Compactness and stability for planar vortex-pairs with prescribed impulse

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Abstract.- Concentration-compactness is used to prove compactness of maximising sequences for a variational problem governing symmetric steady vortex-pairs in a uniform planar ideal fluid flow, where the kinetic energy is to be maximised and the constraint set comprises the set of all equimeasurable rearrangements of a given function (representing vorticity) that have a prescribed impulse (linear momentum). A form of orbital stability is deduced.

Résumé.- On utilise la methode de compacité par concentration pour démonstrer la compacité des séquences de maximisation pour un problème variationnel déscribant des pairs-tourbillons symmétriques dans un écoulement fluide planaire uniforme, où on maximise l'enérgie cinétique dans l'ensemble des réarrangements mesurables d'un fonction (représentant le tourbillon) aux impulsion prescrite (moment linéaire). On en déduit une forme de stabilité orbital.

Key words: variational problem, rearrangements, vortex pairs, convex set, stability, concentration-compactness, weak limit.

1 Introduction.

In this paper we prove compactness (up to translation) of all maximising sequences for a constrained variational problem whose maximisers represent steady axisymmetric vortex-pairs in a planar flow of an ideal fluid of unit density that approaches a uniform flow at infinity. The quantity being maximised is the kinetic energy $E(\zeta)$ due to the vorticity ζ and the constraints are that the vorticity in the upper half-plane should be a rearrangement of a given compactly supported non-negative function and that the impulse (linear momentum parallel to the axis) $I(\zeta)$ should take a prescribed value. This formulation of steady vortex pairs, which is derived from ideas of Arnol′d [1] and Benjamin [2], has the notable feature that all the prescribed data are conserved quantities in the corresponding dynamical problem.

We apply the compactness theorem to orbital stability: we prove that symmetric flows with non-negative upper half-plane vorticity starting close to a maximiser remain close to the set of maximisers for all time, relative to a norm defined by

$$\|\zeta\|_{\mathfrak{X}^p} := |I(\zeta)| + \|\zeta\|_1 + \|\zeta\|_p,$$

where $p > 2$. It must be emphasised that this result does not preclude arbitrarily small perturbations that result in two-signed initial vorticity leading to flows that become distant from the set of maximisers. This stability result is a counterpart for one proved in [6] where kinetic energy penalised by a given constant multiple of impulse was maximised relative to a set of rearrangements.

In the present paper the same norm $\|\cdot\|_{\mathfrak{X}^p}$ is used to measure deviations in both the initial state and the evolved state, whereas in [6] deviations in the evolved state were measured in a weaker norm than deviations in the initial state. The precise formulations of the results are given in Subsection 2.2; the sequences considered in the compactness theorem are in fact slightly more general than maximising sequences.

While it is clear that all maximisers of the variational problem of [6] are maximisers of the problem of the present paper (for appropriate fixed values of the impulse) it is unclear whether the converse holds. In particular, we do not know generally whether the maximisers for either formulation are unique (up to axial translation), or even whether all maximisers for the penalised energy formulation in [6] have the same impulse.

Stability of planar ideal fluid flows has been the subject of many investigations. The outcome can depend on which norm is used in the definition of stability, especially the order of derivatives included, if any, a point that is emphasised in the expository articles by Friedlander and Shnirelman [8] and Friedlander and Yudovich [9].

1.1 Formulation.

We take the flow to be symmetric in the x_1-axis of $\mathbb{R}^2$, so we work in the upper half-plane $\Pi = \{(x_1, x_2) \in \mathbb{R}^2) \mid x_2 > 0\}$ and let G be the Green's function of $-\Delta$ in Π, that is

$$G(x,y) := \frac{1}{2\pi} \log\left(\frac{|x-\overline{y}|}{|x-y|}\right) = \frac{1}{4\pi}\log\left(1 + \frac{4x_2y_2}{|x-y|^2}\right) \tag{1}$$

where $\overline{y} = (y_1, -y_2)$ denotes the reflection in the x_1-axis of $y = (y_1, y_2)$. An operator $\mathscr{G}$ is defined by

$$\mathscr{G}\zeta(x) = \int_\Pi G(x,y)\zeta(y)\mathrm{d}y,$$

when this integral converges (for which it is sufficient that $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ for some $p > 1$). For a flow of an ideal (incompressible, inviscid) fluid of unit density in Π approaching a uniform stream with velocity $(\lambda, 0)$ at infinity, parallel to $e_1 = (1, 0)$ on the axis and having signed scalar vorticity $\zeta(x)$, the stream function is given by $\mathscr{G}\zeta(x) - \lambda x_2$ and the kinetic energy $E(\zeta)$ and impulse $I(\zeta)$ are given by

$$\begin{aligned} E(\zeta) &= \frac{1}{2}\int_\Pi \zeta(x)\mathscr{G}\zeta(x)\mathrm{d}x = \frac{1}{2}\int_\Pi G(x,y)\zeta(x)\zeta(y)\mathrm{d}x\mathrm{d}y, \\ I(\zeta) &= \int_\Pi \zeta(y)y_2\mathrm{d}y. \end{aligned}$$

Note that $E(\zeta) > 0$ if $\zeta \neq 0$.

Consider a fixed compactly supported non-negative $\zeta_0 \in L^p(\Pi)$, for some $2 < p < \infty$ and let $\mathscr{R}(\zeta_0)$ denote the set of all rearrangements of ζ_0 on Π, defined in Sect. 2.1 below. Given a number $i_0 > 0$, any maximiser of $E(\zeta)$ subject to the constraints $\zeta \in \mathscr{R}(\zeta_0)$ and $I(\zeta) = i_0$ satisfies an equation

$$-\Delta\psi(x) = \varphi(\psi(x) - \lambda x_2) \quad \text{in } \Pi \tag{2}$$

where $\psi := K\zeta$ and φ is an (*a priori* unknown) increasing function; if ψ satisfies (2) then $\psi(x_1, x_2) - \lambda x_2$ is the stream function of a steady ideal fluid flow. Viewed in a frame fixed relative to the fluid at infinity, $\overline{\psi}(x_1, x_2, t) := \psi(x_1 + \lambda t, x_2)$, where ψ satisfies (2), is the stream function for a vortex of constant form moving with velocity $(\lambda, 0)$ in an otherwise irrotational fluid that is stationary at infinity.

Our purpose here is to prove that given ζ_0, for all sufficiently large i_0 the maximisers are orbitally stable to non-negative perturbations of vorticity in $\| \|_{\mathfrak{X}^p}$ in the sense that, if ζ is a maximiser and $\omega(t)$ is a flow of a planar ideal fluid whose initial vorticity $\omega(0)$ is non-negative and is close to ζ in $\| \|_{\mathfrak{X}^p}$ then $\omega(t)$ remains close in $\| \|_{\mathfrak{X}^p}$ to the set of maximisers.

We work with a relaxed formulation of the variational problem, having a convex constraint set, and refer to its solutions as *relaxed maximisers*; we show these must always exist. It transpires (see Lemma 10) that, given ζ_0, for all sufficiently large i_0 the relaxed maximisers are in fact solutions to the original unrelaxed problem; it is under these circumstances that we can prove orbital stability.

1.2 Background.

The notion that extrema of kinetic energy relative to a set of equimeasurable vorticities, or "equivortical surface", should provide stable steady flows, was introduced by Arnol′d [1]. Benjamin [2] adapted Arnol′d's ideas to the context of steady axisymmetric vortex-rings in a uniform flow in the whole of $\mathbb{R}^3$ and proposed a programme for proving the existence of energy maximisers subject to the additional constraint of fixed impulse and for proving their stability from this. The problem considered in the present paper is the two-dimensional analogue of Benjamin's problem and this work represents a contribution to its stability theory, existence of solutions having been studied previously in [3].

Arnol′d's stability method proceeds by constructing a Lyapunov functional, derived from the vorticity-stream function relationship, concerning which some degree of detail must therefore be known. However, Benjamin's approach to steady vortices has the novel feature that the functional relationship between the stream function and the vorticity, which is the classical condition for a planar flow to be steady, is not specified *a priori* but is determined *a posteriori*, by contrast with much other work on existence of solutions to semilinear elliptic equations. Our result will therefore not be derived by Arnol′d's method, but rather by the approach to stability envisaged by Benjamin [2].

Burton, Lopes and Lopes [6] proved a stability theorem for a related formulation where the energy is penalised by subtracting a fixed multiple of the impulse and then maximised on a set of rearrangements without any constraint on the impulse, which is the planar analogue of an alternative formulation of vortex rings that had also been proposed by Benjamin [2]. In [6] it was shown that when the initial vorticity is non-negative and close to a maximiser in terms of I and $\| \|_2$, subject to a bound on the area of the vortex-core, the vorticity of the evolving flow remains close in $\| \|_2$ to the set of maximisers. We suggest that the form of stability proved in the present paper is more elegant, in using the same norm to measure the perturbations of both the initial state and the evolved state.

1.3 Methodology.

The proof of stability in [6] proceeded by constructing, given a flow with vorticity $\omega(t)$ for which $\omega(0)$ is close to a maximiser $\widehat{\zeta}$, a "follower" $\zeta(t)$ in the constraint set by advecting $\widehat{\zeta}$ using the transport equation of the velocity field associated with $\omega(t)$; the distance between $\omega(t)$ and $\zeta(t)$ is constant in time. The result was then deduced from a compactness theorem for the maximising sequences of the variational problem, whose proof used the Concentration-Compactness principle expounded by P.-L. Lions [13].

In the present context, this approach gives rise to two difficulties. Firstly, the transport equation need not conserve impulse so its solutions need not remain in the constraint set; this frustrates the construction of a follower. Secondly, the absence of the penalty term on the energy increases the difficulty of bounding the supports of various vorticities arising in the "Dichotomy" case of Concentration-Compactness. We therefore prove directly the compactness of maximising sequences that approach the constraint set but need not be contained within it and, when addressing Dichotomy, we improve the compactness by solving subsidiary constrained variational problems, which may have relaxed solutions only.

2 Preliminaries and Statements of Results.

In this section we review the properties of the set $\mathscr{R}(\zeta_0)$ of rearrangements of a function ζ_0 and we define the constraint set $\mathscr{W}(\zeta_0, \leq i_0)$ for the relaxed variational problem. Then we derive some estimates for the stream function in terms of the vorticity and establish weak continuity properties of the energy, which will be needed for the study of relaxed problem in Section 3. Throughout we use the notation $D(x, R) = \{y \in \mathbb{R}^2 \mid |y - x| < r\}$ and $D_\Pi(x, R) = D(x, R) \cap \Pi$.

