Expanded degenerations for Hilbert schemes of points

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Abstract. The aim of this thesis is to extend the expanded degeneration construction of Li and Wu to obtain good degenerations of Hilbert schemes of points on semistable families of surfaces, as well as to discuss alternative stability conditions and parallels to the GIT construction of Gulbrandsen, Halle and Hulek and logarithmic Hilbert scheme constructions of Maulik and Ranganathan. We construct some good degenerations of Hilbert schemes of points as proper Deligne-Mumford stacks and show that these provide geometrically meaningful examples of constructions arising from the work of Maulik and Ranganathan.

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1 Introduction

The study of moduli spaces is a central topic in algebraic geometry; among moduli spaces, Hilbert schemes form an important class of examples. They have been widely studied in geometric representation theory, enumerative and combinatorial geometry and as the two main examples of hyperkähler manifolds, namely Hilbert schemes of points on K3 surfaces and generalised Kummer varieties. A prominent direction in this area is to understand the local moduli space of such objects and, in particular, the ways in which a degeneration of smooth Hilbert schemes may be given a modular compactification.

For example, we may consider the geometry of relative Hilbert schemes on a degeneration whose central fibre has normal crossing singularities, i.e. singularities which look locally like hyperplanes intersecting transversely. We may then ask how the singularities of such a Hilbert scheme may be resolved while preserving certain of its properties or how it may be expressed as a good moduli space. This then becomes a compactification problem with respect to the boundary given by the singular locus. Historically, an important method used in moduli and compactification problems has been Geometric Invariant Theory (GIT). More recently, the work of Maulik and Ranganthan [MR20] has explored how methods of tropical and logarithmic geometry can be used to address such questions for Hilbert schemes. This builds upon previous work of Li [Li13] and Li and Wu [LW15] on expanded degenerations for Quot schemes and work of Ranganathan [Ran22b] on logarithmic Gromov-Witten theory with expansions.

Briefly stated, the aim of this thesis is to provide explicit examples of such compactifications and explore the connections between these methods.

1.1 Basic setup

Let k be an algebraically closed field of characteristic zero. Let $X \to C$ be a projective family of surfaces over a curve $C \cong \mathbb{A}^1$ such that the total space is smooth and the central fibre X_0 has simple normal crossing singularities. More precisely, we ask that it be locally given by $\operatorname{Spec} k[x,y,z,t]/(xyz-t)$. In other words, the general fibres are smooth and the central fibre X_0 over $0 \in C$ looks locally like three planes intersecting transversely in $\mathbb{A}^3$. These will be denoted Y_1, Y_2 and Y_3 throughout this work, where these are given in local coordinates by $x=0$, $y=0$ and $z=0$ respectively. Let $X^\circ := X \setminus X_0$, which lies over $C^\circ := C \setminus \{0\}$. Given such a family $X \to C$, we will explore how techniques of expanded degenerations may be used to construct good compactifications of the relative Hilbert scheme of m points $\mathrm{Hilb}^m(X^\circ/C^\circ)$. There are two main aims we may consider.

Firstly, one can aim to construct a compactification in which all limit subschemes can be chosen to satisfy some transversality condition in some modification of X_0. In general, transversality will mean that the subschemes should be normal to the codimension 1 strata of the central fibre. This forces any interesting behaviour of the subschemes to occur on the smooth irreducible components of the modifications of X_0. In the case which interests us here, namely Hilbert schemes of points, it will just mean that we would like our subschemes to have support in the smooth loci of the fibres. We shall abuse terminology slightly and refer to this condition throughout this work as the condition that the subschemes have *smooth support*. The problem therefore is to construct expansions (birational modifications of the central fibre of X in a 1-parameter family) in which all limits needed to compactify $\mathrm{Hilb}^m(X^\circ/C^\circ)$ can be chosen with smooth support. This allows us to break down the problem of studying Hilbert schemes of points on X_0 into smaller parts, by studying the products of Hilbert schemes of points on the irreducible components of the modifications of X. Moreover, this approach will allow us to construct compactifications as stacks which have good properties as moduli spaces. In particular, for all the compactifications constructed in this way, the data of each family of length m zero-dimensional subschemes over C is completely determined by its degenerate fibre, i.e. by its limit in the compactification. As we will mention in the following section, the work of Li and Wu only covers the case where the singular locus of X_0 is smooth. We would like to highlight that understanding how these problems work in general for simple normal crossings is quite powerful, as we can always use semistable reduction to reduce to this case.

Secondly, one can consider the more specific case where X is a type III good degeneration of K3 surfaces and try to construct a family of Hilbert schemes of points on X which will be minimal in the sense of the minimal model program, and by this we mean a good or dlt minimal degeneration (see Section 2.3 for

definitions of these terms). The singularities arising in such a degeneration X are of the type described above, i.e. we can restrict ourselves to the local problem where X_0 is thought of as given by $xyz = 0$ in $\mathbb{A}^3$. Among other reasons, Hilbert schemes of points on K3 surfaces are interesting to study because they form a class of examples of hyperkähler varieties.

1.2 Previous work in this area

Expanded degenerations were first introduced by Li [Li13] and then used by Li and Wu [LW15] to study Quot schemes on degenerations $X \to C$, in the case where the singular locus of X_0 is smooth. They construct a stack of expansions $\mathfrak{C}$ and a family $\mathfrak{X}$ over it, which is a stack of modifications of X_0 in an expanded family, subject to some equivalence relations; among others a relation induced by a natural torus action on the modified fibres. They then impose a stability condition which cuts out families of subschemes of $\mathfrak{X}$ which meet the boundary of the special fibre in a transverse way. For each choice of Hilbert polynomial the family thus obtained is a proper Deligne-Mumford stack.

Following on from [LW15], Gulbrandsen, Halle and Hulek [GHH19] present a GIT version of the above construction in the case of Hilbert schemes of points. They construct an explicit expanded degeneration, i.e. a modified family over a larger base, whose fibres correspond to blow-ups of components of X_0 in the family. They present a linearised line bundle on this space for the natural torus action and they are able to show that in this case the Hilbert-Mumford criterion simplifies down to a purely combinatorial criterion. Using this, they impose a GIT stability condition which recovers the transverse zero-dimensional subschemes of Li and Wu and prove that the corresponding stack quotient is isomorphic to that of Li and Wu. A motivation for this work was to construct type II degenerations of Hilbert schemes of points on K3 surfaces. Indeed, type II good degenerations of K3 surfaces present these types of singularities in the special fibre, which is a chain of surfaces intersecting along smooth curves.

There is more recent work of Maulik and Ranganathan [MR20], building upon earlier ideas of Ranganathan [Ran22b] and results of Tevelev [Tev07], in which they use techniques of logarithmic and tropical geometry to construct appropriate expansions of $X \to C$. This allows them to define moduli stacks of transverse subschemes starting from the case where X_0 is any simple normal crossing variety. They show that the stacks thus constructed are proper and Deligne-Mumford. For more details on this, see Section 2.5.

1.3 Main results

Let $X \to C$ be a semistable degeneration of surfaces. In the following sections, we propose explicit constructions of expanded degenerations and stacks of stable length m zero-dimensional subschemes on these expanded families, which we show to have good properties.

We start by making a stack of expansions $\mathfrak{C}$ and family $\mathfrak{X}$ over it, which contains all expansions of X which can be obtained using a specific sequence of blow-ups. The type of expansion of X which can appear in $\mathfrak{X}$ is therefore greatly restricted, which in this case offers significant advantages, as we will see. We then describe how *Li-Wu stability* (abbreviated LW stability) can be extended to this setting and define an alternative notion of stability, called *strict weak stability with smooth support* (abbreviated SWS stability), derived from GIT stability conditions. We may then construct the stacks $\mathfrak{M}^m_{\mathrm{LW}}$ and $\mathfrak{M}^m_{\mathrm{SWS}}$ of LW and SWS stable length m zero-dimensional subschemes on $\mathfrak{X}$. Our first main results are the following.

Theorem 1.3.1. *The stacks $\mathfrak{M}^m_{\mathrm{LW}}$ and $\mathfrak{M}^m_{\mathrm{SWS}}$ are Deligne-Mumford and proper.*

Theorem 1.3.2. *There is an isomorphism of stacks*

$$\mathfrak{M}^m_{\mathrm{LW}} \cong \mathfrak{M}^m_{\mathrm{SWS}}.$$

This construction has the benefit of being very straightforward compared to the other possible constructions solving this problem, as we will discuss later. The restrictive choices made in the construction of $\mathfrak{X}$ mean that LW or SWS stability are already sufficient conditions to make the stacks of stable objects proper. We will see that if we allow for other choices of expansions to occur, this will no longer be the case, and additional steps will need to be taken in order to produce a proper stack.

This leads us to the second part of our work, where we present a stack of expansions $\mathfrak{C}'$ with a family $\mathfrak{X}'$ over it, which contains $\mathfrak{X}$ as a substack, and explore the different choices which can be made in order to construct a proper stack of stable length m zero-dimensional subschemes on this wider range of expansions. We start by extending the LW and SWS stability conditions defined for the first construction to this setting and denote by $\mathfrak{N}^m_{\mathrm{LW}}$ and $\mathfrak{N}^m_{\mathrm{SWS}}$ the corresponding stacks of LW and SWS stable length m zero-dimensional subschemes on $\mathfrak{X}'$. We show that, though both of these stacks are universally closed, they are not separated, as they contain several non-equivalent representatives for the same limit. We consider two main approaches to solve this problem.

