_H^2$$
$$-\frac{g_0}{2}\left\|\sum_{j=0}^{n}(\varphi\circ x)(j)\right\|_H^2\geqslant\operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ x)(j),w(j)\rangle_H. \tag{5.15}$$

To prove part (A), assume that $\varepsilon>0$ and $w=w_0\theta+w_1$, where $w_0\in H$ and $w_1\in l^2(H)$. Combining the identity

$$\frac{g_0}{2}\left\|\sum_{j=0}^{n}(\varphi\circ x)(j)\right\|_H^2+\operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ x)(j),w_0\rangle_H=\frac{g_0}{2}\left\|\sum_{j=0}^{n}(\varphi\circ x)(j)+\frac{w_0}{g_0}\right\|_H^2-\frac{1}{2g_0}\|w_0\|_H^2 \tag{5.16}$$

with (5.15) yields

$$\operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ x)(j),x(j)\rangle_H+\frac{1}{d}\sum_{j=0}^{n}\|(\varphi\circ x)(j)\|_H^2-\varepsilon\sum_{j=0}^{n}\|(\varphi\circ x)(j)\|_H^2$$
$$-\frac{g_0}{2}\left\|\sum_{j=0}^{n}(\varphi\circ x)(j)+\frac{w_0}{g_0}\right\|_H^2\geqslant\operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ x)(j),w_1(j)\rangle_H-\frac{1}{2g_0}\|w_0\|_H^2. \tag{5.17}$$

In the following, $K>0$ is a generic constant which will be suitably adjusted in every step and depends only on ε, d and g, but not on n or w. Invoking the inequality

$$\operatorname{Re}\langle(\varphi\circ x)(j),w_1(j)\rangle_H\geqslant-\|(\varphi\circ x)(j)\|_H\|w_1(j)\|_H\geqslant-\frac{\varepsilon}{2}\|(\varphi\circ x)(j)\|_H^2-\frac{1}{2\varepsilon}\|w_1(j)\|_H^2,$$

we obtain from (5.17) after rearrangement

$$\frac{\varepsilon}{2}\sum_{j=0}^{n}\|(\varphi\circ x)(j)\|_H^2+\frac{g_0}{2}\left\|\sum_{j=0}^{n}(\varphi\circ x)(j)+\frac{w_0}{g_0}\right\|_H^2-\operatorname{Re}\sum_{j=0}^{n}\left\langle(\varphi\circ x)(j),x(j)+\frac{1}{d}(\varphi\circ x)(j)\right\rangle_H$$
$$\leqslant\frac{1}{2\varepsilon}\sum_{j=0}^{n}\|w_1(j)\|_H^2+\frac{1}{2g_0}\|w_0\|_H^2$$
$$\leqslant K\|w\|_{m^2(H)}^2,\quad n\in\mathbb{Z}_+. \tag{5.18}$$

By the sector condition (5.3), we have that

$$-\operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ x)(j),x(j)+\frac{1}{d}(\varphi\circ x)(j)\rangle_H\geqslant 0,\quad n\in\mathbb{Z}_+,$$

showing that the left-hand side of (5.18) is the sum of three nonnegative terms. Inequality (5.18) is the key estimate from which we will derive part (A) of the theorem.

By (5.18) we obtain immediately that

$$\|\varphi \circ x\|_{l^2(H)} + \|\Sigma(\varphi \circ x)\|_{l^\infty(H)} \leqslant K\|w\|_{m^2(H)}. \tag{5.19}$$

Combining (5.18) and (5.19), we conclude that

$$(\|\mathrm{Re}\langle \varphi \circ x, x\rangle_H\|_{l^1})^{1/2} \leqslant K\|w\|_{m^2(H)}. \tag{5.20}$$

Furthermore,

$$\|(g_1 * (\varphi \circ x))(n)\|_H \leqslant \|g_1 * (\varphi \circ x)\|_{l^2(H)} \leqslant \|g_1\|_{l^1}\|\varphi \circ x\|_{l^2(H)}, \quad n \in \mathbb{Z}_+,$$

and thus, invoking (5.9),

$$\begin{aligned}\|x(n)\|_H &\leqslant \|(g_1 * (\varphi \circ x))(n)\|_H + g_0\|(\theta * (\varphi \circ x))(n)\|_H + \|w(n)\|_H \\ &\leqslant \|g_1\|_{l^1}\|\varphi \circ x\|_{l^2(H)} + g_0\|\Sigma(\varphi \circ x)\|_{l^\infty(H)} + \|w\|_{m^2(H)}, \quad n \in \mathbb{Z}_+.\end{aligned} \tag{5.21}$$

It follows from (5.19) and (5.21) that

$$\|x\|_{l^\infty(H)} \leqslant K\|w\|_{m^2(H)}. \tag{5.22}$$

By (5.9) we have that

$$\nabla x = \nabla(g_1 * (\varphi \circ x)) + g_0\nabla(\theta * (\varphi \circ x)) + \nabla w = \nabla(g_1 * (\varphi \circ x)) + g_0(\varphi \circ x) + \nabla w.$$

Since $\varphi \circ x \in l^2(H)$ by (5.19), $g_1 \in l^1$ and $\nabla w \in l^2(H)$, we conclude that $\nabla x \in l^2(H)$. Furthermore, it follows from (5.19) that $\|\nabla x\|_{l^2(H)} \leqslant K\|w\|_{m^2(H)}$. Together with (5.19), (5.20) and (5.22) this yields (5.5), completing the proof of statement (A1).

To prove statement (A2), note that, by (5.12),

$$\begin{aligned}&\frac{g_0}{2}\left\|\sum_{j=0}^{n}(\varphi \circ x)(j)\right\|_H^2 + \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi \circ x)(j), w_0\rangle_H \\ &= \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi \circ x)(j), x(j)\rangle_H - \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi \circ x)(j), w_1(j)\rangle_H - \frac{g_0}{2}\sum_{j=0}^{n}\|(\varphi \circ x)(j)\|_H^2 \\ &\quad - \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi \circ x)(j), (g_1 * (\varphi \circ x))(j)\rangle_H.\end{aligned} \tag{5.23}$$

By completing the square on the left-hand side of (5.23) (see (5.16)), we obtain

$$\begin{aligned}\frac{g_0}{2}\left\|\sum_{j=0}^{n}(\varphi \circ x)(j) + \frac{w_0}{g_0}\right\|_H^2 &= \frac{1}{2g_0}\|w_0\|_H^2 + \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi \circ x)(j), x(j)\rangle_H \\ &\quad - \frac{g_0}{2}\sum_{j=0}^{n}\|(\varphi \circ x)(j)\|_H^2 - \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi \circ x)(j), w_1(j)\rangle_H \\ &\quad - \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi \circ x)(j), (g_1 * (\varphi \circ x))(j)\rangle_H.\end{aligned} \tag{5.24}$$