2.1 Rearrangements and Steiner symmetrisation.

Suppose that f and g are non-negative functions p-integrable (for some $1 \leq p < \infty$) on sets of infinite measure in Euclidean spaces (possibly of different dimensions). We say f is a *rearrangement* of g if the Lebesgue measures of corresponding super-level sets of f and g are equal, that is, if

$$\forall \alpha > 0 \quad |\{x \mid f(x) > \alpha\}| = |\{y \mid g(y) > \alpha\}|$$

where $|\,|$ denotes Lebesgue measure of appropriate dimension. We write $f \preceq g$ if

$$\forall \alpha > 0 \int_U (f(x) - \alpha)_+ \mathrm{d}x \leq \int_V (g(y) - \alpha)_+ \mathrm{d}y. \tag{3}$$

Thus $\preceq$ is transitive, while f is a rearrangement of g if and only if both $f \preceq g$ and $g \preceq f$ hold, since the measures of the super-level sets of f and g can be recovered by right-differentiation of the integrals in (3) with respect to α. Typically we are interested in functions defined on half-planes, planar strips or unbounded intervals in $\mathbb{R}$. Any set of finite positive Lebesgue measure in a Euclidean space (of any dimension) is measure-theoretically isomorphic to an interval of equal linear measure in $\mathbb{R}$, indeed a measure-preserving bimeasurable bijection to the interval less a countable set exists, see [14, Chapter 15], and consequently any set of infinite measure is isomorphic to a half-line and to $\mathbb{R}$. The isomorphisms allow a measurable function on one set to be lifted to a rearrangement on any other set of equal measure. We can therefore be somewhat cavalier about domains in what follows.

The essentially unique *decreasing rearrangement* f^Δ of f can be defined on $(0,\infty)$ and we extend f^Δ to vanish on $(-\infty,0)$. We define the *increasing rearrangement* of f by $f^\nabla(s) = f^\Delta(-s)$ for real s. Clearly f is a rearrangement of g if and only if $f^\Delta = g^\Delta$. The inequality

$$\|f^\Delta - g^\Delta\|_p \leq \|f-g\|_p \tag{4}$$

holds if $f,g \in L^p(\mathbb{R}^n)$ for some $1 \leq p \leq \infty$, see for example [12, Theorem 3.5], thus decreasing rearrangement is continuous in L^p. One can think of f^Δ as capturing the statistics of the values of f without capturing their spatial distribution.

Given non-negative $f \in L^p(\Pi)$, for some $1 \leq p \leq \infty$, and $\beta \in \mathbb{R}$, the *Steiner symmetrisation of f in the line $x_2 = \beta$* is the (essentially unique) non-negative function f^s on Π such that, for almost every $x_2 > 0$, the function $f^s(\cdot, x_2)$ is symmetric decreasing about β and $f^s(\cdot,x_2)$ is a rearrangement of $f(\cdot,x_2)$.

Consider a non-negative p-integrable function ζ_0 (some $2 < p < \infty$) defined on a subset of infinite Lebesgue measure in a Euclidean space and let U be another subset of infinite measure in a Euclidean space. We denote by $\mathscr{R}_U(\zeta_0)$ the set of all rearrangements of ζ_0 on U and define the set $\mathscr{W}_U(\zeta_0)$, which contains $\mathscr{R}_U(\zeta_0)$, by

$$\mathscr{W}_U(\zeta_0) = \left\{0 \leq \zeta \in L^1(U) \;\middle|\; \zeta \preceq \zeta_0\right\},$$

writing in particular $\mathscr{W}(\zeta_0) = \mathscr{W}_\Pi(\zeta_0)$ and $\mathscr{R}(\zeta_0) = \mathscr{R}_\Pi(\zeta_0)$. Then, from the definition, $\mathscr{W}(\zeta_0)$ is convex and Douglas [7] proved that $\mathscr{W}(\zeta_0)$ is the closure of $\mathscr{R}(\zeta_0)$ in the weak topology of $L^p(\Pi)$. Let

$$\begin{aligned}
\mathscr{W}(\zeta_0, \leq i_0) &= \{\zeta \in \mathscr{W}(\zeta_0) \mid I(\zeta) \leq i_0\}\\
\mathscr{W}(\zeta_0, i_0) &= \{\zeta \in \mathscr{W}(\zeta_0) \mid I(\zeta) = i_0\}\\
\mathscr{W}(\zeta_0, < \infty) &= \{\zeta \in \mathscr{W}(\zeta_0) \mid I(\zeta) < \infty\}.
\end{aligned}$$

Then $\mathscr{W}(\zeta_0, \leq i_0)$ is convex and strongly closed in L^p, and therefore weakly closed, whereas $\mathscr{W}(\zeta_0, i_0)$ is not strongly closed. Further if $\zeta \in \mathscr{W}(\zeta_0)$ then $\|\zeta\|_1 \leq \|\zeta_0\|_1$ and $\|\zeta\|_p \leq \|\zeta_0\|_p$.

We write

$$\begin{aligned}
\mathscr{R}^+(\zeta_0) &= \{\zeta 1_A \mid \zeta \in \mathscr{R}(\zeta_0),\ A \subset \Pi \text{ measurable }\},\\
\mathscr{R}\mathscr{C}(\zeta_0) &= \{0 \leq \zeta \in L^1(\Pi) \mid \zeta^\Delta = \zeta_0^\Delta 1_{(0,\ell)},\ \text{some } 0 \leq \ell \leq \infty\}.
\end{aligned}$$

The elements of $\mathscr{R}\mathscr{C}(\zeta_0)$ were called *curtailments of rearrangements* of ζ_0 by Douglas [7], who showed that $\mathscr{R}\mathscr{C}(\zeta_0)$ is the set of extreme points of $\mathscr{W}(\zeta_0)$. The ideas described above form counterparts, for domains of infinite measure, of some results of Ryff [15] for bounded intervals (and hence, by isomorphism, for more general domains of finite measure).

We will consider the relaxed problem of maximising E relative to $\mathscr{W}(\zeta_0, \leq i_0)$ and let M_0 denote the supremum of this problem. It is easy to see from the second form of the Green's function G in (1) that translating any nontrivial $\zeta \geq 0$ in the positive x_2-direction strictly increases E, so any maximiser of E relative to $\mathscr{W}(\zeta_0, \leq i_0)$ must belong to $\mathscr{W}(\zeta_0, i_0)$. The strict convexity of E

shows that maximisers of E relative to $\mathscr{W}(\zeta_0, i_0)$ are extreme points of this set; together with a result of Douglas [7] this will be used to show that all maximisers belong to the set $\mathscr{RC}(\zeta_0)$. This is not quite straightforward because I is an unbounded functional so Douglas's theorem does not apply directly to $\mathscr{W}(\zeta_0, i_0)$.

2.2 Results.

We prove the following compactness theorem, which is more general than compactness of maximising sequences:

Theorem 1 (**Compactness).** *Let $i_0 > 0$ and $\zeta_0 \in L^p(\Pi)$, $p > 2$, be given, such that ζ_0 is nontrivial and non-negative with compact support. Let Σ_0 be the set of maximisers of E relative to $\mathscr{W}(\zeta_0, i_0)$, let M_0 be the value of E on Σ_0 and suppose $\emptyset \neq \Sigma_0 \subset \mathscr{R}(\zeta_0)$. Suppose (ζ_n) is a non-negative sequence in $L^p \cap L^1(\Pi)$ such that $\zeta_n^\Delta \to \zeta_0^\Delta$ in both L^p and L^1, $I(\zeta_n) \to i_0$ and $E(\zeta_n) \to M_0$. Then some subsequence of (ζ_n) converges, after x_1-translation, in both L^p and L^1 to an element of Σ_0.*

Definition. By an L^p*-regular solution* of the vorticity equation we mean $\omega \in L^\infty_{\text{loc}}([0,\infty)), L^1(\Pi)) \cap L^\infty_{\text{loc}}([0,\infty)), L^p(\Pi))$ satisfying, in the sense of distributions,

$$\begin{cases} \partial_t \omega + \operatorname{div}(\omega u) = 0, \\ u = \nabla^\perp \mathscr{G}\omega, \quad (t,x) \in [0,\infty) \times \Pi, \end{cases} \tag{5}$$

such that $E(\omega(t,\cdot))$ and $I(t,\cdot)$ are constant. (The operators div, $\nabla^\perp$ and $\mathscr{G}$ are applied only in the space variable x.)

This differs from the corresponding definition in [6] in omitting the λe_1 term from the formula for u; this change represents viewing the flow in a frame stationary relative to the fluid at infinity instead of one moving with velocity $-\lambda e_1$.

For a discussion of the global existence of L^p-regular solutions of the vorticity equation with given initial vorticity in $L^p(\Pi) \cap L^1(\Pi)$ see [6, Sect. 2]. Note that solutions are not known to be unique, except in the case of compactly supported initial vorticity in L^∞ studied by Yudovich [16].

In stating the following theorem, we regard vorticity ω a function of time t with values in $L^1(\Pi) \cap L^p(\Pi)$ and suppress the space variable x.

Theorem 2 (**Stability).** *Let $2 < p < \infty$, let $\zeta_0 \in L^1(\Pi) \cap L^p(\Pi)$ be non-negative and have compact support and let $i_0 > 0$. Suppose $\emptyset \neq \Sigma_0 \subset \mathscr{R}(\zeta_0)$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\omega(0) \geq 0$ with compact support satisfies $\operatorname{dist}_{\mathfrak{X}^p}(\omega(0), \Sigma_0) < \delta$, then $\operatorname{dist}_{\mathfrak{X}^p}(\omega(t), \Sigma_0) < \varepsilon$ for all $t > 0$, whenever ω is an L^p-regular solution of the vorticity equation with initial vorticity $\omega(0)$.*

The proofs will be given in Section 4.

2.3 Estimates for the stream function and properties of the energy.

In Lemmas 1, 2 and 3 following we refine the calculations of [3, Lemma 1,2,10] to derive estimates for $\mathscr{G}(\zeta)$ that apply uniformly over $\zeta \in \mathscr{W}(\zeta_0, \leq i_0)$ and in Lemmas 4, 5 and 6 we establish weak continuity properties of E. Lemmas 7 and 8 then show we can work in a strip in Π.