The first method to construct separated stacks is to identify all limits of a given object in $\mathfrak{N}^m_{\mathrm{LW}}$ and $\mathfrak{N}^m_{\mathrm{SWS}}$. We will call the resulting stacks $\overline{\mathfrak{N}^m_{\mathrm{LW}}}$ and $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}$. These are no longer algebraic stacks but we may now prove the following theorem.

Theorem 1.3.3. *The stacks $\overline{\mathfrak{N}^m_{\mathrm{LW}}}$ and $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}$ have finite automorphisms and are proper.*

This approach parallels work of Kennedy-Hunt on logarithmic Quot schemes [Ken23].

The second method we use is to define an additional stability condition, called (α, β)-*stability*, which we use to cut out proper substacks of $\mathfrak{N}^m_{\mathrm{LW}}$ and $\mathfrak{N}^m_{\mathrm{SWS}}$. The (α, β)-stability must satisfy some subtle properties in order for the substack it

cuts out to be proper; indeed it must be what we call a *proper LW or SWS stability condition* (see Definition 7.5.8). A proper LW stability condition (α, β) as defined here recovers one of the choices of stability condition arising from the methods of Maulik and Ranganathan [MR20] in the case of Hilbert schemes of points. We denote by $\mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}$ and $\mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)}$ the stacks of stable length m zero-dimensional subschemes for proper LW and SWS stability conditions (α, β). We are then able to prove the following results.

Theorem 1.3.4. *The stacks $\mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}$ and $\mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)}$ are Deligne-Mumford and proper.*

Theorem 1.3.5. *For a fixed (α, β) which defines a proper LW or SWS stability condition, there is an isomorphism of stacks*

$$\mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)} \cong \mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)}.$$

These choices of proper substack distinguish some geometrically meaningful examples among all possible choices arising from the methods of Maulik and Ranganathan. Our examples do not recover every possible choice, because the geometric assumptions we make in the construction of $\mathfrak{X}'$ mean that it does not contain all possible expansions of X. The reason for these assumptions is discussed in Section 7.1. The stacks $\mathfrak{M}^m_{\mathrm{LW}}$ and $\mathfrak{M}^m_{\mathrm{SWS}}$ are special cases of $\mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}$ and $\mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)}$ stacks.

Finally, we are able to show that all proper algebraic stacks constructed here provide good minimal models (see Section 2.3 for definitions of minimality). Moreover, if $X \to C$ is a good type III degeneration of K3 surfaces, the relative holomorphic symplectic 2-form on X induces a nowhere degenerate relative logarithmic 2-form on each of the constructions presented above.

1.4 Organisation

We start, in Section 2, by giving some background on stacks and hyperkähler varieties, the appropriate notion of a minimal degeneration for this context, some relevant definitions in logarithmic and tropical geometry, and an overview of the work of Maulik and Ranganathan from [MR20] which we will want to refer to in later sections.

Then, in Section 3, we set out a first expanded construction on schemes and, in Section 4, we discuss how various GIT stability conditions can be defined on this construction. As this is simpler than our later construction, we take the opportunity to explain in detail the blow-ups, torus action, linearisation and stability conditions, which will facilitate our explanations of similar processes in the second construction. In Section 6, we describe a corresponding stack of expansions and family over it, building on the expanded degenerations we constructed as schemes, and extend our stability conditions to this setting. We are then able to show that the stacks of stable objects defined have the desired Deligne-Mumford and properness properties.

We then make a second, more general construction in Section 7, which we describe also as a stack and adapt the stability conditions from the first part to this context. We then describe two different methods of making proper stacks and how some of our different stacks relate to each other.

Finally, we discuss how all these constructions relate to the minimal model program in Section 8.

2 Background

2.1 Stacks

In this section, we give some background on stacks, as they will feature heavily throughout this work. All information in this section is sourced from the Stacks Project [Stacks]. For further reading see also [Ols16].

Fibred categories. Let $\mathcal{C}$ be a category and $p\colon \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. We start by defining a notion of base change analogous to a fibre product. Indeed, given an object $x \in \mathrm{Ob}(\mathcal{S})$ and $p(x) = U \in \mathrm{Ob}(\mathcal{C})$, we would like any morphism of objects $V \to U$ in $\mathcal{C}$ to lift to a morphism ϕ of objects $\phi\colon x \to y$ in $\mathcal{S}$ which commutes with the existing maps, in the manner of a fibre product.

Remark 2.1.1. Note that it is not actually a fibre product as the horizontal arrows f and ϕ of the corresponding cartesian diagram are morphisms $V \to U$ and $y \to x$, but the vertical arrows just represent y being sent to V and x to U by the map p (we have defined no actual morphism $y \to V$ or $x \to U$).

We formalise this idea with the following definition.

Definition 2.1.2. Let $\mathcal{C}$ be a category and $p\colon \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. A morphism $\phi\colon y \to x$ is *strongly C-cartesian* if for every $z \in \mathrm{Ob}(\mathcal{S})$, the map

$$\mathrm{Mor}_{\mathcal{S}}(z, y) \longrightarrow \mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}}(\phi(z),\phi(x))} \mathrm{Mor}_{\mathcal{C}}(p(z), p(y))$$

given by $\psi \mapsto (\phi \circ \psi, p(\psi))$ is bijective.

Remark 2.1.3. The condition that the map in Definition 2.1.2 be bijective mirrors the universal property for fibre products. Here, as we do not define a morphism $z \to p(y)$, we would need to take instead $p(z) \to p(y)$ in the statement of the corresponding universal property. This will have a similar effect to the case of fibre products, namely that, given $x \in \mathrm{Ob}(\mathcal{S})$ and $f\colon V \to p(x)$, the pair (y, ϕ) such that $\phi\colon y \to x$ is a strongly cartesian morphism and $p(\phi) = f$ is unique up to unique isomorphism. Because of this, for convenience, we may refer to such a strongly cartesian morphism as $f^*x \to x$ since it is a lift of f.

We now see some properties of strongly cartesian morphisms.

Lemma 2.1.4. *Let $\mathcal{C}$ and $p\colon \mathcal{S} \to \mathcal{C}$ as above.*

1. *The composition of two strongly cartesian morphisms is strongly cartesian.*

2. *Any isomorphism of $\mathcal{S}$ is strongly cartesian.*

3. *Any strongly cartesian morphism ϕ such that $p(\phi)$ is an isomorphism, is an isomorphism.*

We are now in a position to define a fibred category. In the following, given $p\colon \mathcal{S} \to \mathcal{C}$ as before, we will use the notation $\mathcal{S}_U$ to denote the objects and morphisms of $\mathcal{S}$ which lie over $U \in \mathrm{Ob}(\mathcal{C})$.

Definition 2.1.5. Let $\mathcal{C}$ and $p\colon \mathcal{S} \to \mathcal{C}$ be as above. We say that $\mathcal{S}$ is a *fibred category over* $\mathcal{C}$ if given any $x \in \mathrm{Ob}(\mathcal{S})$ lying over $U \in \mathrm{Ob}(\mathcal{C})$ and any morphism $f\colon V \to U$ of $\mathcal{C}$, there exists a strongly cartesian morphism $f^*x \to x$ lying over f.

If $\mathcal{S} \to \mathcal{C}$ is a fibred category and all fibre categories (categories $\mathcal{S}_U$ over objects U of $\mathcal{C}$) are groupoids, i.e. categories where every morphism is an isomorphism, then we say that $\mathcal{S} \to \mathcal{C}$ is a *category fibred in groupoids*.

A *category fibred in sets* is a category fibred in groupoids all of whose fibre categories are discrete, i.e. their only morphisms are the identity morphisms.

Now if $\mathcal{S}$ is any fibred category over $\mathcal{C}$, a *choice of pullbacks for* $p\colon \mathcal{S} \to \mathcal{C}$ is given by a choice of a strongly cartesian morphism $f^*x \to x$ lying over f for any morphism $f\colon V \to U$ of $\mathcal{C}$ and any $x \in \mathrm{Ob}(\mathcal{S}_U)$.

Sites. As we will see, a stack consists of a category $p\colon \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$, which satisfies certain properties. As our stack $p\colon \mathcal{S} \to \mathcal{C}$ will be given by some fibration over $\mathcal{C}$, as well as requiring that $p\colon \mathcal{S} \to \mathcal{C}$ be a fibred category, we would like $\mathcal{C}$ to have good properties with respect to covering morphisms in $\mathcal{C}$ itself. This is formalised by requiring that $\mathcal{C}$ be a site, which is defined as follows.

Definition 2.1.6. Let $\mathcal{C}$ be a category. A *family of morphisms with fixed target in* $\mathcal{C}$ is given by an object $U \in \mathrm{Ob}(\mathcal{C})$, a (possibly empty) set I and for each $i \in I$ a morphism $U_i \to U$ of $\mathcal{C}$ with target U. We use the notation $\{U_i \to U\}_{i\in I}$ to indicate this.

