Nonlinear stability for steady vortex pairs

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Abstract

In this article, we prove nonlinear orbital stability for steadily translating vortex pairs, a family of nonlinear waves that are exact solutions of the incompressible, two-dimensional Euler equations. We use an adaptation of Kelvin's variational principle, maximizing kinetic energy penalised by a multiple of momentum among mirror-symmetric isovortical rearrangements. This formulation has the advantage that the functional to be maximized and the constraint set are both invariant under the flow of the time-dependent Euler equations, and this observation is used strongly in the analysis. Previous work on existence yields a wide class of examples to which our result applies.

1 Introduction.

In this paper we prove stability, with respect to symmetric perturbations, of a class of steady symmetric vortex-pairs in a planar irrotational background flow of an ideal fluid that approaches a uniform stream at infinity. The vortex-pairs to which our result applies are maximizers of a functional, comprising kinetic energy penalised with a positive multiple of impulse (linear momentum), relative to an "isovortical surface", that is, the set of all flows whose (scalar) vorticity fields are equimeasurable rearrangements of a single function. This result can be seen as an extension of a result in [7] where it is shown that, for flows in a bounded, simply connected, planar domain, isolated maximizers and minimizers of the kinetic energy relative to an isovortical surface must be stable. That there is a wide class of examples to which our result is applicable follows from the existence theory of [5].

From a mathematical viewpoint, steady vortex pairs are a class of nonlinear waves, travelling wave solutions of the incompressible, two dimensional Euler equations in the full plane. Two special examples are Lamb's circular vortex-pair, see [12, p. 245], and a pair of point vortices with equal magnitude and opposite signs.

The literature of vortex pairs goes back to the work of Pocklington in [24], with contemporary interest beginning from the work of Norbury, Deem & Zabusky and Pierrehumbert, see [8, 21, 23]. The existence (and abundance) of steady vortex pairs has been rigorously established in two different ways, as a nonlinear eigenvalue problem, see [28, 21] or by optimization in rearrangement classes, see [5, 4]. The literature on vortex pairs includes asymptotic studies, see [29], numerical studies, see [25] and experimental work, see [11]. Some analytical results (see [20]) and numerical evidence, [22], suggest orbital stability of steady vortex pairs under appropriate conditions. Still, this stability has been an interesting open problem, see [26].

Vortex pairs are one instance of a large collection of coherent structures found in two dimensional vortex dynamics, for example, single vortices, co-rotating pairs and vortex streets. In the stability theory of such structures, there has long been a view, growing out of ideas of Kelvin [27], that steady fluid flows representing extrema of kinetic energy relative to an "isovortical surface" are stable; this viewpoint is exemplified by the formal arguments of Arnol′d [2] and informs the variational principles for steady vortex-rings in three dimensions proposed by Benjamin [3], which provides the impetus for our work. These ideas grow out of the observation that both the kinetic energy and the isovortical surface constraint set are time-invariant for unsteady flows, as also is impulse, and the speculation that extrema of a variational problem formulated entirely in terms of conserved physical quantities should be stable. The present paper proves the first stability result for vortex-pairs following this approach.

Vortex pairs can be viewed equivalently as the dynamics of vorticity which is odd with respect to a straight line or as general vortex dynamics on a half plane, see [16] for a full discussion. For convenience, we formulate our analysis in terms of steady vortices in a uniform flow in the half-plane, which corresponds in the full plane to stability under symmetric perturbations.

By contrast with the bounded-domain case, in the half-plane the kinetic energy is unbounded above; the introduction of impulse into the functional as a penalty term is to overcome this difficulty. Another difference from the earlier work is the lack of isolated maximizers, for the functional and the constraint set are both invariant under translations parallel to the axis. Therefore we work with the notion of *orbital stability*, meaning that if a flow starts close to a maximizer then it remains close to the set of maximizers; it is unclear whether or not it remains close to any single one.

A particularly interesting example is Lamb's vortex mentioned above, which is an explicit solution having a circular vortex-core. It was shown in [6] that Lamb's vortex and its translations parallel to the axis are precisely the solutions to our maximisation problem for the appropriate data. It is unfortunate that, as will be seen, it does not satisfy the hypotheses of our stability theorem.

The argument is not along the lines envisaged by Arnol′d, but is analogous to that used in [7] for bounded planar domains; the velocity field of a flow with nearby initial vorticity is used to convect the steady state and the differences in energy are estimated.

The vorticity is assumed to be in $L^p \cap L^1$ for some $p > 2$ and a distance between vorticity fields is defined in terms of the 2-norm and the impulse. [1] These results allow discontinuous vorticity, and the solutions studied are known not in closed form, but rather via existence theory. Some stability results in a more symmetric setting, also allowing discontinuous vorticity, have been given by Marchioro & Pulvirenti [18] and Wan & Pulvirenti [30]. Precise definitions and formulations of the theorems are given in Section 2.

Methodologically, a major difficulty is loss of compactness caused by the unbounded domain of the flow. In the existence theory of [5] this was overcome by using Steiner symmetrization to improve the compactness properties of maximizing sequences. However to show stability we have to prove compactness, up to translation, of *all* maximizing sequences, which we achieve by a concentration-compactness argument.

2 Notation and Definitions.

We denote by Π the half-plane

$$\Pi = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}.$$

Let $\mathcal{G}$ denote the inverse for $-\Delta$ in Π, given by

$$\mathcal{G}\xi(x) = \int_\Pi G(x, y)\xi(y)\mathrm{d}y, \tag{1}$$

whenever this integral converges; here G is the Green's function given by

$$G(x, y) = \frac{1}{4\pi} \log\left(\frac{(x_1 - y_1)^2 + (x_2 + y_2)^2}{(x_1 - y_1)^2 + (x_2 - y_2)^2}\right).$$

[1] In [7] it should have been assumed that that the vorticity belonged to L^p where $p \geq 2$, not $p > 4/3$ as stated, since $p \geq 2$ was tacitly assumed in the proof of Lemma 12 on which the main results depended.

It is shown in [6] that finiteness of $\|\xi\|_X := \|\xi\|_2 + I(|\xi|)$ is sufficient for convergence of the integral in (1), where I is defined below.

The kinetic energy due to vorticity ξ is then given by

$$E(\xi) = \frac{1}{2}\int_\Pi \xi(x)\mathcal{G}\xi(x)\mathrm{d}x$$

and the impulse of linear momentum in the x_1-direction is given by

$$I(\xi) = \int_\Pi \xi(x_1, x_2)x_2\mathrm{d}x.$$

It is shown in [6] that E is continuous with respect to $\|\cdot\|_X$. We also make use of $\|\xi\|_Y := \|\xi\|_2 + |I(\xi)|$, which is a non-equivalent, and incomplete, norm on X. The Lebesgue measure, of appropriate dimension, of a measurable set Ω is denoted $|\Omega|$.

The evolution of vorticity ω is governed by the weak form of the vorticity equation

$$\begin{cases} \partial_t\omega + \mathrm{div}(\omega u) = 0, \\ u = \lambda e_1 + \nabla^\perp \mathcal{G}\omega, \qquad (x,t) \in \Pi \times \mathbb{R}, \end{cases} \tag{2}$$

where λe_1 represents the velocity of the fluid at infinity, which is a uniform flow parallel to the x_1-axis and $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$; the stream function is then $-\lambda x_2 + \mathcal{G}\omega(x)$.

