Sobolev Spaces

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Contents

Recommended literature.

1) L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics **19**, American Mathematical Society, 2nd Edn. 2010;

2) R. A. Adams & J.J.F. Fournier, *Sobolev Spaces*, Elsevier, 2nd Edn. 2003;

3) V. G. Maz'ya & T.O. Shaposhnikova, *Sobolev spaces*, Springer, 2nd Edn. 2011. (For reference.)

1 Preliminaries

1.1 Inequalities

μ will always denote a positive measure - think of Lebesgue measure $\mathcal{L}^N$, possibly with a positive density function.

Conjugate exponents. $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$, or $p = \infty$ and $q = 1$, are called *conjugate exponents*.

Young's inequality. If $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ then

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q \tag{1}$$

for all $x \geq 0$ and $y \geq 0$.

Proof. Write (1) as $xy - \frac{1}{p}x^p \leq \frac{1}{q}y^q$. Then maximise LHS over x for fixed y. □

Hölder's Inequality. Let $p, q \in [1, \infty]$ be conjugate exponents, $f \in L^p(X, \mu)$, $g \in L^q(X, \mu)$. Then

$$\int_X |fg| d\mu \leq \|f\|_p \|g\|_q.$$

Proof. Put $x = \frac{|f(z)|}{\|f\|_p}$, $y = \frac{|g(z)|}{\|g\|_q}$ in Young's inequality and integrate. $p = 1$, $q = \infty$ is trivial. □

Minkowski's Inequality. For $1 \le p \le \infty$, $f,\ g \in L^p(X,\mu)$,

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

Proof. Firstly consider the case $p = 1$. We have $|f+g| \le |f| + |g|$ a.e., hence

$$\int_X |f+g| d\mu \le \int_X |f| d\mu + \int_X |g| d\mu.$$

Secondly consider the case $1 < p < \infty$. For $x \in X$ we have

$$|f(x)+g(x)|^p \le (2\max\{|f(x)|, |g(x)|\})^p \le 2^p(|f(x)|^p + |g(x)|^p),$$

hence

$$\int_X |f+g|^p d\mu \le 2^p \left(\int_X |f|^p d\mu + \int_X |g|^p d\mu \right) < \infty.$$

Thus $f+g \in \mathcal{L}^p(X,\Sigma,\mu)$. Let q be the conjugate exponent of p. Then

$$\begin{aligned}
\int_X |f+g|^p d\mu &= \int_X |f+g|\,|f+g|^{p-1} d\mu \\
&\le \int_X |f|\,|f+g|^{p-1} d\mu + \int_X |g|\,|f+g|^{p-1} d\mu \\
&\le \left(\left(\int_X |f|^p d\mu \right)^{1/p} + \left(\int_X |g|^p d\mu \right)^{1/p} \right) \left(\int_X |f+g|^{(p-1)q} d\mu \right)^{1/q} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Hölder's inequality)} \\
&= (||f||_p + ||g||_p) \left(\int_X |f+g|^p d\mu \right)^{1/q}.
\end{aligned}$$

If $||f+g||_p > 0$ we can divide by $||f+g||_p^{p/q}$ to obtain

$$||f+g||_p \le ||f||_p + ||g||_p,$$

whereas if $||f+g||_p = 0$ the result is trivial.

Finally consider the case $p = \infty$. For almost every $x \in X$ we have

$$|f(x)+g(x)| \le |f(x)| + |g(x)| \le \operatorname{esssup}|f| + \operatorname{esssup}|g| = ||f||_\infty + ||g||_\infty.$$

Thus $f+g \in \mathcal{L}^\infty(X,\Sigma,\mu)$ and

$$||f+g||_\infty \le ||f||_\infty + ||g||_\infty.$$

□

Generalised Hölder's Inequality. Suppose $p_1, \dots, p_n \in (1, \infty)$, $\dfrac{1}{p_1} + \dots + \dfrac{1}{p_n} = 1$, $u_i \in L^{p_i}(X, \mu), \quad i = 1, \dots, n$. Then

$$\int_X |u_1 u_2 \cdots u_n| d\mu \leq \|u_1\|_{p_1} \cdots \|u_n\|_{p_n}.$$

Proof. Exercise. $\square$

Interpolation Inequality. Suppose $1 \leq p < q < r < \infty$ and choose $0 < \theta < 1$ such that $\dfrac{1}{q} = \dfrac{\theta}{p} + \dfrac{1-\theta}{r}$. Suppose $u \in L^p(X, \mu) \cap L^r(X, \mu)$. Then

$$\|u\|_q \leq \|u\|_p^\theta \, \|u\|_r^{1-\theta}$$

Proof. Exercise. $\square$

Jensen's Inequality for sums. Let $I \subset \mathbb{R}$ be an open interval, let $\Psi : I \to \mathbb{R}$ be a convex function, let $x_1, \dots, x_n \in I$ and let $\lambda_i \geq 0$ for $1 \leq i \leq n$ with $\lambda_1 + \cdots \lambda_n = 1$. Then

$$\Psi\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i \Psi(x_i).$$

Proof. By induction from the definition of convexity. $\square$

Jensen's Inequality for functions. Let $I \subset \mathbb{R}$ be an open interval, let $\Psi : I \to \mathbb{R}$ be a convex function and let μ be a probability measure on X ($\mu \geq 0$, $\mu(X) = 1$). Then for $u \in L^1(X, \mu)$ taking values in I we have

$$\Psi\left(\int_X u(x) d\mu(x)\right) \leq \int_X \Psi(u(x)) d\mu(x).$$

Proof. Recall that Ψ is everywhere subdifferentiable, that is, for every $x \in I$ there is at least one real α such that

$$\Psi(y) \geq \Psi(x) + \alpha(y - x) \quad \forall y \in I,$$

and so Ψ is the pointwise supremum of all the affine functionals on $\mathbb{R}$ dominated by Ψ.

Suppose firstly that $\alpha, \beta \in \mathbb{R}$ s.t.

$$\varphi(s) = \alpha s + \beta \leq \Psi(s) \quad \forall s \in \mathbb{R}.$$

Then

$$\varphi\left(\int u d\mu\right) = \alpha \int_X u d\mu + \beta = \int_X (\alpha u + \beta) d\mu \leq \int_X \Psi \circ u \, d\mu.$$

Taking the supremum over all such affine functionals φ dominated by Ψ, we obtain

$$\Psi\left(\int_X u d\mu\right) \leq \int_X \Psi \circ u \, d\mu.$$

$\square$

The AM-GM inequality. $(x_1x_2\cdots x_n)^{1/n} \le (x_1+\cdots+x_n)/n$ for positive $x_1,\ldots,x_n$ follows by applying Jensen's inequality for sums to the convex function $-\log$ on $(0,\infty)$.

1.2 Partial Derivatives and Distributions

Integrals are with respect to $\mathcal{L}^N$.

Definition. *The* support *of a real-valued function* f, $\operatorname{supp} f = \overline{\{x \mid f(x) \neq 0\}}$.

Notation for partial derivatives on $\mathbb{R}^N$.
For $1 \le i \le N$ write $D_i = \dfrac{\partial}{\partial x_i}$. Write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Any $\alpha = (\alpha_1,\ldots,\alpha_N) \in \mathbb{N}_0^N$ is called a *multi-index* of *degree* $|\alpha| = \alpha_1+\cdots+\alpha_N$. Write $\alpha! = \alpha_1!\cdots\alpha_N!$ and $D^\alpha = D_1^{\alpha_1}\cdots D_N^{\alpha_N}$.

Note. $0 = (0,\ldots,0) \in \mathbb{N}_0^N$, $|0| = 0$, *and* $D^0u = u$.

If u has continuous partial derivatives of order m, we have equality of cross-derivatives for orders up to m, so the order of differentiation in $D^\alpha u$ for $|\alpha| \le m$ is unimportant.
Leibniz's Theorem. If u, v are m-times continuously differentiable functions of N real variables then, for $0 \le |\alpha| \le m$,

$$D^\alpha uv = \sum_{0\le\beta\le\alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v$$

where

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$$

and $\beta \le \alpha$ signifies $\beta_i \le \alpha_i$ for $i = 1,\ldots,N$.

For $\Omega \subset \mathbb{R}^N$ open, $C^\infty(\Omega)$ denotes the set of real functions on Ω that have continuous partial derivatives of all orders.

$\mathscr{D}(\Omega) = C_c^\infty(\Omega)$ denotes the set of all $u \in C^\infty(\Omega)$ such that $\operatorname{supp} u$ is a compact subset of Ω. Elements of $\mathscr{D}(\Omega)$ are called *test functions*.

Example.

$$J(x) = \begin{cases} ke^{-\frac{1}{1-|x|^2}} & |x| < 1 \\ 0 & |x| \ge 1, \end{cases}$$

k is chosen such that $\displaystyle\int_{\mathbb{R}^N} J = 1$.

$$J_\varepsilon(x) = \varepsilon^{-N} J(\varepsilon^{-1}x), \quad x \in \mathbb{R}^N \text{ and } \varepsilon > 0.$$

Then $J_\varepsilon \in \mathscr{D}(\mathbb{R}^N)$ and is known as the *standard mollifier*.

Convention. If $\varphi \in \mathscr{D}(\Omega)$ then $\varphi = 0$ on $\mathbb{R}^N \setminus \Omega$.

Convergence of test functions.

We say that $\varphi_n \to \varphi_0$ in $\mathscr{D}(\Omega)$ as $n \to \infty$ if there exists a compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi_n \subset K$ for all $n \in \mathbb{N}$ and $D^\alpha \varphi_n \to D^\alpha \varphi_0$ uniformly on K as $n \to \infty$, for every $\alpha \in \mathbb{N}_0^N$.

Definition of Distributions.

Given $\Omega \subset \mathbb{R}^N$ open, a *distribution* on Ω is a real linear functional on $\mathscr{D}(\Omega)$ (sequentially) continuous with respect to convergence of test functions. $\langle u, \varphi \rangle$ denotes the value taken by the distribution u at the test function φ. The set of distributions on Ω is denoted $\mathscr{D}'(\Omega)$.

Remarks

1. See Walter Rudin's *Functional Analysis* for an account of a topology on $\mathscr{D}(\Omega)$ that gives rise to this notion of convergence of test functions. Linear functionals are shown to be continuous iff they are sequentially continuous.
2. If $\varphi \in \mathscr{D}(\Omega)$ and $\alpha \in \mathbb{N}_0^N$ then $D^\alpha \varphi$ is also a test function.
3. $D^\alpha : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$ is linear and sequentially continuous.

Examples

1. We call a measurable function u on Ω *locally integrable* ($u \in L^1_{loc}(\Omega)$) if $\displaystyle\int_K |u| < \infty$ for every $K \subset \Omega$ compact. A locally integrable u gives rise to a distribution by

$$\langle u, \varphi \rangle = \int_\Omega u\varphi \qquad \forall \varphi \in \mathscr{D}(\Omega).$$

This is well defined since φ has compact support and u is integrable on compact sets, and linear. If $\varphi_n \to \varphi$ in $\mathscr{D}(\Omega)$ choose a compact $K \subset \Omega$ containing the supports of all the φ_n. Then

$$|\langle u, \varphi_n \rangle - \langle u, \varphi \rangle| \leq \|u\|_{L^1(K)} \|\varphi_n - \varphi\|_\infty \to 0$$

by uniform convergence on K. Later we'll show that different u give rise to different distributions.

2. Fix $z \in \Omega$ and define

$$\langle \delta_z, \varphi \rangle = \varphi(z) \qquad \forall \varphi \in \mathscr{D}(\Omega).$$

This is well-defined and linear. If $\varphi_n \to \varphi$ in $\mathscr{D}(\Omega)$ then $\varphi_n(z) \to \varphi(z)$, hence δ_z is continuous (Dirac δ-function).

3. Let $\Omega \subset \mathbb{R}^1$, $z \in \Omega$,

$$\langle \delta_z', \varphi \rangle = -\varphi'(z) \qquad \forall \varphi \in \mathscr{D}(\Omega)$$

defines a distribution, called a dipole.

Lemma 1.1. *Suppose $\Omega \subset \mathbb{R}^N$ is open and $u \in C^1(\Omega)$. Then*

$$\int_\Omega (D_i u)\varphi = -\int_\Omega u D_i \varphi \qquad \forall \varphi \in \mathscr{D}(\Omega).$$

Proof. Assume u, φ are zero outside Ω. Then $u\varphi \in C^1(\mathbb{R}^N)$ even if u is not in $C^1(\mathbb{R}^N)$. So

$$\int_\Omega (D_i u)\varphi = \int_{\mathbb{R}^N} (D_i(u\varphi) - uD_i\varphi) = \int_{rB(0)} \operatorname{div}(u\,\varphi e_i) - uD_i\varphi) = 0 - \int_{\mathbb{R}^N} uD_i\varphi = -\int_\Omega uD_i\varphi$$

where e_i is the unit vector in the positive x_i direction and we have applied the Divergence Theorem on a large ball $rB(0)$ whose interior contains the support of φ . ☐

Note. *If $|\alpha| = m$ and $u \in C^m(\Omega)$ then*

$$\int_\Omega D^\alpha u\varphi = (-1)^m \int_\Omega uD^\alpha\varphi \qquad \forall \varphi \in \mathscr{D}(\Omega).$$

Definition. *Let $u \in \mathscr{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^N$. Define $D^\alpha u$ by*

$$\langle D^\alpha u, \varphi\rangle = (-1)^{|\alpha|}\langle u, D^\alpha\varphi\rangle \qquad \forall \varphi \in \mathscr{D}(\Omega).$$

Lemma 1.2. *(i) If $u \in \mathscr{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^N$ then $D^\alpha u \in \mathscr{D}'(\Omega)$.*

(ii) If $\alpha, \beta \in \mathbb{N}_0^N$, $u \in \mathscr{D}'(\Omega)$ then

$$D^\alpha D^\beta u = D^{\alpha+\beta} u = D^\beta D^\alpha u.$$

Proof. (i) $D^\alpha u$ is the composition of $D^\alpha : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$ which is linear and sequentially continuous with $u : \mathscr{D}(\Omega) \to \mathbb{R}$ which is linear and sequentially continuous. So $D^\alpha u \in \mathscr{D}'(\Omega)$.
(ii) Consider $\varphi \in \mathscr{D}(\Omega)$. Then

$$\begin{aligned}
\langle D^\alpha D^\beta u, \varphi\rangle &= (-1)^{|\alpha|}\langle D^\beta u, D^\alpha\varphi\rangle \\
&= (-1)^{|\alpha|+|\beta|}\langle u, D^\beta D^\alpha\varphi\rangle \\
&= (-1)^{|\alpha|+|\beta|}\langle u, D^{\beta+\alpha}\varphi\rangle \quad \text{by equality of cross-derivatives for smooth functions} \\
&= \langle D^{\beta+\alpha} u, \varphi\rangle.
\end{aligned}$$

So $D^\alpha D^\beta u = D^{\beta+\alpha} u = D^{\alpha+\beta} u = D^\beta D^\alpha u$. ☐

Examples.

1. Let

$$u(x) = x_+ = \begin{cases} 0, & x \le 0 \\ x, & x > 0 \end{cases} \quad x \in \mathbb{R}.$$

For $\varphi \in \mathscr{D}(\mathbb{R})$

$$\begin{aligned}
\langle u', \varphi \rangle &= -\langle u, \varphi' \rangle \\
&= -\int_{-\infty}^{\infty} u(x)\varphi'(x)dx \\
&= -\int_{0}^{\infty} x\varphi'(x)dx \\
&= \big[x\varphi(x)\big]_0^\infty + \int_0^\infty 1\varphi(x)dx \qquad \text{(integrating by parts)} \\
&= 0 + \int_0^\infty 1\varphi(x)dx \qquad (\varphi \text{ has compact support}) \\
&= \int_{-\infty}^{\infty} H(x)\varphi(x)dx,
\end{aligned}$$

where

$$H(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases} \qquad \text{(Heaviside Step function).}$$

So $u' = H$.

2. Differentiate H. For $\varphi \in \mathscr{D}(\mathbb{R})$

$$\begin{aligned}
\langle H', \varphi \rangle &= -\langle H, \varphi' \rangle \\
&= -\int_{-\infty}^{\infty} H(x)\varphi'(x)dx \\
&= -\int_{0}^{\infty} 1\varphi'(x)dx \\
&= \big[\varphi(x)\big]_0^\infty = \varphi(0).
\end{aligned}$$

Thus $\langle H', \varphi \rangle = \varphi(0) = \delta_0(\varphi)$. So $H' = \delta_0$ (Dirac delta function).