Definition 2.1.7. A *site* is given by a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i\in I}$, called coverings of $\mathcal{C}$, satisfying the following axioms.

1. If $V \to U$ is an isomorphism, then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.

2. If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.

3. If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of C, then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

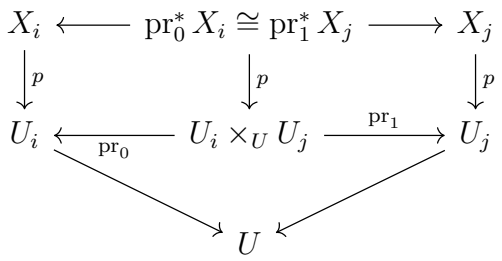
Definition 2.1.8. A morphism $f\colon x \to y$ of a category $\mathcal{C}$ is said to be *representable* if for every morphism $z \to y$ in $\mathcal{C}$, the fibre product $x \times_y z$ exists.

Notice that the first half of the third axiom of Definition 2.1.7 can therefore be rephrased as the requirement that $U_i \to U$ be representable for all $i \in I$.

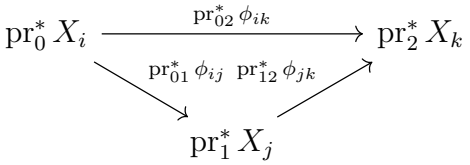
Descent data. We will start by describing a descent datum abstractly and then specify what it means more precisely in the context which interests us, following the approach taken in the Stacks Project [Stacks].

Let $\mathcal{C}$ be a category and let $p\colon \mathcal{S} \to \mathcal{C}$ be a fibred category along with a choice of pullbacks. Let $\mathcal{U} = \{f_i\colon U_i \to U\}_{i\in I}$ be a family of morphisms of $\mathcal{C}$ and assume that all fibre products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ exist. Now, since $p\colon \mathcal{S} \to \mathcal{C}$ is a fibred category, given the projection morphism $\mathrm{pr}_0\colon U_i \times_U U_j \to U_i$ in $\mathcal{C}$ and given $X_i \in \mathrm{Ob}(\mathcal{S})$ such that $p(X_i) = U_i$, we must have a strongly cartesian morphism $\mathrm{pr}_0^* X_i \to X_i$, where we are using the notation of the lift we defined earlier. The object $\mathrm{pr}_0^* X_i$ lies over $U_i \times_U U_j$. Similarly, given the projection morphism $\mathrm{pr}_1\colon U_i \times_U U_j \to U_j$ in $\mathcal{C}$ and $X_j \in \mathrm{Ob}(\mathcal{S})$ such that $p(X_j) = U_j$, there exists a strongly cartesian morphism $\mathrm{pr}_1^* X_j \to X_j$, where $\mathrm{pr}_1^* X_j \in \mathrm{Ob}(\mathcal{S}_{U_i\times_U U_j})$.

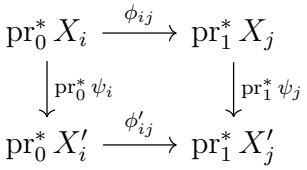
Definition 2.1.9. A *descent datum* (X_i, ϕ_{ij}) *in* $\mathcal{S}$ *relative to the family* $\{f_i\colon U_i \to U\}_{i\in I}$ is given by an object X_i of $\mathcal{S}_{U_i}$ for each $i \in I$ and an isomorphism $\phi_{ij}\colon pr_0^* X_i \to pr_1^* X_j$ in $\mathcal{S}_{U_i\times_U U_j}$ for each pair $(i,j) \in I^2$, as demonstrated in the following commuting diagram.



Moreover, we require that for every triple of indices $(i,j,k) \in I^3$, the diagram



in the category $\mathcal{S}_{U_i\times_U U_j\times_U U_k}$ commutes. A *morphism* $\psi\colon (X_i, \phi_{ij}) \to (X_i', \phi_{ij}')$ *of descent data* is given by a family $\psi = (\psi_i)_{i\in I}$ of morphisms $\psi_i\colon X_i \to X_i'$ in $\mathcal{S}_{U_i}$ such that all the diagrams



in the categories $\mathcal{S}_{U_i\times_U U_j}$ commute.

Here, we will actually want to think of the category $\mathcal{C}$ in Definition 2.1.9 as being a site, so the fibre products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ will exist by condition (3) of Definition 2.1.7.

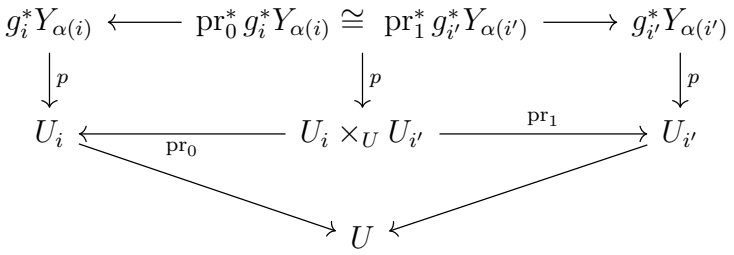
For now, let $\mathcal{C}$ be a category, and let $p\colon \mathcal{S} \to \mathcal{C}$ be a fibred category together with a choice of pullbacks. Let $\mathcal{U} = \{f_i\colon U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ be families of morphisms of $\mathcal{C}$ with fixed target. Assume all the fibre products $U_i \times_U U_{i'}$, $U_i \times_U U_{i'}$, $U_i \times_U U_{i'} \times_U U_{i''}$, $V_j \times_V V_{j'}$ and $V_j \times_V V_{j'} \times_V V_{j''}$ exist. Let $\alpha\colon I \to J$, $h\colon U \to V$ and $g_i\colon U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target. Since $p\colon \mathcal{S} \to \mathcal{C}$ is a fibred category, then, given such a morphism g_i and an object $Y_{\alpha(i)} \in \mathrm{Ob}(\mathcal{S}_{V_{\alpha(i)}})$, there exists an object $g_i^* Y_{\alpha(i)}$ over U_i and a strongly cartesian morphism $g_i^* Y_{\alpha(i)} \to Y_{\alpha(i)}$. Now, again, given the projection morphism $U_i \times_U U_{i'} \to U_i$ and $g_i^* Y_{\alpha(i)} \in \mathrm{Ob}(S_{U_i})$, we have a strongly cartesian morphism $\mathrm{pr}_0^*(g_i^* Y_{\alpha(i)}) \to g_i^* Y_{\alpha(i)}$.

Lemma 2.1.10. *Let the data $(\mathcal{C}, p\colon \mathcal{S} \to \mathcal{C}, \mathcal{U}, \mathcal{V}, \alpha, h, g_i)$ be as above. Let $(Y_j), \phi_{jj'}$ be a descent datum relative to the family $\{V_j \to V\}$. The system*

$$(g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \phi_{\alpha(i)\alpha(i')})$$

is a descent datum relative to $\mathcal{U}$.

The map $(g_i \times g_{i'})^* \phi_{\alpha(i)\alpha(i')}$ gives an isomorphism $\mathrm{pr}_0^* g_i^* Y_{\alpha(i)} \cong \mathrm{pr}_1^* g_{i'}^* Y_{\alpha(i')}$. This can be seen illustrated in the diagram below.



Definition 2.1.11. Let $\mathcal{C}$ be a category and $p\colon \mathcal{S} \to \mathcal{C}$ a fibred category along with a choice of pullback. Let $\mathcal{U} = \{f_i\colon U_i \to U\}_{i\in I}$ be a family of morphisms with target U. Assume all the fibre products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ exist.

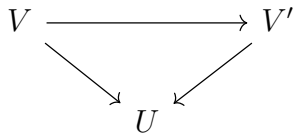
1. Given an object X of $\mathcal{S}_U$, the *trivial descent datum* is the descent datum (X, id_X) with respect to the family $\{\mathrm{id}_U\colon U \to U\}$.

2. Given an object X of $\mathcal{S}_U$ we have a *canonical descent datum* on the family of objects $f_i^* X$ by pulling back the trivial descent datum (X, id_X) via the obvious map $\{f_i\colon U_i \to U\} \to \{\mathrm{id}_U\colon U \to U\}$. We denote this descent datum $(f_i^* X, \mathrm{can})$.

3. A descent datum (X_i, ϕ_{ij}) relative to $\{f_i\colon U_i \to U\}$ is called *effective* if there exists an object X of $\mathcal{S}_U$ such that (X_i, ϕ_{ij}) is isomorphic to $(f_i^* X, \mathrm{can})$.

Presheaves of morphisms and stacks.

Definition 2.1.12. Given a category $\mathcal{C}$ and an object $U \in \mathrm{Ob}(\mathcal{C})$, we define the category of objects over U, denoted $\mathcal{C}/U$, as follows. The objects of $\mathcal{C}/U$ are morphisms $V \to U$ for some $V \in \mathrm{Ob}(\mathcal{C})$. Morphisms between objects $V \to U$ and $V' \to U$ are morphisms $V \to V'$ in $\mathcal{C}$ that make the following diagram commute.