If ξ is a non-negative Lebesgue integrable function on Π, then $\mathcal{R}(\xi)$, the set of (equimeasurable) *rearrangements* of ξ on Π, is defined by

$$\mathcal{R}(\xi) = \left\{0 \le \zeta \in L^1(\Pi) \text{ s.t. } \forall\alpha > 0 \ |\{x : \zeta(x) > \alpha\}| = |\{x : \xi(x) > \alpha\}| \right\}.$$

The set $\mathcal{R}(\xi)$ is strongly closed but not weakly closed in $L^2(\Pi)$; even on a bounded domain oscillations in a sequence of rearrangements can result in weak convergence to a function that is not a rearrangement, and when the domain is unbounded a sequence can escape to infinity in the plane and so converge weakly to 0. This phenomenon was studied by Douglas [10] and we now describe some of his work for later use. Define

$$\mathcal{R}_+(\xi) = \{\zeta 1_\Omega \mid \zeta \in \mathcal{R}(\xi),\ \Omega \subset \Pi \text{ measurable } \}$$

and define the class $\mathcal{RC}(\xi)$ of *curtailments of rearrangements* by

$$\mathcal{RC}(\xi) = \left\{0 \le \eta \in L^1(\Pi) \mid \eta^\Delta = \xi^\Delta 1_{[0,A)} \text{ for some } 0 \le A \le \infty\right\},$$

where $^\Delta$ denotes decreasing rearrangement onto $[0,\infty)$. Then, from the definitions,

$$\mathcal{R}(\xi) \subset \mathcal{RC}(\xi) \subset \mathcal{R}_+(\xi) \subset \overline{\mathcal{R}(\xi)}^{\mathrm{w}}, \tag{3}$$

where $\overline{\mathcal{R}(\xi)}^{\mathrm{w}}$ denotes the closure of $\mathcal{R}(\xi)$ in the weak topology of $L^2(\Pi)$, this last inclusion requiring additionally $\xi \in L^2(\Pi)$. Douglas [10] showed that $\overline{\mathcal{R}(\xi)}^{\mathrm{w}}$ is convex, that $\mathcal{RC}(\xi)$ is the set of extreme points of $\overline{\mathcal{R}(\xi)}^{\mathrm{w}}$ and that

$$\overline{\mathcal{R}(\xi)}^{\mathrm{w}} = \left\{ \zeta \geq 0 \text{ measurable} \mid \forall \alpha > 0 \int_\Pi (\zeta - \alpha)_+ \leq \int_\Pi (\xi - \alpha)_+ \right\}.$$

From this it follows that, if $\zeta \in \overline{\mathcal{R}(\xi)}^{\mathrm{w}}$ and if $U \subset \Pi$ is any measurable set, then $1_U \zeta \in \overline{\mathcal{R}(\xi)}^{\mathrm{w}}$.

For example, in the case of a vortex patch, i.e. $\xi = 1_\Omega$, where Ω is a bounded measurable subset of the half-plane, we have $\mathcal{R}(\xi)$ is the set of all characteristic functions of sets with the same measure as Ω, $\mathcal{RC}(\xi)$ is the set of characteristic functions of sets with measure less than or equal to the measure of Ω, which is the same as $\mathcal{R}_+(\xi)$. The set $\overline{\mathcal{R}(\xi)}^{\mathrm{w}}$ is much larger, a convex set containing, in particular, functions bounded by 1 which are not piecewise constant.

The *(strong) support* $\operatorname{suppt} f$ of a real measurable function f on Π is defined to be the set of points of Lebesgue density 1 for the set $\{x \in \Pi \mid f(x) \neq 0\}$ and is independent of the choice of representative for f.

Our stability results are expressed in terms of *L^p-regular solutions* of the vorticity equation, defined below.

Definition 1. *By an L^p-*regular solution *of the vorticity equation we mean $\zeta \in L^\infty_{\mathrm{loc}}([0,\infty), L^1(\Pi)) \cap L^\infty_{\mathrm{loc}}([0,\infty), L^p(\Pi))$ satisfying, in the sense of distributions,*

$$\begin{cases} \partial_t \zeta + \operatorname{div}(\zeta u) = 0, \\ u = \lambda e_1 + \nabla^\perp \mathcal{G}\zeta, \qquad (x,t) \in \Pi \times \mathbb{R}, \end{cases} \tag{4}$$

such that $E(\zeta(t,\cdot))$ and $I(\zeta(t,\cdot))$ are constant.

Existence of a smooth solution of the initial-boundary-value problem for (4) can be obtained by considering the auxiliary problem

$$\begin{cases} \partial_t v + (v \cdot \nabla) v + \lambda \partial_{x_1} v = -\nabla p, \\ \operatorname{div} v = 0 \\ |v| \to 0 \qquad\qquad \text{as } |x| \to \infty, (x,t) \in \Pi \times \mathbb{R} \end{cases} \tag{5}$$

and taking $\zeta = \operatorname{curl} v$. Indeed, taking the curl of (5) leads to (4) with $u = v + \lambda e_1$.

Now, existence of a smooth solution for (5), when the initial vorticity is compactly supported, is a trivial adaptation of the analogous result for the 2D incompressible Euler equations, see [17, Chapter 3], given that the L^2-norm of v is a conserved quantity under evolution by (5). Once smooth existence has been established, standard weak convergence methods yield existence of weak L^p solutions, see [15, Theorem 2.1], again assuming the initial vorticity has compact support. The only remaining issue, to obtain an L^p-regular solution for compactly supported initial vorticities, is whether E and I are conserved for these weak solutions; this will be the case, easily, if $p > 2$ since, in this case, v is bounded *a priori* in L^r, $2 \leq r \leq \infty$, in terms of L^p and L^1-norms of vorticity. We note, in particular, that L^∞-regular solutions with compactly supported vorticity are unique, by an easy adaptation of the celebrated work of Yudovich, see [31]. Moreover, these L^∞-regular solutions are constant along particle paths associated with the flow u. Our results do not, however, rely on uniqueness.

As we discuss in more detail later, maximizers of $E - \lambda I$ relative to $\mathcal{R}(\zeta_0)$ were shown [5] to exist when $\lambda > 0$ is small, and the first variation condition at a maximum was used to prove that these represented steady flows. Our main result concerns orbital stability of the maximizers and is stated as follows:

Theorem 1. (Stability Theorem.) *Let ζ_0 be a non-negative function whose support has finite positive area πa^2 $(a > 0)$ in the half-plane Π. Suppose $\zeta_0 \in L^p(\Pi)$, for some $2 < p \leq \infty$, and suppose $\lambda > 0$. Let Σ_λ denote the set of maximizers of $E - \lambda I$ relative to $\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$, and suppose $\emptyset \neq \Sigma_\lambda \subset \mathcal{R}(\zeta_0)$. Then Σ_λ is orbitally stable, in the sense that, for every $\varepsilon > 0$ and $A > \pi a^2$, there exists $\delta > 0$ such that, if $\omega(0) \geq 0$ is compactly supported and satisfies $dist_Y(\omega(0), \Sigma_\lambda) < \delta$ and $|suppt(\omega(0))| < A$, then, for all $t \in \mathbb{R}$, we have $dist_2(\omega(t), \Sigma_\lambda) < \varepsilon$, whenever $\omega(t)$ denotes an L^p-regular solution of (2) with initial data $\omega(0)$.*

Theorem 1 is an analogue, for unbounded domains, of [7, Theorem 1], and is deduced, by a similar argument, from the following result:

Theorem 2. (Maximization Theorem.) *Let non-negative $\zeta_0 \in L^p(\Pi)$, for some $2 < p \leq \infty$ and suppose $|suppt(\zeta_0)| = \pi a^2$ for some $0 < a < \infty$. Let $0 < \lambda < \infty$ and suppose that the set Σ_λ in which $E - \lambda I$ attains its supremum S_λ relative to $\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$ satisfies $\Sigma_\lambda \subset \mathcal{R}(\zeta_0)$. Then, in the context of maximizing $E - \lambda I$ relative to $\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$, we have:*

(i) *every maximizing sequence comprising elements of* $\mathcal{R}_+(\zeta_0)$ *contains a subsequence whose elements, after suitable translations in the* x_1*-direction, converge in* $\|\cdot\|_2$ *to an element of* Σ_λ*;*

(ii) *every maximizing sequence* $\{\zeta_n\}_{n=1}^\infty$ *comprising elements of* $\mathcal{R}_+(\zeta_0)$ *satisfies* $dist_2(\zeta_n, \Sigma_\lambda) \to 0$ *as* $n \to \infty$*;*

(iii) Σ_λ *is non-empty;*

(iv) *each element* ζ *of* Σ_λ *is a translate of a function that is even in* x_1*, decreasing in* $|x_1|$*, is compactly supported and satisfies* $\zeta = \varphi \circ (\mathcal{G}\zeta - \lambda x_2)$ *a.e. in* Π *for some increasing function* φ*.*

Remarks. The hypotheses of Theorem 1 exclude 0 as a maximizer, and therefore the supremum is positive.

We also show in Lemma 10 that given non-negative $\zeta_0 \in L^p(\Pi)$, $p > 2$, having compact support, there exists $\Lambda > 0$ such that if $0 < \lambda < \Lambda$ then $\emptyset \neq \Sigma_\lambda \subset \mathcal{R}(\zeta_0)$, so that the hypotheses of Theorem 1 are satisfied. Lemma 10 incidentally provides a mild improvement on the existence result [5, Theorem 16(i)].

It also follows from Theorem 2(i) that Σ_λ comprises a compact set of functions in $L^2(\Pi)$ together with their x_1-translations.