By statement (A1), $\varphi \circ x \in l^2(H)$ and $\mathrm{Re}\langle \varphi \circ x, x\rangle_H \in l^1$. Since $w_1 \in l^2(H)$ and $g_1 \in l^1$, we see that the right-hand side of (5.24) converges as $n \to \infty$ and so the limit

$$l := \lim_{n\to\infty} \left\| \sum_{j=0}^{n} (\varphi \circ x)(j) + \frac{w_0}{g_0} \right\|_H = \lim_{n\to\infty} \left\| (\theta * (\varphi \circ x))(n) + \frac{w_0}{g_0} \right\|_H \tag{5.25}$$

exists and is finite. Now $g_1 * (\varphi \circ x) \in l^2(H)$, implying that $(g_1 * (\varphi \circ x))(n) \to 0$ as $n \to \infty$ and thus, by (5.9),

$$\lim_{n\to\infty} (x(n) - g_0(\theta * (\varphi \circ x))(n)) = w_0. \tag{5.26}$$

Equations (5.25) and (5.26) imply that $\|x(n)\|_H \to g_0 l$ as $n \to \infty$. If $\dim H = 1$, then, since $(\nabla x)(n) \to 0$ as $n \to \infty$ (by statement (A1)), we conclude that $\lim_{n\to\infty} x(n)$ exists, completing the proof of statement (A2).

To prove statement (A3), assume that the additional assumptions (5.2) and (5.6) are satisfied. Since x is bounded, there exists a closed bounded set $B \subset H$ such that $x(n) \in B$ for all $n \in \mathbb{Z}_+$. It follows from (5.2) that $\varphi^{-1}(0) \cap B$ is precompact. Consequently, for given $\eta > 0$, $\varphi^{-1}(0) \cap B$ is contained in a finite union of open balls with radius η, each ball centred at some point in $\mathrm{clos}(\varphi^{-1}(0) \cap B)$. Denoting this union by B_η, we claim that $x(n) \in B_\eta$ for all sufficiently large n. This is trivially true if $B \subset B_\eta$. If not, then the set $C := B \backslash B_\eta$ is nonempty. Moreover, C is bounded and closed with $\varphi^{-1}(0) \cap C = \emptyset$, and so $\inf_{\xi \in C \cap H_0} \|\varphi(\xi)\| > 0$ by (5.6). We know from statement (A1) that $\varphi \circ x \in l^2(H)$, hence $\lim_{n\to\infty} (\varphi \circ x)(n) = 0$, and so also in this case $x(n) \in B_\eta$ for all sufficiently large n. This implies that

$$\lim_{n\to\infty} \mathrm{dist}(x(n), \varphi^{-1}(0) \cap B) = 0$$

and, a fortiori,

$$\lim_{n\to\infty} \mathrm{dist}(x(n), \varphi^{-1}(0)) = 0,$$

completing the proof of statement (A3).

We proceed to prove part (B) of the theorem. To this end, assume that, in (5.4), $\varepsilon = 0$ and that $w = w_0\theta + w_1$, where $w_0 \in H$ and $w_1 \in l^1(H)$. Setting

$$\Phi(n) := \sum_{j=0}^{n} (\varphi \circ x)(j) = (\Sigma(\varphi \circ x))(n), \quad n \in \mathbb{Z}_+, \quad \text{and} \quad \Phi(-1) := 0,$$

we invoke (5.15), with $\varepsilon = 0$, to obtain

$$\mathrm{Re} \sum_{j=0}^{n} \left\langle (\varphi \circ x)(j), x(j) + \frac{1}{d}(\varphi \circ x)(j) \right\rangle_H - \frac{g_0}{2} \|\Phi(n)\|_H^2$$

$$\geqslant \mathrm{Re}\langle \Phi(n), w_0\rangle_H + \mathrm{Re} \sum_{j=0}^{n} \langle \Phi(j) - \Phi(j-1), w_1(j)\rangle_H. \tag{5.27}$$

Partial summation yields the identity

$$\sum_{j=0}^{n} \langle \Phi(j) - \Phi(j-1), w_1(j)\rangle_H = \sum_{j=0}^{n} \langle \Phi(j), w_1(j) - w_1(j+1)\rangle_H + \langle \Phi(n), w_1(n+1)\rangle_H,$$

which, combined with (5.27), leads to

$$\frac{g_0}{2}\|\Phi(n)\|_H^2 - \mathrm{Re}\sum_{j=0}^{n}\left\langle(\varphi\circ x)(j), x(j)+\frac{1}{d}(\varphi\circ x)(j)\right\rangle_H$$
$$\leqslant \|w_0\|_H\|\Phi(n)\|_H + 2\|w_1\|_{l^1(H)}\max_{0\leqslant j\leqslant n}\|\Phi(j)\|_H + \|w_1\|_{l^1(H)}\|\Phi(n)\|_H.$$

Hence,

$$\frac{g_0}{2}\|\Phi(n)\|_H^2 - \mathrm{Re}\sum_{j=0}^{n}\left\langle(\varphi\circ x)(j), x(j)+\frac{1}{d}(\varphi\circ x)(j)\right\rangle_H$$
$$\leqslant 3\|w\|_{m^1(H)}\max_{0\leqslant j\leqslant n}\|\Phi(j)\|_H, \quad n\in\mathbb{Z}_+. \tag{5.28}$$

In the following, $K>0$ is a generic constant which will be suitably adjusted in every step and depends only on d and g, but not on n or w. By the sector condition (5.3),

$$-\mathrm{Re}\sum_{j=0}^{n}\left\langle(\varphi\circ x)(j), x(j)+\frac{1}{d}(\varphi\circ x)(j)\right\rangle_H \geqslant 0, \quad n\in\mathbb{Z}_+,$$

and thus we may conclude from (5.28) that

$$\|\Phi(n)\| \leqslant K\|w\|_{m^1(H)}, \quad n\in\mathbb{Z}_+. \tag{5.29}$$

This implies that

$$\|\Sigma(\varphi\circ x)\|_{l^\infty(H)} = \sup_{n\geqslant 0}\|\Phi(n)\| \leqslant K\|w\|_{m^1(H)}. \tag{5.30}$$

Using (5.29) in (5.28) shows that

$$\left(\left\|\mathrm{Re}\left\langle\varphi\circ x, x+\frac{1}{d}(\varphi\circ x)\right\rangle_H\right\|_{l^1}\right)^{1/2} \leqslant K\|w\|_{m^1(H)}. \tag{5.31}$$

To establish that x is bounded, note that, by (5.9),

$$x = (g_0\delta + g_1 * \theta^{-1}) * (\theta * (\varphi\circ x)) + w. \tag{5.32}$$

It is clear that $g_0\delta + g_1 * \theta^{-1} \in l^1$ and, by (5.30),

$$\sup_{n\geqslant 0}\|(\theta * (\varphi\circ x))(n)\|_H = \|\Sigma(\varphi\circ x)\|_{l^\infty(H)} \leqslant K\|w\|_{m^1(H)}.$$

Consequently, invoking (5.32),

$$\|x\|_{l^\infty(H)} \leqslant \|g_0\delta + g_1 * \theta^{-1}\|_{l^1}\|\theta * (\varphi\circ x)\|_{l^\infty(H)} + \|w\|_{l^\infty(H)} \leqslant K\|w\|_{m^1(H)},$$

which together with (5.30) and (5.31) shows that (5.7) holds, completing the proof of statement (B1).