3. Differentiate δ_0. For $\varphi \in \mathscr{D}(\mathbb{R})$

$$\langle \delta_0', \varphi \rangle = -\langle \delta_0, \varphi' \rangle = -\varphi'(0) \qquad \text{“Dipole”.}$$

4. Le μ be a Radon measure on Ω (Borel measure that assigns finite measure to compact sets). Define

$$\langle \mu, \varphi \rangle = \int_\Omega \varphi d\mu \quad \forall \varphi \in \mathscr{D}(\Omega).$$

Then μ gives rise to a distribution, for if f$\varphi_n \to \varphi$ in then there is a compact $K \subset \Omega$ that contains the supports of the φ_n and φ and $\varphi_n \to \varphi$ uniformly, so

$$\int_\Omega \varphi_n d\mu \to \int_\Omega \varphi d\mu$$

and the linearity follows from properties of the integral.

Connections with classical derivatives.

1. Let $f \in L^1_{loc}(a,b)$, $x_0 \in (a,b)$,

$$F(x) = \int_{x_0}^x f(x)dx, \quad a < x < b.$$

   (a) Then F is continuous and $F' = f$ in the sense of distributions (proved later Proposition 1.4).

   (b) $F' = f$ classically a.e. in (a,b) (tricky - see W. Rudin's Real and Complex Analysis, Ch. 8).

2. Let F be continuous on (a,b).

   (a) If $F' = f \in L^1_{loc}(a,b)$ in the sense of distributions, then

$$F(x) = \int_{x_0}^x f + c \qquad (x_0 \in (a,b))$$

   for some $c \in \mathbb{R}$ (to be proved later).

   (b) If $F' = f \in L^1_{loc}(a,b)$ classically a.e., we *cannot* conclude $F(x) = \displaystyle\int_{x_0}^x f + c$. See Cantor Function (Devil's Staircase) in Rudin's Real and Complex Analysis, Ch. 8.

Lemma 1.3. *Let $\Omega \in \mathbb{R}^N$ be open. Then*

(i) $\mathscr{D}(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$.

(ii) If $u \in L^1_{\text{loc}}(\Omega)$ and

$$\int_\Omega u\varphi = 0 \qquad \forall \varphi \in \mathscr{D}(\Omega)$$

then $u = 0$ a.e.

(Hence different locally integrable functions u give different distributions.)

Proof. Later. ∎

Proposition 1.4. *Let $f \in L^1_{loc}(a,b)$ and $F(x) = \displaystyle\int_{x_0}^x f$, some $x_0 \in (a,b)$. Then $F' = f$ in the sense of distributions.*

Proof. If f is continuous then F is continuously differentiable with $F' = f$ and the result follows from Lemma 1.1.

Now consider the general case. Firstly, choose a sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathscr{D}(a,b)$ converging to f in $L^1([\alpha,\beta])$ for all $[\alpha,\beta] \subset (a,b)$ (by Lemma 1.3(i)). Define $F_n(x) = \int_{x_0}^{x} f_n$ for some fixed $x_0 \in (a,b)$. Then $F_n' = f_n$ both classically and in the sense of distributions. For $\varphi \in \mathscr{D}(a,b)$

$$\begin{aligned}\langle F_n', \varphi\rangle = \langle f_n, \varphi\rangle &= \int_a^b f_n\varphi \\ &\to \int_a^b f\varphi \qquad \text{(by Hölder's ineq. on compact set } \operatorname{supp}\varphi) \\ &= \langle f, \varphi\rangle.\end{aligned}$$

Also,

$$\begin{aligned}\langle F_n', \varphi\rangle &= -\langle F_n, \varphi'\rangle \\ &= -\int_a^b F_n\varphi' \\ &\to -\int_a^b F\varphi' \qquad \text{(since } F_n \to F \text{ uniformly on } \operatorname{supp}\varphi) \\ &= \langle F', \varphi\rangle.\end{aligned}$$

Thus,

$$\langle F', \varphi\rangle = \langle f, \varphi\rangle \qquad \forall \varphi \in \mathscr{D}(a,b).$$

So $F' = f$ as distributions. □

2 Sobolev spaces

Definition. *Let $\Omega \subset \mathbb{R}^N$ be open, $1 \le p \le \infty$, $m \in \mathbb{N}$. Define the Sobolev space $W^{m,p}(\Omega)$ by*

$$W^{m,p}(\Omega) = \left\{u \mid D^\alpha u \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^N \text{ s.t. } 0 \le |\alpha| \le m\right\}.$$

With the obvious real vector space structure, define the norm on $W^{m,p}(\Omega)$ by

$$\|u\|_{m,p} = \left(\sum_{0\le|\alpha|\le m} \int_\Omega |D^\alpha u|^p\right)^{\frac{1}{p}} \qquad 1 \le p < \infty$$

$$\|u\|_{m,\infty} = \max_{0\le|\alpha|\le m} \|D^\alpha u\|_\infty.$$

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be open, $1 \le p \le \infty$, $m \in \mathbb{N}$. Then $W^{m,p}(\Omega)$ is a Banach space.*

Proof. Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. For each $\alpha \in \mathbb{N}_0^N$, $0 \leq |\alpha| \leq m$,

$$\|D^\alpha u_n - D^\alpha u_k\|_p \leq \|u_n - u_k\|_{m,p}.$$

Hence $\{D^\alpha u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^p(\Omega)$, and converges to some $v_\alpha \in L^p(\Omega)$.

Now $u_n \to v_0$, and for $\varphi \in \mathscr{D}(\Omega)$

$$\langle u_n, D^\alpha \varphi \rangle = \int_\Omega u_n D^\alpha \varphi \overset{\text{Hölder}}{\longrightarrow} \int_\Omega v_0 D^\alpha \varphi = \langle v_0, D^\alpha \varphi \rangle$$

and

$$\langle D^\alpha u_n, \varphi \rangle = \int_\Omega D^\alpha u_n \varphi \overset{\text{Hölder}}{\longrightarrow} \int_\Omega v_\alpha \varphi.$$

Since

$$\langle D^\alpha u_n, \varphi \rangle = (-1)^{|\alpha|} \langle u_n, D^\alpha \varphi \rangle$$

we obtain

$$\langle v_\alpha, \varphi \rangle = (-1)^{|\alpha|} \langle v_0, D^\alpha \varphi \rangle = \langle D^\alpha v_0, \varphi \rangle.$$

So, by uniqueness of function representing $D^\alpha v_0$ (Lemma 1.3(ii)),

$$D^\alpha v_0 = v_\alpha.$$

Thus $D^\alpha u_n \to D^\alpha v_0$ in $\| \, \|_p$ for all $0 \leq |\alpha| \leq m$, that is, $u_n \to v_0$ in $W^{m,p}(\Omega)$. $\square$

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$ be open, $m \in \mathbb{N}$.*

(i) If $1 \leq p < \infty$ then $W^{m,p}(\Omega)$ is separable.

(ii) If $1 < p < \infty$ then $W^{m,p}(\Omega)$ is reflexive.

Proof. Write $A = \{\alpha \in \mathbb{N}_0^N \mid 0 \leq |\alpha| \leq m\}$, and write $Y = L^p(\Omega)^A$, that is the set of maps from A to $L^p(\Omega)$, whose members we think of as vectors $(v_\alpha)_{\alpha \in A}$ whose components belong to $L^p(\Omega)$ and are indexed by elements of A. For $v \in Y$ set

$$\|v\|_Y = \left(\sum_{0 \leq |\alpha| \leq m} \|v_\alpha\|_p^p \right)^{\frac{1}{p}} \qquad \text{(note the sum is over } \alpha \in A)$$

which makes Y into a Banach space. The map $T : W^{m,p}(\Omega) \to Y$

$$(Tu)_\alpha = D^\alpha u \qquad \alpha \in A, \text{ for } u \in W^{m,p}(\Omega),$$

is a linear isometry of $W^{m,p}(\Omega)$ to a linear subspace X of Y. Moreover, $W^{m,p}(\Omega)$ is complete, so X is complete with $\| \, \|_Y$, so X is closed in Y.

(i) If $1 \le p < \infty$ then $L^p(\Omega)$ is separable, so Y is separable, so X is separable, so $W^{m,p}(\Omega)$ is separable.

(ii) Suppose $1 < p < \infty$. Then $L^p(\Omega)$ is reflexive ($L^p(\Omega)$ is isometric under the natural map *onto* $L^p(\Omega)^{**}$; equivalently, the closed unit ball of $L^p(\Omega)$ is compact in the weak topology). Hence Y is reflexive, hence X is reflexive (closed linear subspace of a reflexive space), hence $W^{m,p}$ is reflexive. $\square$

2.1 More spaces and boundary values

A proper theory of boundary values for Sobolev functions requires smoothness assumptions on $\partial\Omega$; see "Trace Theorem" later on. A rough-and-ready definition of $W^{m,p}(\Omega)$ functions whose derivatives of orders $0, 1, \ldots, m-1$ vanish on the boundary, is as follows:

Definition. *For $\Omega \subset \mathbb{R}^N$ open, $m \in \mathbb{N}$, define $W_0^{m,p}(\Omega)$ to be the closure of $\mathscr{D}(\Omega)$ in $W^{m,p}(\Omega)$.*

This is frequently a convenient space for studying Dirichlet problems for PDE.

Definition. *$H^m(\Omega) = W^{m,2}(\Omega)$ is a Hilbert space with scalar product*

$$\langle u, v\rangle_m = \sum_{0\le|\alpha|\le m} \int_\Omega D^\alpha u D^\alpha v \qquad u, v \in H^m(\Omega).$$

Write $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^N$ be open, $1 \le p \le \infty$. Then there is a constant $c = c(p, N)$, such that for $u \in W^{1,p}(\Omega)$,*

$$c^{-1}\|u\|_{1,p,\Omega} \le \|u \circ A\|_{1,p,A^{-1}\Omega} \le c\|u\|_{1,p,\Omega}$$

for all $A \in \mathcal{O}(N)$.

Proof. Consider $A = \in \mathcal{O}(N)$, $u \in W^{1,p}(\Omega)$ and $v = u \circ A^T = u \circ A^{-1} \in L^p(\Omega)$. Let $\varphi \in \mathscr{D}(A\Omega)$

and set and $\psi = \varphi \circ A \in \mathscr{D}(\Omega)$. Let e_i denote the unit vector in the positive x_i direction. Then

$$\begin{aligned}
\langle D_i v, \varphi \rangle &= -\int_{A\Omega} v(y) D_i\varphi(y)dy \\
&= -\int_{A\Omega} u(A^T y)(D_i\varphi)(AA^T y)dy \\
&= -\int_{\Omega} u(x) D_i\varphi(Ax)dx \qquad (Ax = y,\ |\det A| = 1) \\
&= -\int_{\Omega} u(x) D_i(\psi \circ A^T)(Ax)dx \\
&= -\int_{\Omega} u(x) D(\psi \circ A^T)(Ax)e_i dx \qquad (D = \text{derivative}) \\
&= -\int_{\Omega} u(x) D(\psi \circ A^T)(Ax)e_i dx \\
&= -\int_{\Omega} u(x) D\psi(x) DA^T(Ax)e_i dx \qquad (\text{Chain rule}) \\
&= -\int_{\Omega} u(x) D\psi(x) A^T e_i dx \\
&= -\int_{\Omega} u(x) e_i^T A\nabla\psi(x)dx \qquad (\text{transposing real integrand}) \\
&= \int_{\Omega} e_i^T A\nabla u(x)\psi(x)dx \\
&= \int_{A\Omega} e_i^T A\nabla u(A^T y)\psi(A^T y)dy
\end{aligned}$$

so

$$\nabla v = A(\nabla u) \circ A^T.$$

So $\nabla v \in L^p(A\Omega)$ and

$$\|\nabla v\|_p \leq c\|\nabla u\|_p,$$

where

$$c = \sup_{A \in \mathcal{O}(N), |\xi|_p = 1} |A\xi|_p$$

from which the result follows. □

Remark. This shows we are free to rotate axes, at the cost of replacing the Sobolev norm by an equivalent norm, bounded by a constant independent of the rotation. Recall - two norms $\| \, \|_1$ and $\| \, \|_2$ are *equivalent* if there is a constant $c > 0$ such that

$$c^{-1}\|x\|_1 \leq \|x\|_2 \leq c\|x\|_1$$

for all $x \in X$. Two norms are equivalent if and only if they give rise to the same convergent sequences.

Theorem 2.4 (Poincaré's Inequality)**.** *Let $\Omega \subset \mathbb{R}^N$ be open, suppose Ω lies between two parallel hyperplanes a distance $l > 0$ apart and let $1 \leq p \leq \infty$, $m \in \mathbb{N}$. Then there exists $c = c(l, p, m, N) > 0$ such that*

$$\|u\|_{m,p} \leq c\Bigg(\sum_{|\alpha|=m} \|D^\alpha u\|_p^p\Bigg)^{\frac{1}{p}} \qquad \textit{when } 1 \leq p < \infty$$

and

$$\|u\|_{m,\infty} \leq c \max_{|\alpha|=m} \|D^\alpha u\|_\infty \qquad \textit{when } p = \infty$$

for all $u \in W_0^{m,p}(\Omega)$.

Proof. Firstly suppose $m = 1$. Consider $u \in \mathscr{D}(\Omega)$. Using Theorem 2.3 we can assume the axes to be chosen in such a way that $\Omega \subset \{(x_1, \dots, x_N) \mid 0 < x_N < l\}$. Then for $x \in \Omega$

$$u(x) = \int_0^{x_N} D_N u(x', \xi_N) d\xi_N \qquad x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R},$$

so

$$|u(x)| \overset{\text{Hölder}}{\leq} \|1_{[0,x_N]}\|_q \|D_N u(x', \cdot)\|_p \qquad q \text{ conjugate to } p.$$

<u>**Case** $1 \leq p < \infty$</u>. Then

$$|u(x)| \leq x_N^{1-\frac{1}{p}} \left(\int_0^l |D_N u(x', \xi_N)|^p d\xi_N\right)^{\frac{1}{p}}$$

So

$$\int_\Omega |u(x)|^p dx \leq \left(\int_0^l x_N^{p-1} dx_N\right) \left(\int_{\mathbb{R}^{N-1}} \int_0^l |D_N u(x', \xi_N)|^p d\xi_N dx'\right) \quad (u = 0 \text{ outside } \Omega)$$

thus

$$\|u\|_p^p \leq \frac{l^p}{p} \|D_N u\|_p^p$$

so

$$\|u\|_p \leq \frac{l}{p^{1/p}} \|D_N u\|_p.$$

<u>**Case** $p = \infty$</u>. We have

$$|u(x)| \leq x_N \|D_N u(x', \cdot)\|_\infty,$$

so taking sup over $x \in \Omega$

$$\|u\|_\infty \leq l \|D_N u\|_\infty.$$

In either case,

$$\|u\|_p \leq l c(p, N) \|\nabla u\|_p.$$

Applying repeatedly, we obtain

$$\|u\|_{1,p} \leq \text{const} \cdot \|\nabla u\|_p$$
$$\vdots$$
$$\|u\|_{m,p} \leq \text{const} \cdot \left(\sum_{|\alpha|=m} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}} \qquad (1 \leq p < \infty)$$
$$\|u\|_{m,\infty} \leq \text{const} \cdot \max_{|\alpha|=m} \|D^\alpha u\|_\infty,$$

for all $u \in \mathscr{D}(\Omega)$. By density the inequality holds for all $u \in W_0^{m,p}(\Omega)$, since both the LHS and RHS are continuous in $\| \ \|_{m,p}$. □

Remark. Poincaré's inequality enables us to define an equivalent norm on $W_0^{m,p}(\Omega)$ when Ω has finite width (in particular when Ω is bounded).

$$\|u\| = \left(\sum_{|\alpha|=m} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}} \qquad (1 \leq p < \infty)$$
$$\|u\| = \max_{|\alpha|=m} \|D^\alpha u\|_\infty \qquad (p = \infty).$$

In particular

$$\langle u, v \rangle = \sum_{|\alpha|=m} \int_\Omega D^\alpha u D^\alpha v$$

defines an equivalent scalar product on $H_0^m(\Omega)$.

2.2 Linear Partial Differential Operators with Constant coefficients.

.

$$L = \sum_{0 \leq |\alpha| \leq m} a^\alpha D^\alpha,$$

where a^α are constants, is a linear partial differential operator of order (at most) m with constant coefficients.

If $f \in \mathscr{D}'(\Omega)$ then $u \in \mathscr{D}'(\Omega)$ is a solution of $Lu = f$ if

$$\langle u, \sum_{0 \leq |\alpha| \leq m} (-1)^{|\alpha|} a^\alpha D^\alpha \varphi \rangle = \langle f, \varphi \rangle \qquad \text{for all } \varphi \in \mathscr{D}(\Omega).$$

The operator

$$L^* = \sum_{0 \leq |\alpha| \leq m} (-1)^{|\alpha|} a^\alpha D^\alpha$$

is the adjoint of L.