Theorem 2(iv) proves that the maximizers have compact support and therefore fit into the context of [5, Theorem 16(i)], which yields the conclusion, repeated above, that $\psi := \mathcal{G}\zeta - \lambda x_2$ satisfies an equation

$$-\Delta\psi = \varphi \circ \psi \text{ in } \Pi$$

which is the classical equation governing stream functions of steady planar ideal fluid flows, so that elements of Σ_λ do indeed represent steady vortices of finite extent.

It has been noted above that Theorem 1 applies to a wide class of examples and that this class does not include Lamb's circular vortex. This is because 0 is a maximizer, relative to the weak closure of the rearrangements, of the relevant variational problem; see [6].

3 Concentration-compactness and Theorem 2.

Here we present a sequence of Lemmas leading to the proof of Theorem 2. The first is a slight reformulation of Lions [14, Lemma 1.1], and we omit the proof:

Lemma 1. (Concentration-Compactness.) *Let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of non-negative elements of $L^1(\mathbb{R}^N)$ and suppose*

$$\limsup_{n\to\infty}\int_{R^N}\xi_n=\mu$$

where $0\le\mu<\infty$. Then, after passing to a subsequence, one of the following holds:
(i) (Compactness) *There exists a sequence $\{y_n\}_{n=1}^{\infty}$ in $\mathbb{R}^N$ such that $\forall\varepsilon>0$ $\exists R>0$ such that*

$$\forall n\quad\int_{y_n+B_R}\xi_n\ge\mu-\varepsilon\ ;$$

(ii) (Vanishing)

$$\forall R>0\quad\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{y+B_R}\xi_n=0\ ;$$

(iii) (Dichotomy) *There exists α, $0<\alpha<\mu$, such that for all $\varepsilon>0$ and all large n, there exist $\xi_n^{(1)}=1_{\Omega_n^{(1)}}\xi_n$ and $\xi_n^{(2)}=1_{\Omega_n^{(2)}}\xi_n$, for some disjoint measurable $\Omega_n^{(1)},\Omega_n^{(2)}\subset\mathbb{R}^N$, such that, for all n,*

$$0\le\int_{R^N}\xi_n-(\xi_n^{(1)}+\xi_n^{(2)})<\varepsilon$$

$$-\varepsilon<\int_{R^N}\xi_n^{(1)}-\alpha<\varepsilon$$

$$-\varepsilon<\int_{R^N}\xi_n^{(2)}-(\mu-\alpha)<\varepsilon$$

$$dist(\Omega_n^{(1)},\Omega_n^{(2)})\to\infty\ \text{as}\ n\to\infty\ .$$

Remarks. Notice that if $\mu=0$ then the whole sequence has the Vanishing Property.

We will apply this result to maximizing sequences of $E-\lambda I$ in $\mathcal{R}_+(\xi)$, for suitable ξ and λ. In this connection, it should be noted that if $\xi\in L^2(\Pi)$ is non-negative and has compact support then, for sequences in $\mathcal{R}_+(\xi)$, it follows from Hölder's inequality and equimeasurability that convergence in $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

The following alternative form of the Green's function will be useful:

$$G(x,y)=\frac{1}{4\pi}\log\left(1+\frac{4x_2y_2}{(x_1-y_1)^2+(x_2-y_2)^2}\right). \tag{6}$$

We also make use of *Steiner symmetrization* in the x_2 axis; if $0\le\xi\in L^p(\Pi)$ for some $1\le p<\infty$ then there is a measurable function $\xi^*\in L^p(\Pi)$ such that for almost every $x_2>0$, the function $\xi^*(\cdot,x_2)$ is even, decreasing on the positive half-line and is a rearrangement of $\xi(\cdot,x_2)$ with respect to 1-dimensional Lebesgue measure. If, additionally, $E(\xi)<\infty$, then $E(\xi^*)\ge E(\xi)$ follows from (6) and the Riesz rearrangement inequality.

The following estimates are derived in Burton [5, Lemmas 1 & 5]:

Lemma 2. *Given $A > 0$, we can choose positive numbers b, c, d, e, and $0 < \beta < 1$, depending only on A, such that if $\xi \in L^2(\Pi)$ satisfies $|suppt(\xi)| \leq A$ then we have*
(i) $|\mathcal{G}\xi(x_1, x_2)| \leq \|\xi\|_2(b + c\log x_2), \quad x_2 > e$;
(ii) $|\mathcal{G}\xi(x_1, x_2)| \leq d|x_2|^\beta \|\xi\|_2, \quad 0 < x_2 < e$.

Lemma 3. (i) *Given $A > 0$ and $2 < p < \infty$ we can choose a positive number N such that $|\nabla\mathcal{G}\xi(x_1, x_2)| \leq N\|\xi\|_p$ for all $\xi \in L^p(\Pi)$ vanishing outside a set of area A.*
(ii) *Given $A > 0$, $Z > 0$ and $2 < p < \infty$, we can choose a positive number f such that if $\xi \in L^p(\Pi)$ is Steiner-symmetric about the x_2-axis, ξ satisfies $|suppt(\xi)| \leq A$ and $\xi(x_1, x_2) = 0$ for $x_2 > Z$ then we have*

$$|\mathcal{G}\xi(x_1, x_2)| \leq f\|\xi\|_p x_2 \min\{1, |x_1|^{-1/2p}\}.$$

The following Lemma shows that $E(\zeta) < \infty$ provided that $\|\zeta\|_1$, $\|\zeta\|_2$ and $I(|\zeta|)$ are all finite. If $I(\zeta) = \infty$ we adopt the convention $E(\zeta) - \lambda I(\zeta) = -\infty$ for $\lambda > 0$.

Lemma 4. *There is a constant $C > 0$ such that*

$$\|\mathcal{G}\zeta\|_\infty \leq C(\|\zeta\|_1 + \|\zeta\|_2 + I(|\zeta|))$$

for all measurable functions ζ, provided that the right-hand side is finite.

Proof. It is enough to consider the case $\zeta \geq 0$. We note the inequality

$$\begin{aligned}\log(a + b + c) &\leq \log(3\max\{a, b, c\}) \\ &\leq \log 3 + (\log a)_+ + (\log b)_+ + (\log c)_+\end{aligned}$$

for positive a, b, c, and write $\rho := |x - y|$ in the formula (6) for G to obtain

$$\int_{y_2 \geq x_2/2} \log\left(1 + \frac{4x_2y_2}{\rho^2}\right)\zeta(y)\mathrm{d}y \leq \int_\Pi \log\left(1 + \frac{8y_2^2}{\rho^2}\right)\zeta(y)\mathrm{d}y.$$

Now

$$\int_\Pi \left(\log(8y_2^2)\right)_+ \zeta(y)\mathrm{d}y \leq \int_\Pi (\log 8 + 2y_2)\zeta(y)\mathrm{d}y = (\log 8)\|\zeta\|_1 + 2I(\zeta),$$

and

$$\begin{aligned}\int_\Pi \left(\log\rho^{-2}\right)_+ \zeta(y)\mathrm{d}y &= \int_{\rho\leq 1}(-2\log\rho)\zeta(y)\mathrm{d}y \\ &\leq \left(\int_{\rho\leq 1} 4(\log\rho)^2\mathrm{d}y\right)^{1/2}\|\zeta\|_2,\end{aligned}$$

hence

$$\int_{y_2 \geq x_2/2} \log\left(1+\frac{4x_2y_2}{\rho^2}\right)\zeta(y)\mathrm{d}y \leq \text{const.}(\|\zeta\|_1 + I(\zeta) + \|\zeta\|_2).$$

Also

$$\begin{aligned}\int_{y_2 \leq x_2/2} \log\left(1+\frac{4x_2y_2}{\rho^2}\right)\zeta(y)\mathrm{d}y &\leq \int_\Pi \log\left(1+\frac{2x_2^2}{x_2^2/4}\right)\zeta(y)\mathrm{d}y \\ &= (\log 9)\|\zeta\|_1,\end{aligned}$$

and the desired inequality follows. □

Lemma 5. *Given positive numbers M_1, M_2, M_3, the functional E is Lipschitz continuous in $\|\cdot\|_2$ relative to*

$$V := \{\zeta \in L^2(\Pi) \mid |\mathit{suppt}(\zeta)| \leq M_1,\ I(|\zeta|) \leq M_2,\ \|\zeta\|_2 \leq M_3\}.$$