Now, let $p\colon \mathcal{S} \to \mathcal{C}$ be a fibred category and let $x, y \in \mathrm{Ob}(\mathcal{S}_U)$. We will define a functor

$$\mathrm{Mor}(x, y)\colon (\mathcal{C}/U)^{\mathrm{op}} \longrightarrow \underline{\mathrm{Sets}},$$

where $(\mathcal{C}/U)^{\mathrm{op}}$ denotes the opposite category of $\mathcal{C}/U$ in which all objects are the same but the morphisms are reversed. We make a choice of pullbacks for $p\colon \mathcal{S} \to \mathcal{C}$. Then for $f\colon V \to U$, we set

$$\mathrm{Mor}(x, y)(f\colon V \to U) = \mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y),$$

where the right hand side denotes the set of V-morphisms in $\mathcal{S}$ from f^*x to f^*y (note that $f^*x, f^*y \in \mathrm{Ob}(\mathcal{S}_V)$ as both $x, y \in \mathrm{Ob}(\mathcal{S}_U)$).

Let $f'\colon V' \to U$ be a second object of $\mathcal{C}/U$. We must now describe what the functor $\mathrm{Mor}(x, y)$ does to morphisms of $\mathcal{C}/U$. Recall from Definition 2.1.12 that a morphism from the object $f'\colon V' \to U$ to the object $f\colon V \to U$ in $\mathcal{C}/U$ is given by a morphism $g\colon V' \to V$ such that $f' = f \circ g$. This gets sent to the map of sets

$$\mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y) \longrightarrow \mathrm{Mor}_{\mathcal{S}'_V}(f'^*x, f'^*y); \quad \phi \mapsto \phi|_{V'},$$

which can be thought of as g^*.

Lemma 2.1.13. *The functor* $\mathrm{Mor}(x, y)$ *defined above is a presheaf.*

We call $\mathrm{Mor}(x, y)$ the *presheaf of morphisms from* x *to* y. The subpresheaf $\mathrm{Isom}(x, y)$ whose value over V is the set of isomorphisms $f^*x \to f^*y$ in the fibre category $\mathcal{S}_V$ is called the *presheaf of isomorphisms from* x *to* y.

Definition 2.1.14. Let $\mathcal{C}$ be a site. A *stack* over $\mathcal{C}$ is a category $p\colon \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ which satisfies the following conditions:

1. $p\colon \mathcal{S} \to \mathcal{C}$ is a fibred category,
2. for any $U \in \mathrm{Ob}(\mathcal{C})$ and any $x, y \in \mathcal{S}_U$, the presheaf $\mathrm{Mor}(x, y)$ is a sheaf on the site $\mathcal{C}/U$, and
3. for any covering $\mathcal{U} = \{f_i\colon U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$, any descent datum in $\mathcal{S}$ relative to $\mathcal{U}$ is effective.

Definition 2.1.15. A *stack in groupoids* over a site $\mathcal{C}$ is a category $p\colon \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ such that

1. $p\colon \mathcal{S} \to \mathcal{C}$ is fibred in groupoids over $\mathcal{C}$,
2. for any $U \in \mathrm{Ob}(\mathcal{C})$ and any $x, y \in \mathcal{S}_U$, the presheaf $\mathrm{Isom}(x, y)$ is a sheaf on the site $\mathcal{C}/U$, and
3. for any covering $\mathcal{U} = \{f_i\colon U_i \to U\}_{i\in I}$ of $\mathcal{C}$, any descent data in $\mathcal{S}$ relative to $\mathcal{U}$ is effective.

The following lemma describes how this definition relates to Definition 2.1.14.

Lemma 2.1.16. *Let $\mathcal{C}$ be a site. Let $p\colon \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. The following are equivalent.*

1. *$\mathcal{S}$ is a stack in groupoids over $\mathcal{C}$,*
2. *$\mathcal{S}$ is a stack over $\mathcal{C}$ and all fibre categories are groupoids, and*
3. *$\mathcal{S}$ is fibred in groupoids over $\mathcal{C}$ and is a stack over $\mathcal{C}$.*

Fppf topology. Before we can define the notion of an algebraic stack, we say a few words about fppf topology. We start by recalling some terminology for morphisms of schemes.

Definition 2.1.17. A morphism of rings $R \to A$ is of finite presentation if A is isomorphic to $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ as an R-algebra for some n, m and some polynomials f_j. Now, let $f\colon X \to S$ be a morphism of schemes. Then we make the following definitions.

1. The morphism f is of *finite presentation at $x \in X$* if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite presentation.
2. We say that f is *locally of finite presentation* if it is of finite presentation at every point of X.
3. We say that f is of *finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated (see Definitions 26.19.1 and 26.21.3 of [Stacks]).

We may now define an fppf covering.

Definition 2.1.18. Let T be a scheme. An *fppf covering of T* is a family of morphisms $\{f_i\colon T_i \to T\}_{i\in I}$ of schemes such that each f_i is flat, locally of finite presentation and such that $T = \bigcup f_i(T_i)$.

The next lemma shows us that an fppf covering on a scheme T satisfies exactly the properties of a site.

Lemma 2.1.19. *Let T be a scheme.*

1. *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fppf covering of T.*
2. *If $\{T_i \to T\}_{i \in I}$ is an fppf covering and for each i we have an fppf covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fppf covering.*
3. *If $\{T_i \to T\}_{i \in I}$ is an fppf covering and $T' \to T$ is a morphism of schemes, then $\{T' \times_T T_i \to T\}_{i \in I}$ is an fppf covering.*

Let $\underline{\mathrm{Sch}}$ be some suitably chosen small category of schemes. For our purposes, it is enough to consider the category of quasi-projective k-schemes. We will say that some category $\underline{\mathrm{Sch}}_{\mathrm{fppf}}$ is a *big fppf site* if it is a site given by the underlying category $\underline{\mathrm{Sch}}$ and a set of fppf coverings of $\underline{\mathrm{Sch}}$.

Algebraic stacks. We may now define algebraic spaces and algebraic stacks. As we will see in Lemma 2.1.25, it is possible to replace fppf by étale in the following.

Definition 2.1.20. Let S be a scheme in $\underline{\mathrm{Sch}}_{\mathrm{fppf}}$. An *algebraic space over S* is a presheaf $F \colon (\underline{\mathrm{Sch}}/S)^{\mathrm{op}}_{\mathrm{fppf}} \to \underline{\mathrm{Sets}}$ with the following properties.

1. The presheaf F is a sheaf.
2. The diagonal morphism $F \to F \times F$ is representable.
3. There exists a scheme $U \in \mathrm{Ob}((\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}})$ and a map $U \to F$ which is surjective and étale.

We now describe what it means for categories and morphisms of categories to be representable.

Definition 2.1.21. Let $\mathcal{C}$ be a category. A category fibred in groupoids $p \colon \mathcal{X} \to \mathcal{C}$ is called *representable* if there exists an object X of $\mathcal{C}$ and an equivalence of stacks $j \colon \mathcal{X} \to \mathcal{C}/X$.

If the category $\mathcal{C}$ in the above definition is actually the site $(\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$ for some scheme $S \in \mathrm{Ob}(\underline{\mathrm{Sch}}_{\mathrm{fppf}})$, then we sometimes say that $p \colon \mathcal{X} \to (\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$ is representable by a scheme. We will now relax this condition slightly to allow for the object X in the above definition to be an algebraic space. In order to do so, we must first construct a category fibred in sets as follows.

Let S be a scheme contained in $(\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$ and let $F \colon (\underline{\mathrm{Sch}}/S)^{\mathrm{op}}_{\mathrm{fppf}} \to \underline{\mathrm{Sets}}$ define an algebraic space. For $f \colon V \to U$ in $(\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$ let us fix the notation $F(f^\top) = f^* \colon F(U) \to F(V)$, where $f^\top$ is the opposite morphism to f in the category $(\underline{\mathrm{Sch}}/S)^{\mathrm{op}}_{\mathrm{fppf}}$. We construct a category $\mathcal{S}_F$ fibred in sets over $(\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$ as follows. Define

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}((\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}), x \in \mathrm{Ob}(F(U))\}.$$

Then, for $(U,x),(V,y) \in \mathrm{Ob}(\mathcal{S}_F)$ we define

$$\mathrm{Mor}_{\mathcal{S}_F}((V,y),(U,x)) = \{f \in \mathrm{Mor}_{(\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}}(V,U) \mid f^*x = y\},$$

where composition is given by $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $(\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$. The category $p_F \colon \mathcal{S}_F \to (\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$ is fibred in sets.

Definition 2.1.22. Let S be a scheme contained in $\underline{\mathrm{Sch}}_{\mathrm{fppf}}$. A category fibred in groupoids $p \colon \mathcal{X} \to (\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$ is called *representable by an algebraic space over* S if there exists an algebraic space F over S and an equivalence $j \colon \mathcal{X} \to \mathcal{S}_F$ of categories over $(\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$.

Now, a 1-morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of categories fibred in groupoids over $(\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$ is called *representable by algebraic spaces* if for any $U \in \mathrm{Ob}((\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}})$ and any $y \colon (\underline{\mathrm{Sch}}/U)_{\mathrm{fppf}} \to \mathcal{Y}$ the category fibred in groupoids $(\underline{\mathrm{Sch}}/U)_{\mathrm{fppf}} \times_y \mathcal{X}$ is representable by an algebraic space over U.