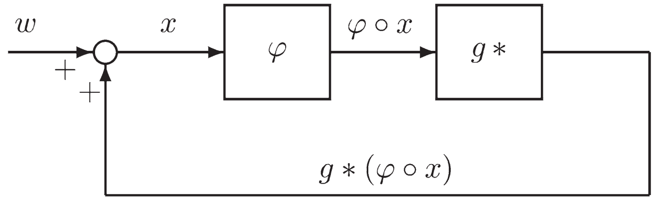

FIG. 1. Equation (5.1) as a feedback system.

To prove statement (B2), assume that the additional assumptions (5.2) and (5.8) are satisfied. Since x is bounded, there exists a closed bounded set $B \subset H$ such that $x(n) \in B$ for all $n \in \mathbb{Z}_+$. It follows from (5.2) that $\varphi^{-1}(0) \cap B$ is precompact. Consequently, for given $\eta > 0$, $\varphi^{-1}(0) \cap B$ is contained in a finite union of open balls with radius η, each ball centred at some point in $\mathrm{clos}(\varphi^{-1}(0) \cap B)$. Denoting this union by B_η, we claim that $x(n) \in B_\eta$ for all sufficiently large n. This is trivially true if $B \subset B_\eta$. If not, then the set $C := B \backslash B_\eta$ is nonempty. Define $\psi\colon H_0 \to (-\infty, 0]$ by

$$\psi(\xi) := \mathrm{Re}\langle \varphi(\xi), \xi + \varphi(\xi)/d \rangle_H .$$

Since C is bounded and closed with $\varphi^{-1}(0) \cap C = \emptyset$, assumption (5.8) implies that

$$\sup_{v \in C \cap H_0} \psi(v) < 0.$$

We know from statement (B1) that $\psi \circ x \in l^1$, hence $\lim_{n\to\infty}(\psi \circ x)(n) = 0$, and so also in this case $x(n) \in B_\eta$ for all sufficiently large n. This implies that

$$\lim_{n\to\infty} \mathrm{dist}(x(n), \varphi^{-1}(0) \cap B) = 0,$$

completing the proof of statement (B2). □

REMARK 5.3 Noting that (5.1) describes the feedback system shown in Fig. 1, Theorem 5.1 can be viewed as a stability result in the spirit of the frequency-domain criteria for feedback system stability (see Corduneanu, 1973; Desoer & Vidyasagar, 1975; Vidyasagar, 1993). Most of the control-theoretic literature on this subject considers continuous-time systems. The two statements of Theorem 5.1 form the discrete-time counterparts to recent continuous-time stability results given in Curtain *et al.* (2003, 2004). Note that the Condition (5.3) allows for a large class of bounded nonlinearities φ. The conjunction of convolution kernels g of the form $g = g_0\theta + g_1$ with $g_0 > 0$ and $g_1 \in l^1$ (so that $g \notin l^1$) and bounded nonlinearities φ is a distinguishing feature of Theorem 5.1.

Next, we derive a version of Theorem 5.1 which yields stability properties of the difference of two solutions of (5.1). In this context, the following incremental sector condition

$$\mathrm{Re}\langle \varphi(n,\xi_1) - \varphi(n,\xi_2), \xi_1 - \xi_2 \rangle_H \leqslant -\|\varphi(n,\xi_1) - \varphi(n,\xi_2)\|_H^2/d, \quad n \in \mathbb{Z}_+, \;\; \xi_1, \xi_2 \in H_0 \qquad (5.33)$$

is relevant. Note that if φ is 'unbiased', i.e.

$$\varphi(n, 0) = 0, \quad n \in \mathbb{Z}_+,$$

then, trivially, (5.3) is implied by (5.33). Moreover, we will see that, if (5.33) holds, but φ is not unbiased, then for the purpose of stability analysis, (5.1) can be be replaced by a modified Volterra equation, the nonlinearity of which is unbiased and satisfies (5.33) (and hence (5.3)). To this end, assume that (5.33) holds, let w^1, w^2 be forcing functions and let x^1 and x^2 be corresponding solutions of (5.1), i.e. $x^i = g * (\varphi \circ x^i) + w^i$ for $i = 1, 2$. Define a new nonlinearity ψ as in Lemma 4.5,

$$\psi\colon \mathbb{Z}_+ \times H_0 \to H, \quad (n, \xi) \mapsto \varphi(n, \xi + x^2(n)) - \varphi(n, x^2(n)). \tag{5.34}$$

Then ψ is unbiased,

$$\psi(n, x^1(n) - x^2(n)) = \varphi(n, x^1(n)) - \varphi(n, x^2(n)), \quad n \in \mathbb{Z}_+,$$

and ψ satisfies the same incremental sector condition as φ. Therefore, in particular,

$$\mathrm{Re}\langle \psi(n, \xi), \xi\rangle_H \leqslant -\|\psi(n, \xi)\|_H^2/d, \quad (n, \xi) \in \mathbb{Z}_+ \times H_0.$$

The difference $x^1 - x^2$ satisfies the Volterra equation

$$x^1 - x^2 = g * (\psi \circ (x^1 - x^2)) + w^1 - w^2.$$

Combining these observations with Theorem 5.1 yields the following result.

COROLLARY 5.4 Let $g = g_0\theta + g_1$, where $g_0 \in (0, \infty)$ and $g_1 \in l^1$, let φ be incrementally sector-bounded in the sense that (5.33) holds for some $d \in (0, \infty]$ and assume that there exists $\varepsilon \geqslant 0$ such that (5.4) is satisfied. Let x^1 and x^2 be solutions of (5.1) corresponding to forcing functions w^1 and w^2, respectively, and set $\Delta x := x^1 - x^2$ and $\Delta w := w^1 - w^2$.