Proof. For $\xi, \eta \in V$ we have

$$\begin{aligned}|E(\xi) - E(\eta)| &= \frac{1}{2}\int_\Pi (\xi+\eta)\mathcal{G}(\xi-\eta) \\ &\leq \|\xi-\eta\|_1\|\mathcal{G}(\xi+\eta)\|_\infty \\ &\leq (2M_1)^{1/2}\|\xi-\eta\|_2 C(\|\xi+\eta\|_1 + \|\xi+\eta\|_2 + I(|\xi+\eta|)) \\ &\leq (2M_1)^{1/2}\|\xi-\eta\|_2 C(2M_1^{1/2}M_3 + 2M_3 + 2M_2),\end{aligned}$$

where C is the constant provided by Lemma 4. □

Lemma 6. *Let $\zeta_0 \in L^2$ be non-negative, suppose $|\mathit{suppt}(\zeta_0)| = \pi a^2$ for some $0 < a < \infty$, and let $\lambda > 0$. Then*
(i) *there exists $Z > 0$ (depending on a, λ and $\|\zeta_0\|_2$ only) such that, for all $\zeta \in \mathcal{R}_+(\zeta_0)$,*

$$\mathcal{G}\zeta(x_1, x_2) - \lambda x_2 < 0 \quad \forall x_2 > Z;$$

(ii) *if $\zeta \in \mathcal{R}_+(\zeta_0)$ and $h = \zeta 1_U$ where U is a set on which $\mathcal{G}\zeta - \lambda x_2$ is nowhere positive, then*

$$(E - \lambda I)(\zeta - h) \geq (E - \lambda I)(\zeta)$$

with strict inequality unless $h = 0$, and in particular we can take $U = \mathbb{R} \times (Z, \infty)$;
(iii) *any maximizer of $E - \lambda I$ relative to $\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$ is supported in $\mathbb{R} \times [0, Z]$;*
(iv) *if $\zeta \in \overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$ with $\|\zeta\|_X < \infty$, and $h = \zeta 1_{\mathbb{R}\times(Z,\infty)}$, then there is a rearrangement h' of h supported in $\mathbb{R} \times [0, Z] \backslash \mathit{suppt}(\zeta)$ such that*

$$(E - \lambda I)(\zeta - h + h') \geq (E - \lambda I)(\zeta);$$

moreover $\zeta - h + h'$ is a rearrangement of ζ.

Proof. (i) follows easily from the estimate of Lemma 2.

For (ii), observe that

$$\begin{aligned}(E-\lambda I)(\zeta-h) &= (E-\lambda I)(\zeta)-\int_\Pi(\mathcal{G}\zeta-\lambda x_2)h+E(h)\\ &= (E-\lambda I)(\zeta)-\int_U(\mathcal{G}\zeta-\lambda x_2)h+E(h)\\ &\geq (E-\lambda I)(\zeta)+E(h),\end{aligned}$$

since on U we have $(\mathcal{G}\zeta-\lambda x_2)\leq 0$ and $h\geq 0$. The result follows since $E(h)>0$ unless $h=0$.

(iii) now follows from (ii), since if $\zeta\in\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$ then $\zeta-h\in\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$ also, using results of Douglas [10].

(iv) is trivial if $h=0$. Suppose therefore that $h\neq 0$. Let

$$\varepsilon=(E-\lambda I)(\zeta-h)-(E-\lambda I)(\zeta)$$

which is positive by (ii). In view of the decay of $\mathcal{G}(\zeta-h)$ at the x_2-axis quantified in Lemma 2(ii) it is enough to form h' by rearranging h on the part of a narrow strip along the x_2-axis outside suppt(ζ); this is justified since any two sets of equal finite positive Lebesgue measure are measure-theoretically equivalent. □

Lemma 7. *Let $\zeta_0\in L^2(\Pi)$ be a non-negative function with support of finite area, and let $\lambda>0$. Then $E-\lambda I$ has the same supremum on all of the sets $\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$, $\mathcal{R}_+(\zeta_0)$, $\mathcal{RC}(\zeta_0)$, and $\mathcal{R}(\zeta_0)$.*

Proof. In view of the inclusions (3) it will be enough to prove equality of the suprema on the first and last sets in the list. Let $\xi\in\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$. Then $\xi':=\xi 1_{\mathbb{R}\times(0,Z)}\in\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$ and, by Lemma 6(ii),

$$(E-\lambda I)(\xi')\geq(E-\lambda I)(\xi).$$

By the monotone convergence theorem, given $\varepsilon>0$ we can choose $R>0$ such that $\xi'':=\xi'1_{(-R,R)\times\mathbb{R}}=\xi 1_{(-R,R)\times(0,Z)}$, which also belongs to $\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$, satisfies

$$(E-\lambda I)(\xi'')>(E-\lambda I)(\xi)-\varepsilon.$$

Now, by compactness of $\mathcal{G}$ as an operator on $L^2((-R,R)\times(0,Z))$, within every weak neighbourhood of ξ'' we can find $\xi'''\in\mathcal{R}_+(\zeta_0)$ supported in $(-R,R)\times(0,Z)$ with

$$-\varepsilon<(E-\lambda I)(\xi''')-(E-\lambda I)(\xi'')<\varepsilon.$$

Finally, if $\xi''' \notin \mathcal{R}(\zeta_0)$, given $\delta > 0$ sufficiently small, we may choose any rearrangement η_δ of $\zeta_0^\Delta - \xi^\Delta$ on a subset of $(1/\delta, \pi a^2/\delta) \times (0,\delta)$ to find that $\xi_\delta := \xi''' + \eta_\delta$ is a rearrangement of ζ_0 supported in a bounded subset of $\mathbb{R} \times (0, Z)$, and that as $\delta \to 0$ we have $\xi_\delta \to \xi'''$ weakly in L^2 and $(E - \lambda I)(\xi_\delta) \to (E - \lambda I)(\xi''')$. □

Lemma 8. (Vanishing excluded.) *Suppose that ζ_0, a, λ and Σ_λ satisfy the hypotheses of the Maximization Theorem 2. Then no maximizing sequence for $E - \lambda I$ relative to $\mathcal{R}_+(\zeta_0)$ has the Vanishing Property of Lemma 1.*

Proof. Suppose $\{\zeta_n\}_{n=1}^\infty$ is a maximizing sequence for $E - \lambda I$ relative to $\mathcal{R}_+(\zeta_0)$ that has the Vanishing Property, reformulated by the equivalence of $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathcal{R}_+(\zeta_0)$ as

$$\forall R > 0 \qquad \lim_{n\to\infty} \sup_{y \in R^N} \int_{y + B_R} \zeta_n^2 = 0.$$

By Lemma 6(ii), we can modify the ζ_n so they are supported in $\mathbb{R} \times [0, Z]$, while remaining a maximizing sequence with the Vanishing Property. Consider any $R > 0$. Then for $x \in \Pi$, relative to $B := x + B_R$, we have

$$\left.\begin{array}{l} \|\zeta_n\|_{L^2(B)} \to 0 \\ \|\zeta_n\|_{L^1(B)} \leq \text{const.}\|\zeta_n\|_{L^2(B)} \to 0 \text{ (by Hölder's inequality)} \\ I(\zeta_n 1_B) \leq Z\|\zeta_n\|_{L^1(B)} \to 0 \end{array}\right\}$$

uniformly over $x \in \Pi$, so $\mathcal{G}(\zeta_n 1_B)(x) \to 0$ as $n \to \infty$ uniformly over $x \in \Pi$, by Lemma 4. By writing the Green's function in the form (6) we estimate

$$\mathcal{G}\left(\zeta_n(1 - 1_B)\right)(x) \leq \frac{Z^2}{\pi R^2}\|\zeta_n\|_1 \leq \frac{Z^2}{\pi R^2}\|\zeta_0\|_1.$$

Hence, as $n \to \infty$,

$$E(\zeta_n) \leq \|\zeta_n\|_1 \left(\frac{Z^2}{\pi R^2}\|\zeta_0\|_1 + o(1)\right) \leq \|\zeta_0\|_1 \left(\frac{Z^2}{\pi R^2}\|\zeta_0\|_1 + o(1)\right).$$

This holds for every $R > 0$, hence $E(\zeta_n) \to 0$ as $n \to \infty$. Hence

$$\limsup_{n\to\infty} (E - \lambda I)(\zeta_n) \leq 0.$$

But the hypotheses of the Lemma, together with Lemma 7 ensure that the supremum of $E - \lambda I$ relative to $\mathcal{R}_+(\zeta_0)$ is positive. Thus Vanishing does not occur. □

Lemma 9. (Dichotomy excluded.) *Suppose that ζ_0, a, λ, Σ_λ and S_λ satisfy the hypotheses of the Maximization Theorem 2. Then no maximizing sequence for $E - \lambda I$ relative to $\mathcal{R}_+(\zeta_0)$ has the Dichotomy Property of Lemma 1.*

Proof. Suppose $\{\zeta_n\}_{n=1}^\infty$ is a maximizing sequence for $E - \lambda I$ relative to $\mathcal{R}_+(\zeta_0)$ that has the Dichotomy Property. In view of the remarks on convergence following Lemma 1 we can assume that, for some $0 < \alpha < \mu$, and some restrictions $\{\zeta_n^{(i)}\}_{i=1}^3$ of ζ_0 to sets $\{\Omega_n^{(i)}\}_{n=1}^3$ partitioning Π, we have

$$\left.\begin{array}{c} \|\zeta_n^{(3)}\|_2 \to 0, \\ \|\zeta_n^{(1)}\|_2^2 \to \alpha, \\ \|\zeta_n^{(2)}\|_2^2 \to \beta := \mu - \alpha, \\ \mathrm{dist}(\mathrm{suppt}(\zeta_n^{(1)}), \mathrm{suppt}(\zeta_n^{(2)})) \to \infty, \\ (E - \lambda I)(\zeta_n^{(1)} + \zeta_n^{(2)} + \zeta_n^{(3)}) \to S_\lambda, \end{array}\right\} \tag{7}$$

as $n \to \infty$.