Definition 2.1.23. Let S be a base scheme contained in $\underline{\mathrm{Sch}}_{\mathrm{fppf}}$. An *algebraic stack* over S is a category $p \colon \mathcal{X} \to (\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$ over $(\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$ satisfying the following properties.

1. The category $\mathcal{X}$ is a stack in groupoids over $(\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}}$.
2. The diagonal morphism $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is representable by algebraic spaces.
3. There exists a scheme $U \in \mathrm{Ob}((\underline{\mathrm{Sch}}/S)_{\mathrm{fppf}})$ and a morphism $(\underline{\mathrm{Sch}}/U)_{\mathrm{fppf}} \to \mathcal{X}$ which is surjective and smooth.

Definition 2.1.24. Let S be a scheme contained in $(\underline{\mathrm{Sch}})_{\mathrm{fppf}}$. Let $\mathcal{X}$ be an algebraic stack over S. We say that $\mathcal{X}$ is a *Deligne-Mumford stack* if there exists a scheme U and a surjective étale morphism $(\underline{\mathrm{Sch}}/U)_{\mathrm{fppf}} \to \mathcal{X}$.

We shall use the following equivalent characterisation of Deligne-Mumford stack in the following sections. An algebraic stack $\mathcal{X}$ over a scheme S is Deligne-Mumford if for every algebraically closed field k and object $X \in \mathcal{X}(k)$, the automorphism group of X is finite. See [Ols16] for details.

In the following, we will use the term *Artin stack* when we want to refer to an algebraic stack which is not Deligne-Mumford. Finally, we note that defining algebraic stacks over the fppf or étale topology comes to the same thing with the following result.

Lemma 2.1.25. *Let $p \colon \mathcal{X} \to (\underline{\mathrm{Sch}}/S)_{\text{étale}}$ be a category over $(\underline{\mathrm{Sch}}/S)_{\text{étale}}$ defined by replacing fppf by étale everywhere in Definition 2.1.23. Then this is also an algebraic stack.*

2.2 K3 surfaces and IHS or hyperkähler manifolds

In order to give an overview of irreducible holomorphic symplectic manifolds, we make a brief detour from algebraic geometry to the category of complex analytic manifolds. This is, in a sense, the natural perspective from which to first approach these objects as we tend to classify them by deformation type.

Definition 2.2.1. An *irreducible holomorphic symplectic (IHS)* or *hyperkähler manifold* X is a simply-connected, compact Kähler manifold whose space of holomorphic 2-forms $H^0(X, \Omega_X^2)$ is spanned by an everywhere nondegenerate form ω.

It follows from the definition that such an X must have even complex dimension $2n$. If $n = 1$, then X is a K3 surface. Moreover, taking an n-fold exterior power of some section of $H^0(X, \Omega_X^2)$ will yield an everywhere nonvanishing section of the canonical bundle, hence all IHS manifolds have trivial canonical bundle. There are broadly speaking four known types of hyperkähler manifold. The first two were constructed by Beauville [Bea83]:

1. the Hilbert scheme of n points on a K3 surface S, denoted $S^{[n]}$ (a K3 surface is a special case of this);

2. the generalised Kummer variety $K^{n+1}(A)$ on a 2-dimensional abelian variety A.

All known examples of hyperkähler manifolds have been shown to be a deformation of one of these two cases, except for two families of examples constructed by O'Grady ([OGr03] and [OGr99]). Note also that, although the representatives of these deformation classes given here happen to be algebraic, all elements of these classes are not.

One of the main motivations for the study of IHS manifolds is the Beauville-Bogomolov decomposition theorem [Bog74], which states that, up to taking a finite étale cover, compact Kähler manifolds with trivial canonical bundle are given as products of abelian varieties, strict Calabi-Yaus and IHS manifolds.

IHS manifolds are also referred to as hyperkähler manifolds (note that our definition of hyperkähler is what is called irreducible hyperkähler in [Huy99]) because they can be alternatively described as compact connected $4n$-dimensional Riemannian manifolds (M, g) with holonomy $\mathrm{Sp}(n)$. For such a manifold, the quaternions $\mathbb{H}$ give rise to endomorphisms on the tangent bundle of M. Moreover, for any imaginary $\lambda \in \mathbb{H}$ such that $\lambda^2 = -1$, the corresponding endomorphism defines an integrable almost complex structure on M with respect to which the metric g is Kähler. There is therefore a sphere S^2 of complex structures on M satisfying these properties. Hyperkähler manifolds described in this way have been shown to be equivalent to IHS manifolds, following Yau's solution of the Calabi conjecture. For further details, see [Huy99].

Moduli spaces of IHS manifolds. For K3 surfaces and higher dimensional IHS manifolds, it is possible to give more or less good descriptions of their moduli spaces in terms of Hodge theory. For any IHS manifold X, the integral part of the degree two cohomology $H^2(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module of finite rank, which carries a nondegenerate integral symmetric bilinear form endowing it with the structure of a lattice L of signature $(3, \mathrm{rk}(L) - 3)$. Given such a lattice, we may define the corresponding *period domain*

$$\Omega_L \coloneqq \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x, x) = 0,\ (x, \overline{x}) > 0\}.$$

A *marking* of X is an isometry $\phi\colon H^2(X,\mathbb{Z}) \to L$. The map ϕ extends by linearity to $H^2(X,\mathbb{C})$ and it is easy to see that given such an isometry, we have $[\phi(\omega)] \in \Omega_L$, where ω is the holomorphic symplectic 2-form defined above. The following Torelli theorem holds for K3 surfaces.

Theorem 2.2.2. *Two K3 surfaces S and S' are isomorphic if and only if there is an isometry $H^2(S',\mathbb{Z}) \to H^2(S,\mathbb{Z})$ whose $\mathbb{C}$-linear extension preserves the Hodge decomposition.*

In fact, a slightly stronger result holds. See [GHS13] for more details. We moreover have the next result, which was first shown by Todorov [Tod89].

Theorem 2.2.3. *The set $\mathrm{O}(L)\backslash\Omega_L$ is in 1:1 correspondence with the set of isomorphism classes of K3 surfaces.*

Note, however, that the action of $\mathrm{O}(L)$ is not well behaved and for a good notion of moduli space for K3 surfaces, we must consider instead polarised K3 surfaces. In the polarised case, we may define a subspace Ω of Ω_L and a subgroup $\widetilde{\mathrm{O}}(L)$ of $\mathrm{O}(L)$ (see [GHS13] for precise details) and show the following.

Theorem 2.2.4. *The variety $\widetilde{\mathrm{O}}(L)\backslash\Omega$ is the moduli space of polarised K3 surfaces of degree $2d$.*

For higher dimensional IHS manifolds, a global Torelli theorem like the one for K3 surfaces no longer holds. A weaker version still exists, formulated by Markman [Mar11], based on results of Verbitsky, see [Huy12] and [Ver13].

Theorem 2.2.5. *Suppose that X and Y are IHS manifolds.*

1. *If $f\colon H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is an isomorphism of integral Hodge structures which is a parallel transport operator then X and Y are bimeromorphic.*
2. *If, moreover, f maps a Kähler class of Y to a Kähler class of X, then X and Y are isomorphic.*

In this situation again, after fixing certain invariants, we may start to describe a moduli space of IHS manifolds, but unlike the K3 case, the degree is no longer the only invariant of a polarisation. It is therefore necessary to fix the dimension $2n$, a lattice L and the *Fujiki invariant* (which together form what is called the *numerical type* N of the IHS manifold) and a *polarisation type*, i.e. an $\mathrm{O}(L)$-orbit of a primitive vector $h \in L$. See [GHS13] for details. A moduli space $\mathcal{M}_{n,N,h}$ of IHS manifolds with this fixed data is known to exist by Viehweg's results [Vie95]. The associated period domain $\Omega_{L_h} \coloneqq h^\perp \cap \Omega_L$ corresponds to the smaller lattice L_h of signature $(2, \mathrm{rk}(L)-3)$. This implies that Ω_{L_h} has two connected components. Fix one and denote it by $\mathcal{D}_{L_h}$.

Theorem 2.2.6. *For every component $\mathcal{M}'_{n,N,h}$ of the moduli space $\mathcal{M}_{n,N,h}$, there exists a finite to one dominant morphism*

$$\psi\colon \mathcal{M}'_{n,N,h} \longrightarrow O^+(L,h)\backslash\mathcal{D}_{L_h},$$

where $\mathrm{O}^+(L,h)$ is the subgroup of real spinor norm 1 of the stabiliser of h in $\mathrm{O}(L)$.

Some further refinements of this result exist, especially for IHS manifolds of the deformation type of Hilbert schemes of points on K3 surfaces. We refer the reader to [GHS13] and [Huy99] for further details.