Lemma 1. *Suppose that $1 < p < \infty$ and $1 < \alpha < \infty$. Then there are positive constants c_1, c_2, c_3, depending only on p and α, such that, if $0 \le \zeta \in L^p(\Pi)$ and $I(\zeta) < \infty$, then*

$$\mathscr{G}\zeta(x) \le x_2^{-1}(c_1 \log x_2 + c_2)I(\zeta) + c_3 x_2^{-1/\alpha} \|\zeta\|_p^{1-1/\alpha} I(\zeta)^{1/\alpha} \quad \forall x_2 \ge 1.$$

Proof. We have

$$\mathscr{G}\zeta(x) = \left(\int_{\rho < x_2/2} + \int_{\rho > x_2/2} \right) G(x,y)\zeta(y)\mathrm{d}y$$

where $\rho = |x - y|$. Now

$$\begin{aligned} \int_{\rho > x_2/2} G(x,y)\zeta(y)\mathrm{d}y &= (4\pi)^{-1} \int_{\rho > x_2/2} \log(1 + 4x_2 y_2 \rho^{-2})\zeta(y)\mathrm{d}y \\ &\le \pi^{-1} \int_{\rho > x_2/2} x_2 y_2 \rho^{-2} \zeta(y)\mathrm{d}y \le \pi^{-1} \int_{\rho > x_2/2} 4x_2^{-1} y_2 \zeta(y)\mathrm{d}y \\ &\le 4\pi^{-1} I(\zeta) x_2^{-1}. \end{aligned}$$

For $0 < \rho < x_2/2$ we have $x_2/2 < y_2 < 3x_2/2$ so

$$\begin{aligned} 4\pi G(x,y) &= \log(1 + 4x_2 y_2/\rho^2) < \log(9x_2 y_2/2\rho^2) \\ &= \log(9y_2/2) + \log x_2 + 2\log \rho^{-1} \\ &\le 2x_2^{-1} y_2 \log(27x_2/4) + 2x_2^{-1} y_2 \log x_2 + 2\log\rho^{-1} \end{aligned}$$

when $x_2 \ge 1$, hence

$$\int_{\rho < x_2/2} G(x,y)\zeta(y)\mathrm{d}y \le (4\pi x_2)^{-1} \left(4\log x_2 + 2\log \tfrac{27}{4}\right) I(\zeta) + 2\int_{\rho < x_2/2} (\log \rho^{-1})\zeta(y)\mathrm{d}y.$$

To treat the last integral in the above inequality, choose $\beta = p/(1 - 1/\alpha)$, which ensures that $\beta > p > 1$ and that

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{p}\left(1 - \frac{1}{\alpha}\right) < \frac{1}{\alpha} + \left(1 - \frac{1}{\alpha}\right) = 1,$$

hence we can choose $\gamma > 1$ such that $\alpha^{-1} + \beta^{-1} + \gamma^{-1} = 1$. Then Hölder's inequality yields

$$\begin{aligned} \int_{\rho < x_2/2} (\log\rho^{-1})\zeta(y)\mathrm{d}y &\le \int_{\rho < x_2/2} (\log\rho^{-1})_+ \, y_2^{-1/\alpha} y_2^{1/\alpha} \zeta(y)\mathrm{d}y \\ &\le 2^{1/\alpha} x_2^{-1/\alpha} \int_{\rho < x_2/2} (\log\rho^{-1})_+ \zeta^{1-1/\alpha} y_2^{1/\alpha} \zeta^{1/\alpha} \mathrm{d}y \\ &\le 2^{1/\alpha} x_2^{-1/\alpha} \|(\log\rho^{-1})_+\|_\gamma \|\zeta^{1-1/\alpha}\|_\beta \|y_2^{1/\alpha}\zeta^{1/\alpha}\|_\alpha \\ &\le 2^{1/\alpha} x_2^{-1/\alpha} \|(\log\rho^{-1})_+\|_\gamma \|\zeta\|_p^{p/\beta} I(\zeta)^{1/\alpha} \\ &\le 2^{1/\alpha} x_2^{-1/\alpha} \|(\log\rho^{-1})_+\|_\gamma \|\zeta\|_p^{1-1/\alpha} I(\zeta)^{1/\alpha} \end{aligned}$$

and we note that

$$\|(\log\rho^{-1})_+\|_\gamma = \left(\int_0^1 2\pi (\log\rho^{-1})^\gamma \rho \mathrm{d}\rho \right)^{1/\gamma}$$

which is a positive real number depending only on α and p since $\gamma = \alpha p/((\alpha - 1)(p - 1))$. □

Lemma 2. *Suppose that $1 < p < \infty$ and $1/p + 1/q = 1$. Then there is a positive constant c_4, depending only on p, such that, if $0 \le \zeta \in L^p(\Pi)$, then*

$$\mathscr{G}\zeta(x) \le c_4(\|\zeta\|_1 + \|\zeta\|_p)(x_2^{1/q} + x_2^2) \quad \forall x \in \Pi.$$

Proof. Write $\rho = |x - y|$. Then $y_2 \le x_2 + \rho$ so by (1) and concavity of log we have firstly

$$\int_{\rho>1} G(x,y)\zeta(y)\mathrm{d}y \le \int_{\rho>1} \frac{x_2 y_2}{\pi\rho^2}\zeta(y)\mathrm{d}y \le \int_{\rho>1} \frac{x_2^2 + x_2\rho}{\pi\rho^2}\zeta(y)\mathrm{d}y \le \frac{1}{\pi}(x_2^2 + x_2)\|\zeta\|_1.$$

Secondly, if $\rho < x_2$ then $|x - \overline{y}| \le 2x_2 + \rho \le 3x_2$, so we have

$$\int_{\rho<x_2} G(x,y)\zeta(y)\mathrm{d}y \le \frac{1}{2\pi}\int_{\rho<x_2} \log\left(\frac{3x_2}{\rho}\right)\zeta(y)\mathrm{d}y \le \frac{1}{2\pi}\left(2\pi x_2^2 \int_0^1 (\log(3s^{-1}))^q \mathrm{d}s\right)^{1/q} \|\zeta\|_p$$

by Hölder's inequality. Thirdly, when $x_2 < 1$ the region $x_2 < \rho < 1$ is nonempty and (1) yields

$$\begin{aligned}\int_{x_2<\rho<1} G(x,y)\zeta(y)\mathrm{d}y &\le \int_{x_2<\rho<1} \frac{x_2(x_2+\rho)}{\pi\rho^2}\zeta(y)\mathrm{d}y \le \int_{x_2<\rho<1} \frac{2x_2}{\pi\rho}\zeta(y)\mathrm{d}y \\ &\le \frac{2}{\pi}\int_{x_2<\rho<1}\left(\frac{x_2}{\rho}\right)^{1/q}\zeta(y)\mathrm{d}y \le \frac{4\|\zeta\|_p}{(2\pi)^{1/p}} x_2^{1/q}\end{aligned}$$

by Hölder's inequality. These three bounds yield the result since $x_2 + x_2^{2/q} \le 2(x_2^{1/q} + x_2^2)$. □

Remark. When $2 < p < \infty$ we can strengthen Lemma 2 as follows:

Lemma 3. *Let $2 < p < \infty$ and $1/q + 1/p = 1$. Then there is a constant $c_5 > 0$ depending only on p, such that if $0 \le \zeta \in L^p(\Pi) \cap L^1(\Pi)$ then, for $x \in \Pi$*

$$\begin{aligned}|\nabla_x \mathscr{G}\zeta(x)| &\le c_5(\|\zeta\|_p + \|\zeta\|_1), \\ \mathscr{G}\zeta(x) &\le c_5 x_2(\|\zeta\|_p + \|\zeta\|_1).\end{aligned}$$

Proof. Let $0 \le \zeta \in L^p(\Pi) \cap L^1(\Pi)$. Then

$$\begin{aligned}\int_\Pi |\nabla_x G(x,y)\zeta(y)|\mathrm{d}y &= \frac{1}{2\pi}\int_\Pi \left|\frac{x-y}{|x-y|^2} - \frac{x-\overline{y}}{|x-\overline{y}|^2}\right|\zeta(y)\mathrm{d}y \\ &\le \frac{1}{2\pi}\int_\Pi \left(\frac{1}{|x-y|} + \frac{1}{|x-\overline{y}|}\right)\zeta(y)\mathrm{d}y \\ &\le \frac{1}{\pi}\int_\Pi \frac{1}{|x-y|}\zeta(y)\mathrm{d}y \\ &= \frac{1}{\pi}\left(\int_{D_\Pi(x,1)} + \int_{\Pi\setminus D(x,1)}\right)\frac{1}{|x-y|}\zeta(y)\mathrm{d}y \\ &\le \frac{1}{\pi}\left(\int_0^1 \frac{2\pi\rho\mathrm{d}\rho}{\rho^q}\right)^{1/q}\|\zeta\|_p + \frac{1}{\pi}\|\zeta\|_1,\end{aligned}$$

hence the first inequality. The second inequality now follows since $\mathscr{G}\zeta(x) \to 0$ as $x_2 \to 0$ uniformly over x_1 by Lemma 2. □

Lemma 4. *Let $1 < p < \infty$, let $L > 0$ and let*

$$\mathscr{L} = \{\xi \in L^1(\Pi) \cap L^p(\Pi) \mid \xi \geq 0,\ I(\xi) \leq L,\ \|\xi\|_1 \leq L,\ \|\xi\|_p \leq L\}.$$

Then E is Lipschitz continuous with respect to $\| \|_1$ relative to $\mathscr{L}$.

Proof. From Lemmas 1 and 2 we can choose a constant K such that $\|\mathscr{G}\xi\|_{\sup} \leq K$ for all $\xi \in \mathscr{L}$. Consider $\xi, \eta \in \mathscr{L}$. We can assume $E(\xi) \geq E(\eta)$ and so

$$0 \leq E(\xi) - E(\eta) = \frac{1}{2}\int_\Pi (\xi - \eta)\mathscr{G}(\xi + \eta) \leq \frac{1}{2}\|\xi - \eta\|_1(\|\mathscr{G}\xi\|_{\sup} + \|\mathscr{G}\eta\|_{\sup}) \leq K\|\xi - \eta\|_1.$$

□

Lemma 5. *Let $2 < p < \infty$. Then there is a constant $c_6 > 0$ such that*

$$\mathscr{G}\zeta(x) \leq c_6(I(\zeta) + \|\zeta\|_1 + \|\zeta\|_p)x_2 \min\{1, |x_1|^{-1/(2p)}\}, \quad x \in \Pi$$

for all non-negative $\zeta \in L^p(\Pi) \cap L^1(\Pi)$ that are Steiner-symmetric in the x_1-axis with $I(\zeta) < \infty$.

Proof. Firstly note that if $\xi \in L^1(0, \infty)$ is decreasing and $0 < b < s$ then the means of ξ on both of the intervals $[s - b, s]$ and $[s, s + b]$ are no greater than the mean of ξ on $[0, s]$ and therefore

$$\frac{1}{2b}\int_{s-b}^{s+b} \xi \leq \frac{1}{s}\int_0^s \xi.$$

Now consider Steiner-symmetric $\zeta \in L^1(\Pi)$ and $x \in \Pi$. Let $0 < b \leq |x|_1$ and apply the above inequality in the y_1 integration to obtain

$$\frac{1}{2b}\int_{|y_1 - x_1| < b} \zeta(y)\mathrm{d}y \leq \frac{1}{2|x_1|}\int_{|y_1| < |x_1|} \zeta(y)\mathrm{d}y$$

and consequently

$$\int_{|y_1 - x_1| < b} \zeta(y)\mathrm{d}y \leq \frac{b}{|x_1|}\int_\Pi \zeta(y)\mathrm{d}y. \tag{6}$$

Suppose that additionally $I(\zeta) < \infty$, fix $x \in \Pi$ with $x_1 \neq 0$, let

$$\zeta_1(y) = \begin{cases} \zeta(y) & \text{if } |y_1 - x_1| < |x_1|^{1/2} \\ 0 & \text{if } |y_1 - x_1| \geq |x_1|^{1/2} \end{cases}$$

and let $\zeta_2 = \zeta - \zeta_1$. We write $\rho = |x - y|$ when $x, y \in \Pi$ and obtain

$$\begin{aligned} \mathscr{G}\zeta_2(x) &\leq \frac{1}{4\pi}\int_{\rho > |x_1|^{1/2}} \log\left(1 + \frac{4x_2y_2}{\rho^2}\right)\zeta_2(y)\mathrm{d}y \leq \frac{1}{4\pi}\int_{\rho > |x_1|^{1/2}} \frac{4x_2y_2}{\rho^2}\zeta_2(y)\mathrm{d}y \\ &\leq \frac{x_2}{\pi|x_1|}\int_\Pi \zeta(y)y_2\mathrm{d}y = \frac{x_2}{\pi|x_1|}I(\zeta). \end{aligned}$$