We may multiply the functions $\{\zeta_n^{(i)}\}_{i=1}^3$ by $1_{\mathbb{R}\times(0,Z)}$, where Z is the number provided by Lemma 6(i), yet still assume the last two lines of (7) to hold, in view of Lemma 6(ii). We also have

$$\begin{aligned} (E - \lambda I)(\zeta_n^{(1)} + \zeta_n^{(2)}) &= (E - \lambda I)(\zeta_n) - \lambda I(\zeta_n^{(3)}) \\ &\quad - \int_\Pi \zeta_n^{(3)} \mathcal{G}(\zeta_n^{(1)} + \zeta_n^{(2)} + \tfrac{1}{2}\zeta_n^{(3)}) \\ &\to S_\lambda \quad \text{as } n \to \infty \end{aligned}$$

by Lemma 4, so we may replace $\zeta_n^{(3)}$ by 0 and suppose $\zeta_n = \zeta_n^{(1)} + \zeta_n^{(2)}$ for all n.

Formula (6) for the Green's function leads to the estimate

$$\begin{aligned} \int_\Pi \zeta_n^{(1)} \mathcal{G} \zeta_n^{(2)} &\leq \pi^{-1} \|\zeta_n^{(1)}\|_1 \|\zeta_n^{(2)}\|_1 Z^2 \mathrm{dist}(\mathrm{suppt}(\zeta_n^{(1)}), \mathrm{suppt}(\zeta_n^{(2)}))^{-2} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Consequently

$$\begin{aligned} (E - \lambda I)(\zeta_n^{(1)}) + (E - \lambda I)(\zeta_n^{(2)}) &= (E - \lambda I)(\zeta_n) - \int_\Pi \zeta_n^{(1)} \mathcal{G} \zeta_n^{(2)} \\ &\to S_\lambda \quad \text{as } n \to \infty. \end{aligned} \tag{8}$$

Let $\zeta_n^{(1)*}$, $\zeta_n^{(2)*}$ denote the Steiner symmetrizations of $\zeta_n^{(1)}$, $\zeta_n^{(2)}$ about the x_2-axis. Then we have

$$(E - \lambda I)(\zeta_n^{(i)*}) \geq (E - \lambda I)(\zeta_n^{(i)}), \quad i = 1, 2, \tag{9}$$

by Riesz's rearrangement inequality in conjunction with formula (6) for G, and the symmetrization-invariance of I.

From the estimate of Lemma 3(ii) (in case $p = \infty$ replacing p by any $2 < p < \infty$), there is a positive number k such that $\mathcal{G}\zeta(x) - \lambda x_2 > 0$ only if $x \in (-k, k) \times (0, Z)$, uniformly over Steiner symmetric $\zeta \in \mathcal{R}_+(\zeta_0)$. Let

$$\zeta_n^{(i)**} = \zeta_n^{(i)*} 1_{\{x | \mathcal{G}\zeta_n^{(i)*}(x) - \lambda x_2 > 0\}}, \quad i = 1, 2.$$

Then from (9) and Lemma 6(ii) we have

$$(E - \lambda I)(\zeta_n^{(i)**}) \geq (E - \lambda I)(\zeta_n^{(i)*}) \geq (E - \lambda I)(\zeta_n^{(i)}), \quad i = 1, 2. \tag{10}$$

Moreover we have

$$\operatorname{suppt}(\zeta_n^{(i)**}) \subset \{x \mid -k < x_1 < k\}, \quad i = 1, 2.$$

Now define

$$\begin{aligned} \zeta_n^{(1)***}(x) &= \zeta_n^{(1)**}(x_1 - k, x_2), \\ \zeta_n^{(2)***}(x) &= \zeta_n^{(2)**}(x_1 + k, x_2), \end{aligned}$$

which are supported in $(0, 2k) \times (0, Z)$ and $(-2k, 0) \times (0, Z)$ respectively. Then, from (10),

$$(E - \lambda I)(\zeta_n^{(i)***}) = (E - \lambda I)(\zeta_n^{(i)**}) \geq (E - \lambda I)(\zeta_n^{(i)}), \quad i = 1, 2.$$

From the definitions and (3) we have $\zeta_n^{(1)***} + \zeta_n^{(2)***} \in \mathcal{R}_+(\zeta_0) \subset \overline{\mathcal{R}(\zeta_0)^{\mathrm{w}}}$ and after passing to a subsequence we can assume that $\zeta_n^{(1)***} \to \overline{\zeta^{(1)}}$ and $\zeta_n^{(2)***} \to \overline{\zeta^{(2)}}$, say, weakly in L^2, and so $\overline{\zeta} := \overline{\zeta^{(1)}} + \overline{\zeta^{(2)}} \in \overline{\mathcal{R}(\zeta_0)^{\mathrm{w}}}$. Now, using compactness of $\mathcal{G}$ as an operator on $L^2((-2k, 2k) \times (0, Z))$, from (8) we have

$$\begin{aligned} (E - \lambda I)(\overline{\zeta^{(1)}}) \;&+\; (E - \lambda I)(\overline{\zeta^{(2)}}) \\ &= \lim_{n \to \infty} \left((E - \lambda I)(\zeta_n^{(1)***}) + (E - \lambda I)(\zeta_n^{(2)***}) \right) \\ &\geq S_\lambda \end{aligned}$$

and therefore

$$\begin{aligned} (E - \lambda I)(\overline{\zeta}) &= (E - \lambda I)(\overline{\zeta^{(1)}}) + (E - \lambda I)(\overline{\zeta^{(2)}}) + \int_\Pi \overline{\zeta^{(1)}} \mathcal{G} \overline{\zeta^{(2)}} \\ &\geq (E - \lambda I)(\overline{\zeta^{(1)}}) + (E - \lambda I)(\overline{\zeta^{(2)}}) \qquad (11) \\ &\geq S_\lambda. \end{aligned}$$

Therefore $\overline{\zeta} \in \Sigma_\lambda$ so, by hypothesis, $\overline{\zeta} \in \mathcal{R}(\zeta_0)$. It follows that

$$\|\overline{\zeta^{(1)}}\|_2^2 + \|\overline{\zeta^{(2)}}\|_2^2 = \|\overline{\zeta}\|^2 = \mu.$$

Since $\|\overline{\zeta^{(1)}}\|_2^2 \le \alpha$ and $\|\overline{\zeta^{(2)}}\|_2^2 \le \beta$, we deduce $\|\overline{\zeta^{(1)}}\|_2^2 = \alpha$ and $\|\overline{\zeta^{(2)}}\|_2^2 = \beta$, so both $\overline{\zeta^{(1)}}$ and $\overline{\zeta^{(2)}}$ are non-zero. Hence

$$\int_\Pi \overline{\zeta^{(1)}}\mathcal{G}\overline{\zeta^{(2)}} > 0.$$

Therefore strict inequality holds in (11) contradicting the definition of S_λ. □

Proof of Theorem 2. There is no loss of generality in supposing $2 < p < \infty$.

To prove (i), we first consider a maximizing sequence $\{\zeta_n\}_{n=1}^\infty$ for $E - \lambda I$ comprising elements of $\mathcal{R}_+(\zeta_0)$ and having the Compactness Property. There is thus a sequence $\{y_n\}_{n=1}^\infty$ in Π such that

$$\forall \varepsilon > 0 \ \exists R > 0 \text{ s.t. } \forall n \ \|\zeta_n 1_{\Pi\setminus(y_n+B_R)}\|_2^2 < \varepsilon. \tag{12}$$

We assume the y_n to lie on the x_2 axis, which results in no loss of generality in view of the invariance of $E - \lambda I$ under translations in the x_1-direction.