2.3 Minimal degenerations

There is a natural way to compactify moduli spaces of IHS manifolds called the Baily-Borel compactification. Boundary components in the Baily-Borel compactification correspond to orbits of totally isotropic subspaces $E \subset L_h$. Since L_h has signature $(2, n)$, the dimension of E is 1 or 2, corresponding to dimension 0 and dimension 1 boundary components respectively. See [GHS13] for more details. The different types of degenerations, see Definition 2.3.4, allow us to situate local moduli problems within these compactified moduli spaces. Type I corresponds to degenerations whose central fibre still has a component which is an IHS manifold. Type II and type III correspond to degenerations whose central fibre lies over a general point of a dimension 1 boundary component and a dimension 0 boundary component respectively.

As mentioned in the introduction, there are two main goals we aimed to achieve with our constructions. The first one was to find a good (i.e. proper and Deligne-Mumford) compactification of the space $\mathrm{Hilb}^m(X^\circ/C^\circ)$ such that all limit subschemes in this compactification have smooth support. The second was to study specifically the case where $X \to C$ is a good type III degeneration of K3 surfaces. In this context, the objective was to construct compactifications of $\mathrm{Hilb}^m(X^\circ/C^\circ)$ which give us type III degeneration of Hilbert schemes of points on K3 surfaces satisfying some minimality condition in the sense of the minimal model program. In this section, we will discuss what these types of degeneration mean and what a good notion of minimality is in this setting.

An important notion in these definitions is that of the *dual complex of a fibre*. Given a scheme $Y = \bigcup Y_i$ of dimension d, where the Y_i are irreducible and meet transversely, we define the dual intersection complex to be the regular cell complex Σ defined as follows. Each irreducible component of Y of dimension d corresponds to a vertex of Σ and, more generally, an irreducible component $W \subseteq \bigcap_{i\in I} Y_i$ corresponds to an $|I|-1$ dimensional cell of Σ, which we denote by v_W. For every irreducible component W' with $W \subseteq W'$, the cell corresponding to W' is contained in v_W. See [FKX17] for further details. For example, for the type of normal crossing singularity described above, the dual complex will look like a triangle, whose vertices are given by Y_1, Y_2 and Y_3. The dual complex of a good type III degeneration of Hilbert schemes of points on K3 surfaces is therefore expressed as a triangulated sphere.

The minimality condition first considered by Kulikov [Kul77] for K3 surfaces and Enriques surfaces is that of a semistable degeneration being good. This definition is used by Nagai [Nag08] when constructing good type II degenerations of Hilbert schemes of n points on K3 surfaces by resolving the singularities of the relative Hilbert scheme of n points on good type II degenerations of K3 surfaces.

In a similar direction, there are some computations in the thesis of Hanke [Han15] on type II and type III degenerations of Hilbert schemes of 2 points on K3 surfaces. We define what good means in this context, as it is given in [Nag08]. Note that the term *good minimal model* used in Section 8 is meant in the more general sense of Definitions 2.3.7 and 2.3.8.

Definition 2.3.1. A *good semistable degeneration* of compact symplectic Kähler manifolds is a degeneration $\pi\colon X \to C$ of relative dimension $2n$ satisfying

1. π is semistable, i.e. the total space X is smooth and the special fibre $X_0 = \pi^{-1}(0)$ is a reduced simple normal crossing divisor as a scheme theoretic fibre.

2. There exists a relative logarithmic 2-form $\omega_\pi \in H^0(X, \Omega^2_{X/C}(\log X))$ such that $\wedge^n \omega_\pi \in H^0(X, K_{X/C})$ is nowhere vanishing.

Kulikov requires this property when describing minimal degenerations of K3 surfaces in terms of the dual complexes of their special fibres. In fact, this condition is simply called *good*, as, in this case, semistable degenerations seem to be a natural property to require. It is now accepted that this is slightly too strong a property to demand in general and for higher dimensional problems, de Fernex, Kollár and Xu [FKX17] define the appropriate generalisation of the semistable condition, which is the following.

Definition 2.3.2. A pair (X, D) is *log canonical* (or *lc*) if the discrepancy $a(E, X, D) \geq -1$ for every divisor $E \subset Y$ and every birational morphism $f\colon Y \to X$.

A log canonical pair (X, D) is *divisorial log terminal* (or *dlt*) if for every divisor E over (X, D) with discrepancy -1, the pair (X, D) is simple normal crossing at the generic point of $\mathrm{center}_X(E)$, where $\mathrm{center}_X(E)$ is the image of E in X.

See [FKX17] for more details.

Definition 2.3.3. We say that $X \to C$ is a *semistable minimal degeneration*, respectively a *dlt minimal degeneration of hyperkähler manifolds* if $K_{X/C}$ is trivial, and $X \to C$ is semistable or respectively (X, X_0) is a dlt pair.

Note that every semistable degeneration is dlt. Moreover, the condition that $K_{X/C}$ is trivial can be identified with the condition that there exists an everywhere nondegenerate relative holomorphic logarithmic 2-form, as in the second point of Definition 2.3.1. Semistable and dlt minimal models are not unique and it is possible to construct many such models with different types of singularities. However, the topology of the dual complex is fixed for any dlt minimal degeneration. Let us now define what we mean exactly by the type of a degeneration (as defined in [KLSV18]).

Definition 2.3.4. Let $X \to C$ be a projective degeneration of hyperkähler manifolds (including the K3 case). Let $\nu \in \{1, 2, 3\}$ be the nilpotency index for the associated monodromy operator N on $H^2(X_t)$ (i.e. $N = \log T_u$, where T_u is the unipotent part of the monodromy $T = T_s T_u$). We say that the degeneration is *of type I, II, or III* respectively if $\nu = 1, 2, 3$ respectively.

Moreover, the following theorem of Kollár, Laza, Saccà and Voisin [KLSV18] tells us that given a dlt minimal degeneration, the type of the degeneration is dictated by the dimension of the dual complex.

Theorem 2.3.5. *Let $X \to C$ be a minimal dlt degeneration of $2n$-dimensional hyperkähler manifolds. Let Σ denote the dual complex of the central fibre (and $|\Sigma|$ its topological realisation). Then*

1. $\dim |\Sigma|$ *is* $0, n$ *or* $2n$ *if and only if the type of the degeneration is I, II, or III respectively, i.e.* $\dim |\Sigma| = (\nu - 1)n$*, where* ν *is as above.*

2. *If the degeneration is of type III, then* $|\Sigma|$ *is a simply connected closed pseudo-manifold (see [NX16] for details), which is a rational homology* $\mathbb{CP}^n$*.*

We will now extend the previous notions of semistable and dlt to stacks.

Definition 2.3.6. Let $\mathfrak{Y}$ be a Deligne-Mumford stack which is flat and locally of finite type over C. For any C-scheme S with a (non-constant) étale morphism $\xi\colon S \to C$ and any $Y \in \mathrm{Ob}(\mathfrak{Y}_C)$ over C, we denote the composition map $f\colon \xi^*Y \to C$. If every pair $(\xi^*Y, f^{-1}(0))$ constructed in this way is dlt, we say that $\mathfrak{Y} \to C$ is *dlt*. We say that $\mathfrak{Y} \to C$ is *semistable* if every such composition map f is semistable.

We may generalise the notion of a semistable or dlt minimal degeneration of hyperkähler manifolds to the stack setting in a similar way by adding the requirement that the relative canonical bundle be trivial. We will see that working with stacks enables us to better understand how hyperkähler varieties degenerate, as it will allow us to construct a logarithmic 2-form which captures the data of how the holomorphic symplectic 2-form degenerates. We also extend the notion of a *good minimal model* to describe degenerations whose fibres are not necessarily hyperkähler manifolds.

Definition 2.3.7. A semistable or dlt model $Y \to C$ is a good minimal model if Y is $\mathbb{Q}$-factorial and the divisor $K_Y + (Y_0)_{\mathrm{red}}$ is semi-ample.

This generalises to the stack setting in the following way.

Definition 2.3.8. Let $\mathfrak{Y}$ be a Deligne-Mumford stack which is flat and locally of finite type over C. For any C-scheme S with a (non-constant) étale morphism $\xi\colon S \to C$ and any $Y \in \mathrm{Ob}(\mathfrak{Y}_C)$ over C, let $W \coloneqq \xi^*Y$ and denote by $f\colon W \to C$ the composition map. We say that $\mathfrak{Y} \to C$ is a *good minimal model* if for every such W and f, the divisor $K_{W/C} + f^{-1}(0)$ is semi-ample.

2.4 Logarithmic and tropical geometry

Motivation. Logarithmic geometry arises as a natural solution in the study of moduli spaces. Indeed, in our search to describe and classify varieties with fixed invariants, it is essential that we understand the spaces parametrising such varieties. Often, for example in the case of the moduli space of smooth curves

of genus g, this is not compact. In order to compactify such a space, we must allow for certain singular varieties to appear at the boundary of our moduli space. Logarithmic geometry addresses this problem by extending the notion of smoothness to allow for mild singularities, e.g. normal crossing singularities. Moreover, a logarithmic scheme is endowed with additional structure which remembers the smoothings of these boundary objects. For more details see the article [Log], lecture notes [Ran22a], as well as the first section of [MR20].

Logarithmic structure. Let $(X, \mathcal{O}_X)$ be a scheme and let $\mathcal{M}_X$ be some sheaf of monoids of X.