Example.

$$\Delta = \sum_{i=1}^{N} D_i^2.$$

For a distribution u and test function φ

$$\langle \Delta u, \varphi\rangle = -\sum_{i=1}^{N} \langle D_i u, D_i \varphi\rangle = \langle u, \Delta\varphi\rangle.$$

Application. Suppose $\Omega \subset \mathbb{R}^N$ is a bounded open set, $f \in L^2(\Omega)$. Show that the boundary value problem

$$\left.\begin{aligned} -\Delta u &= f \\ u &\in H_0^1(\Omega) \end{aligned}\right\} \tag{BVP}$$

has exactly one solution.

Write $H = H_0^1(\Omega)$ and set

$$\langle u, v\rangle_H = \int_\Omega \nabla u \cdot \nabla v \qquad u, v \in H$$

which defines an equivalent scalar product on H.

For $u \in H$,

$$-\Delta u = f$$

if and only if

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi \qquad \forall \varphi \in \mathscr{D}(\Omega)$$

if and only if

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \qquad \forall v \in H$$

by density of $\mathscr{D}(\Omega)$ in H, since LHS is the scalar product of H, and the RHS defines a bounded linear functional of $v \in H$; to see this, put

$$\Lambda(v) = \int_\Omega f v \qquad \forall v \in H$$

then

$$|\Lambda(v)| \le \int_\Omega |f||v| \le \|f\|_2 \|v\|_2 \le \text{const} \cdot \|f\|_2 \|v\|_H$$

by Poincaré's inequality. So $\Lambda \in H^*$, and the Riesz Representation Theorem for Hilbert spaces shows

$$\Lambda(v) = \langle u_0, v\rangle_H \qquad \forall v \in H$$

for exactly one $u_0 \in H$. Now u_0 is the unique solution of the BVP.

Remark. Δ is a second order partial differential operator, but $u_0 \in H_0^1(\Omega)$ at first sight only has first order derivatives. The question "Does u_0 have second order derivatives?" belongs to Regularity Theory. In fact $u_0 \in H^2_{loc}(\Omega)$ in general, and $u_0 \in H^2(\Omega)$ if the boundary is sufficiently smooth. This is typical of elliptic PDE. The situation is not so good for hyperbolic PDE (e.g. the wave equation).

2.3 Sobolev embeddings

Theorem 2.5. *Let $-\infty < a < b < \infty$, $1 \leq p \leq \infty$. Then every element of $W_0^{1,p}(a,b)$ has a continuous representative, and the following embeddings are well-defined bounded linear maps:*

$$\begin{aligned} W_0^{1,1}(a,b) &\hookrightarrow C([a,b]) \\ W_0^{1,\infty}(a,b) &\hookrightarrow C^{0,1}([a,b]) \qquad (\textit{Lipschitz continuous functions}) \\ W_0^{1,p}(a,b) &\hookrightarrow C^{0,\alpha}([a,b]) \qquad (\textit{Hölder continuous functions}),\ \alpha = 1-\frac{1}{p}, 1<p<\infty. \end{aligned}$$

Proof. **Case $p = 1$.**

For $\varphi \in \mathscr{D}(a,b)$

$$|\varphi(x)| = \left| \int_a^x \varphi' \right| \leq \|\varphi'\|_1 \qquad (a < x < b)$$

so $\|\varphi\|_{\sup} \leq \|\varphi\|_{1,1}$.

Case $p = \infty$.

For $\varphi \in \mathscr{D}(a,b)$

$$|\varphi(x) - \varphi(y)| = \left| \int_x^y \varphi' \right| \leq |y-x| \|\varphi'\|_\infty$$

so

$$\|\varphi\|_{C^{0,1}} = \|\varphi\|_{\sup} + \sup_{a<x<y<b} \frac{|\varphi(x)-\varphi(y)|}{|x-y|} \leq \|\varphi\|_\infty + \|\varphi'\|_\infty \leq 2\|\varphi\|_{1,\infty}.$$

Case $1 < p < \infty$.

For $\varphi \in \mathscr{D}(\Omega)$, $a < x < y < b$,

$$\begin{aligned} |\varphi(x) - \varphi(y)| &\leq \int_x^y |\varphi'| \overset{\text{Hölder}}{\leq} |x-y|^{\frac{1}{q}} \|\varphi'\|_p \\ |\varphi(x)| &\leq (b-a)^{\frac{1}{q}} \|\varphi'\|_p \end{aligned}$$

so with $\alpha = \dfrac{1}{q} = 1 - \dfrac{1}{p}$

$$\|\varphi\|_{C^{0,\alpha}} = \|\varphi\|_{\sup} + \sup_{a<x<y<b} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha} \leq \text{const} \cdot \|\varphi\|_{1,p}.$$

We have proved inequalities of the form

$$\|\varphi\|_X \leq c\|\varphi\|_{1,p} \qquad \varphi \in \mathscr{D}(a,b)$$

with

$$\begin{aligned} X &= C([a,b]) \qquad \text{when } p = 1 \\ X &= C^{0,1}([a,b]) \qquad \text{when } p = \infty \\ X &= C^{0,\alpha}([a,b]) \qquad \text{when } 1 < p < \infty \text{ and } \alpha = 1 - \frac{1}{p}. \end{aligned}$$

For general $u \in W_0^{1,p}(\Omega)$, choose a sequence $\{\varphi_n\}$ in $\mathscr{D}(\Omega)$ converging to u in $\| \|_{1,p}$. Then $\|\varphi_n - u\|_p \to 0$, so passing to a subsequence $\varphi_n \to u$ a.e. Also $\{\varphi_n\}$ is Cauchy in $\| \|_{1,p}$, and by above inequalities Cauchy in $\| \|_X$. Then by completeness $\{\varphi_n\}_{n=1}^\infty$ converges in X to v say. Then $v \in X$, so v is (uniformly) continuous, and $\varphi_n \to v$ uniformly, so $v = u$ a.e. Thus v is a continuous representative for u, hence $W_0^{1,p}(a,b) \subset X$. Finally, $\| \|_X$ and $\| \|_{1,p}$ are continuous functions in $\| \|_X$ and $\| \|_{1,p}$ respectively and $\varphi_n \to u$ in both norms, so the inequality $\| \|_X \leq c\| \|_{1,p}$ holds on the whole of $W_0^{1,p}(a,b)$. □

- In the results proved above for $N = 1$ the restrictions to bounded intervals and $W_0^{m,p}$ can be avoided.
- In higher dimensions we don't generally get continuous functions;
- The embeddings are bounded linear operators, which for certain domains, and for certain values of p, are compact.
- Some results in higher dimensions require regularity assumptions on the boundary.
- Some results require boundedness of the domain.

We now consider the higher-dimensional cases.

Theorem 2.6 (Sobolev's Inequality). *Let $m \geq 1$, $N \geq 2$, $p \geq 1$, $mp < N$, $p^* = \dfrac{Np}{N-mp}$. Then*

$$\|u\|_{p^*} \leq c\Big(\sum_{|\alpha|=m} \|D^\alpha u\|_p^p\Big)^{\frac{1}{p}}$$

for all $u \in C_c^m(\mathbb{R}^N)$ $(\supset \mathscr{D}(\mathbb{R}^N))$.

Proof of the inequality. We consider the following cases:

- **Case $m = 1$, $p = 1$, $p^* = N/(N-1)$.** For $u \in C_c^1(\mathbb{R}^N)$, $x \in \mathbb{R}^N$,

$$u(x) = \int_{-\infty}^{x_j} D_j u(x_1, \ldots, \xi_j, \ldots, x_N) d\xi_j$$

so

$$|u(x)| \leq \int_{-\infty}^{\infty} |D_j u(x)| dx_j$$

so

$$|u(x)|^{\frac{N}{N-1}} \leq \prod_{1 \leq j \leq N} \Big(\int_{-\infty}^{\infty} |D_j u(x)| dx_j\Big)^{\frac{1}{N-1}}.$$

The first term of the product is independent of x_1 and the remaining terms are each functions of $N-1$ variables including x_1. So

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{N}{N-1}} dx_1 \leq \Big(\int_{-\infty}^{\infty} |D_1 u(x)| dx_1\Big)^{\frac{1}{N-1}} \cdot \int_{-\infty}^{\infty} \prod_{j \neq 1} \Big(\int_{-\infty}^{\infty} |D_j u(x)| dx_j\Big)^{\frac{1}{N-1}} dx_1.$$

On the RHS the second term is the integral of a product of $N-1$ functions. Applying the generalised Hölder inequality

$$\int v_1 \cdots v_{N-1} \leq \|v_1\|_{N-1} \cdots \|v_{N-1}\|_{N-1}$$

we obtain

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{N}{N-1}} dx_1 \leq \Big(\int_{-\infty}^{\infty} |D_1 u(x)| dx_1\Big)^{\frac{1}{N-1}} \cdot \prod_{j \neq 1} \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |D_j u(x)| dx_j dx_1\Big)^{\frac{1}{N-1}},$$

thus we have taken the product outside the integral. We repeat this process over all values of j; at each step one factor in the RHS is independent of x_j, and we apply the generalised Hölder inequality to the integral of the product of the remaining $N-1$ factors. We end up with

$$\int_{\mathbb{R}^N} |u|^{\frac{N}{N-1}} \leq \prod_{j=1}^{N} \Big(\int_{\mathbb{R}^N} |D_j u|\Big)^{\frac{1}{N-1}}$$

so taking the $(N-1)/N$-th power yields

$$\|u\|_{p^*} \leq \prod_{j=1}^{N} \Big(\int_{\mathbb{R}^N} |D_j u|\Big)^{\frac{1}{N}}.$$

Now by the AM-GM inequality

$$\|u\|_{p^*} \le \frac{1}{N} \sum_{j=1}^{N} \|D_j u\|_1.$$

This proves the case $m = 1$, $p = 1$.

• **Case** $m = 1$**,** $1 < p < N$**,** $p^* = \dfrac{Np}{N-p}$**.** Let $u \in C_c^1(\mathbb{R}^N)$. Let $v = |u|^s$ where $s > 1$ is to be chosen later; note that $v \in C_c^1(\mathbb{R}^N)$. Applying the above inequality to v,

$$\left(\int_{\mathbb{R}^N} |u|^{\frac{sN}{N-1}} \right)^{\frac{N-1}{N}} \le c. \int_{\mathbb{R}^N} |\nabla v|_1 = c. \int_{\mathbb{R}^N} |u|^{s-1} |\nabla u|_1 \le c \|\nabla u\|_p \left(\int_{\mathbb{R}^N} |u|^{(s-1)q} \right)^{\frac{1}{q}}$$

where $\dfrac{1}{q} + \dfrac{1}{p} = 1$. We choose s so that $\dfrac{sN}{N-1} = (s-1)q$, which yields

$$s = \frac{(N-1)p}{N-p}.$$

Thus

$$\left(\int_{\mathbb{R}^N} |u|^{\frac{sN}{N-1}} \right)^{\frac{N-1}{N} - \frac{1}{q}} \le c \|\nabla u\|_p.$$

Then

$$\frac{sN}{N-1} = \frac{Np}{N-p} = p^* \qquad \text{and} \qquad \frac{N-1}{N} - \frac{1}{q} = \frac{N-p}{Np} = \frac{1}{p*}$$

so

$$\|u\|_{p^*} \le c \|\nabla u\|_p.$$

This completes the case $m = 1$, $1 < p < N$.

• **General case.** Induction on m. The initial case $m = 1$ is done. Assume true for $m - 1$. Consider $\alpha \in \mathbb{N}_0^N$, $|\alpha| = m - 1$, $u \in C_C^m(\mathbb{R}^N)$. Then by the initial case

$$\|D^\alpha u\|_{\frac{Np}{N-p}} \le c \|\nabla D^\alpha u\|_p.$$

Thus by the inductive hypothesis

$$\|u\|_{\frac{NNp/(N-p)}{N-(m-1)Np/(N-p)}} \le c \sum_{|\alpha|=m-1} \|D^\alpha u\|_{\frac{Np}{N-p}} \le c \sum_{|\beta|=m} \|D^\beta u\|_p$$

that is

$$\|u\|_{\frac{Np}{N-mp}} \le c \sum_{|\beta|=m} \|D^\beta u\|_p$$

as required, since all norms on a Euclidean space are equivalent. This completes the inductive step and we are done. □

Corollary 2.7. *Let $N \ge 2$, $m \ge 1$, $mp < N$, $p^* = \dfrac{Np}{N-mp}$, $\varnothing \ne \Omega \subset \mathbb{R}^N$ open. Then $W_0^{m,p}(\Omega)$ is embedded in $L^{p^*}(\Omega)$ and the embedding is a bounded linear map.*

Proof. Let c be the constant in the Sobolev inequality for the given N, m, p. Thus

$$\|\varphi\|_{p^*} \leq c \left(\sum_{|\alpha|=m} \|D^\alpha \varphi\|_p^p \right)^{\frac{1}{p}} \leq c\|\varphi\|_{m,p} \qquad \forall \varphi \in \mathscr{D}(\Omega).$$

Consider $u \in W_0^{m,p}(\Omega)$; so u is the limit in $\| \, \|_{m,p}$ of a sequence $\{\varphi_n\}$ of test functions. We can also assume $\varphi_n \to u$ a.e. Now $\{\varphi_n\}$ is Cauchy in $\| \, \|_{m,p}$ and therefore Cauchy in $\| \, \|_{p^*}$, so $\{\varphi_n\}$ converges in L^{p^*}, and the limit must equal u a.e. Thus $u \in L^{p^*}$. Continuity of $\| \, \|_{p^*}$ on L^{p^*} and $\| \, \|_{m,p}$ on $W^{m,p}$ now ensure

$$\|u\|_{p^*} \leq c\|u\|_{m,p}. \qquad \square$$

Definitions. *Let $\varnothing \neq \Omega \subset \mathbb{R}^N$ be open, $m \in \mathbb{N}$, $0 < \lambda \leq 1$.*

$$\begin{aligned}
C(\Omega) &= \{\textit{continuous functions on } \Omega\} \\
C^m(\Omega) &= \{\textit{functions } m\textit{-times continuously differentiable on } \Omega\} \\
C_B(\Omega) &= \{\textit{bounded continuous functions on } \Omega\} \\
C_B^m(\Omega) &= \{u \mid D^\alpha u \in C_B(\Omega), 0 \leq |\alpha| \leq m\} \\
C(\overline{\Omega}) &= \{\textit{bounded uniformly continuous functions on } \Omega\} \\
C^m(\overline{\Omega}) &= \{u \mid D^\alpha u \in C(\overline{\Omega}), 0 \leq |\alpha| \leq m\} \\
C^{0,\lambda}(\Omega) &= \{\textit{functions on } \Omega \textit{ of Hölder class } \lambda\} \\
C^{m,\lambda}(\Omega) &= \{u \mid D^\alpha u \in C^{0,\lambda}(\Omega), 0 \leq |\alpha| \leq m\}
\end{aligned}$$

Then

$$C^{m,\lambda}(\Omega) \subset C^m(\overline{\Omega}) \subset C_B^m(\Omega) \subset C^m(\Omega),$$

functions $u \in C^m(\overline{\Omega})$ have $D^\alpha u$, $0 \leq |\alpha| \leq m$, continuously extendable to $\overline{\Omega}$, Hölder continuous in case $C^{m,\lambda}$. Note $C(\overline{\mathbb{R}^N}) \neq C(\mathbb{R}^N)$. Then $C_B^m(\Omega)$ and $C^m(\overline{\Omega})$ are Banach spaces with

$$\|u\|_{C_B^m(\Omega)} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{\sup}.$$

$C^{m,\lambda}(\Omega)$ is a Banach space with

$$\|u\|_{C^{m,\lambda}(\Omega)} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{\sup} + \max_{0 \leq |\alpha| \leq m} \sup_{x,y \in \Omega, x \neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\lambda}.$$

Theorem 2.8 (Morrey's Inequality)**.** *Suppose $N \geq 2$, $N < p < \infty$. Then there is a constant $c = c(N,p)$ such that*

$$\|u\|_{C^{0,\lambda}} \leq c\|u\|_{1,p} \qquad \forall u \in C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N),$$

where $\lambda = 1 - \dfrac{N}{p}$.

Proof. **Step 1**. We show that

$$\fint_{B(x,r)} |u(y) - u(x)| dy \leq c \int_{B(x,r)} \frac{|\nabla u(y)|}{|y - x|^{N-1}} dy$$

where $\fint$ denotes the mean.