Let $\zeta_n^0 = \zeta_n 1_{\mathbb{R}\times(0,Z)}$ and $\zeta_n^R = \zeta_n 1_{(-R,R)\times(0,Z)}$, where Z is the number given in Lemma 6(i) and $R > 0$ is arbitrary. Then $\{\zeta_n^0\}_{n=1}^\infty$ is also a maximizing sequence by Lemma 6(ii), and (12) has the consequence that

$$\forall \varepsilon > 0 \ \exists R > 0 \text{ s.t. } \forall n \ \|\zeta_n^0 - \zeta_n^R\|_2^2 < \varepsilon. \tag{13}$$

After passing to a subsequence, we can suppose that $\{\zeta_n^0\}_{n=1}^\infty$ converges weakly in $L^2(\Pi)$ to a limit ζ^0, hence $\zeta_n^R \to \zeta^R := \zeta^0 1_{(-R,R)\times(0,Z)}$ weakly as $n \to \infty$. With this notation (13) takes the form

$$\|\zeta_n^0 - \zeta_n^R\|_2 \to 0 \text{ as } R \to \infty, \text{ uniformly over } n. \tag{14}$$

Now $E(\zeta_n^R) \to E(\zeta^R)$ as $n \to \infty$, for each R. We have

$$E(\zeta_n^0) = E(\zeta_n^R + (\zeta_n^0 - \zeta_n^R)) = E(\zeta_n^R) + E(\zeta_n^0 - \zeta_n^R) + \int_\Pi \zeta_n^R \mathcal{G}(\zeta_n^0 - \zeta_n^R),$$

whence

$$|E(\zeta_n^0) - E(\zeta_n^R)| \le \text{const.}\|\zeta_n^0 - \zeta_n^R\|_2 \tag{15}$$

by Lemma 5, and the constant is independent of R and n. Now from (14) and (15) we have

$$|E(\zeta_n^0) - E(\zeta_n^R)| \to 0 \text{ as } R \to \infty, \text{ uniformly over } n. \tag{16}$$

But $E(\zeta_n^R) \to E(\zeta^R)$ as $n \to \infty$ for each fixed R by weak continuity, and $E(\zeta^R) \to E(\zeta^0)$ as $R \to \infty$ by the Monotone Convergence Theorem, and in conjunction with (16) this yields $E(\zeta_n^0) \to E(\zeta^0)$.

We have

$$|I(\zeta_n^0) - I(\zeta_n^R)| \le R\|\zeta_n^0 - \zeta_n^R\|_1. \tag{17}$$

Let $\varepsilon > 0$. Now

$$|I(\zeta_n^0) - I(\zeta^0)| \le |I(\zeta_n^0) - I(\zeta_n^R)| + |I(\zeta_n^R) - I(\zeta^R)| + |I(\zeta^R) - I(\zeta^0)|;$$

we may choose $R > 0$ to make the first term less than $\varepsilon/3$ for all n by (14) and (17), and the last term less than $\varepsilon/3$ for all n by the Monotone Convergence Theorem, then the middle term is less than $\varepsilon/3$ for all sufficiently large n by weak convergence. Hence $I(\zeta_n^0) \to I(\zeta^0)$ as $n \to \infty$.

Thus $(E - \lambda I)(\zeta_n^0) \to (E - \lambda I)(\zeta^0)$ as $n \to \infty$, so $(E - \lambda I)(\zeta^0) = S_\lambda$. Therefore $\zeta^0 \in \mathcal{R}(\zeta_0)$ by hypothesis, so $\zeta_n^0 \to \zeta^0$ strongly in $L^2(\Pi)$ by uniform convexity.

Since ζ_n^0 and $\zeta_n - \zeta_n^0$ are supported on disjoint sets,

$$\|\zeta_n - \zeta_n^0\|_2^2 = \|\zeta_n\|_2^2 - \|\zeta_n^0\|_2^2 \le \|\zeta_0\|_2^2 - \|\zeta_n^0\|_2^2 \to 0,$$

so $\zeta_n \to \zeta^0$ as $n \to \infty$. We have thus proved that a compact maximizing sequence has a subsequence which, after suitable translations in the x_1-direction, converges strongly in $L^2(\Pi)$ to an element of Σ_λ. Now Lemmas 1, 8 and 9 show that every maximizing sequence contains a subsequence having the Compactness Property, and (i) follows.

To prove (ii), let $\{\zeta_n\}_{n=1}^\infty$ be a maximizing sequence for $E - \lambda I$ relative to $\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$ comprising elements of $\mathcal{R}_+(\zeta_0)$. Suppose that $\mathrm{dist}_2(\zeta_n, \Sigma_\lambda) \not\to 0$ as $n \to \infty$. Then, after discarding a subsequence, we can suppose

$$\mathrm{dist}_2(\zeta_n, \Sigma_\lambda) > \delta \quad \forall n, \tag{18}$$

where δ is some positive number.

But, by (i), $\{\zeta_n\}_{n=1}^\infty$ can be replaced by a subsequence that, after suitable translations in the x_1-direction, converges in $\|\cdot\|_2$ to an element of Σ_λ. This contradicts the supposition (18), showing that $\mathrm{dist}_2(\zeta_n, \Sigma_\lambda) \to 0$ as $n \to \infty$, proving (ii).

To prove (iii) observe that Lemma 7 shows the existence of maximizing sequences in $\mathcal{R}_+(\zeta_0)$, which can, by (i), be assumed to contain subsequences converging to elements of Σ_λ, which is therefore non-empty.

Finally, to prove (iv) observe that, for fixed x_2 and y_2, formula (6) shows that $G(x,y)$ is a positive strictly decreasing function of $|x_1 - y_1|$ alone, so we can apply the one-dimensional case of Lieb's analysis [13, Lemma 3] of equality in Riesz's rearrangement inequality, on pairs of lines parallel to the x_1-axis, to deduce that every $\zeta \in \Sigma_\lambda$ is, after a translation, Steiner-symmetric about the x_2-axis. From Lemma 6(ii) we know that every $\zeta \in \Sigma_\lambda$ is supported in the set where $\mathcal{G}\zeta(x) - \lambda x_2 > 0$, which is bounded by Lemma 2(i) and Lemma 3(ii). The functional relationship $\zeta = \varphi \circ (\mathcal{G}\zeta - \lambda x_2)$, where $\zeta \in \Sigma_\lambda$ is any element and φ is some (*a priori* unknown) increasing function, is given by [5, Theorem 16(i)] (where it forms the first variation condition at a maximum). $\square$

4 Existence, Transport and Theorem 1.

Lemma 10. *Let $0 \le \zeta_0 \in L^p(\Pi)$, for some $2 < p \le \infty$ and suppose $|suppt(\zeta_0)| = \pi a^2$ for some $0 < a < \infty$. For $0 < \lambda < \infty$ let S_λ be the supremum of $E - \lambda I$ relative to $\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$ and let Σ_λ be the set of maximizers of $E - \lambda I$ relative to $\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$. Then, there exists $\Lambda > 0$ such that, if $0 < \lambda < \Lambda$, then $\emptyset \ne \Sigma_\lambda \subset \mathcal{R}(\zeta_0)$.*

Proof. There is no loss of generality in supposing $2 < p < \infty$. Since E is unbounded above on $\mathcal{R}(\zeta_0)$, which may be seen by translating ζ_0 away to infinity in the x_2-direction, we have $S_\lambda \to \infty$ as $\lambda \to 0$. Therefore we can choose $\Lambda > 0$ such that, if $0 < \lambda < \Lambda$ then $S_\lambda > M$, where M is a positive number to be chosen later.