When $|x_1| > 1$ we have from (6)

$$\begin{aligned} \int_\Pi \zeta_1^1 &\leq \frac{|x_1|^{1/2}}{|x_1|}\int_\Pi \zeta = |x_1|^{-1/2}\int_\Pi \zeta, \\ \int_\Pi \zeta_1^p &\leq \frac{|x_1|^{1/2}}{|x_1|}\int_\Pi \zeta^p = |x_1|^{-1/2}\int_\Pi \zeta^p \end{aligned}$$

so by Lemma 3 we have

$$\mathscr{G}\zeta_1(x) \leq c_5 x_2(\|\zeta_1\|_1 + \|\zeta_1\|_p) \leq c_5 x_2(\|\zeta\|_1 |x_1|^{-1/2} + \|\zeta\|_p |x_1|^{-1/(2p)}).$$

The result follows from the above inequalities for $\mathscr{G}\zeta_1(x)$ and $\mathscr{G}\zeta_2(x)$ when $|x_1| > 1$ and from Lemma 3 when $|x_1| \leq 1$. □

Lemma 6. *Let $2 < p < \infty$ and let (ζ_n) be a sequence of Steiner symmetric non-negative functions on Π, suppose $\|\zeta_n\|_1$ and $I(\zeta_n)$ are bounded and suppose $\zeta_n \to \zeta$ weakly in $L^p(\Pi)$. Then $E(\zeta_n) \to E(\zeta)$ as $n \to \infty$.*

Proof. Fix $T > 0$, to be chosen later, and consider Steiner symmetric $\xi \in L^1(\Pi) \cap L^p(\Pi)$ with $I(\xi) < \infty$. Define

$$\begin{aligned} \Pi_0 &= \{x \in \Pi \mid x_2 > T\}, \\ \Pi_1 &= \{x \in \Pi \mid x_2 < T,\ |x_1| > T\}, \\ \Pi_2 &= \{x \in \Pi \mid x_2 < T,\ |x_1| < T\} \end{aligned}$$

and let $\xi_k = \xi 1_{\Pi_k}$ for $k = 1, 2, 3$. Then we have

$$E(\xi) = \frac{1}{2}\int_\Pi \xi_2 \mathscr{G}\xi_2 + \int_\Pi \xi_2 \mathscr{G}(\xi_0 + \xi_1) + \frac{1}{2}\int_\Pi (\xi_0 + \xi_1)\mathscr{G}(\xi_0 + \xi_1). \tag{7}$$

There is a positive constant C, depending only on p, such that the inequalities

$$\begin{aligned} \mathscr{G}\xi(x) &\leq C(I(\xi) + \|\xi\|_1 + \|\xi\|_p) & (8) \\ \mathscr{G}\xi(x) &\leq C(I(\xi) + \|\xi\|_1 + \|\xi\|_p) x_2 \min\{1, |x_1|^{-1/(2p)}\} & (9) \end{aligned}$$

hold for all $x \in \Pi$, from Lemmas 1 and 2 in the case of (8) and from Lemma 5 in the case of (9), the Steiner symmetry being employed only for (9). From (8) and (9) we obtain

$$\begin{aligned} \int_\Pi \xi_2 \mathscr{G}(\xi_0 + \xi_1) &+ \frac{1}{2}\int_\Pi (\xi_0 + \xi_1)\mathscr{G}(\xi_0 + \xi_1) \leq \int_\Pi \xi_0 \mathscr{G}\xi + \xi_1 \mathscr{G}\xi \\ &\leq \|\mathscr{G}\xi\|_{\sup} \int_{x_2 > T} \xi + \int_{|x_1| > T} \xi \mathscr{G}\xi \\ &\leq C(I(\xi) + \|\xi\|_1 + \|\xi\|_p)\left(\int_{x_2 > T} T^{-1} x_2 \xi(x)\mathrm{d}x + \int_{|x_1| > T} x_2 |x_1|^{-1/(2p)} \xi(x)\mathrm{d}x\right) \\ &\leq C(I(\xi) + \|\xi\|_1 + \|\xi\|_p) I(\xi)(T^{-1} + T^{-1/(2p)}). \qquad (10) \end{aligned}$$

Now consider the sequence (ζ_n), which must be bounded in $L^p(\Pi)$. Let $\varepsilon > 0$, write $Q = (-T, T) \times (0, T)$ and use (7) and (10) to choose $T > 0$ such that

$$0 \leq E(\zeta_n) - E(1_Q \zeta_n) < \varepsilon$$

for all n and

$$0 \leq E(\zeta) - E(1_Q \zeta) < \varepsilon.$$

In view of the compactness of $\mathscr{G}$ as an operator from $L^p(Q)$ to $L^q(Q)$ we have

$$E(1_Q \zeta_n) \to E(1_Q \zeta)$$

as $n \to \infty$. It follows that

$$E(\zeta_n) \to E(\zeta).$$

□

Lemma 7. *Let $1 < p < \infty$, let $i_0 > 0$ and let ζ_0 be non-negative and have compact support. Let $\zeta \in \mathscr{W}(\zeta_0, \le i_0)$. Then ζ is the weak limit in L^p of a sequence (ζ_n) in $\mathscr{R}^+(\zeta_0)$ having $I(\zeta_n) \le i_0$ for each n.*

Proof. Firstly consider the case when $I(\zeta) < i_0$. Since $\mathscr{R}(\zeta_0)$ is weakly dense in $\mathscr{W}(\zeta_0)$ by the results of Douglas [7] we may choose a sequence (ξ_k) in $\mathscr{R}(\zeta_0)$ converging weakly to ζ in $L^p(\Pi)$.

Given $g \in L^q(\Pi)$, where $1/p + 1/q = 1$, we have

$$\left| \int_{D_\Pi(0,n)} \xi_k g - \int_\Pi \zeta g \right| \le \left| \int_\Pi \xi_k g - \int_\Pi \zeta g \right| + \|\zeta_0\|_p \|g\|_{L^q(\Pi \setminus D(0,n))}$$

for all $k, n \in \mathbb{N}$. Hence if $(k(n))$ is any strictly increasing sequence in $\mathbb{N}$ then $\xi_{k(n)} 1_{D(0,n)} \to \zeta$ weakly in L^p as $n \to \infty$.

Further, for each fixed $n \in \mathbb{N}$ we have $I(\xi_k 1_{D(0,n)}) \to I(\zeta 1_{D(0,n)})$ as $k \to \infty$, so by a diagonal sequence argument we can chose a strictly increasing sequence $(k(n))$ of positive integers such that $I(\xi_{k(n)} 1_{D(0,n)}) - I(\zeta 1_{D(0,n)}) \to 0$ as $n \to \infty$. Since $I(\zeta 1_{D(0,n)}) \to I(\zeta)$ by the monotone convergence theorem we now have $I(\xi_{k(n)} 1_{D(0,n)}) \to I(\zeta)$ as $n \to \infty$.

Therefore $\xi_{k(n)} 1_{D(0,n)} \in \mathscr{R}^+(\zeta_0)$ satisfies $I(\xi_{k(n)} 1_{D(0,n)}) < i_0$ for all sufficiently large n and $\xi_{k(n)} 1_{D(0,n)} \to \zeta$ weakly in L^p. This completes the proof in the case $I(\zeta) < i_0$.

Secondly, suppose $I(\zeta) = i_0 > 0$. Then, by truncation, ζ can we written as the strong limit of elements $\xi_n = \zeta 1_{\{x | x_2 < r_n\}}$ of $\mathscr{W}(\zeta_0)$ with $I(\xi_n) < i_0$, where (r_n) is an increasing sequence of positive numbers, so each ξ_n is in the weak closure of $\mathscr{R}^+(\zeta_0) \cap I^{-1}([0, i_0])$ by the above argument, hence ζ is in the weak closure of $\mathscr{R}^+(\zeta_0) \cap I^{-1}([0, i_0])$. □

Lemma 8. *Let $i_0 > 0$ and $0 \le \zeta_0 \in L^p(\Pi)$ for some $1 < p < \infty$, compactly supported. Then there exists $Z > 0$ such that, if (ζ_n) is any maximising sequence for E relative to $\mathscr{W}(\zeta_0, i_0)$ comprising elements of $\mathscr{R}^+(\zeta_0)$ then $\left(\zeta_n 1_{\mathbb{R}\times(0,Z)}\right)$ is also a maximising sequence and $\left\|\zeta_n 1_{\mathbb{R}\times(Z,\infty)}\right\|_1 \to 0$.*

Proof. Let M_0 be the supremum of E relative to $\mathscr{W}(\zeta_0, \le i_0)$, so $M_0 < \infty$ by Lemma 4. Consider $\zeta \in \mathscr{W}(\zeta_0, \le i_0) \cap \mathscr{R}^+(\zeta_0)$ such that $E(\zeta) \ge M_0/2$; such a ζ must exist if any such sequences (ζ_n) exist. Write $S = \{x \mid \zeta(x) > 0\}$ and $S_0 = \{x \mid \zeta_0(x) > 0\}$.

Then $\|\zeta\|_1 \le \|\zeta_0\|_1$ since $\zeta \in \mathscr{R}^+(\zeta_0)$, hence $\|\mathscr{G}\zeta\|_{\sup} \ge M_0/\|\zeta_0\|_1$ by Hölder's inequality. The estimate of Lemma 1 shows there exists $Z_0 > 0$ such that every function $\xi \in \mathscr{W}(\zeta_0, \le i_0)$ obeys $\mathscr{G}\xi(x) < M_0/(2\|\zeta_0\|_1)$ if $x_2 > Z_0$. Choose $x^* \in \Pi$ at which $\mathscr{G}\zeta$ achieves its supremum; thus $x_2^* \le Z_0$.