Preliminary calculation

$$\begin{aligned}
\int_{\partial B(0,1)} |u(x + sw) - u(x)| dw &= \int_{\partial B(0,1)} \left| \int_0^s \frac{d}{dt} u(x + tw) dt \right| dw \\
&\leq \int_{\partial B(0,1)} \int_0^s |\nabla u(x + tw) \cdot w| dt dw \\
&\leq \int_{\partial B(0,1)} \int_0^s |\nabla u(x + tw)| dt dw \\
&= \int_0^s \int_{\partial B(0,1)} |\nabla u(x + tw)| dw dt \\
&= \int_0^s \int_{\partial B(0,t)} |\nabla u(x + w)| \frac{1}{t^{N-1}} dw dt \\
&= \int_{B(0,s)} \frac{|\nabla u(x + w)|}{|w|^{N-1}} dt \\
&= \int_{B(x,s)} \frac{|\nabla u(y)|}{|x - y|^{N-1}} dy.
\end{aligned}$$

Now

$$\begin{aligned}
\int_{B(x,r)} |u(y) - u(x)| dy &= \int_0^r \int_{\partial B(0,s)} |u(x + w) - u(x)| dw ds \\
&= \int_0^r s^{N-1} \int_{\partial B(0,1)} |u(x + sw) - u(x)| dw ds \\
&\leq \int_0^r s^{N-1} \int_{B(x,s)} \frac{|\nabla u(y)|}{|y - x|^{N-1}} dy ds \\
&\leq \left(\int_0^r s^{N-1} ds \right) \left(\int_{B(x,r)} \frac{|\nabla u(y)|}{|y - x|^{N-1}} dy \right) \\
&= \frac{r^N}{N} \int_{B(x,r)} \frac{|\nabla u(y)|}{|y - x|^{N-1}} dy.
\end{aligned}$$

So dividing by r^N

$$\fint_{B(x,r)} |u(y) - u(x)| dy \leq c \int_{B(x,r)} \frac{|\nabla u(y)|}{|y - x|^{N-1}} dy.$$

Step 2. Estimate $\|u\|_{\sup}$.

For $x \in \mathbb{R}^N$

$$
\begin{aligned}
|u(x)| &\le \fint_{B(x,1)} |u(x)-u(y)|dy + \fint_{B(x,1)} |u(y)|dy \\
&\le c\int_{B(x,1)} \frac{|\nabla u(y)|}{|x-y|^{N-1}}dy + |B(x,1)|^{\frac{1}{q}-1}\|u\|_p \qquad \text{(by Step 1 and Hölder } \tfrac{1}{q}+\tfrac{1}{p}=1\text{)} \\
&\le \|\nabla u\|_p \left(\int_{B(x,1)} |x-y|^{-(N-1)q}dy\right)^{\frac{1}{q}} + c\|u\|_p \qquad \text{(since } (N-1)q<N\text{)} \\
&\le c\|\nabla u\|_p + c\|u\|_p \\
&\le c\|u\|_{1,p}.
\end{aligned}
$$

Step 3. Hölder estimate for $|u(x)-u(y)|$.
Consider $x, y \in \mathbb{R}^N$, $|x-y| = r > 0$. For any $z \in \mathbb{R}^N$

$$|u(x)-u(y)| \le |u(x)-u(z)| + |u(z)-u(y)|.$$

So averaging over a region W of finite positive measure

$$|u(x)-u(y)| \le \fint_W |u(x)-u(z)|dz + \fint_W |u(z)-u(y)|dz.$$

Choose $W_r = B(x,r) \cap B(y,r)$ (c.f. Figure 1).

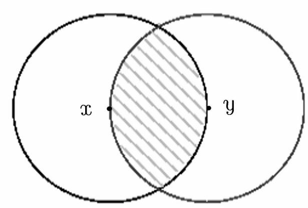

Figure 1: $W_r = B(x,r) \cap B(y,r)$

Notice that W_r is similar to $W_1 = B(0,1) \cap B(e,1)$ where e is any unit vector. So

$$|W_r| = r^N |W_1|.$$

Now

$$|W_r| \fint_{W_r} |u(x)-u(z)|d\zeta \le |B(x,r)| \fint_{B(x,r)} |u(x)-u(z)|dz.$$

So

$$\fint_{W_r} |u(x)-u(z)|dz \le \frac{|B(x,r)|}{|W_r|} \fint_{B(x,r)} |u(x)-u(z)|dz$$

thus

$$\fint_{W_r} |u(x)-u(z)|dz \le c\fint_{B(x,r)} |u(x)-u(z)|dz \qquad \text{(since } \tfrac{|B(x,r)|}{|W_r|} \text{ is independent of } r\text{)}.$$

Now using Step 1,

$$\fint_{W_r} |u(x)-u(z)|dz \le \text{const}\cdot \int_{B(x,r)} \frac{|\nabla u(z)|}{|x-z|^{N-1}}dz \le c\|\nabla u\|_p \left(\int_{B(x,r)} |x-z|^{-(N-1)q}dz\right)^{\frac{1}{q}}.$$

Now

$$\begin{aligned}\int_{B(x,r)} |x-z|^{-(N-1)q}dz &= \int_0^r \int_{\partial B(x,s)} s^{-(N-1)q}dzds = c\int_0^r s^{N-1}s^{-(N-1)q}ds \\ &= cr^{(N-1)(1-q)+1} = cr^{\frac{p-N}{p-1}},\end{aligned}$$

since

$$(N-1)(1-q)+1 = (N-1)\left(1-\frac{p}{p-1}\right)+1 = (N-1)\frac{(-1)}{p-1}+1 = \frac{p-N}{p-1}.$$

So

$$\fint_{W_r} |u(x)-u(z)|dz \le c\|\nabla u\|_p\, r^{\frac{p-N}{p-1}\cdot\frac{1}{q}} = c\|\nabla u\|_p\, r^{1-\frac{N}{p}},$$

since

$$\frac{p-N}{p-1}\cdot\frac{1}{q} = \frac{p-N}{p-1}\left(1-\frac{1}{p}\right) = \frac{p-N}{p} = 1-\frac{N}{p}.$$

Similarly

$$\fint_{W_r} |u(z)-u(y)|dz \le c\|\nabla u\|_p\, r^{1-\frac{N}{p}}$$

so

$$|u(x)-u(y)| \le c\|\nabla u\|_p |x-y|^{1-\frac{N}{p}}.$$

That is,

$$\frac{|u(x)-u(y)|}{|x-y|^\lambda} \le c\|\nabla u\|_p.$$

Now

$$\|u\|_{C^{0,\lambda}} = \|u\|_{\sup} + \sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^\lambda} \le c\|u\|_{W^{1,p}(\mathbb{R}^N)}. \qquad \square$$

Theorem 2.9. *Let $N \ge 2$, $m \in \mathbb{N}$, $m < N$, $mp = N$, $1 \le q < \infty$. Then there exists a constant $c = c(N,m,q)$ such that*

(i) $\|u\|_q \le c|\Omega|^{1/q} \sum_{|\alpha|=m} \|D^\alpha u\|_p$ for all $u \in C_c^m(\mathbb{R}^N)$, where $\Omega = \{x \in \mathbb{R}^N \mid u(x) \neq 0\}$;

(ii) $\|u\|_q \le c\|u\|_{m,p}$, for all $u \in C^m(\mathbb{R}^N) \cap W^{m,p}(\mathbb{R}^N)$ and $q > p$.

Proof. <u>**(i) Case** $\frac{N}{N-m} \le q < \infty$</u>. Choose r, $1 \le r < p$, such that $r^* = \frac{Nr}{N-mr} = q$. Then by the Sobolev inequality, for $u \in C_c^m(\mathbb{R}^N)$

$$\|u\|_q \le c \sum_{|\alpha|=m} \|D^\alpha u\|_r.$$

Now

$$\int_{\mathbb{R}^N} |D^\alpha u|^r \le \|1_\Omega\|_{s'} \left(\int_{\mathbb{R}^N} |D^\alpha u|^{rs} \right)^{1/s}$$

where $rs = p$, $1/s' + 1/s = 1$, $rs' = pr/(p-r) = (Nr/m)/(N/m - r) = (Nr)/(N-mr) = q$ and $\Omega = \{x \mid u(x) \neq 0\}$. So

$$\|D^\alpha u\|_r \le |\Omega|^{1/rs'} \|D^\alpha u\|_p = |\Omega|^{1/q} \|D^\alpha u\|_p.$$

Thus

$$\|u\|_q \le c|\Omega|^{1/q} \sum_{|\alpha|=m} \|D^\alpha u\|_p.$$

<u>**(i) Case** $1 \le q < \frac{N}{N-m} = t$</u>. We have

$$\int_{\mathbb{R}^N} |u|^q \le \|1_\Omega\|_{s'} \||u|^q\|_s \qquad \text{(where } qs = t \text{ and } 1/s' + 1/s = 1)$$

so

$$\|u\|_q \le |\Omega|^{1/s'q} \|u\|_t = |\Omega|^{1/q - 1/t} \|u\|_t.$$

Using the previous case to estimate $\|u\|_t$ we get

$$\|u\|_q \le c|\Omega|^{1/q-1/t} |\Omega|^{1/t} \sum_{|\alpha|=m} \|D^\alpha u\|_p = c|\Omega|^{1/q} \sum_{|\alpha|=m} \|D^\alpha u\|_p.$$

<u>**(ii)**</u> Suppose $u \in C^m(\mathbb{R}^N) \cap W^{m,p}(\mathbb{R}^N)$, $q > p$. Construct a partition of unity as follows. Let $\Phi \in C^\infty(\mathbb{R})$ satisfy $\Phi(\xi) > 0$ for $-1 < \xi < 1$ and $\Phi(\xi) = 0$ for $|\xi| \ge 1$. For $k = (k_1, \ldots, k_N) \in \mathbb{Z}^N$ let

$$\Phi_k(x) = \prod_{i=1}^N \Phi(x_i - k_i)$$

which lives on

$$Q_k = (-1 + k_1, 1 + k_1) \times \cdots \times (-1 + k_N, 1 + k_N).$$

Note almost every $x \in \mathbb{R}^N$ belongs to 2^N of the cubes Q_k, and all points belong to at least one. Define

$$\varphi_k = \frac{\Phi_k}{\sum_{l \in \mathbb{Z}^N} \Phi_l}$$

which are smooth functions adding up to the constant function 1, and all but finitely many vanish outside any bounded set. Thus

$$u = \sum_{k\in\mathbb{Z}^N} \varphi_k u.$$

Now

$$|u(x)|^q = \left|\sum_k \varphi_k(x)u(x)\right|^q = 2^{Nq}\left|\sum_k 2^{-N}\varphi_k(x)u(x)\right|^q \leq 2^{Nq}\sum_k 2^{-N}|\varphi_k(x)u(x)|^q$$

by Jensen's inequality, so

$$\int_{\mathbb{R}^N} |u|^q \leq 2^{N(q-1)} \int_{\mathbb{R}^N} \sum_k |\varphi_k u|^q.$$

Now

$$\begin{aligned}
\int_{Q_k} |\varphi_k u|^q &\leq c\left(\sum_{|\alpha|=m}\int_{Q_k} |D^\alpha(\varphi_k u)|^p\right)^{q/p} \qquad \text{(by (i))}\\
&\leq c\left(\sum_{0\leq|\alpha|\leq m}\int_{Q_k}\left(\sum_{0\leq\beta\leq\alpha}|D^\beta u|\right)^p\right)^{q/p}\\
&\leq c\left(\sum_{0\leq|\beta|\leq m}\int_{Q_k}|D^\beta u|^p\right)^{q/p}
\end{aligned}$$

where we have differentiated by Leibniz's theorem and used the independence of $\|D^{\alpha-\beta}\varphi_k\|_{\sup}$ from k for each α, β, then applied Jensen's inequality.

Then

$$\begin{aligned}
\int_{\mathbb{R}^N}|u|^q \leq c\sum_{k\in\mathbb{Z}^N}\left(\sum_{0\leq|\beta|\leq m}\int_{Q_k}|D^\beta u|^p\right)^{q/p} &\leq c\left(\sum_{k\in\mathbb{Z}^N}\sum_{0\leq|\beta|\leq m}\int_{Q_k}|D^\beta u|^p\right)\|u\|_{m,p,\mathbb{R}^N}^{q-p}\\
= c\left(2^N\sum_{0\leq|\beta|\leq m}\int_{\mathbb{R}^N}|D^\beta u|^p\right)\|u\|_{m,p,\mathbb{R}^N}^{q-p} &= c\|u\|_{m,p,\mathbb{R}^N}^q
\end{aligned}$$

since the family $\{Q_k\}_{k\in\mathbb{Z}^N}$ forms a 2^N-fold covering of $\mathbb{R}^N$ except for a set of zero measure. Thus

$$\|u\|_q \leq c\|u\|_{m,p}. \qquad \square$$

Theorem 2.10. *Let $I = (a_1, b_1)\times\cdots\times(a_N, b_N)$ be a rectangle in $\mathbb{R}^N$. Then there is a constant c, depending only on the edge-lengths of I, such that*

(i) $\|u\|_{\sup} \leq c\|u\|_{N,1}$ for all $u \in C^N(I)\cap W^{N,1}(I)$;

(ii) $\|u\|_{\sup} \leq c\|u\|_{N,1}$ *and* $u(x) \to 0$ *as* $|x| \to \infty$ *for all* $u \in C^N(\mathbb{R}^N) \cap W^{N,1}(\mathbb{R}^N)$.

Proof. **Case $N = 1$**. For $u \in C^1(a_1, b_1) \cap W^{1,1}(a_1, b_1)$, $x, y \in I = (a_1, b_1)$

$$|u(x) - u(y)| \leq \int_{a_1}^{b_1} |u'|$$

and, by continuity,there exists $\bar{x} \in I$ such that

$$u(\bar{x}) = (b_1 - a_1)^{-1} \int_{a_1}^{b_1} u.$$

So

$$|u(y)| \leq |u(\bar{x})| + \int_{a_1}^{b_1} |u'| \leq (b_1 - a_1)^{-1} \int_{a_1}^{b_1} |u| + \int_{a_1}^{b_1} |u'| \leq \max\{(b_1 - a_1)^{-1}, 1\}\|u\|_{1,1}.$$

Inductive step. Assume true in dimension $N - 1$. Consider $x, y \in I$ and suppose initially that x and y differ in one coordinate only, say the last. Write $x = (x', x_N), y = (x', y_N)$ where $x' \in \mathbb{R}^{N-1}$ and $x_N, y_N \in (a_N, b_N)$. Then

$$\begin{aligned} |u(x) - u(y)| &\leq \int_{a_N}^{b_N} |D_N u(x', \xi)| d\xi \leq c(\ell_1, \ldots, \ell_{N-1}) \int_{a_N}^{b_N} \|D_N u(\cdot, \xi)\|_{N-1,1}\, d\xi \\ &\leq c(\ell_1, \ldots, \ell_{N-1})\|u\|_{N,1} \end{aligned}$$

where the $W^{N-1,1}$-norm is taken over an $(N - 1)$-dimensional rectangle and $\ell_j = b_j - a_j$.

In the general case we can choose points $x = x^0, x^1, \ldots, x^N = y$ such that $x^i - x^{i-1}$ is parallel to the i-th coordinate axis, and apply the above calculation to obtain

$$|u(x) - u(y)| \leq \sum_{i=1}^{N} |u(x^i - u(x^{i-1})| \leq c(\ell_1, \ldots, \ell_N)\|u\|_{N,1}.$$

We can choose $\bar{x} \in I$ such that $u(\bar{x}) = |I|^{-1} \int_I u$. Then, for all $y \in I$,

$$\begin{aligned} |u(y)| &\leq |u(\bar{x})| + |u(y) - u(\bar{x})| \leq |u(\bar{x})| + c(\ell_1, \ldots, \ell_N)\|u\|_{N,1} \\ &\leq (\ell_1 \ell_2 \cdots \ell_N)^{-1} \int_I |u| + c(\ell_1, \ldots, \ell_N)\|u\|_{N,1} \\ &\leq c(\ell_1, \ldots, \ell_N)\|u\|_{N,1}. \end{aligned}$$

This completes the inductive step.

The remaining parts of Theorem 2.10 are an exercise. $\square$

Lemma 2.11. *If* $\Omega \subset \mathbb{R}^N$ *is open and* $0 < \alpha < \beta \leq 1$*, then the embedding* $C^{0,\beta}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$ *is bounded. [Exercise]*

Theorem 2.12 (Sobolev Embedding Theorem for $W_0^{m,p}(\Omega)$)**.** *Suppose $N \geq 2$, $\Omega \subset \mathbb{R}^N$ is open, $1 \leq p < \infty$, $m \in \mathbb{N}$. Then the following embeddings are well-defined bounded linear maps:*

(i) $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, *$mp < N$, $p \leq q \leq p^*$, $p^* = \dfrac{Np}{N-mp}$;*

(ii) $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, *$mp = N$, $1 < p \leq q < \infty$;*

(iii) $W_0^{N,1}(\Omega) \hookrightarrow C(\overline{\Omega})$ *and* $W_0^{N,1}(\Omega) \hookrightarrow L^q(\Omega)$ *for $1 \leq q \leq \infty$;*

(iv) $W_0^{m,p}(\Omega) \hookrightarrow C^{0,\lambda}(\Omega)$, *$mp > N > (m-1)p$, $0 < \lambda \leq m - \dfrac{N}{p}$;*

(v) $W_0^{m,p}(\Omega) \hookrightarrow C^{0,\lambda}(\Omega)$, *$(m-1)p = N$, $0 < \lambda < 1$.*

Proof. First check for test functions u.

