

§3.1 Localisation

To adapt Theorem 3.3 to a more general domain Ω , we run into the problem that $J_\varepsilon * u(x)$, when u is defined on Ω and $x \in \Omega$, may require values of u at points outside Ω ; setting $u = 0$ outside Ω results in a jump across $\partial\Omega$ causing large derivatives of $J_\varepsilon * u$ near $\partial\Omega$. As a consequence we need to restrict attention to, say a compact subset of Ω .

3.4 Lemma Let $\Omega \subset \mathbb{R}^N$ be nonempty and open, Ω_0 a nonempty bounded open set with $\overline{\Omega}_0 \subset \Omega$, and let $0 < \varepsilon < \text{dist}(\Omega_0, \mathbb{R}^N \setminus \Omega)$. Then

(i) For $1 \leq p \leq \infty$ we have, for $u \in L^p(\Omega)$,

$$\|J_\varepsilon * u\|_{p, \Omega_0} \leq \|u\|_{p, \Omega_0 + B(0, \varepsilon)} \quad \text{and, for } u \in W^{m,p}(\Omega),$$

$$\|J_\varepsilon * u\|_{m,p, \Omega_0} \leq \|u\|_{m,p, \Omega_0 + B(0, \varepsilon)},$$

(ii) for $1 \leq p < \infty$ we have $\|J_\varepsilon * u\|_{p, \Omega_0} \rightarrow 0$ as $\varepsilon \rightarrow 0$ for $u \in L^p(\Omega)$,

$$\|J_\varepsilon * u\|_{m,p, \Omega_0} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ for } u \in W^{m,p}(\Omega).$$

Proof (i) For $0 < \varepsilon < \varepsilon' < \text{dist}(\Omega_0, \mathbb{R}^N \setminus \Omega)$,

choose $\psi \in \mathcal{D}(\Omega)$ s.t. $0 \leq \psi \leq 1$ on Ω , $\psi = 1$ on $\Omega_0 + B^\circ(0, \varepsilon)$, $\psi = 0$ on $\mathbb{R}^N \setminus (\Omega_0 + B^\circ(0, \varepsilon'))$.

Then by Thm 3.3

$$\begin{aligned}\|J_\varepsilon * u\|_{p, \Omega_0} &\leq \|J_\varepsilon * (\psi u)\|_{p, \mathbb{R}^N} \leq \|\psi u\|_{p, \mathbb{R}^N} \\ &\leq \|u\|_{p, \Omega_0 + B^\circ(0, \varepsilon')} \leq \|u\|_{p, \Omega}.\end{aligned}$$

The rest follows by similar arguments. \square

3.5 Theorem Fundamental Thm of Calculus

Suppose $\Omega \subset \mathbb{R}^N$ is nonempty, connected and open, suppose

$u \in W_{loc}^{1,1}(\Omega)$ and that

$D_i u = 0$ a.e. in Ω for $i=1, \dots, N$.

Then u is almost everywhere equal to a constant on Ω .

Proof. Consider an open ball B s.t.

$\bar{B} \subset \Omega$. Then for small $\varepsilon > 0$,

$$\forall x \in B \quad (D_i J_\varepsilon * u)(x) = (J_\varepsilon * D_i u)(x) = 0.$$

So $J_\varepsilon * u$ is smooth and has vanishing 1st order partials in B , so $J_\varepsilon * u$ is constant in B . As $\varepsilon \rightarrow 0$, $J_\varepsilon * u \rightarrow u$ in $L^1(B)$ so u is a.e. equal to a constant in B .

For $c \in \mathbb{R}$ let $\Omega(c)$ be the union of all the open balls $B \subset \Omega$ with $u = c$ a.e. in B . Then

$$\Omega = \bigcup_{c \in \mathbb{R}} \Omega(c)$$

and $\Omega(c_1) \cap \Omega(c_2) = \emptyset$ for $c_1 \neq c_2$. By connectedness of Ω , only one $\Omega(c)$ is nonempty. \square

We can now prove Lemma 3.3:

3.6 Lemma Let $\Omega \subset \mathbb{R}^N$ be open.

(i) Let $1 \leq p < \infty$. Then $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.