Consider $y \in \Pi$ satisfying $y_2 > 2Z_0$. Then we have $|x^* - y| \ge y_2 - x_2^* > y_2/2$ so

$$G(x^*, y) = \frac{1}{4\pi} \log\left(1 + \frac{4x_2^* y_2}{|x^* - y|^2}\right) < \frac{1}{4\pi} \log\left(1 + \frac{16Z_0}{y_2}\right).$$

We choose $Z_1 > 2Z_0$ such that $Z_1^2 \ge |S_0|$ and

$$\frac{1}{4\pi} \log\left(1 + \frac{16Z_0}{Z_1}\right) \le \frac{M_0}{2\|\zeta_0\|_1^2}.$$

This choice of Z_1 ensures that if $y_2 > Z_1$ then we have

$$G(x^*, y) < \frac{1}{4\pi} \log\left(1 + \frac{16Z_0}{Z_1}\right) \le \frac{M_0}{2\|\zeta_0\|_1^2}$$

whereas if $y_2 \le Z_1$ and $|y_1 - x_1^*| > Z_1$ then we have

$$G(x^*, y) < \frac{1}{4\pi} \log\left(1 + \frac{4Z_0}{Z_1}\right) \le \frac{M_0}{2\|\zeta_0\|_1^2}.$$

Write $Q = (x_1^* - Z_1, x_1^* + Z_1) \times (0, Z_1)$; thus $|Q| = 2Z_1^2 \ge 2|S_0|$. Then, by the above bounds,

$$\int_{\Pi \setminus Q} G(x^*, y)\zeta(y)\mathrm{d}y \le \frac{M_0}{2\|\zeta_0\|_1}$$

and therefore, since the choice of x^* ensures $\mathscr{G}(x^*) \ge M_0/\|\zeta_0\|_1$, we have

$$\int_Q G(x^*, y)\zeta(y)\mathrm{d}y \ge \frac{M_0}{2\|\zeta_0\|_1}.$$

It follows by Hölder's inequality that

$$\int_Q G(x^*, y)\zeta(y)\mathrm{d}y \le \|G(x^*, \cdot)\|_{L^q(Q)}\|\zeta\|_{L^p(Q)} \le g\|\zeta\|_{L^p(Q)}$$

where $1/q + 1/p = 1$ and we may find a suitable value for the constant g from the rearrangement inequality for the integral of a product of two functions, as follows. We have

$$G(x^*, y) \le \frac{1}{4\pi} \log\left(1 + \frac{8Z_1^2}{|x^* - y|^2}\right), \quad y \in Q,$$

so, denoting by D the disc with centre x^* and radius Z_1, denoting by Q^* the disc with centre x^* and area $|Q|$ and noting that $|D| \ge |Q^*|$, we have

$$\begin{aligned} \|G(x^*, \cdot)\|_{L^q(Q)}^q = \int 1_Q(y)G(x^*, y)^q\mathrm{d}y &\le \int 1_{Q^*}(y)\left(\frac{1}{4\pi} \log\left(1 + \frac{8Z_1^2}{|x^* - y|^2}\right)\right)^q \mathrm{d}y \\ &\le \int_D \left(\frac{1}{4\pi} \log\left(1 + \frac{8Z_1^2}{|x^* - y|^2}\right)\right)^q \mathrm{d}y \\ &\le \int_0^{Z_1} \left(\frac{1}{2\pi} \log\left(\frac{3Z_1}{\rho}\right)\right)^q 2\pi\rho\mathrm{d}\rho =: g^q. \end{aligned}$$

Hence

$$\|\zeta\|_{L^p(Q)} \ge \frac{M_0}{2g\|\zeta_0\|_1} =: m.$$

Choose $\theta > 0$ such that

$$\int_0^\theta (\zeta_0^\Delta)^p < \frac{m^p}{2}.$$

Then we have

$$|Q \cap S| \ge 2\theta.$$

Let $Q_0 = (x_1^* - Z_1, x_1^* + Z_1) \times (0, \eta)$ where $\eta = \theta/(2Z_1)$. Then $|Q_0| = \theta$ and therefore

$$\int_{Q \setminus Q_0} \zeta \geq \inf_{U \subset S, |U|=\theta} \int_U \zeta \geq \inf_{U \subset S_0, |U|=\theta} \int_U \zeta_0 \geq \int_0^\theta \zeta_0^\nabla =: \nu.$$

If $x, y \in Q \setminus Q_0$ then

$$G(x,y) \geq \frac{1}{4\pi} \log\left(1 + \frac{4\eta^2}{5Z_1^2}\right) =: \mu.$$

Hence for all $x \in Q \setminus Q_0$ we have $\mathscr{G}\zeta(x) \geq \mu\nu$. Moreover

$$|Q \setminus Q_0| = |Q| - \theta \geq \frac{|Q|}{2} \geq |S_0|.$$

Choose $Z > Z_1$ such that $x_2 > Z$ implies $\mathscr{G}\xi(x) < \mu\nu/2$ for all $\xi \in \mathscr{W}(\zeta_0, \leq i_0)$, by Lemma 1. Let $h = \zeta 1_{\mathbb{R}\times(Z,\infty)}$ and consider the possibility that h is non-trivial; then $h^{-1}(0,\infty)$ is a set of finite positive planar Lebesgue measure and is therefore measure-theoretically isomorphic to a bounded interval in $\mathbb{R}$ with linear Lebesgue measure. Similarly $Q \setminus (Q_0 \cup S)$, which has planar Lebesgue measure greater than that of $S \setminus Q$, is therefore isomorphic to another interval, of greater length. It follows that we can choose a rearrangement h' of h supported in $Q \setminus (Q_0 \cup S)$. Then $\zeta + h' - h$ lies in $\mathscr{W}(\zeta_0, \leq i_0) \cap \mathscr{R}^+(\zeta_0)$ and

$$E(\zeta + h' - h) = E(\zeta) + \int_\Pi (\mathscr{G}\zeta)(h' - h) + E(h' - h) \geq E(\zeta) + \mu\nu\|h'\|_1 - \frac{\mu\nu}{2}\|h\|_1 = E(\zeta) + \frac{\mu\nu}{2}\|h\|_1,$$

so

$$\|h\|_1 \leq \frac{2}{\mu\nu}(M_0 - E(\zeta)).$$

Hence if (ζ_n) is a maximising sequence belonging to $\mathscr{W}(\zeta_0, \leq i_0) \cap \mathscr{R}^+(\zeta_0)$ and $h_n = \zeta_n 1_{\{x \in \Pi | x_2 > Z\}}$ then $\|h_n\|_1 \to 0$ so

$$E(\zeta_n - h_n) = E(\zeta_n) - \int_\Pi (\mathscr{G}\zeta_n)h_n + E(h_n) \geq E(\zeta_n) - \mu\nu\|h_n\|_1 \to M_0.$$

That is, $(\zeta_n 1_{\mathbb{R}\times(0,Z)})$ is also a maximising sequence of functions in $\mathscr{W}(\zeta_0, \leq i_0) \cap \mathscr{R}^+(\zeta_0)$. □

3 Existence of relaxed maximisers and first variation condition.

In this section we use the estimates of Section 2 to extend the existence theory of [3, Theorem 16(ii)] to the relaxed problem, since in the proofs of the main results we will have to consider subsidiary variational problems where only relaxed solutions might exist under the hypotheses that apply (situations where solutions to an unrelaxed problem of this type fail to exist can be found in the study of Lamb's vortex [5, Corollary 1]). The first variation condition at a maximum gives rise to a functional relationship between the vorticity and the stream function that shows the maximisers represent steady flows. Finally we show that for large values of the impulse, the relaxed solutions are indeed solutions of the unrelaxed problem. This will be needed to show that the Stability Theorem applies in a wide range of cases.

Lemma 9. *Let $2 < p < \infty$, let $i_0 > 0$ and let $0 \le \zeta_0 \in L^1(\Pi) \cap L^p(\Pi)$ have compact support. Then*
(i) *there exist maximisers for E relative to $\mathscr{W}(\zeta_0, \le i_0)$ and all maximisers belong to $\mathscr{W}(\zeta_0, i_0)$,*
(ii) *all maximisers are Steiner-symmetric elements of $\mathscr{RC}(\zeta_0)$,*
(iii) *for every maximiser ζ there exists an increasing function φ and a number $\lambda > 0$ such that $\zeta = \varphi \circ (\mathscr{G}\zeta - \lambda x_2)$ almost everywhere in Π and ζ vanishes almost everywhere in the set $\{x \in \Pi \mid \psi(x) - \lambda x_2 \le 0\}$,*
(iv) *every maximiser vanishes outside a bounded subset of $\mathbb{R} \times (0, Z)$, where Z is the number, depending on i_0, ζ_0 and p only, provided by Lemma 8.*

Proof. To prove (i) we choose a maximising sequence (ζ_n) for E relative to $\mathscr{W}(\zeta_0, \le i_0)$. It follows from the 1-dimensional case of the Riesz rearrangement inequality that Steiner symmetrisation about the x_2-axis does not decrease E. Therefore let us assume (ζ_n) to comprise Steiner-symmetric functions. The sequence (ζ_n) is bounded in both $\| \|_1$ and $\| \|_p$. We may therefore pass to a subsequence and assume (ζ_n) converges weakly in $L^p(\Pi)$ to a limit $\widehat{\zeta}$. Then $\widehat{\zeta} \in \mathscr{W}(\zeta_0, \le i_0)$ and $E(\widehat{\zeta}) = M_0$ by Lemma 6. Thus $\widehat{\zeta}$ is a maximiser.

If $\zeta \in \mathscr{W}(\zeta_0, \le i_0)$ is any element then translation of ζ in the positive x_2 direction strictly increases $E(\zeta)$. Therefore every maximiser ζ of E relative to $\mathscr{W}(\zeta_0, \le i_0)$ must satisfy $I(\zeta) = i_0$.

To prove (ii) consider a maximiser ζ. We can write

$$E(\zeta) = \int_0^\infty \int_0^\infty J(\zeta, x_2, y_2) \mathrm{d}x_2 \mathrm{d}y_2$$

where

$$J(\zeta, x_2, y_2) = \int_{-\infty}^\infty \int_{-\infty}^\infty \zeta(x_1, x_2) G(x_1, x_2, y_1, y_2) \zeta(y_1, y_2) \mathrm{d}x_1 \mathrm{d}y_1$$

with similar expressions for $E(\zeta^s)$ and $J(\zeta^s, x_2, y_2)$. Since $G(x_1, x_2, y_1, y_2)$ is a decreasing function of $|x_1 - y_1|$ for fixed x_2 and y_2, the Riesz rearrangement inequality shows that

$$J(\zeta, x_2, y_2) \le J(\zeta^s, x_2, y_2) \quad \forall x_2 > 0, y_2 > 0$$

and hence $E(\zeta) \le E(\zeta^s)$, so $E(\zeta) = E(\zeta^s)$ by maximality. Thus

$$\int_0^\infty \int_0^\infty (J(\zeta^s, x_2, y_2) - J(\zeta, x_2, y_2)) \mathrm{d}x_2 \mathrm{d}y_2 = 0$$

and since the integrand is non-negative we must have

$$J(\zeta, x_2, y_2) = J(\zeta^s, x_2, y_2) < \infty$$

for almost all pairs (x_2, y_2) of positive numbers. We can now apply the one-dimensional case of Lieb's analysis[1] [11, Lemma 3] of equality in the Riesz rearrangement inequality to conclude that, for almost every pair (x_2, y_2) of positive real numbers, the functions $\zeta(\cdot, x_2)$ and $\zeta(\cdot, y_2)$ are symmetric decreasing about the same real number β say, and then that β is independent of x_2 and y_2. Thus ζ is Steiner symmetric about the line $x_2 = \beta$.

[1] Lieb's result applies to the case when one of the functions is strictly symmetric decreasing, which is the situation here, and has an unstated but necessary assumption that the integrals in question are finite. In the proof, m should be defined by $m := f * \overline{h}$, where $\overline{h}(y) = h(-y)$, so that m is invariant under equal translations of f and h.