(A) If $\varepsilon > 0$ and $\Delta w \in m^2(H)$, then the following statements hold:

(A1) There exists a constant $K > 0$ (depending only on ε, d and g, but not on w^1 and w^2) such that

$$\|\Delta x\|_{l^\infty(H)} + \|\nabla(\Delta x)\|_{l^2(H)} + \|\varphi \circ x^1 - \varphi \circ x^2\|_{l^2(H)} + \|\Sigma(\varphi \circ x^1 - \varphi \circ x^2)\|_{l^\infty(H)}$$
$$+ (\|\mathrm{Re}\langle \varphi \circ x^1 - \varphi \circ x^2, \Delta x\rangle_H\|_{l^1})^{1/2} \leqslant K\|\Delta w\|_{m^2(H)}.$$

(A2) The limit $\lim_{n\to\infty} \|(\Delta x)(n)\|_H$ exists and is finite; in particular, if $\dim H = 1$, then $\lim_{n\to\infty}(\Delta x)(n)$ exists.

(B) If $\varepsilon = 0$ and $\Delta w \in m^1(H)$, then there exists a constant $K > 0$ (depending only on d and g, but not on w^1 and w^2) such that

$$\|\Delta x\|_{l^\infty(H)} + \left(\left\|\mathrm{Re}\left\langle \varphi \circ x^1 - \varphi \circ x^2, \Delta x + \frac{1}{d}(\varphi \circ x^1 - \varphi \circ x^2)\right\rangle_H\right\|_{l^1}\right)^{1/2}$$
$$+ \|\Sigma(\varphi \circ x^1 - \varphi \circ x^2)\|_{l^\infty(H)} \leqslant K\|\Delta w\|_{m^1(H)}.$$

6. Application of Theorem 5.1 and Corollary 5.4 to linear multistep stability

In this section, we use Theorem 5.1 and Corollary 5.4 to derive results on the asymptotic behaviour of the solutions of (1.1). Recall that a rational function A (with possibly complex coefficients) is called ‘positive’ if

$$\mathrm{Re}\, A(z) \geqslant 0 \quad \text{for all } z \in \mathbb{C} \text{ with } |z| > 1 \text{ and such that } z \text{ is not a pole of } A.$$

The following lemma lists some simple (and well-known) properties of positive rational functions.

LEMMA 6.1 Let A be a rational function such that $A(z) \not\equiv 0$. If A is positive, then the following statements hold:

(a) If $z \in \mathbb{C}$ is a zero of A, then $|z| \leqslant 1$.
(b) If $z \in \mathbb{C}$ is a pole of A, then $|z| \leqslant 1$.
(c) $\lim_{|z|\to\infty} A(z)$ exists, is finite and not equal to zero.
(d) If $e^{i\nu}$ is a zero of A for some $\nu \in [0, 2\pi)$, then it is simple (i.e. $A'(e^{i\nu}) \neq 0$).
(e) If $e^{i\nu}$ is a pole of A for some $\nu \in [0, 2\pi)$, then it is simple (i.e. $(1/A)'(e^{i\nu}) \neq 0$) and $e^{-i\nu}\,\mathrm{res}(A, e^{i\nu}) > 0$, where $\mathrm{res}(A, e^{i\nu})$ denotes the residue of A at $z = e^{i\nu}$.

Note that part (c) says that a nontrivial positive rational function has no zeros or poles at ∞.

Proof of Lemma 6.1. The proof is essentially routine and therefore we do not give a complete proof here. We only derive part (e) (parts (a)–(d) are even easier to prove). To this end, assume that $e^{i\nu}$ is a pole of A. Then

$$A(z) = (z - e^{i\nu})^{-m} B(z),$$

where m is a positive integer and B is a rational function which is holomorphic at $z = e^{i\nu}$ and $B(e^{i\nu}) \neq 0$. For $\varepsilon > 0$ and $\omega \in [-\pi/2, \pi/2]$, we define

$$z_{\varepsilon,\omega} := e^{i\nu}(1 + \varepsilon\, e^{i\omega}).$$

It is clear that $|z_{\varepsilon,\omega}| > 1$ and so, for $\varepsilon > 0$ and $\omega \in [-\pi/2, \pi/2]$,

$$0 \leqslant \varepsilon^m \mathrm{Re}\, A(z_{\varepsilon,\omega}) = \cos(m\omega)\mathrm{Re}(e^{-im\nu} B(z_{\varepsilon,\omega})) + \sin(m\omega)\mathrm{Im}(e^{-im\nu} B(z_{\varepsilon,\omega})). \tag{6.1}$$

Since $B(z_{\varepsilon,\omega}) \to B(e^{i\nu})$ as $\varepsilon \to 0$ (uniformly in ω), it follows from (6.1) that $m = 1$ (because otherwise there would exist $\varepsilon > 0$ and $\omega \in [-\pi/2, \pi/2]$ such that the right-hand side of (6.1) is negative). Consequently, the pole at $z = e^{i\nu}$ is simple. Considering (6.1) when $m = 1$, choosing ω to be $-\pi/2$, 0 and $\pi/2$ and letting $\varepsilon \to 0$, shows that $\mathrm{Im}(e^{-i\nu} B(e^{i\nu})) = 0$ and $\mathrm{Re}(e^{-i\nu} B(e^{i\nu})) \geqslant 0$. Since $B(e^{i\nu}) \neq 0$, we conclude that $e^{-i\nu} B(e^{i\nu}) > 0$. The claim now follows from the fact that $\mathrm{res}(A, e^{i\nu}) = B(e^{i\nu})$. □

The next result shows that the inclusion of a disk of the form $\mathbb{B}(-c, c)$ (where $c > 0$) in the linear stability domain of (1.1) is equivalent to the positivity of the rational function $1/(2c) + \sigma/\rho$, where ρ and σ are given by (3.1).

LEMMA 6.2 Let $c > 0$. Then

$$\mathbb{B}(-c, c) \subset S \Longleftrightarrow 1/(2c) + \sigma/\rho \text{ is positive.}$$

Moreover, if $\mathbb{B}(-c, c) \subset S$, then the following statements hold:

(a) Method (1.1) is zero-stable.
(b) If $\rho(e^{i\nu}) = 0$ for some $\nu \in [0, 2\pi)$, then $e^{-i\nu}\sigma(e^{i\nu})/\rho'(e^{i\nu}) > 0$.

Proof. A straightforward calculation shows that, for all $z \in \mathbb{C}$ such that $\rho(z) \neq 0$, the following equivalences hold:

$$|\rho(z)/\sigma(z) + c| \geqslant c \Leftrightarrow (1/c^2)|\rho(z)|^2|1 + c\sigma(z)/\rho(z)|^2 \geqslant |\sigma(z)|^2$$

$$\Leftrightarrow 1/(2c) + \mathrm{Re}(\sigma(z)/\rho(z)) \geqslant 0.$$

Now, by Lemma 4.1, $\mathbb{B}(-c, c) \subset S$ if and only if $\inf_{|z|\geqslant 1} |\rho(z)/\sigma(z)+c| \geqslant c$, showing that the inclusion $\mathbb{B}(-c, c) \subset S$ is equivalent to the positivity of $1/(2c) + \sigma/\rho$. Statements (a) and (b) are now immediate consequences of Lemma 6.1. □

The following proposition shows that if $\mathbb{B}(-c, c) \subset S$, then $\overline{\mathbb{B}}(-c, c) \subset S$.