(i) $mp < N$. We have $\|u\|_{p^*} \leq c\|u\|_{m,p}$ by the Sobolev inequality and $\|u\|_p \leq \|u\|_{m,p}$. We get $L^{p^*} \hookrightarrow L^q$ by interpolation: assume $p < q < p^*$ and write

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*} \qquad \text{for some } 0 < \theta < 1;$$

then

$$\|u\|_q \leq \|u\|_p^\theta \|u\|_{p^*}^{1-\theta}$$

hence

$$\|u\|_q \leq c\|u\|_p^\theta \|u\|_{m,p}^{1-\theta} \leq c\|u\|_{m,p} \qquad \text{for } u \in \mathscr{D}(\Omega).$$

(ii) $mp = N$, $1 < p < q < \infty$. Theorem 2.9 shows $\|u\|_q \leq c\|u\|_{m,p}$ for $u \in \mathscr{D}(\Omega) \subset C^m(\overline{\Omega})$.

(iii) $m = N$, $p = 1$.
Theorem 2.10(ii) shows $\|u\|_{\sup} \leq c\|u\|_{N,1}$ for $u \in \mathscr{D}(\Omega)$ $(\subset W^{N,1}(\mathbb{R}^N) \cap C^N(\mathbb{R}^N))$ (and hence $\|u\|_\infty \leq c\|u\|_{N,1}$).
If $1 < q < \infty$ then $\|u\|_q^q \leq \|u\|_1 \|u\|_{\sup}^{q-1} \leq c\|u\|_{N,1}^q$.

(iv) $mp > N > (m-1)p$. We want to use the Morrey inequality

$$\|v\|_{C^{0,\lambda_0}} \leq c\|v\|_{1,p_0} \qquad \text{for } v \in \mathscr{D}(\Omega)$$

where

$$p_0 = \frac{Np}{(N-(m-1))p} \; (> N \text{ since } mp > N) \qquad \text{and} \qquad \lambda_0 = 1 - \frac{N}{p_0}$$

and the Sobolev inequality

$$\|v\|_{p_0} \leq c\|v\|_{m-1,p} \qquad (\text{since } (m-1)p < N). \tag{2}$$

So for test functions u

$$\begin{aligned}\|\nabla u\|_{p_0} &\le c\|\nabla u\|_{m-1,p} \qquad \text{(from (2))}\\ &\le c\|u\|_{m,p}\end{aligned}$$

and

$$\|u\|_{p_0} \le c\|u\|_{m,p} \qquad \text{(from (2))}$$

so

$$\|u\|_{1,p_0} \le c\|u\|_{m,p}.$$

Morrey now gives

$$\|u\|_{C^{0,\lambda_0}} \le c\|u\|_{m,p}.$$

By Lemma 2.11, for $0 < \lambda \le \lambda_0$

$$\|u\|_{C^{0,\lambda}} \le c\|u\|_{m,p}.$$

Finally, note that

$$\lambda_0 = 1 - \frac{N}{p_0} = m - \frac{N}{p}.$$

(v) $N = (m-1)p$. Then, for $p \le q < \infty$

$$\begin{aligned}\|\nabla u\|_q &\le c\|\nabla u\|_{m-1,p} \qquad \text{(Theorem 2.9)}\\ &\le c\|u\|_{m,p}\end{aligned}$$

$$\|u\|_q \le c\|u\|_{m-1,p} \le c\|u\|_{m,p}$$

so

$$\|u\|_{1,q} \le c\|u\|_{m,p}. \tag{3}$$

For $q > N$

$$\|u\|_{C^{0,\lambda}} \le c\|u\|_{1,q} \qquad \text{(Morrey)} \tag{4}$$

where $\lambda = 1 - \dfrac{N}{q}$; by varying q in the range $N < q < \infty$ we can make λ take any value, $0 < \lambda < 1$. Thus from (3) and (4) we have

$$\|u\|_{C^{0,\lambda}} \le c\|u\|_{m,p} \qquad \text{if } 0 < \lambda < 1 \text{ and } u \in \mathscr{D}(\Omega).$$

So in each of the above cases we have an inequality

$$\|u\|_X \le c\|u\|_{m,p} \qquad \text{for all } u \in \mathscr{D}(\Omega) \tag{5}$$

where X is $L^q(\Omega)$, $C(\overline{\Omega})$, or $C^{0,\lambda}(\Omega)$ as appropriate.

For general $u \in W_0^{m,p}(\Omega)$ choose a sequence $\{u_n\}$ of test functions converging in $\| \, \|_{m,p}$ to u. Then $\{u_n\}$ is Cauchy in $\| \, \|_{m,p}$, and therefore Cauchy in $\| \, \|_X$, so $\{u_n\}$ converges in X to $\overline{u}$ say. Passing to a subsequence, $u_n \to u$ a.e., and either $u_n \to \overline{u}$ uniformly, or after passing to a subsequence $u_n \to \overline{u}$ a.e. So $u = \overline{u}$ a.e. Each side of (5) is continuous on X or $W^{m,p}$ as appropriate. So (5) also holds for u. □

3 Regularisation and approximation

Definition. *The **convolution** of two measurable functions u, v on $\mathbb{R}^N$*

$$u * v(x) = \int_{\mathbb{R}^N} u(y)v(x-y)dy$$

when this exists.

Lemma 3.1. *(i) If $u, v \in L^1(\mathbb{R}^N)$ then $u * v$ is defined a.e. on $\mathbb{R}^N$ and*

$$\|u * v\|_1 \leq \|u\|_1 \|v\|_1.$$

(ii) If $u \in L^1_{loc}(\mathbb{R}^N)$ and $v \in L^1(\mathbb{R}^N)$ has compact support, then $u * v$ and $v * u$ exist a.e. and $u * v = v * u$ a.e., and is locally L^1.

(iii) If $u \in L^1_{loc}(\mathbb{R}^N)$, $v, w \in L^1(\Omega)$, v, w have compact support then

$$(u * v) * w = u * (v * w) \quad a.e.$$

(iv) If $u \in L^1_{loc}(\mathbb{R}^N)$ and $v \in C_c(\mathbb{R}^N)$ then $u * v$ is continuous.

Proof. Not given, by Fubini. Part (iv) exercise. □

Lemma 3.2. *Suppose $u \in L^1_{loc}(\mathbb{R}^N)$ and $\varphi \in \mathscr{D}(\mathbb{R}^N)$. Then*

(i) $D^\alpha(u * \varphi) = u * D^\alpha \varphi$ for all $\alpha \in \mathbb{N}_0^N$ and is continuous, hence $u * \varphi \in C^\infty(\mathbb{R}^N)$.

(ii) If $D^\alpha u \in L^1_{loc}(\mathbb{R}^N)$ then

$$D^\alpha(u * \varphi) = (D^\alpha u) * \varphi.$$

Proof. (i) Consider first order partial derivatives; say e is the unit vector in the x_i direction for some i. If $0 < |h| < 1$, then

$$\begin{aligned} \frac{(u * \varphi)(x + he) - (u * \varphi)(x)}{h} &= \int_{\mathbb{R}^N} u(y) \frac{\varphi(x + he - y) - \varphi(x - y)}{h} dy \\ &= \int_U u(y) \frac{\varphi(x + he - y) - \varphi(x - y)}{h} dy \end{aligned}$$

where $U = B(x,1) - \operatorname{supp}\varphi$. The integrand converges pointwise to $uD_i\varphi$, and is dominated by $|u(x)|\|D_i\varphi\|_{\sup}$ which is integrable on the compact set U so we can pass to the limit using the Dominated Convergence Theorem to get

$$D_i(u * \varphi) = u * D_i\varphi.$$

Repeated applications give the result for D^α.

$D^\alpha\varphi$ is continuous and has compact support, hence $u * D^\alpha\varphi$ is continuous for every $\alpha \in \mathbb{N}_0^N$, hence $u * \varphi \in C^\infty(\mathbb{R}^N)$.

(ii) Assume $D^\alpha u \in L^1_{loc}(\mathbb{R}^N)$, for some $\alpha \in \mathbb{N}_0^N$. For $\varphi \in \mathscr{D}(\mathbb{R}^N)$

$$\begin{aligned}
D^\alpha(u * \varphi)(x) &= \int_{\mathbb{R}^N} u(y)(D^\alpha\varphi)(x-y)dy \qquad \text{(by (i))} \\
&= \int_{\mathbb{R}^N} u(y)(-1)^{|\alpha|}D^\alpha\overline{\varphi}_x(y)dy \qquad (\overline{\varphi}_x(y) = \varphi(x-y)) \\
&= (-1)^{|\alpha|}\langle u, D^\alpha\overline{\varphi}_x\rangle \\
&= \langle D^\alpha u, \overline{\varphi}_x\rangle \\
&= \int_{\mathbb{R}^N} D^\alpha u(y)\varphi(x-y)dy \qquad (\text{since } D^\alpha u \in L^1_{loc}) \\
&= (D^\alpha u) * \varphi \qquad (6)
\end{aligned}$$

□

Reminder.

$$J(x) = \begin{cases} ke^{-\frac{1}{1-|x|^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \qquad x \in \mathbb{R}^N$$

with $k > 0$ chosen so that $\int_{\mathbb{R}^N} J = 1$. Take

$$J_\varepsilon(x) = \varepsilon^{-N}J(\varepsilon^{-1}x) \qquad \text{(Standard mollifier)}$$

so $\int_{\mathbb{R}^N} J_\varepsilon = 1$ and $\operatorname{supp} J_\varepsilon = \overline{B}(0,\varepsilon)$.

Definition. *If $u \in L^1_{loc}(\mathbb{R}^N)$ we call $J_\varepsilon * u$ the **mollification** or **regularisation** of u.*

Note.

(1) $J_\varepsilon * u \in C^\infty(\mathbb{R}^N)$.

(2) $J_\varepsilon * u(x) = \int_{\mathbb{R}^N} u(y)J_\varepsilon(x-y)dy = \int_{B(x,\varepsilon)} u(y)J_\varepsilon(x-y)dy$.
So $J_\varepsilon * u(x)$ is a weighted mean of u over $B(x,\varepsilon)$.

(3) $D^\alpha(J_\varepsilon * u) = (D^\alpha J_\varepsilon) * u$.

(4) If $D^\alpha u \in L^1_{loc}(\mathbb{R}^N)$ then $D^\alpha(J_\varepsilon * u) = J_\varepsilon * D^\alpha u$.

Theorem 3.3. *Let $1 \le p \le \infty$.*

(i) $\|J_\varepsilon * u\|_p \le \|u\|_p$ for $u \in L^p(\mathbb{R}^N)$ and $\|J_\varepsilon * u\|_{m,p} \le \|u\|_{m,p}$ for $u \in W^{m,p}(\mathbb{R}^N)$.

(ii) If $1 \le p < \infty$ then $\|J_\varepsilon * u - u\|_p \to 0$ as $\varepsilon \to 0$ for $u \in L^p(\mathbb{R}^N)$ and $\|J_\varepsilon * u - u\|_{m,p} \to 0$ as $\varepsilon \to 0$ for $u \in W^{m,p}(\mathbb{R}^N)$.

(iii) if $1 \le p < \infty$ then $\mathscr{D}(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ and in $W^{m,p}(\mathbb{R}^N)$ so $W_0^{m,p}(\mathbb{R}^N) = W^{m,p}(\mathbb{R}^N)$.

Proof. (i) Case $1 \le p < \infty$.

$$\begin{aligned} |J_\varepsilon * u(x)|^p &= \left| \int_{\mathbb{R}^N} u(y) J_\varepsilon(x-y)\,dy \right|^p \\ &\le \int_{\mathbb{R}^N} |u(y)|^p J_\varepsilon(x-y)\,dy \qquad \text{(Jensen's Inequality)}. \end{aligned}$$

So

$$\begin{aligned} \int_{\mathbb{R}^N} |J_\varepsilon * u(x)|^p dx &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(y)|^p J_\varepsilon(x-y)\,dydx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(y)|^p J_\varepsilon(x-y)\,dxdy \\ &= \int_{\mathbb{R}^N} |u(y)|^p dy \qquad \left(\text{since } \int_{\mathbb{R}^N} J_\varepsilon = 1\right) \end{aligned}$$

thus $\|J_\varepsilon * u\|_p \le \|u\|_p$.

Case $p = \infty$.

$$|J_\varepsilon * u(x)| = \left| \int_{\mathbb{R}^N} u(y) J_\varepsilon(x-y)\,dy \right| \le \|u\|_\infty \int_{\mathbb{R}^N} J_\varepsilon(x-y)\,dy = \|u\|_\infty.$$

In either case, the inequality $\|J_\varepsilon * u\|_{m,p} \le \|u\|_{m,p}$ follows by applying the above to each $D^\alpha u$.

(ii) Suppose $1 \le p < \infty$.

Simple case. $u = 1_Q$ where Q is a rectangle. Then

$$J_\varepsilon * u(x) = \begin{cases} 1 & \text{for all small } \varepsilon > 0 \text{ if } x \in Q^\circ \\ 0 & \text{for all small } \varepsilon > 0 \text{ if } x \in \mathbb{R}^N \setminus \overline{Q} \end{cases}$$

$$0 \le J_\varepsilon * u(x) \le 1 \text{ for all } x, \qquad \text{and} \qquad J_\varepsilon(x) = 0 \text{ if } x \notin Q + B(0,1), \varepsilon < 1.$$

So $J_\varepsilon * u \to u$ a.e. and $0 \le J_\varepsilon * u \le 1_{Q+B(0,1)}$. So the Dominated Convergence Theorem shows

$$\begin{aligned}\int_{\mathbb{R}^N} |J_\varepsilon * u - u|^p &= \int_{Q+B(0,1)} |J_\varepsilon * u - u|^p \qquad (\varepsilon < 1)\\ &\to 0 \quad \text{as } \varepsilon \to 0.\end{aligned}$$

General case. Let $u \in L^p(\mathbb{R}^N)$ and $\eta > 0$. We can choose rectangles $Q_1, \ldots, Q_k$ and constants $c_1, \ldots, c_k$ such that $\|u - u_0\|_p < \eta$ where

$$u_0 = \sum_{n=1}^{k} c_k 1_{Q_k}.$$

Now $J_\varepsilon * u_0 \to u_0$ as $\varepsilon \to 0$ by the above case plus the triangle inequality; choose $\varepsilon_0 > 0$ such that $\|J_\varepsilon * u_0 - u_0\|_p < \eta$ for $0 < \varepsilon < \varepsilon_0$. Then

$$\begin{aligned}\|J_\varepsilon * u - u\|_p &\le \|u - u_0\|_p + \|u_0 - J_\varepsilon * u_0\|_p + \|J_\varepsilon * u_0 - J_\varepsilon * u\|_p\\ &\le \eta + \|u_0 - J_\varepsilon * u_0\|_p + \eta \qquad \text{(by (i))}\\ &< 3\eta \qquad \text{provided } \varepsilon < \varepsilon_0.\end{aligned}$$

Thus $\|J_\varepsilon * u - u\|_p \to 0$ as $\varepsilon \to 0$.

For $u \in W^{m,p}(\mathbb{R}^N)$ we obtain $\|J_\varepsilon * u - u\|_{m,p} \to 0$ by applying the above to each $D^\alpha u$.

<u>(iii)</u> Let $1 \le p < \infty$.

Density of $\mathscr{D}(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$ actually follows from the proof of (ii) since $J_\varepsilon * u_0 \in \mathscr{D}(\mathbb{R}^N)$.

Let $\eta > 0$. Using a result from the Problem Sheets (Sheet 4 Q2), given $u \in W^{m,p}$ we can choose $v \in W^{m,p}$ with compact support such that $\|u - v\|_{m,p} < \eta$. Now $J_\varepsilon * v \to v$ in $W^{m,p}$ by (ii), and $J_\varepsilon * v \in \mathscr{D}(\mathbb{R}^N)$, so we can choose ε with $\|J_\varepsilon * v - v\|_{m,p} < \eta$, so $\|u - J_\varepsilon * v\|_{m,p} < 2\eta$.

Therefore $W_0^{m,p}(\mathbb{R}^N) = W^{m,p}(\mathbb{R}^N)$. $\square$

Remark. In general $W^{m,p}(\Omega) \neq W_0^{m,p}(\Omega)$.

3.1 Localisation

We want analogues of the results of Theorem 3.3 for a general domain Ω. For a function $u \in L^1_{loc}(\Omega)$,

$$u * J_\varepsilon(x) = \int_{\mathbb{R}^N} u(y) J_\varepsilon(x - y)\,dy$$

which requires values of u at points of $B(x, \varepsilon)$ which might be outside Ω. If we set $u = 0$ outside Ω then the resulting discontinuity of u will be reflected in large derivatives of $J_\varepsilon * u$ near $\partial\Omega$.