Lemmas 7 and 8 show that ζ is a weak limit in L^p of functions $\zeta_n \in \mathscr{R}^+(\zeta_0)$ satisfying $I(\zeta_n) \le i_0$ and that $\zeta_n 1_{\mathbb{R}\times(Z,\infty)} \to 0$ in L^1. It follows that ζ vanishes outside $\mathbb{R}\times(0,Z)$. Any maximiser ζ satisfies $I(\zeta) = i_0$ and the strict convexity of E shows that ζ must be an extreme point of $\mathscr{W}_{\mathbb{R}\times(0,Z)}(\zeta_0, i_0)$. Since I is a bounded linear functional relative to $L^1(\mathbb{R}\times(0,Z))$, we can apply Douglas's characterisation [7, Theorem 2.1(ii)] of the extreme points of the intersection of $\mathscr{W}(\zeta_0)$ with a closed hyperplane to deduce that $\zeta \in \mathscr{RC}(\zeta_0)$ (the statement of Douglas's result excludes L^1 but the proof of part (ii) is also valid for L^1).

To prove (iii) and (iv) let ζ be a maximiser and $\psi = \mathscr{G}\zeta$. From convexity of E it follows that ζ maximises $\int_\Pi \psi\xi$ subject to $\xi \in \mathscr{W}(\zeta_0, \le i_0)$. Define the "value function" f by

$$f(i) = \sup_{\xi\in\mathscr{W}(\zeta_0,\le i)} \int_\Pi \psi\xi \quad \text{for all } i \ge 0.$$

Since ψ is bounded it follows that f is finite-valued. Since $\mathscr{W}(\zeta_0, \le i_1) \subset \mathscr{W}(\zeta_0, \le i_2)$ if $0 \le i_1 \le i_2$ it follows that f is increasing. The convexity of $\mathscr{W}(\zeta_0, < \infty)$ ensures that

$$(1-\theta)\mathscr{W}(\zeta_0, \le i_1) + \theta\mathscr{W}(\zeta_0, \le i_2) \subset \mathscr{W}(\zeta_0, \le (1-\theta)i_1 + \theta i_2) \quad \forall i_1 \ge 0,\ i_2 \ge 0,\ 0 \le \theta \le 1,$$

hence $-f$ is a convex function. Therefore $-f$ is continuous and subdifferentiable on $(0,\infty)$.

Now ζ maximises $\int_\Pi \psi\xi - f(I(\xi))$ subject to $\xi \in \mathscr{W}(\zeta_0, <\infty)$ so we have the subdifferential condition that, for some $-\lambda \in \partial(-f)(i_0)$,

$$\zeta \text{ maximises } \int_\Pi \psi\xi - \lambda I(\xi) \left(= \int_\Pi (\psi - \lambda x_2)\xi \right) \text{ subject to } \xi \in \mathscr{W}(\zeta_0, <\infty). \tag{11}$$

Since $-f$ is decreasing we have $-\lambda \le 0$ so $\psi - \lambda x_2$ is bounded above, from Lemma 1.

In order to derive a functional relationship between ζ and $\Psi := \psi - \lambda x_2$ from (11), for each $n \in \mathbb{N}$ we now write $Q(n)$ for the planar rectangle $(-n,n)\times(0,n)$ and consider the consequences of rearranging ζ within $Q(n)$ while fixing ζ outside $Q(n)$. Thus $\zeta 1_{Q(n)}$ maximises $\int_{Q(n)} \Psi\xi$ subject to $\xi \in \mathscr{R}_{Q(n)}(\zeta 1_{Q(n)})$ so by [4, Lemma 2.15] there exists an increasing function φ_n such that $\zeta = \varphi_n \circ \Psi$ almost everywhere in $Q(n)$. We can assume φ_n to be defined on an interval I_n such that $\Psi(Q(n))^\circ \subset I_n \subset \overline{(\Psi(Q(n))}$. We claim that all the φ_n can be assumed to be restrictions of a single increasing function φ defined on the interval $I = \bigcup I_n$. To see this, firstly define $S_n = \{s \in I_n \mid \varphi_n(s) \ne \varphi_{n+1}(s)\}$. Then S_n must have empty interior relative to I_n, otherwise $\varphi_n \circ \Psi \ne \varphi_{n+1} \circ \Psi$ throughout a nonempty open subset of $Q(n)$, which must have positive measure. Hence $\varphi_n = \varphi_{n+1}$ on a dense subset of I_n and therefore at all points where φ_n and φ_{n+1} are both continuous relative to I_n. Let $D \subset I$ comprise all discontinuities of the φ_n, $n \in \mathbb{N}$, which is a countable set by monotonicity; then $\varphi_k(s) = \varphi_n(s)$ provided that $s \in I_n \setminus D$ and $k > n$. Let T be the set of $s \in I$ for which $\Psi^{-1}(s)$ has positive measure, which is also countable. If $s \in T$ and n is the least number for which $\Psi^{-1}(s) \cap Q(n)$ has positive measure then we must have $\varphi_k(s) = \varphi_n(s)$ for all $k > n$, because $\varphi_k \circ \Psi = \varphi_n \circ \Psi$ almost everywhere on $Q(n)$; hence $\varphi_k(\Psi(x)) = \varphi_n(s)$ for almost every $x \in Q(k) \cap \Psi^{-1}(s)$, for every $k \in \mathbb{N}$. For $s \in I \setminus (D \setminus T)$ we can now define $\varphi(s) = \varphi_n(s)$ for all sufficiently large n and find that φ is increasing and $\zeta = \varphi \circ \Psi$ almost everywhere on $\Pi \setminus \Psi^{-1}(D \setminus T)$. Since $\Psi^{-1}(D \setminus T)$ has zero measure, we can complete the construction of φ by adopting any definition of $\varphi(s)$ for $s \in D \setminus T$ that makes φ increasing on I.

We defer the proof that λ is strictly positive until after (iv).

For (iv), let ζ, ψ, φ and $\lambda \geq 0$ be as above. Since ψ is Steiner symmetric about the x_2-axis, Lemma 5 yields a constant $c_6 > 0$ such that

$$\psi(x) \leq c_6(I(\zeta) + \|\zeta\|_1 + \|\zeta\|_p)x_2 \min\{1, |x_1|^{-1/(2p)}\}, \quad x \in \Pi. \tag{12}$$

Since ζ maximises $\int_\Pi(\psi - \lambda x_2)\xi$ subject to $\xi \in \mathscr{W}(\zeta_0, < \infty)$ we must have $\zeta = 0$ almost everywhere in the set $A = \{x \in \Pi \mid \psi(x) - \lambda x_2 < 0\}$, otherwise

$$\int(\psi - \lambda x_2)\zeta < \int(\psi - \lambda x_2)\zeta 1_{\Pi\setminus A}$$

which is impossible since $\zeta 1_{\Pi\setminus A} \preceq \zeta$ so $\zeta 1_{\Pi\setminus A} \in \mathscr{W}(\zeta_0, < \infty)$. The set of x where $\zeta(x)(= -\Delta\psi(x))$ is positive and $\psi(x) - \lambda x_2 = 0$ necessarily has zero measure.

If $\lambda > 0$ it now follows from (12) that ζ vanishes almost everywhere outside the region where $x_1^{1/(2p)} \leq c(I(\zeta) + \|\zeta\|_1 + \|\zeta\|_p)/\lambda$, so the support of ζ is also bounded in the x_1 direction.

If $\lambda = 0$ then $\zeta = \varphi \circ \psi$ almost everywhere in Π so, since φ is increasing, there exists $\kappa \geq 0$ such that

$$\{x \in \Pi \mid \psi(x) > \kappa\} \subset \{x \in \Pi \mid \zeta(x) > 0\} \subset \{x \in \Pi \mid \psi(x) \geq \kappa\}$$

apart from sets of measure zero. Thus $\kappa > 0$, for otherwise $\zeta > 0$ almost everywhere on Π, whereas ζ_0 vanishes outside a set of finite measure and $\zeta \in \mathscr{RC}(\zeta_0)$. It then follows from (12) that the support of ζ lies within the region $\kappa x_1^{1/(2p)} \leq c_6(I(\zeta) + \|\zeta\|_1 + \|\zeta\|_p)Z$. Thus ζ vanishes outside a bounded region in the case $\lambda = 0$ also.

We now return to (iii) and let ζ, φ and $\lambda \geq 0$ be as above. We can assume φ to be non-negative throughout $\mathbb{R}$ and to vanish on $(-\infty, 0]$, we define $\Phi(s) = \int_{-\infty}^s \varphi$ and we note that $\Phi(s) > 0$ if and only if $\varphi(s) > 0$, because φ is increasing. From [3, Lemma 9] we have

$$2\int_Q \Phi(\mathscr{G}\zeta(x) - \lambda x_2)\mathrm{d}x - \lambda\int_Q \zeta(x)x_2\mathrm{d}x = \int_{\partial Q} \Phi(\mathscr{G}\zeta(x) - \lambda x_2)(x \cdot \mathfrak{n})\mathrm{d}x$$

where $Q = [-R, R] \times [0, R]$ is a rectangle containing the support of ζ and $\mathfrak{n}$ is the outward unit normal. Since $\varphi \circ (\mathscr{G}\zeta - \lambda x_2)$ vanishes outside Q, so too does $\Phi \circ (\mathscr{G}\zeta - \lambda x_2)$ and we deduce that

$$2\int_\Pi \Phi(\mathscr{G}\zeta(x) - \lambda x_2)\mathrm{d}x = \lambda I(\zeta). \tag{13}$$

Since $\Phi(\mathscr{G}\zeta(x) - \lambda x_2) > 0$ almost always when $\zeta(x) > 0$ we deduce from (13) that $\lambda > 0$. □

The next result shows that the hypotheses of Theorems 1 and 2 below are satisfied in a wide range of situations.

Lemma 10. *Let $2 < p < \infty$, let $\zeta_0 \in L^1(\Pi) \cap L^p(\Pi)$ be non-negative and have compact support, let $i > 0$ and let Σ_0 be the set of maximisers of E relative to $\mathscr{W}(\zeta_0, \leq i)$. Then for all sufficiently large i we have $\Sigma_0 \subset \mathscr{R}(\zeta_0)$.*

Proof. We use the arguments from the corresponding part of the proof of [3, Theorem 16(ii)]. Consider $i > 0$, let Σ_i denote the set of maximisers of E relative to $\mathscr{W}(\zeta_0, \leq i)$ and let M_i denote the maximum value. Consider $\zeta \in \Sigma_i$.

Then, from Lemma 9, $\zeta \in \mathscr{RC}(\zeta_0)$ and there exist $\lambda > 0$ and an increasing function φ such that $\zeta(x) = \varphi(\mathscr{G}\zeta(x) - \lambda x_2)$ except for a set of x having measure zero. There must be a number

β such that $\varphi(s) > 0$ for $s > \beta$ and $\varphi(s) = 0$ for $s < \beta$. Moreover $\beta \geq 0$, for if $\beta < 0$ then $\mathscr{G}\zeta(x) - \lambda x_2 > \beta$ for almost all x satisfying $0 < x_2 < -\beta/\lambda$ whereas ζ vanishes outside a set of finite measure.