PROPOSITION 6.3 Let $c > 0$. If $\mathbb{B}(-c, c) \subset S$, then $\overline{\mathbb{B}}(-c, c) \subset S$.

Proof. Let ζ be on the boundary of $\mathbb{B}(-c, c)$, i.e. $|\zeta + c| = c$. We have to show that $\zeta \in S$. To this end, let $z_0 \in \mathbb{C}$ be such that

$$\rho(z_0) - \zeta\sigma(z_0) = 0. \tag{6.2}$$

Since $\mathbb{B}(-c, c) \subset S$, it is clear that $|z_0| \leqslant 1$. Therefore, it is sufficient to prove that if $|z_0| = 1$, then

$$\rho'(z_0) - \zeta\sigma'(z_0) \neq 0. \tag{6.3}$$

So let us assume that $|z_0| = 1$. If $\zeta = 0$, then $\rho(z_0) = 0$. By part (b) of Lemma 6.2, $\rho'(z_0) - \zeta\sigma'(z_0) = \rho'(z_0) \neq 0$, and so (6.3) holds. Hence, w.l.o.g. we may assume that $\zeta \neq 0$. Using that $|\zeta + c| = c$, a routine calculation shows that

$$\mathrm{Re}(1/\zeta) = -1/(2c).$$

Invoking Lemma 6.2, we see that the rational function $A := \sigma/\rho - 1/\zeta$ is positive. It follows from (6.2) that $A(z_0) = 0$. By part (d) of Lemma 6.1, $A'(z_0) \neq 0$ and so

$$0 \neq A'(z_0) = \frac{(\rho(z_0) - \zeta\sigma(z_0))\sigma'(z_0) - (\rho'(z_0) - \zeta\sigma'(z_0))\sigma(z_0)}{\rho^2(z_0)} = -\frac{(\rho'(z_0) - \zeta\sigma'(z_0))\sigma(z_0)}{\rho^2(z_0)},$$

showing that (6.3) holds. □

For later purposes, it is convenient to introduce the following assumption on f.

$$\left.\begin{array}{l}\text{The function } f \text{ does not depend on } t \text{ and } f^{-1}(0) \cap B \\ \text{is precompact for every bounded set } B \subset H.\end{array}\right\} \tag{6.4}$$

It will be explicitly stated when (6.4) is assumed to hold.

We are now in the position to formulate the main result of this section. Recall that the function $f_h \colon \mathbb{Z}_+ \times H_0 \to H$ is defined in (4.6). As in Section 3, we use the (functional) notation $u := (U_n)_{n\in\mathbb{Z}_+} \in \mathcal{S}(H)$ (i.e. $u(n) = U_n$ for all $n \in \mathbb{Z}_+$) for a solution $(U_n)_{n\in\mathbb{Z}_+}$ of (1.1).

THEOREM 6.4 Assume that method (1.1) satisfies the following two conditions: $\rho(1) = 0$ and $\rho(\mathrm{e}^{\mathrm{i}\omega}) \neq 0$ for all $\omega \in (0, 2\pi)$.

(A) Assume that there exists $0 < c < \infty$ such that

$$\|hf(nh, \xi) + c\xi\|_H \leqslant c\|\xi\|_H, \quad (n, \xi) \in \mathbb{Z}_+ \times H_0. \tag{6.5}$$

Under these conditions the following statements hold:

(A1) If $\mathbb{B}(-c_0, c_0) \subset S$ for some $c_0 > c$, then there exists a constant $K > 0$ (depending only on c_0, c and (ρ, σ)) such that, for every solution $(U_n)_{n\in\mathbb{Z}_+} =: u$ of (1.1),

$$\|u\|_{l^\infty(H)} + \|\nabla u\|_{l^2(H)} + \|f_h \circ u\|_{l^2(H)} + (\|\mathrm{Re}\langle f_h \circ u, u\rangle_H\|_{l^1})^{1/2} + \|\Sigma(f_h \circ u)\|_{l^\infty(H)}$$
$$\leqslant K\left(\sum_{k=0}^{q-1} \|u(k)\|_H^2\right)^{1/2}.$$

Furthermore, the limit $\lim_{n\to\infty} \|u(n)\|_H$ exists and is finite (in particular, if $\dim H = 1$, $u(n)$ converges as $n \to \infty$). If (6.4) holds and $\inf_{\xi\in B\cap H_0} \|f(\xi)\| > 0$ for every bounded closed set $B \subset H$ such that $f^{-1}(0) \cap B = \emptyset$, then $\lim_{n\to\infty} \operatorname{dist}(u(n), f^{-1}(0)) = 0$.

(A2) If $\mathbb{B}(-c, c) \subset S$, then there exists a constant $K > 0$ (depending only on c and (ρ, σ)) such that, for every solution $(U_n)_{n\in\mathbb{Z}_+} =: u$ of (1.1),

$$\|u\|_{l^\infty(H)} + \left(\left\|\mathrm{Re}\left\langle f_h \circ u, u + \frac{1}{2c}(f_h \circ u)\right\rangle_H\right\|_{l^1}\right)^{1/2} + \|\Sigma(f_h \circ u)\|_{l^\infty(H)}$$
$$\leqslant K \sum_{k=0}^{q-1} \|u(k)\|_H.$$

If (6.4) holds and if $\sup_{\xi\in B\cap H_0} \mathrm{Re}\langle f(\xi), \xi + hf(\xi)/(2c)\rangle < 0$ for every bounded closed set $B \subset H$ such that $f^{-1}(0) \cap B = \emptyset$, then $\lim_{n\to\infty} \operatorname{dist}(u(n), f^{-1}(0)) = 0$.

(B) Assume that

$$\mathrm{Re}\langle f(nh, \xi), \xi\rangle_H \leqslant 0, \quad (n, \xi) \in \mathbb{Z}_+ \times H_0. \tag{6.6}$$

If $\{z \in \mathbb{C}\colon \mathrm{Re}\, z < 0\} \subset S$, then there exists a constant $K > 0$ (depending only on (ρ, σ)) such that, for every solution $(U_n)_{n\in\mathbb{Z}_+} =: u$ of (1.1),

$$\|u\|_{l^\infty(H)} + (\|\mathrm{Re}\langle f_h \circ u, u\rangle_H\|_{l^1})^{1/2} + \|\Sigma(f_h \circ u)\|_{l^\infty(H)}$$
$$\leqslant K \sum_{k=0}^{q-1} (\|u(k)\|_H + \|hf(kh, u(k))\|_H).$$

If (6.4) holds and if $\sup_{\xi\in B\cap H_0} \mathrm{Re}\langle f(\xi), \xi\rangle < 0$ for every bounded closed set $B \subset H$ such that $f^{-1}(0) \cap B = \emptyset$, then $\lim_{n\to\infty} \operatorname{dist}(u(n), f^{-1}(0)) = 0$.