Definition 2.4.1. A *pre-logarithmic structure* on X is given by a homomorphism of monoids

$$\alpha \colon \mathcal{M}_X \longrightarrow \mathcal{O}_X,$$

where we consider $\mathcal{O}_X$ as a sheaf of monoids with respect to its multiplicative structure. This pre-logarithmic structure will be called a *logarithmic structure* if

$$\alpha^{-1}(\mathcal{O}_X^*) \longrightarrow \mathcal{O}_X^*$$

is an isomorphism. This implies $\mathcal{M}_X^* \cong \mathcal{O}_X^*$. A *logarithmic scheme* $(X, \mathcal{M}_X)$ is a scheme X together with a logarithmic structure $\mathcal{M}_X$.

We may construct examples of logarithmic structures in the following way. Let D be a closed algebraic or analytic subset of X and denote by $X^* \coloneqq X \setminus D$ its complement. Let $j \colon X^* \hookrightarrow X$ and $i \colon D \hookrightarrow X$ be the inclusions. We then define the sheaf of monoids on X

$$\mathcal{M}_{X^*/X} \coloneqq \{f \in \mathcal{O}_X \mid j^*(f) \in \mathcal{O}_{X^*}^*\} \subseteq \mathcal{O}_X.$$

We may then, for example, define the trivial logarithmic structure $\mathcal{M}_{X/X} \cong \mathcal{O}_X^*$ or the empty logarithmic structure $\mathcal{M}_{\emptyset/X} \cong \mathcal{O}_X$. The logarithmic structure which will be of interest to us in this work is the following.

We let D be a simple normal crossing divisor in X. Then, for an open subset $U \subseteq X$, we have

$$\mathcal{M}_{X^*/X}(U) \coloneqq \{f \in \mathcal{O}_X(U) \mid f|_{U \setminus D} \in \mathcal{O}_X^*(U \setminus D)\}.$$

Definition 2.4.2. A morphism of logarithmic schemes $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a morphism of the underlying schemes $X \to Y$ along with a morphism of sheaves of monoids $f^*\mathcal{M}_Y \to \mathcal{M}_X$ compatible with the maps to $\mathcal{O}_X$.

Note that the morphism $f^*\mathcal{M}_Y \to \mathcal{M}_X$ is given by the composition

$$f^{-1}\mathcal{M}_Y \longrightarrow f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X.$$

Definition 2.4.3. The *characteristic sheaf* is the quotient sheaf $\overline{\mathcal{M}}_X \coloneqq \mathcal{M}_X/\mathcal{O}_X^*$.

Tropical geometry. We now see how the above ideas tie into toric and tropical geometry. Let X be toric, with dense torus $T \cong \mathbb{G}_m^n$ over an algebraically closed field k of characteristic 0. Let

$$M = \mathrm{Hom}(T, \mathbb{G}_m) \quad \text{and}$$
$$N = \mathrm{Hom}(\mathbb{G}_m, T)$$

be the character and cocharacter lattices.

Definition 2.4.4. A *cone* in $N_\mathbb{R} \coloneqq N \otimes_\mathbb{Z} \mathbb{R}$ is given by the span of a finite set of elements of N over $\mathbb{R}_{\geq 0}$. A *face* τ of this cone is the intersection of a supporting hyperplane with the cone itself, spanned by a subset of this set over $\mathbb{R}_{\geq 0}$ such that whenever $u + v \in \tau$ then we must have both $u \in \tau$ and $v \in \tau$. A *fan* is a collection of cones such that the intersection of any two cones is a face of each cone, and each face of each cone in the fan is also a cone in the fan.

Toric geometry gives a correspondence between fans in $N_\mathbb{R}$ and normal equivariant compactifications of T. Indeed, given a fan Σ in $N_\mathbb{R}$ and a cone $\sigma \in \Sigma$, we may write $X(\sigma) \coloneqq \mathrm{Spec}\, k[\sigma^\vee \cap M]$. If cones σ and σ' intersect along a face τ we may glue the varieties $X(\sigma)$ and $X(\sigma')$ along the open immersions

$$X(\sigma) \longleftarrow X(\tau) \hookrightarrow X(\sigma').$$

Given a fan Σ, this produces a scheme $X(\Sigma)$ and many properties of this scheme can be easily read off the fan Σ. For example, $X(\Sigma)$ is smooth if and only if every cone $\sigma \subset \Sigma$ has $\mathbb{Z}$-independent primitive generators. For a general reference on toric geometry, see [Ful93].

More generally, if $(X, \mathcal{M}_X)$ is a logarithmic scheme, where $\mathcal{M}_X$ gives the divisorial logarithmic structure described before, then we may give the corresponding notion of a fan $\Sigma(X)$ in the following way. Recall that the characteristic sheaf $\overline{\mathcal{M}}_X$ for the divisorial logarithmic structure is defined by

$$\overline{\mathcal{M}}_X(U) = \{f \in \mathcal{O}_X(U) \mid f|_{U \setminus D} \in \mathcal{O}_X^*(U \setminus D),\ f|_D = 0\},$$

i.e. it records the data of monomials which vanish at the divisor D. The idea of $\Sigma(X)$ will be to keep track of the corresponding degrees of vanishing. We let $\Sigma(X) \coloneqq \mathrm{colim}_{x \in X} (\overline{\mathcal{M}}_{X,x})^\vee$. This will be contained in $\mathbb{R}^r_{\geq 0}$, where r is the number of components of D, since D is a simple normal crossing divisor. We call $\Sigma(X)$ the *tropicalisation* of X.

The Artin fan. In the following work, we will want to study possible birational modifications of our scheme X around the divisor D. In the tropical language, these are expressed as subdivisions.

Definition 2.4.5. Let Υ be a fan, let $|\Upsilon| = \bigcup \Upsilon$ be its support and v be a continuous map

$$v \colon |\Upsilon| \longrightarrow \Sigma(X)$$

such that the image of every cone in Υ is contained in a cone of Σ and that is given by an integral linear map when restricted to each cone in Υ. We say that υ is a *subdivision* if it is injective on the support of Υ and the integral points of the image of each cone $\tau \in \Upsilon$ are exactly the intersection of the integral points of $\Sigma(X)$ with τ.

Remark 2.4.6. This definition is based on Definition 1.3.1 of [MR20] but here, for simplicity, we define subdivisions on fans as that is sufficient for our purposes.

A subdivision of the tropicalisation defines a birational modification of X in the following way. The subdivision

$$\Upsilon \hookrightarrow \Sigma(X) \hookrightarrow \mathbb{R}^r_{\geq 0}$$

has an associated toric variety $\mathbb{A}_\Upsilon$, which comes with a $\mathbb{G}_m^r$-equivariant birational map $\mathbb{A}_\Upsilon \to \mathbb{A}^r$. Then we have an induced morphism of quotient stacks

$$[\mathbb{A}_\Upsilon/\mathbb{G}_m^r] \longrightarrow [\mathbb{A}^r/\mathbb{G}_m^r]$$

and we may define the induced birational modification of X to be

$$X_\Upsilon := X \times_{[\mathbb{A}^r/\mathbb{G}_m^r]} [\mathbb{A}_\Upsilon/\mathbb{G}_m^r].$$

We shall call such a birational modification an *expansion* of X.

In order to study these subdivisions, or corresponding expansions, we define the *Artin fan*

$$A(X) := \operatorname{colim}_{x \in X} \operatorname{Spec} k[\overline{\mathcal{M}}_{X,x}/T],$$

where T is the dense torus of X. This is an Artin stack which encodes the same data as the tropicalisation but there is now a tautological map $X \to A(X)$. The logarithmic structure on X is pulled back from $A(X)$ and the logarithmic properties of X are given by the properties of the map $X \to A(X)$. For example, X is logarithmically smooth if the map $X \to A(X)$ is smooth; X is logarithmically flat if $X \to A(X)$ is flat; etc. Moreover, the Artin fan parametrises expansions of X.

2.5 Maulik-Ranganathan construction

In this subsection, we briefly recall the relevant aspects of the Maulik-Ranganathan construction for logarithmic Hilbert schemes from [MR20] so as to give some context and so that we may refer to them later.

Motivations and setup. Let X be a smooth threefold with simple normal crossing divisor D. We shall refer to D as the *boundary divisor*. The aim is to study the moduli space of ideal sheaves of fixed numerical type which meet the boundary divisor transversely. Some key motivations for the study of such an object come from enumerative geometry. For example, a common method used to address problems of curve counting in a given smooth variety is to degenerate

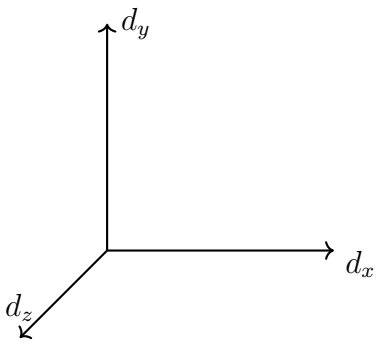


Figure 1: Tropicalisation of X.

this variety to a singular union of simpler irreducible components. The property of transversality is then crucial to ensure that all interesting behaviour of the ideal sheaves on the degenerate object occurs with support in the interior of the simpler irreducible components, which allows us to study it with more ease. One of the main difficulties with this approach is that often, as in this setting, the space of transverse ideal sheaves with respect to D is non-compact. In [MR20], Maulik and Ranganathan formulate the Donaldson-Thomas theory of the pair (X, D), starting by constructing compactifications of the space of ideal sheaves in X transverse to D.