Consider λ with $0 < \lambda < \Lambda$, and consider $\overline{\zeta} \in \Sigma_\lambda$. Arguing as in Douglas [10, Theorem 4.1], the strict convexity of $E - \lambda I$ ensures that $\overline{\zeta}$ is an extreme point of $\overline{\mathcal{R}(\zeta_0)}^{w}$ so $\overline{\zeta} \in \mathcal{RC}(\zeta_0) \subset \mathcal{R}_+(\zeta_0)$ by [10, Theorem 2.1]. Let $m := \sup_{x\in\Pi}(\frac{1}{2}\mathcal{G}\overline{\zeta}(x) - \lambda x_2)$. Then

$$M < S_\lambda = \int_\Pi \overline{\zeta}(x)\left(\tfrac{1}{2}\mathcal{G}\overline{\zeta}(x) - \lambda x_2\right)\mathrm{d}x \le m\|\overline{\zeta}\|_1 \le m\|\zeta_0\|_1,$$

so $m > M/\|\zeta_0\|_1$. From Lemma 3(i) we have

$$|\nabla\mathcal{G}\overline{\zeta}(x)| \le N\|\overline{\zeta}\|_p \quad \forall x \in \Pi$$

where N is a positive constant independent of λ and M, hence if $x \in \Pi$ is such that $\frac{1}{2}\mathcal{G}\overline{\zeta}(x) - \lambda x_2 > m/2$, and $y \in \Pi$ is such that $|y - x| < 2a$ and $y_2 < x_2$, then

$$\begin{aligned}\tfrac{1}{2}\mathcal{G}\overline{\zeta}(y) - \lambda y_2 &> \tfrac{1}{2}\left(\mathcal{G}\overline{\zeta}(x) - 2aN\|\zeta_0\|_1\right) - \lambda x_2 \\ &> \frac{m}{2} - 2aN\|\zeta_0\|_1 > \frac{M}{2\|\zeta_0\|_1} - 2aN\|\zeta_0\|_1 > 0,\end{aligned}$$

provided we choose $M = 5aN\|\zeta_0\|_1^2$; note this shows $x_2 > a$ because $\frac{1}{2}\mathcal{G}\overline{\zeta}(y) - \lambda y_2$ vanishes when $y_2 = 0$. Hence

$$|\{y \in \Pi \mid \mathcal{G}\overline{\zeta}(y) - \lambda y_2 > 0\}| > |\{y \in \Pi \mid \tfrac{1}{2}\mathcal{G}\overline{\zeta}(y) - \lambda y_2 > 0\}| > 2\pi a^2.$$

It follows that if $\overline{\zeta}$ is a proper curtailment of a rearrangement of ζ_0, then we have the freedom to choose ζ_1 supported in $\{y \in \Pi \mid \mathcal{G}\overline{\zeta}(y) - \lambda y_2 > 0\} \setminus \mathrm{suppt}(\overline{\zeta})$ such that $\overline{\zeta} + \zeta_1 \in \mathcal{R}(\zeta_0)$, and then

$$(E - \lambda I)(\overline{\zeta} + \zeta_1) = (E - \lambda I)(\overline{\zeta}) + E(\zeta_1) + \int_\Pi (\mathcal{G}\overline{\zeta} - \lambda x_2)\zeta_1 > (E - \lambda I)(\overline{\zeta}),$$

contradiction. So every maximizer belongs to $\mathcal{R}(\zeta_0)$. $\square$

Recall the definition of an L^p-regular solution given in Definition 1.

Lemma 11. *Let $2 < p < \infty$ and let ζ be an L^p-regular solution of the vorticity equation. Let $\psi(t,x) = \mathcal{G}\zeta(t,x) - \lambda x_2$ and set $u(t,x) = \nabla^\perp\psi(t,x) - \lambda e_1$. Suppose $\omega_0 \in L^p(\Pi)$. Then the initial value problem for the linear transport equation*

$$\begin{cases} \partial_t\omega + \operatorname{div}(\omega u) = 0 \\ \omega(0) = \omega_0 \end{cases} \tag{19}$$

has a unique weak solution $\omega \in L^\infty_{\mathrm{loc}}([0,\infty), L^p(\Pi))$. Moreover, $\omega \in C([0,\infty), L^p(\Pi))$ and ω satisfies the renormalisation property in the form $\omega(t) \in \mathcal{R}(\omega_0)$ for almost all $t > 0$.

Proof. We begin by extending ψ to the whole of $\mathbb{R}^2$ as a function odd in x_2, which is accomplished by allowing arbitrary $x \in \mathbb{R}^2$ in the formula (1); then extending ζ to $\mathbb{R}^2$ as a function odd in x_2 gives $-\Delta\psi = \zeta$ throughout $\mathbb{R}^2$ and we take $u = \nabla^\perp\psi + \lambda e_1$ throughout $\mathbb{R}^2$. Similarly we extend ω_0 to $\mathbb{R}^2$ as a function odd in x_2. Write $\psi_0(x) = \psi(x) + \lambda x_2$.

Now $\|\psi_0(t,\cdot)\|_\infty$ is bounded by Lemma 4 and it then follows from elliptic regularity theory, specifically Agmon [1, Thm. 6.1], that $\|\psi_0(t,\cdot)\|_{2,p,B}$ is bounded over all unit balls B uniformly over every bounded interval of t. Hence $\|u(t,\cdot)\|_\infty$ is bounded on every bounded interval of t.

The DiPerna-Lions theory of transport equations [9, Thm. II.2, Cor. II.1 and Cor. II.2] assures us of the existence of a unique solution $\omega \in L^\infty_{\text{loc}}([0,\infty), L^p(\mathbb{R}^2))$ to (19), that $\omega \in C([0,\infty), L^p(\mathbb{R}^2))$ and that ω has the renormalisation property

$$\partial_t(\beta\circ\omega) + \operatorname{div}((\beta\circ\omega)u) = 0$$

for every $\beta \in C^1(\mathbb{R})$ that satisfies $|\beta'(s)| \le \text{const.}(1+|s|^{p/2})$. Moreover ω is odd in x_2 from the uniqueness.

Now consider a test function of the form $\chi(t)\varphi_R(x)$ where $\chi \in \mathscr{D}(\mathbb{R})$ and $\varphi_R \in \mathscr{D}(\mathbb{R}^2)$ satisfies $0 \le \varphi_R \le 1$ everywhere, $\varphi_R(x) = 1$ if $|x| < R$, $\varphi_R(x) = 0$ if $|x| > 2R$ and $|\nabla\varphi_R| < 2/R$ everywhere. Then, for any β as above,

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}} \chi'(t)\varphi_R(x)\beta(\omega(t,x))\mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}^2}\int_{\mathbb{R}} \chi(t)\nabla\varphi_R(x)\cdot u(t,x)\beta(\omega(t,x))\mathrm{d}t\mathrm{d}x = 0. \quad (20)$$

We now suppose further that $|\beta(s)| \le \text{const.}|s|^p$ for all s, choose $\sigma > 2$ such that $1/2 + 1/p + 1/\sigma = 1$ and deduce

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \beta(\omega(t,x))\nabla\varphi_R(x)\cdot u(t,x)\mathrm{d}x\right| &\le \|\beta(\omega(t,\cdot)\|_p\|u(t,\cdot)\|_2\|\nabla\varphi_R\|_\sigma \\ &\le \text{const.}\|\omega(t,\cdot)\|_p\|u(t,\cdot)\|_2 R^{2/\sigma-1} \\ &\to 0 \text{ as } R\to\infty \text{ uniformly over } t\in\operatorname{suppt}\chi.\end{aligned}$$

From (20) we now have

$$\int_{\mathbb{R}^2}\beta(\omega(t,x))\mathrm{d}x = \text{const.}$$

for each β; taking β to be a mollification of $1_{[\alpha,\infty)}$ for $\alpha > 0$ we deduce that the positive part of $\omega(t)$ is a rearrangement of the positive part of ω_0 and similarly for the negative parts. □

Remark Note that it follows from Lemma 11, in particular, that $\|\zeta(t,\cdot)\|_p$ and $\|\zeta(t,\cdot)\|_1$ are conserved if ζ is an L^p-regular solution.

We also observe that a version of Lemma 11, in the case $\lambda = 0$, was established in [19, Proposition 1].

Proof of Theorem 1. There is no loss of generality in supposing $2 < p < \infty$. Choose $Z > 0$ such that $\mathcal{G}\omega(x) - \lambda x_2 < 0$ for $x_2 > Z$ provided

$$\left.\begin{array}{rcl} \omega & \geq & 0 \\ |\mathrm{suppt}(\omega)| & < & A \\ \|\omega\|_2 & \leq & \|\zeta_0\|_2 + 1 \\ I(\omega) & \leq & \sup I(\Sigma_\lambda) + 1 \end{array}\right\} \tag{21}$$

by Lemma 4. Write

$$\widetilde{\omega} = \omega 1_{\mathbb{R}\times(0,Z)}.$$

Recall the notation S_λ introduced in Theorem 2, as the supremum of $E - \lambda I$ relative to $\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$.

Then we have

$$(E - \lambda I)(\widetilde{\omega}) \geq (E - \lambda I)(\omega)$$

and

$$(E - \lambda I)(\omega) \to S_\lambda \quad \text{as} \quad \mathrm{dist}_Y(\omega, \Sigma_\lambda) \to 0,$$

for ω satisfying (21), by Lemma 5.

