Sobolev Spaces – Autumn 2008 – Analysis Background

Remark. This is more than you need to know, but is made available for reference.

Definition. A metric space (X, d) is called *totally bounded* if, for every $\varepsilon > 0$, X can be covered by finitely many sets of diameter less than ε.

Theorem. A metric space is compact if, and only if, it is complete and totally bounded.

Definition. Let X, Y be normed vector spaces and $T : X \to Y$ a bounded linear operator. We call T *compact* if $\overline{T(B))}$ is a compact subset of Y, where B denotes the closed unit ball of X.

Definition. Let $\mathcal{F}$ be a family of continuous functions from a metric space (X, d) to a metric space (Y, e). $\mathcal{F}$ is called *equicontinuous* if for every $\varepsilon > 0$ and $x \in X$ there exists $\delta > 0$ such that, for every $f \in \mathcal{F}$, if $z \in X$ satisfies $d(z, x) < \delta$ then $e(f(z), f(x)) < \varepsilon$.

Arzelà-Ascoli Theorem - first version. Let (X, d) and (Y, e) be metric spaces and suppose X is separable. Let $\{f_n\}_{n=1}^{\infty}$ an equicontinuous sequence of functions from X to Y, and suppose that for each $x \in X$ there is a compact subset of Y containing $\{f(x) \mid f \in \mathcal{F}\}$. Then $\{f_n\}_{n=1}^{\infty}$ has a subsequence converging pointwise to a continuous function, and the convergence is uniform on compact subsets of X.

Arzelà-Ascoli Theorem - second version. Let (X, d) be a compact metric space, and let $\mathcal{F} \subset C(X, \mathbb{R})$. Then $\mathcal{F}$ is relatively compact in $(C(X, \mathbb{R}), || \cdot ||_{\text{sup}})$ if and only if $\mathcal{F}$ is equicontinuous and bounded in $|| \cdot ||_{\text{sup}}$.

Definition. If X is a normed vector space the *dual space* X^* comprises all bounded linear functionals on X, and has norm

$$||\xi||_* = \sup\{\xi(x) \mid x \in X, \, ||x|| = 1\}.$$

The *canonical isometric embedding* of X in X^{**} is the map $x \mapsto \widehat{x}$ where $\widehat{x}(\xi) = \xi(x)$ for $x \in X$ and $\xi \in X^*$. We call X *reflexive* if $\widehat{X} = X^{**}$.

Riesz Representation Theorem for Hilbert spaces. Let H be a (real) Hilbert space. Then the map $v \mapsto \Lambda_v$, where $\Lambda_v(u) = \langle u, v \rangle$, is an isometric isomorphism of H onto H^*.

Riesz Representation Theorem for L^p. If $1 \le p < \infty$ and q is the conjugate exponent of p, then $L^q(\Omega)$ is isometrically isomorphic to $L^p(\Omega)^*$ under the map $f \mapsto \Lambda_f$ where

$$\Lambda_f(u) = \int_\Omega uf.$$

Definitions. Let X be a normed vector space. The *weak topology* on X is the weakest topology that make all elements of X^* continuous. The *weak* topology* on X^* is the weakest topology that makes all elements of $\widehat{X}$ continuous.

Remarks.

• $x_n \to x$ weakly in X iff $\xi(x_n) \to \xi(x)$ for all $\xi \in X^*$.

• $\xi_n \to \xi$ weak* in X^* iff $\xi_n(x) \to \xi(x)$ for all $x \in X$.

• If $x_n \to x$ weakly in X then $\liminf_{n\to\infty} ||x_n|| \leq ||x||$.

• If X, Y are normed vector spaces, $T : X \to Y$ a bounded linear operator and $x_n \to x$ weakly in X then $Tx_n \to TX$ weakly in Y.

• If X, Y are normed vector spaces, $T : X \to Y$ a compact linear operator and $x_n \to x$ weakly in X then $Tx_n \to Tx$ strongly in Y.

• X^* is necessarily complete, and is therefore a Banach space, even if X is not.

• The canonical embedding is a homeomorphism from X with its weak topology to $\widehat{X}$ with its relative weak* topology.

• A closed linear subspace of a reflexive Banach space is reflexive.

• A product of finitely many reflexive Banach spaces is reflexive.

• Hilbert spaces are reflexive.

• If $1 < p < \infty$ then $L^p(\Omega)$ is reflexive, and hence $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are reflexive.

Goldstine's Theorem. If X is a normed vector space then $\widehat{X}$ is weak* dense in X^{**}.

Banach-Alaoglu Theorem. If X is a normed vector space, then the closed unit ball of X^* is compact in the weak* topology.

Sequential Banach-Alaoglu Theorem. If X is a separable normed vector space, then the closed unit ball of X^* is sequentially compact in the weak* topology.

Corollary. A Banach space X is reflexive if and only if its closed unit ball is compact in the weak topology.

Eberlein-Šmulyan Theorem. A Banach space is reflexive if and only if its closed unit ball is weakly sequentially compact.

Example. Let $S = \{u \in L^\infty(\Omega) \mid |u| \leq 1 \text{ a.e. }\}$. Since $L^\infty(\Omega)$ can be identified with the dual space of $L^1(\Omega)$, which is separable, we deduce that any sequence $\{u_n\}_{n=1}^\infty$ in S (which is the closed unit ball of $L^\infty(\Omega)$) has a subsequence $\{u_{n_j}\}_{j=1}^\infty$ converging in the induced weak* topology to some $u \in S$, that is,

$$\int_\Omega u_{n_j} f \to \int_\Omega uf \quad \forall f \in L^1(\Omega).$$