(ii) Let $u \in L^1_{loc}(\Omega)$ and suppose

$$\int_{\Omega} u \varphi = 0 \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Then $u = 0$ a.e. in Ω .

Proof

(i) When $\Omega = \mathbb{R}^N$ this was proved in Thm 3.3. For general Ω and $u \in L^p(\Omega)$, let $\varepsilon > 0$ and choose bounded open Ω_0 s.t. $\bar{\Omega}_0 \subset \Omega$, s.t. $\|u - u \chi_{\Omega_0}\|_{p, \Omega} < \varepsilon$.

For $\eta > 0$ small enough we have

$$J_\eta \in \mathcal{D}(\Omega)$$

$$\|J_\eta * (\chi_{\Omega_0} u) - \chi_{\Omega_0} u\|_{p, \Omega} < \varepsilon$$

$$\text{Thus } \|u - J_\eta * (\chi_{\Omega_0} u)\|_{p, \Omega} < 2\varepsilon.$$

(ii) Consider bounded open Ω_0 with
 $\overline{\Omega}_0 \subset \Omega$, $0 < \varepsilon < \text{dist}(\Omega_0, \overline{\Omega} \setminus \Omega)$

Then

$$J_\varepsilon * u(x) = \int_{\mathbb{R}^N} u(y) J_\varepsilon(x-y) dy$$

$= 0$ for $x \in \Omega_0$
since $y \mapsto J_\varepsilon(x-y)$ is a test function
on Ω . Let $\varepsilon \rightarrow 0$ to get
 $J_\varepsilon * u \rightarrow u$ in $L^1(\Omega_0)$ so $u=0$ a.e.
in Ω_0 . Hence $u=0$ a.e. in Ω . \square

3.7 Lemma
Exercise Let $\Omega \subset \mathbb{R}^N$ be open,
 $u \in C(\Omega)$. Then $J_\varepsilon * u \rightarrow u$
uniformly on compact subsets of Ω .

3.8 Lemma Let $\phi \neq \Omega \subset \mathbb{R}^N$ be open.

Then there is a sequence $\{\phi_n\}$ in $C_c(\Omega)$
s.t.

(i) $0 \leq \phi_n \leq 1$ for every n and

$$\sum_{n=1}^{\infty} \phi_n \equiv 1 \text{ on } \Omega \text{ (partition of unity),}$$

(ii) every point of Ω has a neighbourhood
on which all except finitely many of the
 ϕ_n vanish identically (*local finiteness*),

(iii) (by local finiteness) any compact subset
of Ω only intersects the supports of
only finitely many ϕ_n .

Proof. For each $n \in \mathbb{N}$ write

$$\mathcal{S}_n = \{x \in \mathcal{S} \mid |x| < n, \text{dist}(x, \mathbb{R}^N \setminus \mathcal{S}) > \frac{2}{n}\}$$

so each \mathcal{S}_n is a bounded open set

$$\text{with } \overline{\mathcal{S}_n} \subset \mathcal{S}, \quad \mathcal{S}_n \subset \mathcal{S}_{n+1}$$

$$\text{and } \mathcal{S} = \bigcup_n \mathcal{S}_n -$$

- Let $S_n = \overline{\mathcal{S}_n} \setminus \mathcal{S}_{n-1}$ for $n \geq 2$
and $S_1 = \mathcal{S}_1$. Write

$$\psi_n = J_{1/n} * \chi_{S_n},$$

so $\psi_n \in \mathcal{D}(\mathcal{S})$, $\text{supp } \psi_n \subset S_n + \overline{B}(0, \frac{1}{n})$.

Consider $x \in \mathcal{S}$, so $B^\circ(x, r) \subset \mathcal{S}_n$ for some $n \in \mathbb{N}$ and $r > 0$. If $k > n$ then $B^\circ(x, r) \cap S_k = \emptyset$ so

$$B^\circ(x, \frac{r}{2}) \cap (S_k + B^\circ(0, \frac{1}{k})) = \emptyset.$$

- Thus $B^\circ(x, \frac{r}{2}) \cap \text{supp } \psi_k = \emptyset$
if $k > \max\{n, 2/r\}$. Thus the ψ_n are locally finite. A covering argument shows that each compact set only meets the supports of finitely many ψ_n .