This necessitates restricting attention to subsets of Ω, typically compact ones.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^N$ be open and nonempty, Ω_0 open, $\overline{\Omega_0}$ a compact subset of Ω. Let $0 < \varepsilon < dist(\Omega_0, \mathbb{R}^N \setminus \Omega)$, then*

(i) For $1 \le p \le \infty$ we have $\|J_\varepsilon * u\|_{p,\Omega_0} \le \|u\|_{p,\Omega_0+B^\circ(0,\varepsilon)}$ if $u \in L^p(\Omega)$ and $\|J_\varepsilon * u\|_{m,p,\Omega_0} \le \|u\|_{m,p,\Omega_0+B^\circ(0,\varepsilon)}$ if $u \in W^{m,p}(\Omega)$.

(ii) For $1 \le p < \infty$ we have $\|J_\varepsilon * u - u\|_{p,\Omega_0} \to 0$ as $\varepsilon \to 0$ if $u \in L^p(\Omega)$ and $\|J_\varepsilon * u - u\|_{m,p,\Omega_0} \to 0$ as $\varepsilon \to 0$ if $u \in W^{m,p}(\Omega)$.

Proof. (i) For $0 < \varepsilon < \varepsilon' < dist(\Omega_0, \mathbb{R}^N \setminus \Omega)$ we can choose $\psi \in \mathscr{D}(\Omega)$ with $0 \le \psi \le 1$ everywhere, $\psi = 1$ on $\Omega_0 + B^\circ(0,\varepsilon)$ and $\psi = 0$ outside $\Omega_0 + B^\circ(0,\varepsilon')$. Then by Theorem 3.3

$$\|J_\varepsilon * u\|_{p,\Omega_0} \le \|J_\varepsilon * (\psi u)\|_{p,\mathbb{R}^N} \le \|\psi u\|_{p,\mathbb{R}^N} \le \|u\|_{p,\Omega+B^\circ(0,\varepsilon')} \to \|u\|_{p,\Omega+B^\circ(0,\varepsilon)} \text{ as } \varepsilon' \to \varepsilon.$$

The remaining parts use similar arguments, applied to the partial derivatives where necessary.

□

Remark. If $u \in W_0^{m,p}(\Omega)$ then $u \in W^{m,p}(\mathbb{R}^N)$ (take $u = 0$ outside Ω). So $J_\varepsilon * u$ makes sense and we have $\|J_\varepsilon * u\|_{p,\Omega} \le \|u\|_{p,\mathbb{R}^N} = u\|_{p,\Omega+B^\circ(0,\varepsilon)}$ etc.

Theorem 3.5 (Fundamental Theorem of Calculus)**.** *Suppose $\Omega \subset \mathbb{R}^N$ is a nonempty, connected open set, $u \in W^{1,1}_{loc}(\Omega)$, and $D_i u = 0$ a.e. in Ω for $i = 1, \ldots, N$. Then u is essentially constant on Ω.*

Proof. Consider a ball B such that $\overline{B} \subset \Omega$. Then, for all small $\varepsilon > 0$,

$$D_i(J_\varepsilon * u)(x) = J_\varepsilon * D_i u(x) = 0 \qquad \text{for all } x \in B,$$

for $i = 1, \ldots, N$. Hence $J_\varepsilon * u$ is constant in B. As $\varepsilon \to 0$, $J_\varepsilon * u \to u$ in $L^1(B)$, so $u =$ const. a.e. in B.

Take $S(c)$ to be the union of all the open balls B such that $\overline{B} \subset \Omega$ and $u = c$ a.e. on B, for $c \in \mathbb{R}$. Then $\Omega = \bigcup_{c\in\mathbb{R}} S(c)$, and the $S(c)$ are open and disjoint, so by connectedness $\Omega = S(c)$ for one particular value of c. □

We are now in a position to prove Lemma 1.3:

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^N$ be open.*
(i) Let $1 \le p < \infty$. Then $\mathscr{D}(\Omega)$ is dense in $L^p(\Omega)$.
(ii) Let $u \in L^1_{loc}(\Omega)$ with $\displaystyle\int_\Omega u\varphi = 0$ for all $\varphi \in \mathscr{D}(\Omega)$. Then $u = 0$ a.e. in Ω.

Proof. (i) We addressed the case $\Omega = \mathbb{R}^N$ in Theorem 3.3. For general Ω, choose bounded open Ω_0 such that $\overline{\Omega}_0 \subset \Omega$, and such that $\|1_{\Omega_0}u - u\|_p < \varepsilon$ ($u \in L^p(\Omega)$, $\varepsilon > 0$, having been given). Then, for $\eta > 0$ small enough, $J_\eta * (1_{\Omega_0}u)$ is a test function on Ω and $\|J_\eta * (1_{\Omega_0}u) - 1_{\Omega_0}u\|_p < \varepsilon$. So $\|u - J_\eta * (1_{\Omega_0}u)\|_p < 2\varepsilon$.

(ii) Consider bounded open Ω_0 with $\overline{\Omega}_0 \subset \Omega$ and take $0 < \varepsilon < dist(\overline{\Omega}_0, \mathbb{R}^N \setminus \Omega)$. Then

$$J_\varepsilon * u(x) = \int_{\mathbb{R}^N} u(y) J_\varepsilon(x - y)dy = 0 \qquad \forall x \in \Omega_0$$

since $y \mapsto J_\varepsilon(x - y)$ is a test function on Ω. Letting $\varepsilon \to 0$ we get $J_\varepsilon * u \to u$ in $L^1(\Omega_0)$, so $u = 0$ a.e. in Ω_0. Hence $u = 0$ a.e. in Ω. $\square$

Remark. This shows that different locally integrable functions represent different distributions and in particular, if $D^\alpha u \in L^1_{loc}$, then the function representing $D^\alpha u$ is unique.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^N$ be open, $u \in C(\Omega)$. Then $J_\varepsilon * u \to u$ uniformly on compact subsets of Ω. [Exercise]*

Lemma 3.8. *Let $\varnothing \neq \Omega \subset \mathbb{R}^N$ be open. Then there is a sequence $\{\varphi_n\}$ in $\mathscr{D}(\Omega)$ such that:*

(i) $0 \leq \varphi_n \leq 1$ for every n, and $\sum_{n=1}^{\infty} \varphi_n = 1$ on Ω ("partition of unity");

(ii) every point of Ω has a neighbourhood on which all except finitely many φ_n vanish identically ("local finiteness");

(iii) local finiteness has the consequence that any compact subset of Ω intersects the supports of only finitely many φ_n.

Proof. For $n \in \mathbb{N}$ define

$$\Omega_n = \{x \in \Omega \mid |x| < n \text{ and } dist(x, \mathbb{R}^N \setminus \Omega) > 2/n\}.$$

Then Ω_n is open and bounded, $\overline{\Omega}_n \subset \Omega$, $\Omega_n \subset \Omega_{n+1}$ and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. Set $S_n = \overline{\Omega}_n \setminus \Omega_{n-1}$ for $n \geq 2$ with $S_1 = \overline{\Omega}_1$, and write

$$\psi_n = J_{1/n} * 1_{S_n},$$

so that $\psi_n \in \mathscr{D}(\Omega)$ and $\mathrm{supp}(\psi_n) = S_n + \overline{B}(0, 1/n)$.

Consider $x \in \Omega$, so $B^\circ(x, r) \subset \Omega_n$ for some $n \in \mathbb{N}$ and $r > 0$. If $k > n$ then $B^\circ(x, r) \cap S_k = \varnothing$ so $B^\circ(x, r/2) \cap (S_k + \overline{B}(0, r/2)) = \varnothing$. Hence $B^\circ(x, r/2) \cap \mathrm{supp}(\psi_k) = \varnothing$ if $k > \max\{n, 2/r\}$. It follows by covering that if $K \subset \Omega$ is compact then K meets the supports of only finitely many ψ_n.

Let $x \in \Omega$; we claim $\psi_n(x) > 0$ for some $n \in \mathbb{N}$. We have $x \in \Omega_m$ for some $m \in \mathbb{N}$ and then $\Omega_m \subset S_1 \cup \cdots \cup S_m$. We can choose r, $0 < r < 1/m$, such that $B^\circ(x,r) \subset \Omega_m$ and then $S_n \cap B^\circ(x,r)$ has positive measure for some $n \in \{1, \ldots, m\}$ so

$$\psi_n(x) = \int_{S_n} J_{1/n}(x-y)dy \geq \int_{S_n \cap B^\circ(x,r)} J_{1/n}(x-y)dy > 0.$$

Set

$$\varphi_n = \frac{\psi_n}{\sum_{k \in \mathbb{N}} \psi_k}.$$

Then every point of Ω has a neighbourhood on which the above sum involves only finitely many functions, hence φ_n is smooth and $\displaystyle\sum_{n \in \mathbb{N}} \varphi_n = 1$.

If $K \subset \Omega$ is compact, then K intersects the supports of only finitely many v_n. For, each point of K is the centre of an open ball that intersects only finitely many $\operatorname{supp} v_n$, and K can be covered by finitely many such balls. $\square$

Theorem 3.9 (Meyers-Serrin "$H = W$" Theorem)**.** *Let $\varnothing \neq \Omega \subset \mathbb{R}^N$ be open, $m \in \mathbb{N}$, $1 \leq p < \infty$. Then $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.*

Proof. Choose a locally finite, countable partition of unity into test functions on Ω, $\{\varphi_n\}_{n=1}^\infty$, as provided by Lemma 3.8. Consider $\delta > 0$, $u \in W^{m,p}(\Omega)$.

For each $n \in \mathbb{N}$ choose $0 < \varepsilon_n < 1/n$ such that $\varepsilon_n < dist(\operatorname{supp} \varphi_n, \mathbb{R}^N \setminus \Omega)$ so $v_n = J_{\varepsilon_n} * (\varphi_n u) \in \mathscr{D}(\Omega)$, and such that $\|v_n - \varphi_n u\|_{m,p} < \delta 2^{-n}$.

Consider $x \in \Omega$. Then $r > 0$ can be chosen such that $B^\circ(x,r) \cap \operatorname{supp} \varphi_n = \varnothing$ for all except finitely many n, so $B^\circ(x, \frac{1}{2}r) \cap \left(\operatorname{supp} \varphi_n + \overline{B}(0, \frac{1}{2}r)\right) = \varnothing$ except for finitely many n, hence $B^\circ(x, \frac{1}{2}r) \cap \operatorname{supp} v_n = \varnothing$ for all sufficiently large n. Thus the family $\{v_n\}_{n=1}^\infty$ is locally finite.

Take $v = \displaystyle\sum_{k=1}^\infty v_k \in C^\infty(\Omega)$ by local finiteness.

Choose $\{\Omega_n\}_{n=1}^\infty$ to be an increasing family of bounded open sets with $\overline{\Omega}_n \subset \Omega$, and $\bigcup_{n=1}^\infty \Omega_n = \Omega$.

By local finiteness, each $\overline{\Omega}_n$ intersects the supports of only finitely many φ_k and v_k and since $u = \sum_k \varphi_k u$ we have

$$\|v - u\|_{m,p,\Omega_n} = \left\| \sum_{k=1}^\infty (v_k - \varphi_k u \right\|_{m,p,\Omega_n};$$

the above sum involves only finitely many functions so there are no convergence problems.

Therefore

$$\int_{\Omega_n} \sum_{0\le|\alpha|\le m} |D^\alpha v - D^\alpha u|^p = \left\| \sum_{k=1}^{\infty} (v_k - \varphi_k u) \right\|_{m,p,\Omega_n}^p$$
$$\le \left(\sum_{k=1}^{\infty} \|v_k - \varphi_k u\|_{m,p,\Omega_n} \right)^p$$
$$< \left(\sum_{k=1}^{\infty} \delta 2^{-k} \right)^p = \delta^p$$

and we can let $n \to \infty$ and apply the Monotone Convergence Theorem to LHS to get

$$\int_{\Omega} \sum_{0\le|\alpha|\le m} |D^\alpha v - D^\alpha u|^p \le \delta^p$$

i.e.

$$\|v - u\|_{m,p,\Omega} \le \delta.$$

Now $v = u + (v - u) \in W^{m,p}(\Omega)$ so $v \in C^\infty(\Omega) \cap W^{m,p}(\Omega)$. □

Remarks. This result says nothing about the behaviour of the approximating smooth functions near the boundary, so it cannot be used to define boundary values of Sobolev functions.

Note that $p < \infty$ cannot be avoided.

Theorem 3.10. *Let Θ, Ω be nonempty, bounded, open sets in $\mathbb{R}^N$ and suppose $F : \Theta \to \Omega$ is a bijection satisfying $F \in C^1(\overline{\Theta})$ and $F^{-1} \in C^1(\overline{\Omega})$. Then, for $1 \le p < \infty$, the map $v \mapsto v \circ F$ is an invertible bounded linear operator from $W^{1,p}(\Omega)$ onto $W^{1,p}(\Theta)$.*

Proof. First consider $u = v \circ F$, $v \in C^1(\Omega) \cap W^{1,p}(\Omega)$. Then

$$\int_\Theta |D_j u(x)|^p dx = \int_\Theta \left| \frac{\partial}{\partial x_j} v(F(x)) \right|^p dx = \int_\Theta \left| \sum_{k=1}^N D_k v(F(x)) D_j F_k(x) \right|^p dx$$
$$\le \text{const} \cdot \int_\Theta \sum_{k=1}^N \left| D_k v(F(x)) \right|^p dx$$
$$= \text{const} \cdot \int_\Omega \sum_{k=1}^N \left| D_k v(y) \right|^p |JF^{-1}(y)| dy \qquad (JF^{-1} \text{ Jacobian})$$
$$\le \text{const} \cdot \int_\Omega \sum_{k=1}^N |D_k v(y)|^p dy$$

hence

$$\|u\|_{1,p,\Theta} \le \text{const} \cdot \|v\|_{1,p,\Omega}.$$

A similar inequality holds in the reverse direction, and by density (Meyers-Serrin) these inequalities hold throughout $W^{1,p}(\Omega)$. □

Lemma 3.11. *Let $B = B^\circ(0,r) \subset \mathbb{R}^N$, $B^\pm = \{(x', x_N) \in B \mid \pm x_N > 0\}$, $u \in W^{1,1}(B^+)$. Then*

(i) $\displaystyle\int_{B^+} (D_j u)\varphi = -\int_{B^+} u(D_j\varphi)$ *for all* $\varphi \in \mathscr{D}(B)$, $1 \le j \le N-1$;

(ii) $\displaystyle\int_{B^+} (D_N u)\varphi = -\int_{B^+} u(D_N\varphi)$ *for all* $\varphi \in \mathscr{D}(B)$ *such that* $\varphi(x', 0) = 0$ *if* $x' \in B_{N-1}$;

(iii) defining $\bar{u}(x', x_N) = u(x', |x_N|)$ *we have* $\bar{u} \in W^{1,1}(B)$ *with* $D_j\bar{u}(x', x_N) = D_j(x', |x_N|)$ *for* $1 \le j \le N-1$ *and* $D_N\bar{u}(x', x_N) = \operatorname{sgn}(x_N) D_N u(x', |x_N|)$ *a.e.*

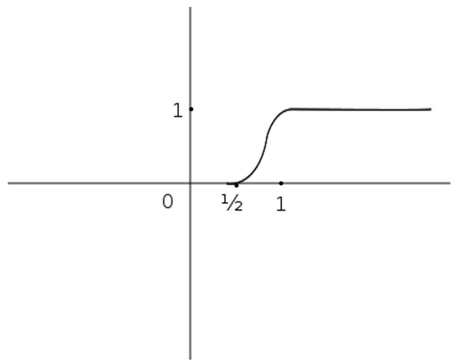

Figure 2: $1_{[1,\infty)} \le \psi \le 1_{[\frac{1}{2},\infty)}$

Proof. For (i) and (ii) choose increasing $\psi \in C^\infty(\mathbb{R})$ such that $1_{[1,\infty)} \le \psi \le 1_{[\frac{1}{2},\infty)}$ (c.f. Figure 2), e.g. $\psi = J_{1/4} * 1_{[\frac{3}{4},\infty)}$. Define $\psi_\varepsilon(s) = \psi\left(\frac{s}{\varepsilon}\right)$ for $s \in \mathbb{R}$. Thus $\psi_\varepsilon(x_N)\varphi(x', x_N)$ defines an element of $\mathscr{D}(B^+)$.

(i) For $1 \le j \le N-1$

$$\begin{aligned}\int_{B^+} (D_j u)(x)\psi_\varepsilon(x_N)\varphi(x')dx &= -\int_{B^+} u(x) D_j(\psi_\varepsilon(x_N)\varphi(x))dx \\ &= -\int_{B^+} u(x)\psi_\varepsilon(x_N) D_j\varphi(x)dx.\end{aligned}$$

For $x_N > 0$ we have $0 \le \psi_\varepsilon(x) \le 1$ and $\psi_\varepsilon \to 1$ as $\varepsilon \to 0$, so we can apply the Dominated Convergence Theorem to deduce (i).