Now, from the definitions,

$$\int \zeta(x)(\mathscr{G}\zeta(x) - \lambda x_2)\mathrm{d}x = 2E(\zeta) - \lambda I(\zeta)$$

and, by [3, Lemma 9],

$$2E(\zeta) = \int \zeta\mathscr{G}\zeta \geq \frac{3}{2}\lambda I(\zeta) + \beta\|\zeta\|_1 \geq \frac{3}{2}\lambda I(\zeta)$$

so we obtain

$$\int \zeta(x)(\mathscr{G}\zeta(x) - \lambda x_2)\mathrm{d}x \geq \frac{2}{3}E(\zeta) = \frac{2}{3}M_i.$$

Since $\|\zeta\|_1 \leq \|\zeta_0\|_1$ we deduce

$$S := \sup\{\mathscr{G}\zeta(x) - \lambda x_2 \mid x \in \Pi\} \geq \frac{2M_i}{3\|\zeta_0\|_1}.$$

Let z be a point where $\mathscr{G}\zeta(x) - \lambda x_2$ achieves its supremum. Then, for x with $x_2 < z_2$ we may apply the mean value inequality along the line segment $[z, x]$ and use Lemma 3 to obtain

$$\mathscr{G}\zeta(x) - \lambda x_2 \geq \mathscr{G}\zeta(x) - \lambda z_2 \geq \mathscr{G}\zeta(z) - \lambda z_2 - c_5(\|\zeta_0\|_1 + \|\zeta_0\|_p)|x - z|$$

and this is positive provided $|x - z| < S/(c_5(\|\zeta_0\|_1 + \|\zeta_0\|_p))$, for which it is sufficient that $|z - x| \leq 2M_i/(3c_5\|\zeta_0\|_1(\|\zeta_0\|_1 + \|\zeta_0\|_p))$. Since $M_i \to \infty$ as $i \to \infty$ it follows that we can choose $i_1 > 0$ such that if $i > i_1$ then the area of the set $\{x \mid \mathscr{G}\zeta(x) - \lambda x_2 > 0\}$ is greater than the area of the set $\{x \mid \zeta(x) > 0\}$.

If $i > i_1$ we claim that $\zeta \in \mathscr{R}(\zeta_0)$. Suppose not; then the supports of ζ^Δ and ζ_0^Δ would be intervals $[0, s]$ and $[0, t]$ respectively with $s < t$. Then, for some $s < r < t$ the isomorphism construction described in Section 2.1 would yield a rearrangement η of $\zeta_0^\Delta 1_{[s,r]}$ on a subset of $\{x \mid \mathscr{G}\zeta(x) - \lambda x_2 > 0,\ \zeta(x) = 0\}$ and then we would have $\xi := \zeta + \eta \in \mathscr{R}\mathscr{C}(\zeta_0)$ and

$$\int \xi(x)(\mathscr{G}\zeta(x) - \lambda x_2) > \int \zeta(x)(\mathscr{G}\zeta(x) - \lambda x_2).$$

From this it would follow that $E(\xi) > E(\zeta)$. This would be impossible, so $\zeta \in \mathscr{R}(\zeta_0)$ as claimed. □

4 Proofs of the Compactness and Stability Theorems

4.1 Proof of Theorem 1

We note that Σ_0 is equal to the set of maximisers of E relative to $\mathscr{W}(\zeta_0, \leq i_0)$, from Lemma 9(i). We write

$$\beta = \int_\Pi \zeta_0 = \lim_{n\to\infty} \int_\Pi \zeta_n.$$

By concentration-compactness [13, Lemma I.1], we can replace (ζ_n) by a subsequence having one of the following properties:
Dichotomy: For each $n \in \mathbb{N}$ there is a partition of Π into measurable sets Ω_n^1, Ω_n^2 and Ω_n^3 in such a way that $\zeta_n^k = \zeta_n 1_{\Omega_n^k}$, $k = 1, 2, 3$, satisfy

$$\begin{aligned}\int_\Pi \zeta_n^1 &\to \alpha,\\ \int_\Pi \zeta_n^2 &\to \beta - \alpha,\\ \int_\Pi \zeta_n^3 &\to 0,\\ \mathrm{dist}(\Omega_n^1, \Omega_n^2) &\to 0\end{aligned}$$

as $n \to \infty$, where $0 < \alpha < \beta$.
Vanishing:

$$\forall R > 0 \quad \lim_{n\to\infty} \sup_{y\in\Pi} \int_{D_\Pi(y,R)} \zeta_n = 0;$$

Compactness: there exists a sequence (y^n) in $\overline{\Pi}$ such that

$$\forall \varepsilon > 0 \exists R > 0 \text{ s.t. } \forall n \in \mathbb{N} \int_{D_\Pi(y^n,R)} \zeta_n > \beta - \varepsilon.$$

We show that Dichotomy and Vanishing cannot occur and deduce the result from Compactness.

Excluding Dichotomy.
We have

$$\int_\Pi \zeta_n^3(\mathscr{G}\zeta_n) \to 0 \text{ as } n \to \infty$$

because $\mathscr{G}\zeta_n$ is uniformly bounded by Lemmas 1 and 3 and $\|\zeta_n^3\|_1 \to 0$. Now

$$E(\zeta_n) \geq E(\zeta_n^1 + \zeta_n^2) = E(\zeta_n - \zeta_n^3) = E(\zeta_n) + E(\zeta_n^3) - \int_\Pi (\mathscr{G}\zeta_n)\zeta_n^3 \geq E(\zeta_n) - \int_\Pi (\mathscr{G}\zeta_n)\zeta_n^3 \to M_0$$

and since $E(\zeta_n) \to M_0$ it follows that

$$E(\zeta_n^1 + \zeta_n^2) \to M_0. \tag{14}$$

We claim

$$\int_\Pi \zeta_n^1 \mathscr{G}\zeta_n^2 \to 0. \tag{15}$$

To prove (15) note firstly that, given $\varepsilon > 0$, we can by Lemma 1 choose $W > 0$ independent of n such that $\mathscr{G}\zeta_n^1(x) < \varepsilon$ and $\mathscr{G}\zeta_n^2(x) < \varepsilon$ if $x_2 > W$, thus

$$\begin{aligned}\int_{y_2>W} \zeta_n^2(y)\mathscr{G}\zeta_n^1(y)\mathrm{d}y &\leq \varepsilon\|\zeta_n\|_1,\\ \int_{x_2>W} \zeta_n^1(x)\mathscr{G}\zeta_n^2(x)\mathrm{d}x &\leq \varepsilon\|\zeta_n\|_1.\end{aligned}$$

The remaining term in (15) is

$$\int_{x_2<W}\int_{y_2<W} G(x,y)\zeta_n^1(x)\zeta_n^2(y)\mathrm{d}x\mathrm{d}y \le \frac{W^2}{\pi\,\mathrm{dist}(\Omega_n^1,\Omega_n^2)^2}\|\zeta_n\|_1^2 \to 0 \text{ as } n\to\infty.$$

Hence (15), which, together with (14), shows that

$$E(\zeta_n^1)+E(\zeta_n^2)\to M_0.$$

For $k=1,2$ let ζ_n^{k*} denote the Steiner symmetrisation of ζ_n^k about the x_2-axis, so $E(\zeta_n^{k*})\ge E(\zeta_n^k)$ and $I(\zeta_n^{k*})=I(\zeta_n^k)=i_n^k$ say. We can pass to a subsequence and suppose that $\zeta_n^{k*}\to\zeta^k$ say, weakly in L^p. Now E is continuous with respect to L^p weak convergence of Steiner-symmetric sequences when $\|\ \|_1$ and I are bounded, by Lemma 6, so $E(\zeta^1)+E(\zeta^2)\ge M_0$, whereas $I(\zeta^1)+I(\zeta^2)\le i_0$ by weak lower semicontinuity of I relative to non-negative functions in L^p. The decreasing rearrangements $\zeta_n^{1\Delta}$ and $\zeta_n^{2\Delta}$ are both dominated by ζ_n^Δ which converges in L^1 to ζ_0^Δ, so a variant of Helly's Selection Principle for monotonic functions (see [10]) shows that we can pass to a subsequence and suppose $\zeta_n^{1\Delta}$ and $\zeta_n^{2\nabla}$ converge pointwise, and then deduce that they converge strongly in L^1, to non-negative functions ξ^1 and ξ^2, say, dominated by ζ_0^Δ and ζ_0^∇ respectively, thus the ξ^k are supported on bounded intervals.

Since ζ_n^{k*} is a rearrangement of $\zeta_n^{k\Delta}$ we have $\zeta_n^{k*}\preceq\zeta_n^{k\Delta}$ and since the right-hand integral in (3) is strongly continuous in L^1 whereas the left-hand integral is weakly lower semicontinuous in L^p, we deduce $\zeta^k\preceq\xi^k$. On the other hand $\zeta_n^1+\zeta_n^2\preceq\zeta_0$ so $\zeta_n^{1\Delta}+\zeta_n^{2\nabla}\preceq\zeta_0$, since the left-hand integral of (3) is additive over two functions that are simultaneously positive almost nowhere, thus once more we can pass to the limit in (3) to obtain $\xi^1+\xi^2\preceq\zeta_0$.

Let $i^k=I(\zeta^k)$ for $k=1,2$ so $i^1+i^2\le i_0$ and let $\widetilde{\xi}^k$ be a maximiser for E relative to $\mathscr{W}(\xi^k,\le i^k)$, so that $E(\widetilde{\xi}^k)\ge E(\zeta^k)$. Lemma 9 shows that the $\widetilde{\xi}^k$ exist and have compact supports, say in a common rectangle $[-Q,Q]\times[0,Q]$. Then define $\widehat{\xi}^1(x_1,x_2):=\widetilde{\xi}^1(x_1+Q,x_2)$ and $\widehat{\xi}^2(x_1,x_2):=\widetilde{\xi}^2(x_1-Q,x_2)$, which are simultaneously positive almost nowhere and satisfy $\widehat{\xi}^k\preceq\xi^k$. Again the additivity of the integrals in (3) ensures that $\widehat{\xi}^1+\widehat{\xi}^2\preceq\xi^1+\xi^2\preceq\zeta_0$ so $\widehat{\xi}^1+\widehat{\xi}^2\in\mathscr{W}(\zeta_0,\le i_0)$. Now

$$E(\widehat{\xi}^1+\widehat{\xi}^2)=E(\widehat{\xi}^1)+E(\widehat{\xi}^2)+\int_\Pi\widehat{\xi}^1\mathscr{G}\widehat{\xi}^2\ge E(\widehat{\xi}^1)+E(\widehat{\xi}^2)\ge M_0 \tag{16}$$

proving that $\widehat{\xi}^1+\widehat{\xi}^2$ is a maximiser for E relative to $\mathscr{W}(\zeta_0,\le i_0)$. Since $\Sigma_0\subset\mathscr{R}(\zeta_0)$ by hypothesis, we now have $\widehat{\xi}^1+\widehat{\xi}^2\in\mathscr{R}(\zeta_0)$.

We have

$$\begin{aligned}\int_\Pi\widehat{\xi}^1\le\int_\Pi\xi^1 &\le \lim_{n\to\infty}\int_\Pi\zeta_n^1=\alpha,\\ \int_\Pi\widehat{\xi}^2\le\int_\Pi\xi^2 &\le \lim_{n\to\infty}\int_\Pi\zeta_n^2=\beta-\alpha.\end{aligned}$$

If $\widehat{\xi}^1=0$ or $\widehat{\xi}^2=0$ then

$$\int_\Pi\widehat{\xi}^1+\widehat{\xi}^2<\alpha+(\beta-\alpha)=\beta=\int_\Pi\zeta_0$$

contradicting $\widehat{\xi}^1+\widehat{\xi}^2\in\mathscr{R}(\zeta_0)$. Therefore $\widehat{\xi}^1$ and $\widehat{\xi}^2$ are both nonzero so the first inequality of (16) is strict, which is impossible. Thus Dichotomy does not occur.