Let $x \in \mathbb{R}$, we claim $\psi_n(x) > 0$ for some $n \in \mathbb{N}$. We have $x \in S_m$ for some $m \in \mathbb{N}$ and then

$S_m \subset S_1 \cup \dots \cup S_m$. Choose r , $0 < r < y_m$, s.t. $B^o(x, r) \subset S_m$ and then $S_n \cap B^o(x, r)$ has positive measure for some n , $1 \leq n \leq m$. Then

$$\begin{aligned}\psi_n(x) &= \int_{S_n} J_{1/n}(x-y) dy \\ &\geq \int_{S_n \cap B^o(x, r)} J_{1/n}(x-y) dy > 0\end{aligned}$$

since $J_{1/n}(x-y) > 0$ for $y \in B^o(x, r)$

Hence $\sum \psi_n > 0$ throughout \mathbb{R} ,
so we can define

$$\phi_n = \frac{\psi_n}{\sum_k \psi_k}$$

and the ϕ_n have the desired properties \square

3.9 Meyers-Serrin H=W Theorem

Let $\emptyset \neq \Omega \subset \mathbb{R}^N$ be open and $1 \leq p < \infty$, $m \in \mathbb{N}$. Then

$C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$ (in $\| \cdot \|_{m,p,\Omega}$).

Proof Choose a countable partition of unity into test functions on Ω as provided by Lemma 3.8, say $\{\varphi_n\}$. Consider $u \in W^{m,p}(\Omega)$ and $\delta > 0$.

For each $n \in \mathbb{N}$ choose $0 < \varepsilon_n < \frac{1}{n}$ s.t. $\varepsilon_n < \text{dist}(\text{supp } \varphi_n, \mathbb{R}^N \setminus \Omega)$.

- Let $v_n = \int_{\varepsilon_n} (\varphi_n u) \in \mathcal{W}(n)$

We can also suppose ε_n has been chosen small enough that

$$\|v_n - \varphi_n u\|_{m,p} < \delta 2^{-n}$$

Consider $x \in \Omega$ and choose $r > 0$ s.t. $B^o(x, r) \cap \text{supp } \varphi_n = \emptyset$ for all except finitely many n . Then

$$B^o(x, \frac{r}{2}) \cap (\text{supp } \varphi_n + B^o(0, \frac{r}{2})) = \emptyset$$

- for all except finitely many n , so

$B^o(x, \frac{r}{2}) \cap \text{supp } v_n$ for all suff large n . Thus $\{v_n\}$ is a locally finite family.

Take $v = \sum_{k=1}^{\infty} v_k$. By local finiteness $v \in C^{\infty}(\Omega)$.

Choose $\{\Omega_n\}$ to be an increasing family of nonempty bounded open sets with $\Omega_n \subset \Omega$, and

$$\Omega = \bigcup_n \Omega_n.$$

By local finiteness, each $\overline{\mathcal{S}_n}$ intersects the supports of only finitely many v_n . Since $u = \sum_k \varphi_k u$ we have

$$\begin{aligned} \|v - u\|_{m,p,\mathcal{S}_n} &= \left\| \sum_{k=1}^{\infty} (v_k - \varphi_k u) \right\|_{m,p,\mathcal{S}_n} \\ &\leq \sum_{k=1}^{\infty} \|v_k - \varphi_k u\|_{m,p,\mathcal{S}_n} \\ &< \sum_{k=1}^{\infty} \delta 2^{-k} = \delta. \end{aligned}$$

That is

$$\int_{\mathcal{S}_n} \sum_{0 \leq |\alpha| \leq m} |\mathcal{D}_v^\alpha - \mathcal{D}^{\alpha} u|^p$$

$$\leq \delta^p$$

Letting $n \rightarrow \infty$ and using the MCT,

$$\|v - u\|_{m,p,\mathcal{S}} \leq \delta$$

so $v = u + (v - u) \in W^{m,p}(\mathcal{S})$
and $v \in C^\infty(\mathcal{S})$. □