REMARK 6.5

(a) In the scalar case (i.e. $H = \mathbb{C}$) the inequality (6.5) is equivalent to the condition that $hf(nh, \xi)/\xi \in \overline{\mathbb{B}}(-c, c)$ for all $(n, \xi) \in \mathbb{Z}_+ \times \mathbb{C}$ with $\xi \neq 0$.

(b) Assume that there exist $c_1, h_1 > 0$ such that

$$\|h_1 f(t, \xi) + c_1\xi\|_H \leqslant c_1\|\xi\|_H, \quad (t, \xi) \in \mathbb{R}_+ \times H_0 \tag{6.7}$$

and $\mathbb{B}(-c_1, c_1) \subset S$, so that the conclusions of part (A2) hold for $c = c_1$ and $h = h_1$. Letting $h_2 \in (0, h_1)$ and setting $c_2 := c_1h_2/h_1 < c_1$, it follows trivially from (6.7) that

$$\|h_2 f(t, \xi) + c_2\xi\|_H \leqslant c_2\|\xi\|_H, \quad (t, \xi) \in \mathbb{R}_+ \times H_0,$$

showing that the conclusions of part (A1) hold for $c = c_2$, $h = h_2$ and $c_0 = c_1$.

(c) We emphasise that Theorem 6.4 deals with a situation which is not captured by Theorem 4.3. In particular, Theorem 4.3 cannot be applied if f is bounded (see Remark 4.4(c)), while the assumptions in Theorem 6.4 allow for a large class of bounded functions f.

Proof of Theorem 6.4. Since $u(n) := U_n$, it follows from Section 3 that

$$u = g * (f_h \circ u) + r^{-1} * v, \tag{6.8}$$

where $g := r^{-1} * s$ and

$$v(n) := \begin{cases} (r * u)(n) - (s * (f_h \circ u))(n), & 0 \leqslant n \leqslant q-1, \\ 0, & n \geqslant q. \end{cases}$$

Clearly, (6.8) is of the form (5.1): in order to apply Theorem 5.1 (with $g = r^{-1} * s$, $\varphi = f_h$, $w = r^{-1} * v$ and $d = 2c$), we need to verify the relevant assumptions. It follows from the hypotheses on ρ and S, via part (a) of Lemma 6.2 that (1.1) is strictly zero-stable. The residue of $\hat{g}$ at $z = 1$ is given by $g_0 := \sigma(1)/\rho'(1)$ and, by part (b) of Lemma 6.2, $g_0 > 0$. Invoking part (b) of Corollary 2.3, we obtain that $g_1 := g - g_0\theta \in l^1$. Consequently, g is of the form required for an application of Theorem 5.1. Furthermore, Lemma 6.2 shows that

$$1/(2c) + \operatorname{Re} \hat{g}(e^{i\omega}) = 1/(2c) + \operatorname{Re}(\sigma(e^{i\omega})/\rho(e^{i\omega})) \geqslant \varepsilon, \quad \omega \in (0, 2\pi), \tag{6.9}$$

with $\varepsilon > 0$ under the assumptions of (A1) and with $\varepsilon = 0$ under the assumptions of (A2). Also note that under the assumptions of (B), (6.9) remains true with $\varepsilon = 0$ and $c = \infty$. Consequently, (5.4) in Theorem 5.1 holds under the assumptions of both part (A) and part (B). Next we observe that (6.5) is equivalent to

$$\operatorname{Re}\langle f_h(n,\xi), \xi\rangle_H \leqslant -1/(2c)\|f_h(n,\xi)\|_H^2, \quad (n,\xi) \in \mathbb{Z}_+ \times H_0. \tag{6.10}$$

Note that for $c = \infty$, inequality (6.10) is equivalent to (6.6). Therefore, $\varphi = f_h$ satisfies the sector condition (5.3) in Theorem 5.1. To show that $w = r^{-1} * v$ satisfies the required assumption, we note that by strict zero-stability, combined with statement (b) of Corollary 2.3, there exists $r_1 \in l^1$ such that

$$r^{-1} = \gamma\theta + r_1, \quad \text{where } \gamma := 1/\rho'(1).$$

Hence,

$$w = r^{-1} * v = \gamma\theta * v + r_1 * v = \gamma\Sigma v + r_1 * v.$$

Setting

$$w_0 := \gamma \sum_{k=0}^{q-1} v(k), \quad w_1 := w - w_0\theta,$$

it is clear that $w_1 \in l^1(H) \subset l^2(H)$. Consequently, $w \in m^1(H) \subset m^2(H)$, so that w satisfies the requirement of Theorem 5.1. Furthermore,

$$\|w_0\|_H \leqslant |\gamma|\|v\|_{l^1(H)} \leqslant |\gamma|\sqrt{q}\|v\|_{l^2(H)} \tag{6.11}$$

and

$$\|\gamma\theta * v - w_0\theta\|_{l^1(H)} \leqslant |\gamma|(q-1)\|v\|_{l^1(H)}, \quad \|\gamma\theta * v - w_0\theta\|_{l^2(H)} \leqslant |\gamma|(q-1)\|v\|_{l^2(H)}.$$

The last two inequalities imply

$$\|w_1\|_{l^1(H)} \leqslant (\|r_1\|_{l^1} + |\gamma|(q-1))\|v\|_{l^1(H)}, \quad \|w_1\|_{l^2(H)} \leqslant (\|r_1\|_{\mathcal{L}(l^2)} + |\gamma|(q-1))\|v\|_{l^2(H)}. \tag{6.12}$$

To estimate $\|w\|_{m^p(H)}$ in terms of $u(0), u(1), \ldots, u(q-1)$, we consider parts (A) and (B) separately.

To prove part (A), we use (6.5) to obtain

$$\|v\|_{l^p(H)} \leqslant L_p \left(\sum_{k=0}^{q-1} \|u(k)\|_H^p \right)^{1/p}, \quad p = 1, 2, \tag{6.13}$$

where $L_1 := \|r\|_{l^1(H)} + 2c\|s\|_{l^1(H)}$ and

$$L_2 := \|r\|_{\mathcal{L}(l^2)} + 2c\|s\|_{\mathcal{L}(l^2)} = \max_{\omega\in[0,2\pi)} |\rho(\mathrm{e}^{\mathrm{i}\omega})| + 2c \max_{\omega\in[0,2\pi)} |\sigma(\mathrm{e}^{\mathrm{i}\omega})|.$$

Inequality (6.13) together with (6.11) yields

$$\|w_0\|_H \leqslant L_1|\gamma| \sum_{k=0}^{q-1} \|u(k)\|_H, \quad \|w_0\|_H \leqslant L_2|\gamma|\sqrt{q} \left(\sum_{k=0}^{q-1} \|u(k)\|_H^2 \right)^{1/2}. \tag{6.14}$$

Furthermore, by (6.12)–(6.14),

$$\|w\|_{m^p(H)} \leqslant M_p \left(\sum_{k=0}^{q-1} \|u(k)\|_H^p \right)^{1/p}, \quad p = 1, 2,$$

for suitable $M_p > 0$. Part (A) follows now from Theorem 5.1.