Suppose, to seek a contradiction, that $\{\omega^n(\cdot)\}_{n=1}^\infty$ are non-negative L^p-regular solutions of the vorticity equation (2) for which $\omega^n(0)$ is compactly supported for each n,

$$\mathrm{dist}_Y(\omega^n(0), \Sigma_\lambda) \to 0$$

as $n \to \infty$, but

$$\sup_{t>0} \mathrm{dist}_2(\omega^n(t), \Sigma_\lambda) > \theta \quad \forall n, \tag{22}$$

where $\theta > 0$. For each n choose $t_n > 0$ such that

$$\mathrm{dist}_2(\omega^n(t_n), \Sigma_\lambda) > \theta, \tag{23}$$

and choose $\zeta_0^n \in \Sigma_\lambda$ such that

$$\|\zeta_0^n - \omega^n(0)\|_2 \to 0 \text{ as } n \to \infty.$$

In view of Theorem 2(i), after translating in the x_1-direction, and passing to a subsequence, we may additionally suppose that $\{\zeta_0^n\}_{n=1}^\infty$ converges in $L^2(\Pi)$ to a limit in Σ_λ. Reassigning the label ζ_0 we shall suppose

$$\zeta_0^n \to \zeta_0 \in \Sigma_\lambda \text{ as } n \to \infty \text{ in } \|\cdot\|_2.$$

Now

$$\inf_{t>0}(E - \lambda I)(\widetilde{\omega}^n(t)) \geq S_\lambda - o(1) \text{ as } n \to \infty. \tag{24}$$

Let $\zeta(\cdot) = \zeta^n(\cdot)$ be the solution of the transport equation

$$\begin{cases} \partial_t \zeta + \operatorname{div}(\zeta u) = 0, \\ u = \lambda e_1 + \nabla^\perp \mathcal{G}\omega^n \end{cases}$$

with initial data ζ_0^n. Then, continuing to use $\widetilde{}$ for restriction to $\mathbb{R} \times (0, Z)$, we have

$$\begin{aligned} |I(\widetilde{\zeta}^n(t)) - I(\widetilde{\omega}^n(t))| &\leq Z\|\widetilde{\zeta}^n(t) - \widetilde{\omega}^n(t)\|_1 \\ &\leq Z\|\zeta_0^n - \omega^n(0)\|_1 \\ &\leq Z(\pi a^2 + A)^{1/2}\|\zeta_0^n - \omega^n(0)\|_2 \end{aligned}$$

and

$$\|\widetilde{\zeta}^n(t) - \widetilde{\omega}^n(t)\|_2 \leq \|\zeta^n(t) - \omega^n(t)\|_2 = \|\zeta_0^n - \omega^n(0)\|_2,$$

hence

$$\begin{aligned} |E(\widetilde{\zeta}^n(t)) - E(\widetilde{\omega}^n(t))| &\leq \text{const.}\|\widetilde{\zeta}^n(t) - \widetilde{\omega}^n(t)\|_2 \\ &\leq \text{const.}\|\zeta_0^n - \omega^n(0)\|_2 \end{aligned} \tag{25}$$

by Lemma 5. It now follows from (24) and (25) that $\{\widetilde{\zeta}^n(t_n)\}_{n=1}^\infty$ is a maximizing sequence for $E - \lambda I$ relative to $\overline{\mathcal{R}(\zeta_0)}^{\mathrm{w}}$. It follows from Theorem 2 that

$$\operatorname{dist}_2(\widetilde{\zeta}^n(t_n), \Sigma_\lambda) \to 0$$

as $n \to \infty$.

From this it follows that

$$\|\widetilde{\zeta}^n(t_n) - \zeta^n(t_n)\|_2 \to 0,$$

since the functions $\widetilde{\zeta}^n(t_n) - \zeta^n(t_n)$ and $\widetilde{\zeta}^n(t_n)$ have disjoint supports and are therefore orthogonal in L^2, so

$$\|\widetilde{\zeta}^n(t_n) - \zeta^n(t_n)\|_2^2 = \|\zeta^n(t_n)\|_2^2 - \|\widetilde{\zeta}^n(t_n)\|_2^2 = \|\zeta_0\|_2^2 - \|\widetilde{\zeta}^n(t_n)\|_2^2 \to 0.$$

Hence

$$\operatorname{dist}_2(\zeta^n(t_n), \Sigma_\lambda) \to 0.$$

Since

$$\|\zeta^n(t_n) - \omega^n(t_n)\|_2 = \|\zeta_0^n - \omega^n(0)\|_2 \to 0$$

we deduce

$$\mathrm{dist}_2(\omega^n(t_n), \Sigma_\lambda) \to 0,$$

and this contradicts the choices of θ and t_n made in (22) and (23). $\square$

5 Further Remarks.

If we only consider perturbations formed by adding non-negative vorticity to a maximizer then we can prove the following variant of Theorem 1 concerning stability in $\|\cdot\|_Y$:

Theorem 3. *Let ζ_0 be a non-negative function whose support has finite positive area πa^2 ($a > 0$) in the half-plane Π, suppose $\zeta_0 \in L^p(\Pi)$ for some $2 < p \le \infty$, and suppose $\lambda > 0$. Let Σ_λ denote the set of maximizers of $E - \lambda I$ relative to $\overline{\mathcal{R}(\zeta_0)^{\mathrm{w}}}$, and suppose $\emptyset \ne \Sigma_\lambda \subset \mathcal{R}(\zeta_0)$. Then Σ_λ is orbitally stable, in the sense that, for every $\varepsilon > 0$ and $A > \pi a^2$, there exists $\delta > 0$ such that, if $\omega(0)$ is compactly supported, satisfies $\omega(0) \ge \zeta_0$ for some element of Σ_λ, again denoted ζ_0, and if $dist_Y(\omega(0), \Sigma_\lambda) < \delta$ and $|suppt(\omega(0))| < A$, then for all $t \in \mathbb{R}$, then we have $dist_Y(\omega(t), \Sigma_\lambda) < \varepsilon$, whenever $\omega(t)$ denotes an L^p-regular solution of (2) with initial data $\omega(0)$.*

Proof. We indicate the modifications that should be made to the proof of Theorem 1. We have $I(\zeta^n(t_n)) - I(\omega^n(t_n)) \le 0$. Therefore

$$\begin{aligned} S_\lambda - (E - \lambda I)(\omega^n(t_n) &\ge (E - \lambda I)(\zeta^n(t_n) - (E - \lambda I)(\omega^n(t_n)) \\ &\ge E(\zeta^n(t_n)) - E(\omega^n(t_n)). \end{aligned}$$

Now $E(\zeta^n(t_n)) - E(\omega^n(t_n)) \to 0$ by Lemma 5, whereas, by conservation of the impulse and energy of $\omega^n(t)$, we have

$$(E - \lambda I)(\omega^n(t_n)) = (E - \lambda I)(\omega^n(0)) \to S_\lambda,$$

using Lemma 5.

What is now required is to choose a maximizer σ_n close to $\omega^n(t_n)$ in $\|\cdot\|_2$, and then after taking a subsequence, and suitably translating the σ_n in the x_1-direction to σ_n', obtain convergence in $\|\cdot\|_2$, say to σ_0. Then $E(\omega^n(t_n)) \to E(\sigma_0)$ and

$$(E - \lambda I)(\omega^n(t_n)) \to S_\lambda = (E - \lambda I)(\sigma_0)$$

so $I(\omega^n(t_n) \to I(\sigma_0)$. For large n, we then have a contradiction to the choice of θ and t_n in (3) and (22), which in this case would have been

$$\sup_{t>0} \operatorname{dist}_Y(\omega^n(t), \Sigma_\lambda) > \theta, \quad \operatorname{dist}_Y(\omega^n(t_n), \Sigma_\lambda) > \theta.$$

Hence $\sup_t \operatorname{dist}_Y(\omega(t), \Sigma_\lambda) \to 0$ as $\operatorname{dist}_Y(\omega(0), \Sigma_\lambda) \to 0$, as desired. $\square$

We conclude this article with some final remarks. Although we describe our result as a nonlinear stability theorem, it falls short of what one would desire because we only show that, for a class of steady vortex pairs ζ_0, the perturbed trajectories stay close to the set $\Sigma_\lambda(\zeta_0)$, but not necessarily to the orbit of the unperturbed steady wave $\{\zeta_0(\cdot - (\lambda t, 0)), t \in \mathbb{R}\}$. In consequence, the most natural problem raised by this work is to further investigate the structure of Σ_λ. Another issue that bears further scrutiny is that the notions of closeness employed for the perturbation of initial vorticity and the change in the evolving vorticity are slightly different. An extension of this work in any way which would include Lamb's circular vortex-pair (the one case where we have a precise characterization of Σ_λ) would also be very interesting.

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