Excluding Vanishing.
Let $\varepsilon > 0$ and choose by Lemma 1 $W > 0$ large enough that $\mathscr{G}\zeta_n(x_1, x_2) < \varepsilon$ for all n if $x_2 > W$. We write

$$\begin{aligned} \Pi_W &= \{(x_1, x_2) \mid 0 < x_2 < W\} \\ \Pi^W &= \{(x_1, x_2) \mid x_2 > W\} \end{aligned}$$

and deduce that

$$\int_\Pi \int_{\Pi^W} G(x, y)\zeta_n(x)\zeta_n(y)\mathrm{d}x\mathrm{d}y \le \varepsilon \|\zeta_n\|_1.$$

Then for $x \in \Pi_W$ and $R > 0$, writing $\rho = |x - y|$,

$$\begin{aligned} \int_{\Pi_W \setminus D(x,R)} G(x, y)\zeta_n(y)\mathrm{d}y &\le \int_{\Pi_W \setminus D(x,R)} \frac{1}{4\pi} \log\left(1 + \frac{4x_2y_2}{\rho^2}\right) \zeta_n(y)\mathrm{d}y \\ &\le \int_{\Pi_W \setminus D(x,R)} \frac{1}{4\pi} \log\left(1 + \frac{4x_2(x_2 + \rho)}{\rho^2}\right) \zeta_n(y)\mathrm{d}y \\ &\le \frac{1}{4\pi} \log\left(1 + \frac{4W(W + R)}{R^2}\right) \|\zeta_n\|_1 \le \varepsilon\|\zeta_n\|_1 \end{aligned}$$

provided we choose R suitably large, independently of n. Again for $x \in \Pi_W$ and $R > 0$ chosen as above we have

$$\begin{aligned} \int_{\Pi_W \cap D(x,R)} G(x, y)\zeta_n(y)\mathrm{d}y &\le \int_{\Pi_W \cap D(x,R)} \frac{1}{4\pi} \log\left(1 + \frac{4W(W + R)}{\rho^2}\right) \zeta_n(y)\mathrm{d}y \\ &\le \left(\int_{D(x,R)} \left(\log\left(1 + \frac{4W(W + R)}{\rho^2}\right)\right)^r\right)^{1/r} \|\zeta_n\|^\theta_{L^1(D_\Pi(x,R))}\|\zeta_n\|_p^{1-\theta}, \end{aligned}$$

where $1/r+1/s = 1$, $1 < s < p$ and $\theta+(1-\theta)/p = 1/s$, by Hölder's inequality and the interpolation inequality. Since $\|\zeta_n\|_{L^1(D_\Pi(x,R))} \to 0$ as $n \to \infty$ uniformly over $x \in \Pi$ by assumption of Vanishing and $\|\zeta_n\|_p$ is bounded we now have

$$\int_{\Pi_W \cap D(x,R)} G(x, y)\zeta_n(y)\mathrm{d}y < \varepsilon$$

for all sufficiently large n, uniformly over $x \in \Pi_W$.

Thus

$$\int_{\Pi_W} G(x, y)\zeta_n(y)\mathrm{d}y < \varepsilon\|\zeta_n\|_1 + \varepsilon \quad \text{for all } x \in \Pi_W$$

for all sufficiently large n and therefore

$$\int_{\Pi_W} \int_{\Pi_W} G(x, y)\zeta_n(y)\zeta_n(x)\mathrm{d}y\mathrm{d}x < \varepsilon(\|\zeta_n\|_1^2 + \|\zeta_n\|_1)$$

for all sufficiently large n. Now

$$\begin{aligned} E(\zeta_n) &= \left(\frac{1}{2}\int_{\Pi_W}\int_{\Pi_W} + \int_{\Pi_W}\int_{\Pi^W} + \frac{1}{2}\int_{\Pi^W}\int_{\Pi^W}\right) G(x, y)\zeta_n(x)\zeta_n(y)\mathrm{d}x\mathrm{d}y \\ &\le \left(\frac{1}{2}\int_{\Pi_W}\int_{\Pi_W} + \int_{\Pi}\int_{\Pi^W}\right) G(x, y)\zeta_n(x)\zeta_n(y)\mathrm{d}x\mathrm{d}y \le \varepsilon\left(\frac{1}{2}\|\zeta_n\|_1^2 + \frac{1}{2}\|\zeta_n\|_1 + \|\zeta_n\|_1\right) \end{aligned}$$

for all sufficiently large n, hence $E(\zeta_n) \to 0$ as $n \to \infty$. Since M_0 is positive, Vanishing cannot occur for a maximising sequence.

Exploiting Compactness.
If $y_2^n \to \infty$ as $n \to \infty$ then, for each fixed $R > 0$, we would have

$$\int_{D_\Pi(y^n,R)} \zeta_n \le (y_2^n - R)^{-1} I(\zeta_n) \to 0$$

as $n \to \infty$, contradicting the assumption of Compactness. Therefore, after passing to a further subsequence if necessary, we can suppose that $y_2^n < W$ for all n, where $W > 0$ is fixed.

The Compactness assumption ensures that

$$\sup_{n\in\mathbb{N}} \int_{\Pi\setminus D(y^n,R)} \zeta_n \to 0 \text{ as } R \to \infty. \tag{17}$$

We have $D((y_1^n, y_2^n), R) \subset D((y_1^n, 0), R + W)$ so there is no loss in supposing that $y_2^n = 0$ for all n. Since $\mathscr{G}\zeta_n$ is bounded in L^∞ uniformly over n we deduce

$$\sup_{n\in\mathbb{N}} \int_{\Pi\setminus D(y^n,R)} \zeta_n \mathscr{G}\zeta_n \to 0 \text{ as } R \to \infty.$$

We write $\overline{\zeta}_n(x_1, x_2) := \zeta_n(x_1 - y_1^n, x_2)$. It follows that

$$\sup_{n\in\mathbb{N}} \int_{\Pi\setminus D(0,R)} \int_\Pi G(x,y)\overline{\zeta}_n(x)\overline{\zeta}_n(y)\mathrm{d}x\mathrm{d}y \to 0 \text{ as } R \to \infty. \tag{18}$$

Now, for fixed $R > 0$, $\mathscr{G}$ followed by restriction to $D_\Pi(0, R)$ acts as a compact operator from $L^p(D_\Pi(0, R))$ to $L^q(D_\Pi(0, R))$, where $p^{-1} + q^{-1} = 1$, and we can further pass to a subsequence to ensure $\overline{\zeta}_n \to \overline{\zeta} \in \mathscr{W}(\zeta_0, \le i_0)$ say, weakly in $L^p(\Pi)$. Then, for each fixed $R > 0$,

$$\int_{D_\Pi(0,R)} \int_{D_\Pi(0,R)} G(x,y)\overline{\zeta}_n(x)\overline{\zeta}_n(y)\mathrm{d}x\mathrm{d}y \to \int_{D_\Pi(0,R)} \int_{D_\Pi(0,R)} G(x,y)\overline{\zeta}(x)\overline{\zeta}(y)\mathrm{d}x\mathrm{d}y \text{ as } n \to \infty. \tag{19}$$

From (18) and (19) we deduce

$$E(\overline{\zeta}_n) \to E(\overline{\zeta}).$$

Thus $\overline{\zeta}$ is a maximiser for E relative to $\mathscr{W}(\zeta_0, \le i_0)$. Therefore $\overline{\zeta} \in \mathscr{R}(\zeta_0)$ so

$$\|\overline{\zeta}\|_p = \|\zeta_0\|_p = \lim_{n\to\infty} \|\overline{\zeta}_n\|_p$$

hence by uniform convexity $\overline{\zeta}_n \to \overline{\zeta}$ strongly in $L^p(\Pi)$.

It follows that $\overline{\zeta}_n \to \overline{\zeta}$ strongly in $L^1(D_\Pi(0, R))$ for each $R > 0$. In view of (17) it now follows that $\overline{\zeta}_n \to \overline{\zeta}$ strongly in $L^1(\Pi)$. Since $\overline{\zeta}$ is a maximiser we have $I(\overline{\zeta}) = i_0$ so $I(\overline{\zeta}_n) \to I(\overline{\zeta})$. □

4.2 Proof of Theorem 2

As previously stated, we view vorticity ω as a function of time t taking values in $L^1(\Pi) \cap L^p(\Pi)$ and suppress the space variable x.

Suppose the result fails. Then there exists $\varepsilon > 0$ such that, for all sufficiently large $n \in \mathbb{N}$, we can choose a solution $\omega_n(\cdot)$ of the vorticity equation and a time t_n, such that $\mathrm{dist}_{\mathfrak{X}^p}(\omega_n(0), \Sigma_0) < 1/n$ but $\mathrm{dist}_{\mathfrak{X}^p}(\omega_n(t_n), \Sigma_0) \geq \varepsilon$.

Observe that, by the conservation properties of the vorticity equation

$$I(\omega_n(t_n)) = I(\omega_n(0)) \to i_0,$$

$$\mathrm{dist}_1(\omega_n(t_n), \mathscr{R}(\zeta_0)) = \mathrm{dist}_1(\omega_n(0), \mathscr{R}(\zeta_0)) \leq \mathrm{dist}_1(\omega_n(0), \Sigma_0) \to 0,$$
$$\mathrm{dist}_p(\omega_n(t_n), \mathscr{R}(\zeta_0)) = \mathrm{dist}_p(\omega_n(0), \mathscr{R}(\zeta_0)) \leq \mathrm{dist}_p(\omega_n(0), \Sigma_0) \to 0$$

as $n \to \infty$, hence using inequality (4),

$$\begin{aligned} \|\omega_n(t_n)^\Delta - \zeta_0^\Delta\|_p &\to 0, \\ \|\omega_n(t_n)^\Delta - \zeta_0^\Delta\|_1 &\to 0. \end{aligned}$$

Moreover, using Lemma 4 in addition,

$$E(\omega_n(t_n)) = E(\omega_n(0)) \to M_0$$

as $n \to \infty$. Theorem 1 now ensures that, after passing to a subsequence, we can choose an x_1 translation ξ_n of each $\omega_n(t_n)$ such that the sequence (ξ_n) converges to an element ξ_0 of Σ_0 in $\| \|_1 + \| \|_p$. Since x_1 translations preserve Σ_0 and $\| \|_{\mathfrak{X}^p}$ we have

$$\mathrm{dist}_{\mathfrak{X}^p}(\omega_n(t_n), \Sigma_0) = \mathrm{dist}_{\mathfrak{X}^p}(\xi_n, \Sigma_0) \leq \|\xi_n - \xi_0\|_{\mathfrak{X}^p} \to 0$$

and this contradicts the choice of the ω_n and t_n, completing the proof. □

Examples.

Given arbitrary compactly supported nontrivial non-negative ζ_0 in $L^p(\Pi)$ for some finite $p > 2$, the hypotheses of Theorem 2 are satisfied for all sufficiently large i_0, by Lemma 10.

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