To prove part (B), we estimate

$$\|v\|_{l^1(H)} \leqslant L \sum_{k=0}^{q-1} (\|u(k)\|_H + \|hf(kh, u(k))\|_H), \tag{6.15}$$

where $L := \max(\|r\|_{l^1}, \|s\|_{l^1})$. Therefore, by (6.11),

$$\|w_0\|_H \leqslant L|\gamma| \sum_{k=0}^{q-1} (\|u(k)\|_H + \|hf(kh, u(k))\|_H). \tag{6.16}$$

Furthermore, by (6.12), (6.15) and (6.16),

$$\|w\|_{m^1(H)} \leqslant M \sum_{k=0}^{q-1} (\|u(k)\|_H + \|hf(kh, u(k))\|_H),$$

for some suitable $M > 0$. Part (B) follows now from Theorem 5.1. □

EXAMPLE 6.6

(a) Let ρ and σ be given by $\rho(z) = z - 1$ and $\sigma(z) = 1$ (Euler's method). It is clear that $S = \overline{\mathbb{B}}(-1, 1)$. Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(\xi) := \begin{cases} -2\xi, & \xi \geqslant -2, \\ 4, & \xi < -2, \end{cases}$$

so that $f(\xi)/\xi \in \overline{\mathbb{B}}(-1, 1)$ for all $\xi \neq 0$. It follows from part (a) of Remark 6.5 that the conclusions of statements (A1) of Theorem 6.4 hold if $h \in (0, 1)$, while statements (A2) of the same theorem applies if $h = 1$. In particular, the numerical solution $(U_n)_{n\in\mathbb{Z}_+}$ converges to 0 as $n \to \infty$ for every choice of $h \in (0, 1)$. If $h = 1$, then $(U_n)_{n\in\mathbb{Z}_+}$ does not converge in general as $n \to \infty$ (e.g. if $U_0 = 2$, then $U_n = (-1)^n 2$ for all $n \in \mathbb{Z}_+$).

(b) Let ρ and σ be given by $\rho(z) = (3/2)z^2 - 2z + 1/2$ and $\sigma(z) = z^2$ (two-step backwards differentiation method). Define $f: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(\xi_1, \xi_2) := (-2\xi_1^3 + \xi_2, -\xi_1 - \xi_2^3).$$

Then $\{z \in \mathbb{C}: \operatorname{Re} z < 0\} \subset S$ and $\langle f(\xi), \xi\rangle < 0$ for all $\xi \in \mathbb{R}^2\backslash\{0\}$. Hence, the conclusions of statement (B) of Theorem 6.4 hold for every $h > 0$. In particular, the numerical solution $(U_n)_{n\in\mathbb{Z}_+}$ converges to 0 as $n \to \infty$ for every $h > 0$.

While Theorem 6.4 has some overlap with results by Nevanlinna (1977a,b), there are also considerable differences. Theorem 6.4 assumes ρ to be strictly zero-stable, rather than merely zero-stable as in Nevanlinna (1977a,b); on the other hand, numerous aspects of Theorem 6.4 are more general than in Nevanlinna (1977a,b), the assumptions in Theorem 6.4 are easier to check than those in Nevanlinna (1977a,b) and some of the conclusions are stronger as compared to Nevanlinna (1977a,b). The following remark gives more details.

REMARK 6.7

(a) The same comment as in statement (a1) of Remark 4.4 applies in the context of comparing Theorem 6.4 with the main result in Nevanlinna (1977b).

(b) The situation considered in part (B) of Theorem 6.4 is dealt with in Nevanlinna (1977b) under the additional assumption that the method is strictly stable at infinity, i.e. the roots of σ are contained in the open unit disk $\mathbb{B}(0, 1)$.

(c) The constant K in Theorem 6.4 does not depend on h in contrast to Theorem 3.1 in Nevanlinna (1977b) (see Remark 3.1 in Nevanlinna, 1977b). Moreover, Theorem 3.1 in Nevanlinna (1977b) does not give estimates for $\|\nabla u\|_{l^2(H)}$, $\|f_h \circ u\|_{l^2(H)}$, $\|\operatorname{Re}\langle f_h \circ u, u\rangle_H\|_{l^1}$ and $\|\sum(f_h \circ u)\|_{l^\infty(H)}$.

(d) Let ρ, σ and f be as in part (a) of Example 6.6. Let $a \in \mathbb{R}$ be such that $1/a \in \operatorname{int}(S)$, which holds if and only if $a \in (-\infty, -1/2)$. A routine calculation shows that for every $a \in (-\infty, -1/2)$

$$b := -\inf_{|z|\geqslant 1} \frac{\rho(z)}{\sigma(z) - a\rho(z)} = 0.$$

Set $c := -1/(2a)$. Then, using the notation of Nevanlinna (1977b), $D(a, b) = D(a, 0) = \mathbb{B}(-c, c)$. For given $a \in (-\infty, -1/2)$, Theorem 3.1 in Nevanlinna (1977b) applies for all $0 < h < -1/(2a)$. Note that if $h = -1/(2a)$, then

$$(I - ahf)(\xi) = \xi + f(\xi)/2 = 0, \quad \xi \geqslant -2,$$

so that $(I - ahf)$ is not invertible. Consequently, Theorem 3.1 in Nevanlinna (1977b) does not apply in this case (see part (a) of this remark). Furthermore, if $a = -1/2$, then $1/a = -2 \notin \operatorname{int}(S)$. It follows that Theorem 3.1 in Nevanlinna (1977b) is not applicable (see p. 60 in Nevanlinna, 1977b). In particular, Theorem 3.1 in Nevanlinna (1977b) does not give a result for the stepsize $h = 1$.

(e) Using Theorem 6.4, it was shown in Example 6.6 that (for certain values of h) the numerical solution $(U_n)_{n\in\mathbb{Z}_+}$ converges to 0 as $n \to \infty$. It seems to be difficult to obtain these convergence properties by applying the results in Nevanlinna (1977a) on the behaviour of $(U_n)_{n\in\mathbb{Z}_+}$ at infinity. In part (a) of Example 6.6, the method is not A-stable, while A-stability is assumed throughout in Nevanlinna (1977a). Furthermore, in part (b) of Example 6.6 the nonlinearity f is not a gradient field, and hence Corollary 2 in Nevanlinna (1977a) (which assumes that the nonlinearity is a gradient mapping) cannot be used.