(ii) For $j = N$,

$$\int_{B^+} (D_N u(x))\psi_\varepsilon(x_N)\varphi(x)dx = -\int_{B^+} u(x)\left(\varepsilon^{-1}\psi'\left(\frac{x_N}{\varepsilon}\right)\varphi(x) + \psi_\varepsilon(x_N) D_N\varphi(x)\right)dx.$$

Now on B^+ we have $|\varphi(x)| \le c_1 x_N$ where $c_1 = \|D_N\varphi\|_{\sup}$, since $\varphi(x) = 0$ when $x_N = 0$, hence

$$\left|\varepsilon^{-1}\psi'\left(\frac{x_N}{\varepsilon}\right)\varphi(x', x_N)\right| \le c_1\left(\frac{x_N}{\varepsilon}\right)\psi'\left(\frac{x_N}{\varepsilon}\right),$$

which is bounded above by c_1c_2 when $0 < x_N < \varepsilon$ and vanishes for $x_N \geq \varepsilon$, where $c_2 = \|\psi'\|_{\sup}$. Therefore $\varepsilon^{-1}\psi'\left(\frac{x_N}{\varepsilon}\right)\varphi(x', x_N)$ is uniformly bounded and tends to 0 pointwise as $\varepsilon \to 0$. We now deduce (ii) using the Dominated Convergence Theorem.

(iii) If $1 \leq j \leq N-1$ and $\varphi \in \mathscr{D}(B)$ then

$$\begin{aligned}\int_B \bar{u}(x) D_j\varphi(x)dx &= \int_{B^+} u(x', x_N) D_j\varphi(x', x_N)dx + \int_{B^-} u(x', -x_N) D_j\varphi(x', x_N)dx \\ &= \int_{B^+} u(x', x_N)\left[D_j\varphi(x', x_N) + D_j\varphi(x', -x_N)\right]dx \qquad \text{(by (i) with } \bar{\varphi}) \\ &= -\int_B \left(D_j u(x', |x_N|)\right)\varphi(x', x_N)dx.\end{aligned}$$

For $j = N$ we have

$$\begin{aligned}\int_B \bar{u}(x) D_N\varphi(x)dx &= \int_{B^+} u(x', x_N) D_N\varphi(x', x_N)dx + \int_{B^-} u(x', -x_N) D_N\varphi(x', x_N)dx \\ &= \int_{B^+} u(x', x_N)\left[D_N\varphi(x', x_N) - \frac{\partial}{\partial x_N}\varphi(x', -x_N)\right]dx \\ &= -\int_{B^+} (D_N u(x))\left[\varphi(x', x_N) - \varphi(x', -x_N)\right]dx \\ &\qquad \text{(by (ii) since, if } x_N = 0 \text{ then } \varphi(x', x_N) - \varphi(x', -x_N) = 0) \\ &= -\int_B \operatorname{sgn}(x_N)\left(D_N u(x', |x_N|)\right)\varphi(x)dx\end{aligned}$$

□

Terminology. Let $\varnothing \neq \Omega \subset \mathbb{R}^N$, $N \geq 2$. A **$\mathbf{C^1}$ chart** for $\partial\Omega$ is an open set $U = rB_{N-1} \times (-a, a)$ (with respect to some local Cartesian coordinates in $\mathbb{R}^N$) and $f \in C^1(\overline{rB_{N-1}})$ such that $\|f\|_{\sup} < a$ and such that

$$\Omega \cap U = \{(x', x_N) \in U \mid x_N < f(x')\}.$$

We say $\partial\Omega$ is of class C^1 if there is a C^1 chart for $\partial\Omega$ in a neighbourhood of every point.

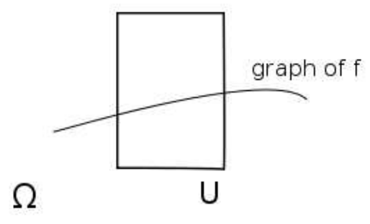

Figure 3: C^1 chart

Theorem 3.12 (Extension Theorem)**.** *Let $\varnothing \neq \Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be bounded with C^1 boundary, and $1 \leq p < \infty$. Then there exists a bounded open set $V \supset \overline{\Omega}$ and a bounded linear operator*

$$E : W^{1,p}(\Omega) \hookrightarrow W_0^{1,p}(V)$$

such that $Eu = u$ almost everywhere in Ω for all $u \in W^{1,p}(\Omega)$ and $Eu \in C(\overline{V})$ for all $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$.

Proof. Consider a chart (U, f) where $U = rB_{N-1} \times (-a, a)$. Define

$$F(x', x_N) = (x', x_N - f(x')),$$

(c.f. Figure 4) which is a bijection from U to an open set W such that $F \in C^1(\overline{U})$ and $F^{-1} \in C^1(\overline{W})$ given by

$$F^{-1}(y', y_N) = (y', y_N + f(y')).$$

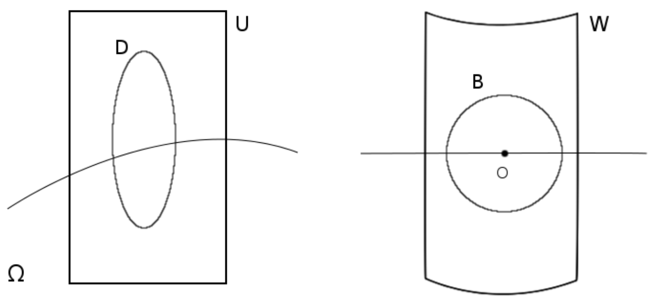

Figure 4: $F \in C^1(\overline{U})$

Choose a ball B with $\overline{B} \subset W$, centre O (which lies on $F((\partial\Omega) \cap U)$ and set

$$B^{\pm} = \{(y', y_N) \in B \mid \pm y_N > 0\}.$$

Then Lemma 3.11 provides an extension operator $T : W^{1,p}(B^+)$ to $W^{1,p}(B)$. Note that if $u \in W^{1,p}(B^+) \cap C(\overline{B^+})$ then by construction $Tu \in W^{1,p}(B) \cap C(\overline{B})$.

Now, assuming $p < \infty$, the operator

$$L : W^{1,p}(D) \to W^{1,p}(B), \qquad \text{defined by } Lu = u \circ F^{-1},$$

where $D = F^{-1}(B)$, $D^{\pm} = F^{-1}(B^{\pm})$, is bounded and has a bounded inverse. Define

$$K : W^{1,p}(D^-) \to W^{1,p}(D) \qquad \text{by } Ku = L^{-1}TLu, \quad u \in W^{1,p}(D^-).$$

Then K is bounded and is an extension operator of the desired form for D^-.

Now cover $\partial\Omega$ with finitely many bounded open sets $D_1, \dots, D_n$, each having an extension operator $K_i : W^{1,p}(D_i \cap \Omega) \to W^{1,p}(D_i)$. Choose an open set D_0, $\overline{D_0} \subset \Omega$, such that $D_0, \dots, D_n$ cover $\overline{\Omega}$.

Choose $\varphi_i \in \mathscr{D}(D_i)$ such that $\sum_{i=0}^{n} \varphi_i \equiv 1$ on a set whose interior contains $\overline{\Omega}$. Define

$$Eu(x) = \sum_{i=1}^{n} \varphi_i(x) K_i(u|_{D_i})(x) + \varphi_0 u(x).$$

Then $E(x) = \sum_{i=0}^{n} \varphi_i(x)u(x) = u(x)$ for $x \in \Omega$, and $Eu \in W_0^{1,p}(V)$ where $V = \bigcup_{i=0}^{n} D_i$. Moreover if $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ then $Eu \in C_c(V)$. $\square$

Remark. For smoother boundaries, extension operators for $W^{m,p}$ can be defined.

Theorem 3.13 (Trace Theorem)**.** *Let $\varnothing \neq \Omega \subset \mathbb{R}^N$ be a bounded domain with C^1 boundary. Let $1 \leq p < \infty$. Then there is a bounded linear operator* $\operatorname{Tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ *such that if $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ and $\overline{u}$ denotes the uniformly continuous extension of u to $\overline{\Omega}$ then*

$$\operatorname{Tr} u(x) = \overline{u}(x) \qquad \textit{for all } x \in \partial\Omega.$$

Proof. Consider $u \in W^{1,p}(\Omega) \cap C^1(\overline{\Omega})$. Consider a chart for $\partial\Omega$, say (U, f) where $U = rB_{N-1} \times (-a, a)$. Consider $\psi \in C^\infty(\mathbb{R})$ such that

$$\psi(-a) = 0 \qquad \text{and} \qquad \psi(s) = 1 \quad \text{for } s > \frac{1}{2}\left(-a - \|f\|_{\sup}\right).$$

Then

$$\begin{aligned}
\int_{U\cap\partial\Omega} |u|^p &= \int_{U\cap\partial\Omega} \psi(x_N)|u(x)|^p dS(x) \\
&= \int_{B_{N-1}} \psi(f(x'))|u(x', f(x'))|^p (1 + |\nabla f(x')|^2)^{\frac{1}{2}} dx' \\
&= \int_{B_{N-1}} \int_{-a}^{f(x')} D_N\big(\psi(x_N)|u(x', x_N)|^p\big)(1 + |\nabla f(x')|^2)^{\frac{1}{2}} dx_N dx' \text{ if } p > 1 \text{ (*)} \\
&\leq c \int_{U\cap\Omega} |\psi'(x_N)||u(x)|^p + p|\psi(x_N)||u(x)|^{p-1}|D_N u(x)| dx \\
&\leq c \int_{U\cap\Omega} |u|^p + |u|^{p-1}|D_N u| dx \\
&\leq c \int_{u\cap\Omega} |u|^p + |D_N u|^p dx
\end{aligned}$$

where we have used Young's inequality in the last line to obtain

$$|u|^{p-1}|D_N u| \leq \frac{|u|^{(p-1)q}}{q} + \frac{|D_N u|^p}{p}$$

with $1/p + 1/q = 1$, so $(p-1)q = p$, hence

$$\int_{U \cap \partial\Omega} |u|^p \leq c\|u\|_{W^{1,p}(U\cap\Omega)}.$$

The case $p = 1$ is similar, using at (*) the inequality

$$|\psi(x', f(x'))u(x', f(x'))| \leq \int_{-a}^{f(x')} |D_N(\psi(x', \xi_N)u(x', \xi_N))| dx_N.$$

Now, covering $\partial\Omega$ with finitely many charts $\{(U_k, f_k)\}_{k=1}^n$ then

$$\|u\|_{L^p(\partial\Omega)} \leq \sum_{k=1}^n \|u\|_{L^p(U_k\cap\Omega)} \leq \sum_{k=1}^n c_k\|u\|_{W^{1,p}(U_k\cap\Omega)} \leq c\|u\|_{W^{1,p}(\Omega)}.$$

To deal with the case of general $u \in W^{1,p}(\Omega)$, it must be shown that u can be approximated in $\| \|_{1,p}$ by such functions. First extend u to $Eu \in W_0^{1,p}(V)$, then use density to approximate Eu in $\| \|_{1,p}$ by a sequence $\{u_n\}$ in $\mathscr{D}(V)$. The restrictions of the u_n to Ω form the desired approximating sequence. If now $\{u_n\}$ is any sequence in $C^1(\overline{\Omega}) \cap W^{1,p}(\Omega)$ converging in $\| \|_{1,p}$ to u, then their boundary traces form a Cauchy sequence in $L^p(\partial\Omega)$ converging to a limit, denoted $\mathrm{Tr}(u)$, which is independent of the choice of approximating sequence (any two such sequences can be interlaced to give another one, whose boundary traces must also converge).

Finally, let $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$; we have to check that the above definition agrees with $\overline{u}|_{\partial\Omega}$. Note that $Eu|_{\overline{\Omega}}$ is a uniformly continuous extension of u to $\overline{\Omega}$, so $\overline{u} = Eu|_{\overline{\Omega}}$ and therefore $\overline{u}|_{\partial\Omega} = Eu|_{\partial\Omega}$. Now, as $\varepsilon \to 0$, we have $J_\varepsilon * Eu \to Eu$ on V both uniformly and in $\| \|_{1,p}$, so $J_\varepsilon * Eu|_{\partial\Omega}$ converges uniformly to $Eu|_{\partial\Omega}$ and converges in $L^p(\partial\Omega)$ to some limit which must therefore be $\mathrm{Tr}\, u$. Hence $\mathrm{Tr}(u) = Eu|_{\partial\Omega} = \overline{u}|_{\partial\Omega}$ as required. $\square$

4 Embeddings on Smooth Bounded Domains

Theorem 4.1 (Sobolev Embedding Theorem for smooth bounded domains)**.** *Let $N \geq 2$, $\varnothing \neq \Omega \subset \mathbb{R}^N$ be open and bounded with C^1 boundary, $m \in \mathbb{N}$ and $1 \leq p < \infty$. Then the following embeddings are bounded:*

(i) $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p \leq q \leq p^* := \dfrac{Np}{N-mp}$ if $mp < N$;

(ii) $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p \leq q < \infty$ if $m < N$, $mp = N$;

(iii) $W^{N,1}(\Omega) \hookrightarrow C(\overline{\Omega})$;

(iv) $W^{m,p}(\Omega) \hookrightarrow C^{0,\lambda}(\Omega)$ *for* $0 < \lambda \leq m - \dfrac{N}{p}$ *if* $mp > N > (m-1)p$;

(v) $W^{m,p}(\Omega) \hookrightarrow C^{0,\lambda}(\Omega)$ *for* $0 < \lambda < 1$ *if* $(m-1)p = N$.

Proof. When $m = 1$ cases (i), (ii), (iv), (v) follow by using Theorem 3.12 to choose an extension operator $E : W^{1,p}(\Omega) \to W_0^{1,p}(V)$ for some bounded open $V \supset \overline{\Omega}$, and applying the embedding theorem for $W_0^{1,p}(\Omega)$ (Theorem 2.12). We leave case (iii) to the end, and proceed to describe the inductive step in the other cases.

(i) Suppose the result holds for some $m \geq 1$ and all p with $mp < N$. Let p satisfy $(m+1)p < N$. Let $p_1 = \dfrac{Np}{N - mp}$ and $p_2 = \dfrac{Np}{N-(m+1)p} = \dfrac{Np_1}{N - p_1}$. For $u \in W^{m+1,p}(\Omega)$ we now have $\nabla u \in W^{m,p}(\Omega)$ and thence

$$\begin{aligned} \|\nabla u\|_{p_1} &\leq c\|\nabla u\|_{m,p} \leq c\|u\|_{m+1,p} \\ \|u\|_{p_1} &\leq c\|u\|_{m,p} \leq \|u\|_{m+1,p}, \\ \text{so} \qquad \|u\|_{1,p_1} &\leq c\|u\|_{m+1,p}, \\ \text{so} \qquad \|u\|_{p_2} &\leq c\|u\|_{m+1,p} \end{aligned}$$

from the initial case $W^{1,p_1} \hookrightarrow L^{p_2}$. The case $q < p^*$ follows by interpolation, completing the inductive step.

(ii) Suppose the result holds for some $m \geq 1$ with $m < N$. Suppose $m + 1 < N$ and let $p = N/(m+1)$; then $mp < N$ and $Np/(N - mp) = N$. For $u \in W^{m+1,p}(\Omega)$ we have $\nabla u \in W^{m,p}(\Omega)$ hence using (i)

$$\begin{aligned} \|\nabla u\|_N &\leq c\|\nabla u\|_{m,p} \leq c\|u\|_{m+1,p}, \\ \|u\|_N &\leq \|u\|_{m,p} \leq c\|u\|_{m+1,p}, \\ \|u\|_q &\leq c\|u\|_{1,N} \leq c\|u\|_{m+1,p}, \end{aligned}$$

where the first inequality of the last line comes from the initial case of (ii). This completes the inductive step of (ii).

(iv) Suppose $m \geq 2$ and $mp > N > (m-1)p$. Consider $u \in W^{m,p}(\Omega)$. Then from (i) we have, writing $p_0 = \dfrac{Np}{N-(m-1)p} > N$,

$$\begin{aligned} \|\nabla u\|_{p_0} &\leq c\|\nabla u\|_{m-1,p} \leq c\|u\|_{m,p}, \\ \|u\|_{p_0} &\leq c\|u\|_{m-1,p} \leq c\|u\|_{m,p}, \\ \text{so} \qquad \|u\|_{1,p_0} &\leq c\|u\|_{m,p}. \end{aligned}$$

Now apply the initial case of (iv) together with the above inequality to obtain, writing $\lambda_0 = 1 - \frac{N}{p_0} = m - \frac{N}{p}$,

$$\|u\|_{C^{0,\lambda_0}} \le c\|u\|_{1,p_0} \le c\|u\|_{m,p}.$$

When $0 < \lambda < \lambda_0$ we can apply the embedding $C^{0,\lambda_0}(\Omega) \hookrightarrow C^{0,\lambda}(\Omega)$ (Lemma 2.11) to obtain

$$\|u\|_{C^{0,\lambda}} \le c\|u\|_{m,p}$$

establishing the higher-order cases of (iv).