The following result is a version of Theorem 6.4 which yields stability properties of the difference of two solutions of method (1.1).

COROLLARY 6.8 Assume that the method (1.1) satisfies the following two conditions: $\rho(1) = 0$ and $\rho(\mathrm{e}^{\mathrm{i}\omega}) \neq 0$ for all $\omega \in (0, 2\pi)$.

(A) Assume that there exists $0 < c < \infty$ such that

$$\|hf(nh,\xi_1) - hf(nh,\xi_2) + c(\xi_1 - \xi_2)\|_H \leqslant c\|\xi_1 - \xi_2\|_H, \quad n \in \mathbb{Z}_+, \ \ \xi_1, \xi_2 \in H_0.$$

Let $(U_n^1)_{n\in\mathbb{Z}_+} =: u^1$ and $(U_n^2)_{n\in\mathbb{Z}_+} =: u^2$ be solutions of (1.1) and set $\varDelta u := u^1 - u^2$. Then the following statements hold:

(A1) If $\mathbb{B}(-c_0, c_0) \subset S$ for some $c_0 > c$, then there exists a constant $K > 0$ (depending only on c_0, c and (ρ, σ)) such that

$$\begin{aligned}
&\|\varDelta u\|_{l^\infty(H)} + \|\nabla(\varDelta u)\|_{l^2(H)} + \|f_h \circ u^1 - f_h \circ u^2\|_{l^2(H)} \\
&\quad + \|\Sigma(f_h \circ u^1 - f_h \circ u^2)\|_{l^\infty(H)} + (\|\mathrm{Re}\langle f_h \circ u^1 - f_h \circ u^2, \varDelta u\rangle_H\|_{l^1})^{1/2} \\
&\quad \leqslant K\left(\sum_{k=0}^{q-1}\|(\varDelta u)(k)\|_H^2\right)^{1/2}.
\end{aligned}$$

Furthermore, the limit $\lim_{n\to\infty}\|(\varDelta u)(n)\|_H$ exists and is finite; in particular, if $\dim H = 1$, then $\lim_{n\to\infty}(\varDelta u)(n)$ exists.

(A2) If $\mathbb{B}(-c, c) \subset S$, then there exists a constant $K > 0$ (depending only on c and (ρ, σ)) such that

$$\begin{aligned}
&\|\varDelta u\|_{l^\infty(H)} + \left(\left\|\mathrm{Re}\left\langle f_h \circ u^1 - f_h \circ u^2, \varDelta u + \frac{1}{2c}(f_h \circ u^1 - f_h \circ u^2)\right\rangle_H\right\|_{l^1}\right)^{1/2} \\
&\quad + \|\Sigma(f_h \circ u^1 - f_h \circ u^2)\|_{l^\infty(H)} \leqslant K\sum_{k=0}^{q-1}\|(\varDelta u)(k)\|_H.
\end{aligned}$$

(B) Assume that

$$\mathrm{Re}\langle f(nh,\xi_1) - f(nh,\xi_2), \xi_1 - \xi_2\rangle_H \leqslant 0, \quad n \in \mathbb{Z}_+, \ \ \xi_1, \xi_2 \in H_0.$$

Let $(U_n^1)_{n\in\mathbb{Z}_+} =: u^1$ and $(U_n^2)_{n\in\mathbb{Z}_+} =: u^2$ be solutions of (1.1) and set $\varDelta u := u^1 - u^2$. If $\{z \in \mathbb{C}\colon \mathrm{Re}\, z < 0\} \subset S$, then there exists a constant $K > 0$ (depending only on (ρ, σ))

such that

$$\|\varDelta u\|_{l^\infty(H)} + (\|\mathrm{Re}\langle f_h \circ u^1 - f_h \circ u^2, \varDelta u\rangle_H\|_{l^1})^{1/2} + \|\varSigma(f_h \circ u^1 - f_h \circ u^2)\|_{l^\infty(H)}$$
$$\leqslant K \sum_{k=0}^{q-1} (\|(\varDelta u)(k)\|_H + \|hf(kh, u^1(k)) - hf(kh, u^2(k))\|_H).$$

Corollary 6.8 follows from an application of Corollary 5.4 and arguments are similar to those used in the proof of Theorem 6.4.

Finally, noting that for f in part (a) of Example 6.6, $(f(\xi_1) - f(\xi_2))/(\xi_1 - \xi_2) \in \overline{\mathbb{B}}(-1, 1)$ for all $\xi_1, \xi_2 \in \mathbb{R}$, $\xi_1 \neq \xi_2$, and, for f in part (b) of Example 6.6, $\langle f(\xi_1) - f(\xi_2), \xi_1 - \xi_2\rangle < 0$ for all $\xi_1, \xi_2 \in \mathbb{R}^2$, $\xi_1 \neq \xi_2$, it follows that a suitably modified version of Remark 6.7 holds for Corollary 6.8.

Acknowledgement

This work was supported in part by the UK EPSRC (Grant GR/S94582/01).

REFERENCES

CONWAY, J. B. (1978) *Functions of One Complex Variable*, 2nd edn. New York: Springer.

CORDUNEANU, C. (1973) *Integral Equations and Stability of Feedback Systems*. New York: Academic Press.

CURTAIN, R. F., LOGEMANN, H. & STAFFANS, O. J. (2003) Stability results of Popov type for infinite-dimensional systems with applications to integral control. *Proc. Lond. Math. Soc.*, **86**, 779–816.

CURTAIN, R. F., LOGEMANN, H. & STAFFANS, O. J. (2004) Absolute stability results in infinite dimensions. *Proc. R. Soc. Lond. A*, **460**, 2171–2196.

DAHLQUIST, G. (1978) G-stability is equivalent to A-stability. *BIT*, **18**, 384–401.

DESOER, C. A. & VIDYASAGAR, M. (1975) *Feedback Systems: Input-Output Properties*. New York: Academic Press.

NEVANLINNA, O. (1977a) On the behaviour of global errors at infinity in the numerical integration of stable initial value problems. *Numer. Math.*, **28**, 445–454.

NEVANLINNA, O. (1977b) On the numerical integration of nonlinear initial value problems by linear multistep methods. *BIT*, **17**, 58–71.

NEVANLINNA, O. & ODEH, F. (1981) Multiplier techniques for linear multistep methods. *Numer. Funct. Anal. Optim.*, **3**, 377–423.

PARTINGTON, J. R. (1997) *Interpolation, Identification and Sampling*. Oxford: Clarendon Press.

POPOV, V. M. (1962) Absolute stability of nonlinear systems of automatic control. *Autom. Remote Control*, **22**, 857–875.

VIDYASAGAR, M. (1993) *Nonlinear Systems Analysis*, 2nd edn. Englewood Cliffs, NJ: Prentice Hall.