(v) Suppose $N = (m-1)p$ with $m \ge 2$, let $q > p$ and let $u \in W^{m,p}(\Omega)$. Then (ii) yields

$$\begin{aligned} \|\nabla u\|_q &\le c\|\nabla u\|_{m-1,p} \le c\|u\|_{m,p}, \\ \|u\|_q &\le c\|u\|_{m-1,p} \le c\|u\|_{m,p}, \\ \text{so} \qquad \|u\|_{1,q} &\le c\|u\|_{m,p}. \end{aligned}$$

When $q > N$ (so $q > p$) and $\lambda(q) = 1 - \frac{N}{q}$, the preliminary case of (v) yields

$$\|u\|_{C^{0,\lambda(q)}} \le c\|u\|_{1,q},$$

and for $0 < \lambda < 1$ we can apply this inequality with $q > \frac{N}{1-\lambda}$ together with the embedding $C^{0,\lambda(q)}(\Omega) \hookrightarrow C^{0,\lambda}(\Omega)$ to deduce

$$\|u\|_{C^{0,\lambda}} \le c\|u\|_{C^{0,\lambda(q)}} \le c\|u\|_{m,p},$$

establishing the higher-order cases of (v).

(iii) Recall the estimate

$$\|u\|_{\sup} \le c\|u\|_{N,1} \quad \forall u \in W^{N,1}(Q)$$

where the constant c depends on the dimensions of the rectangle Q but not on its position or orientation; this holds for $u \in C^N(Q) \cap W^{N,1}(Q)$ by Theorems 2.10 and 2.3, and follows for general $u \in W^{N,1}(Q)$ by Meyers-Serrin.

Consider a chart (U, f) for $\partial\Omega$, where $U = rB_{N-1} \times (-a, a)$ and $f \in C^1(\overline{rB_{N-1}})$. Let $n = \|n\|\widehat{n} = (\nabla f(0), -1)$ which is the inward normal to $\partial\Omega$ at $(0, f(0))$. Let

$$g(\xi) = \nabla f(0)\xi - \alpha|\xi| \qquad \text{for } \xi \in \mathbb{R}^{N-1},$$

where $\alpha > 0$ is to be chosen later. Let $v_i = (v_i', v_{i,N})$, $i = 1, \dots, N$ be the vertices adjacent to 0 of a (unit) cube Q with diagonal $[0, \sqrt{N}\widehat{n}]$. Then the vertices $(0, f(0)) + v_i$ lie below the tangent hyperplane to $\partial\Omega$ at $(0, f(0))$, so $\nabla f(0)v_i' > v_{i,N}$. We choose r and α small enough that $g(v_i') > v_{i,N}$ and $\nabla f(x')\xi > g(\xi)$ for all $x' \in rB_{N-1}$ and $0 \neq \xi \in \mathbb{R}^{N-1}$.

If $x' \neq x' + \xi$ both belong to rB_{N-1} then for $0 < t < 1$ the forward directional derivative satisfies

$$\frac{d}{dt+}(f(x' + t\xi) - g(t\xi)) = \nabla f(x' + t\xi) - g(\xi) > 0$$

and it follows that

$$f(x' + \xi) > f(x') + g(\xi).$$

It now follows that

$$((0, f(x')) + Q) \cap \overline{U} \subset \overline{\Omega} \qquad \text{for all } x' \in rB_{N-1}.$$

Hence $x + \delta Q \subset \overline{\Omega}$ for all $x \in \partial\Omega$ within distance $\delta > 0$ of $(0, f(0))$ provided that δ is chosen sufficiently small. Then every point of $\overline{\Omega}$ sufficiently close to $(0, f(0))$ lies in a cube of edge δ contained in $\overline{\Omega}$.

A compactness argument now shows that, for some $\varepsilon > 0$, every point $x \in \overline{\Omega}$ lies in a (closed) cube Q_x of side ε contained in $\overline{\Omega}$, and so

$$|u(x)| \leq \|u\|_{\sup, Q_x} \leq c\|u\|_{N,1,Q_x^\circ} \leq c\|u\|_{N,1,\Omega}. \qquad \square$$

Remarks.

1) Boundedness of Ω can be avoided, at the expense of a more complicated proof and carefully chosen regularity assumptions on $\partial\Omega$.

2) The smoothness of $\partial\Omega$ can be weakened somewhat. See Adams's book.

Theorem 4.2. *Let $\varnothing \neq \Omega \subset \mathbb{R}^N$ be open, $1 \leq p < \infty$, and $K \subset L^p(\Omega)$. Then K is relatively compact in $L^p(\Omega)$ if and only if K is bounded in $L^p(\Omega)$ and $\forall \eta > 0$ $\exists \delta > 0$ and $\exists G \subset \Omega$ compact such that*

(i) $\displaystyle\int_{\Omega \setminus G} |u|^p < \eta^p$ *for all $u \in K$, and*

(ii) $\displaystyle\int_{\Omega} |u(x+h) - u(x)|^p dx < \eta^p$ *(taking $u = 0$ outside Ω) for all $u \in K$ and all $h \in \mathbb{R}^N$, satisfying $|h| < \delta$.*

Proof. ($\Leftarrow$ only will be proved.) It is enough to suppose $\Omega = \mathbb{R}^N$, extending $u = 0$ outside Ω.
<u>Claim 1</u>. If $\varepsilon > 0$ and $G \subset \mathbb{R}^N$ then $K(G, \varepsilon) := \{1_G J_\varepsilon * u \mid u \in K\}$ is relatively compact in $C(G)$, and therefore in $L^p(G)$.

For, writing $B = \overline{B}(0,1)$ and taking q to be conjugate to p,

$$\|J_\varepsilon * u\|_{C(G)} \leq \|J_\varepsilon\|_{\sup}\|u\|_{L^1(G+\varepsilon B)} \leq \|J_\varepsilon\|_{\sup}\|u\|_p\|1_{G+\varepsilon B}\|_q,$$

so $K(G,\varepsilon)$ is uniformly bounded on G. Further, if $x \in G$ and $|h| < 1$ then

$$\begin{aligned}|J_\varepsilon * u(x+h) - J_\varepsilon * u(x)| &\leq \int_{\mathbb{R}^N} \big|J_\varepsilon(x+h-y) - J_\varepsilon(x-y)\big||u(y)|dy \\ &\leq |h|\|\nabla J_\varepsilon\|_{\sup}\|u\|_{L^1(G+(1+\varepsilon)B)},\end{aligned}$$

whence $K(\varepsilon)$ is equicontinuous. Relative compactness in $C(G)$ follows by Arzelà-Ascoli, and relative compactness in $L^p(G)$ follows from this.

Claim 2. $\|J_\varepsilon * u - u\|_p \to 0$ as $\varepsilon \to 0$ uniformly over $u \in K$. For

$$\begin{aligned}\|J_\varepsilon * u - u\|_p &= \int_{\mathbb{R}^N} \left|\int_{\mathbb{R}^N} J_\varepsilon(h)(u(x-h) - u(x))dh\right|^p dx \\ &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J_\varepsilon(h)\big|(u(x-h) - u(x))\big|^p dhdx \quad \text{(Jensen's inequality)} \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J_\varepsilon(h)\big|(u(x-h) - u(x))\big|^p dxdh \\ &\leq \sup_{|h|<\varepsilon} \int_{\mathbb{R}^N} \big|(u(x-h) - u(x))\big|^p dx \to 0 \text{ as } \varepsilon \to 0, \text{ uniformly over } u \in K \text{ by (ii).}\end{aligned}$$

Claim 3. $\forall \eta > 0\ \exists G \subset \mathbb{R}^N$ compact and $\exists \varepsilon > 0$ such that $\forall u \in K$ $\|u - 1_G J_\varepsilon * u\|_p < \eta$.
For, let $\eta > 0$ and let G' be the compact set provided by (i). Then, for $G = G' + \varepsilon B$, $u \in K$,

$$\begin{aligned}\|u - 1_G J_\varepsilon * u\|_p &\leq \|u - 1_{G'}\|_p + \|1_{G'}(u - J_\varepsilon * u)\|_p + \|(1_G - 1_{G'})J_\varepsilon * u\|_p \\ &< \eta + \|u - J_\varepsilon * u\|_p + \eta\|(1_G - 1_{G'})\|_p \qquad \text{(if } 0 < \varepsilon < 1) \\ &< 3\eta\end{aligned}$$

for $\varepsilon > 0$ small enough, independent of $u \in K$.

Claim 4. K is totally bounded.
For, let $\eta > 0$ and choose $\varepsilon > 0$ and compact $G \subset \mathbb{R}^N$ such that

$$\|u - 1_G J_\varepsilon * u\|_p < \eta \qquad \text{for all } u \in K.$$

By Claim 1,

$$K(G,\varepsilon) := \{1_G J_\varepsilon * u \mid u \in K\}$$

is relatively compact in L^p; let $S_1, \ldots, S_n$ be sets of diameter less than η covering $K(G,\varepsilon)$. Then $\{S_k + \overline{B}(0,\eta)\}_{k=1}^n$ is a finite collection of sets of diameter less than 2η covering K. $\square$

Lemma 4.3. *Let $\varnothing \neq \Omega \subset \mathbb{R}^N$ be bounded and open and let $0 < \alpha < \beta \leq 1$. Then the embedding $C^{0,\beta}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$ is compact.*

Proof. Exercise on Sheet 5. □

Theorem 4.4 (Rellich-Kondrachov Compact Embedding Theorem)**.** *Let $N \geq 2$, $\varnothing \neq \Omega \subset \mathbb{R}^N$ be open and bounded, $m \in \mathbb{N}$ and $1 \leq p < \infty$. Then the following embeddings are compact:*

(i) $W_0^{m,p}(\Omega) \hookrightarrow L^p(\Omega)$ *for* $p \leq q < p^* := \dfrac{Np}{N - mp}$*, if* $mp < N$*;*

(ii) $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for* $p \leq q < \infty$*, if* $m < N$, $mp = N$*;*

(iii) $W_0^{m,p}(\Omega) \hookrightarrow C^{0,\lambda}(\Omega)$ *for* $0 < \lambda < m - \dfrac{N}{p}$*, if* $mp > N > (m-1)p$*;*

(iv) $W_0^{m,p}(\Omega) \hookrightarrow C^{0,\lambda}(\Omega)$ *for* $0 < \lambda < 1$*, if* $(m-1)p = N$*.*

If $\partial\Omega$ is of class C^1 then $W_0^{m,p}(\Omega)$ can be replaced by $W^{m,p}(\Omega)$.

Proof. We firstly assume $1 \leq p \leq N$ and show $W_0^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$ is compact. Let S denote the unit ball in $W_0^{1,p}(\Omega)$. Fix q_0, $p < q_0 < \dfrac{Np}{N-p}$ if $p < N$, or $p < q_0 < \infty$ if $p = N$. Let $\varepsilon > 0$, $G \subset \Omega$ a measurable set; then

$$\int_{\Omega \setminus G} |u| \leq \left(\int_{\Omega \setminus G} |u|^{q_0} \right)^{\frac{1}{q_0}} \left|\Omega \setminus G\right|^{1 - \frac{1}{q_0}}.$$

We can now choose compact G such that

$$\int_{\Omega \setminus G} |u| \leq \varepsilon \qquad \forall u \in S,$$

since S is bounded in $L^{q_0}(\Omega)$.

Consider $u \in \mathscr{D}(\Omega)$, $h \in \mathbb{R}^N$. Then

$$\begin{aligned}
\int_\Omega |u(x+h) - u(x)| dx &\leq \int_\Omega \int_0^1 \left| \frac{d}{dt} u(x+th) \right| dt dx \\
&\leq \int_\Omega \int_0^1 |\nabla u(x+th)||h| dt dx \\
&= \int_0^1 \int_\Omega |\nabla u(x+th)||h| dx dt \\
&\leq |h| \int_\Omega |\nabla u| \leq c|h| \|u\|_{1,p}.
\end{aligned}$$

By density this inequality holds for all $u \in W_0^{1,p}(\Omega)$. We can now choose $\delta > 0$ such that

$$\int_\Omega |u(x+h) - u(x)| dx < \varepsilon \qquad \forall u \in S, |h| < \delta.$$

If $\partial\Omega$ is C^1, using the Extension Theorem we can prove the above for $u \in W^{1,p}(\Omega)$. Using Theorem 4.2 it follows that S is relatively compact in $L^1(\Omega)$.

Case (i) Choose q_1, $q < q_1 < p^*$. Choose λ, $0 < \lambda < 1$, such that $\dfrac{1}{q} = \dfrac{\lambda}{1} + \dfrac{1-\lambda}{q_1}$. Then

$$\|u\|_q \leq \|u\|_1^\lambda \|u\|_{q_1}^{1-\lambda} \tag{7}$$

for $u \in L^{q_1}(\Omega)$ $(\subset L^1(\Omega))$, and so for $u \in W_0^{m,p}(\Omega)$. Consider a bounded sequence $\{u_n\}$ in $W_0^{m,p}(\Omega)$. Then $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, and by above has a subsequence, also denoted $\{u_n\}$, converging in $L^1(\Omega)$. From (7), together with boundedness of $\{u_n\}$ in $L^{q_1}(\Omega)$, we deduce that $\{u_n\}$ converges in $L^q(\Omega)$. Hence compactness of the embedding $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$.

Case (ii) Essentially the same; choose q_1 such that $q < q_1 < \infty$.

Case (iii) Consider λ, $0 < \lambda < m - \dfrac{N}{p}$, and choose μ, $\lambda < \mu < m - \dfrac{N}{p}$. Then $W_0^{m,p}(\Omega) \hookrightarrow C^{0,\mu}(\Omega)$ is bounded and $C^{0,\mu}(\Omega) \hookrightarrow C^{0,\lambda}(\Omega)$ is compact by Lemma 4.3, hence $W_0^{m,p}(\Omega) \hookrightarrow C^{0,\lambda}(\Omega)$ is compact. Case (iv) Similar: given $0 < \lambda < 1$, choose $\lambda < \mu < 1$. When $\partial\Omega$ is of class C^1, identical arguments apply except at the stage indicated (compactness into L^1). $\square$

Remarks

1) The assumption that Ω is bounded is unavoidable. For example, consider $u_0 \in W_0^{m,p}(\mathbb{R}^N)$, $u_0 \neq 0$. Set

$$u_n(x) = u_0(x + nh)$$

where h is a fixed unit vector. Then $\{u_n\}$ is a bounded sequence in $W_0^{m,p}(\mathbb{R}^N)$, and

$$\int_\Omega |u_n|^q \to 0 \text{ a.e. as } n \to \infty$$

for every bounded domain $\Omega \in \mathbb{R}^N$, so no subsequence of $\{u_n\}$ can converge to a nonzero limit in $\|\ \|_q$. But $\|u_n\|_q = \|u\|_q$ so no subsequence of $\{u_n\}$ tends to 0 in L^q. So $W_0^{m,p}(\mathbb{R}^N)$ is not compactly embedded in $L^q(\mathbb{R}^N)$ for all q.

2) $W^{N,1}(\Omega) \hookrightarrow C(\overline{\Omega})$ is not compact. For the case $N = 1$ see Problem Sheet 9 Q1.

3) $W_0^{m,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, $p^* = \dfrac{Np}{N - mp}$, $mp < N$ is not compact (Problem Sheet 4 Q3). For suppose $B(0,1) \subset \Omega$, choose $0 \neq \varphi \in \mathscr{D}(B(0,1))$ and let

$$\varphi_\varepsilon(x) = \varepsilon^{-\frac{N-p}{p}} \varphi\left(\frac{x}{\varepsilon}\right), \qquad 0 < \varepsilon < 1.$$

Then $\{\varphi_\varepsilon\}_\varepsilon$ is bounded in $W_0^{1,p}(\Omega)$, and $\varphi_\varepsilon(x) \to 0$ as $\varepsilon \to 0$ for all $x \neq 0$, but $\|\varphi_\varepsilon\|_{p^*} \not\to 0$ as $\varepsilon \to 0$ through any subsequence.

4) In dimension 2, if $p > 2$ then $\alpha > 0$, $\beta > 0$ can be chosen such that $u(x, y) = x^\alpha$ and

$$\Omega = \left\{(x, y) \mid 0 < x < 1 \text{ and } 0 < y < x^\beta\right\}$$

satisfy $u \in W^{1,2}(\Omega) \setminus L^p(\Omega)$, showing $p^* = 2$ is best possible for this case of the embedding theorem when the boundary is not assumed smooth (Problem Sheet 6 Q2).