Key ideas. We will discuss the methods of Maulik and Ranganathan specifically with respect to the case which interest us here, namely where $X \to C$ is locally given by $\operatorname{Spec} k[x, y, z, t]/(xyz - t)$ and we seek to study the moduli space of ideal sheaves with fixed constant Hilbert polynomial m, for some $m \in \mathbb{N}$ with respect to the boundary divisor $D \coloneqq X_0$.

The key idea is to construct the tropicalisation of X, denoted Σ_X, and a corresponding tropicalisation map, which we will use to understand how to obtain the desired transversality properties in our compactifications. Recall from Section 2.4 that, given a divisorial logarithmic structure on X, the tropicalisation is a fan or cone complex which for each defining function of the divisor records the degree of vanishing of this function in X. Here we are considering X with the divisorial logarithmic structure corresponding to D. The functions vanishing at D will be x, y and z, therefore we may represent Σ_X as a fan in $\mathbb{R}^3_{\geq 0}$, in this case the positive orthant and its faces, as can be seen in Figure 1.

The tropicalisation of D can be visualised by taking a hyperplane slice through the cone in Figure 1; this yields a triangle with vertices corresponding to Y_1, Y_2 and Y_3 in X_0, edges between these vertices corresponding to the lines $Y_i \cap Y_j$, and 2-dimensional interior corresponding to the point $Y_1 \cap Y_2 \cap Y_3$, as pictured in Figure 2. This is the dual complex of X_0 as defined in Section 2.3. We shall refer to the tropicalisation of X_0 as $\operatorname{trop}(X_0)$. Placing the triangle at different heights within the cone $\mathbb{R}^3_{\geq 0}$ can be thought of as making a finite base change on X.

We may construct a tropicalisation map which takes points of X° to Σ_X, as

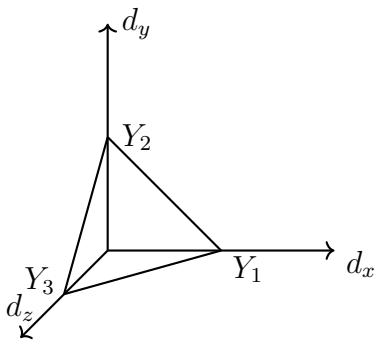


Figure 2: Tropicalisation of X_0.

in Section 1.4 of [MR20]. We recall the details of this map here. We assume that $\mathcal{K}$ is a valued field extending k. First, we take a point of $X^\circ(\mathcal{K})$, given by some morphism $\operatorname{Spec}\mathcal{K} \to X^\circ$. By the properness of X, this extends to a morphism $\operatorname{Spec} R \to X$ for some valuation ring R. Now, let $P \in X$ denote the image of the closed point by the second morphism. The stalk of the characteristic sheaf at P is given by $\mathbb{N}^r$, where r is the number of linearly independent vanishing equations of D at the point P. For example, in our context, if $P \in Y_1 \subset X_0$, then $r = 1$ and $\mathbb{N}$ is generated by the function x; if $P \in Y_1 \cap Y_2$, then $r = 2$ with $\mathbb{N}^2$ generated by the functions x and y; etc.

Each element of $\mathbb{N}^r$ corresponds to a function f on X in the neighbourhood of P up to multiplication by a unit and we may then evaluate f with respect to the valuation map associated to $\mathcal{K}$. This determines an element of

$$[\mathbb{N}^r \to \mathbb{R}_{\geq 0}] \in \operatorname{Hom}(\mathbb{N}^r, \mathbb{R}_{\geq 0}) \longhookrightarrow \Sigma_X.$$

This gives rise to a morphism

$$\operatorname{trop}\colon X^\circ \longrightarrow \Sigma_X$$

called the tropicalisation map. Now let the valuation map $\mathcal{K} \to \mathbb{R}$ be surjective and let $Z^\circ \subset X^\circ$ be an open subscheme. We denote by $\operatorname{trop}(Z^\circ)$ the image of the map trop restricted to $Z^\circ(\mathcal{K})$. Maulik and Ranganathan are then able to show, based on previous work of Tevelev [Tev07] for the toric case, that given such an open subscheme $Z^\circ \subset X^\circ$, the subset $\operatorname{trop}(Z^\circ)$ gives rise to an expansion X' of X in which the closure Z of Z° has the required transversality properties. This gives us a convenient dictionary to move back and forth between the geometric and combinatorial points of view.

The possible tropicalisations of such subschemes, corresponding to expansions on the geometric side, are captured on the combinatorial side by the notion of *1-complexes* embedding into Σ_X. See [MR20] for precises definitions.

Stability conditions. Let us now make precise what the correct notion of transversality and stability are in this context.

Definition 2.5.1. Let $Z^\circ \subset X_0^\circ \times C^\circ$ be a flat family of subschemes over C°. Let $\mathcal{Y} \to X_0 \times C$ be an expansion over C such that the closure Z of Z° in $\mathcal{Y}$ has the following properties:

1. The scheme Z is proper and flat over C,
2. The scheme Z has nonempty intersection with each stratum of $\mathcal{Y}$,
3. The scheme Z intersects all the strata of $\mathcal{Y}$ in the expected dimension.

Then we say that the family Z is *dimensionally transverse*.

Maulik and Ranganathan actually need a slightly stronger condition, resembling the admissibility of Li and Wu.

Definition 2.5.2. In the situation of Definition 2.5.1, we say that Z is *strongly transverse* if it is dimensionally transverse and for every divisor stratum $\mathcal{S} \subset \mathcal{Y}$ the induced map

$$\mathcal{I}_Z \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{S} \to \mathcal{O}_\mathcal{S}$$

is injective.

Note that in the specific case of Hilbert schemes of points the definition of dimensional transversality happens to be equivalent to the condition that for every divisor stratum $\mathcal{S} \subset \mathcal{Y}$ the map

$$\mathcal{I}_Z \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{S} \to \mathcal{O}_\mathcal{S}$$

be injective paired with the requirement for finite automorphisms, i.e. it is equivalent to Li-Wu stability (see Section 5.2 for a definition of this stability condition). In general for higher dimensional subschemes this will not be the case, however.

As mentioned above, for an open subscheme $Z^\circ \subset X^\circ$, we may consider its image $\mathrm{trop}(Z^\circ)$ under the tropicalisation map. Now recall from Section 2.4 that a subdivision of the tropicalisation Σ_X defines an expansion of X. The expansion defined by $\mathrm{trop}(Z^\circ)$ will in general not correspond to a blow-up but to some birational modification given by a series a blow-ups and contractions. We note here that while contracting components of a fibre over a 1-parameter family will in general be flat, this is no longer the case when this operation is made over a larger base. It will therefore be necessary, for each possible $\mathrm{trop}(Z^\circ)$, to make a choice of subdivision corresponding to an actual blow-up on X. Maulik and Ranganathan prove that, given Z°, an expansion can be constructed from $\mathrm{trop}(Z^\circ)$ which has the required stability condition and good existence and uniqueness properties. Indeed we recall the statement of Proposition 2.5.2 of [MR20] here for convenience.

Proposition 2.5.3. *Let Z_η° be a flat family of subschemes of X° over $\operatorname{Spec}\mathcal{K}$ whose closure in X is strongly transverse. Then there exists a canonical triple $(R', \mathcal{Y}', Z')$ consisting of a ramified base change $R \subset R'$ with fraction field $\mathcal{K}'$ and expansion of X*

$$\mathcal{Y}' \to \operatorname{Spec} R'$$

such that the closure Z' of $Z^\circ_\eta \otimes_{\mathcal{K}} \mathcal{K}'$ in $\mathcal{Y}$ is strongly transverse, satisfying the following uniqueness property:

For any other choice $(R'', \mathcal{Y}'', Z'')$ satisying these requirements there exists a unique toroidal birational morphism

$$\mathcal{Y}'' \to \mathcal{Y}''_1$$

over $\operatorname{Spec} R''$ *with subscheme Z''_1, and a ramified covering* $\operatorname{Spec} R'' \to \operatorname{Spec} R$, *such that $Z''_1 \subset \mathcal{Y}''_1$ is obtained from $Z' \subset \mathcal{Y}'$ by base change.*

In fact, in our case, since we are only considering Hilbert schemes of points and the stability conditions simplify as mentioned above, we may use Propositions 2.3.1 and 2.4.1 of [MR20] instead.

Construction of the stacks of expansions. Through these methods, one can obtain existence and uniqueness properties for these flat limits. To build a moduli space of such subschemes, Maulik and Ranganathan start by constructing a moduli space of possible expansions arising from Tevelev's procedure. Let us denote the set of isomorphism classes of 1-complexes which embed into Σ_X by $|T(\Sigma_X)|$. Some subtleties arise at this point, namely that in general the space constructed will not be representable as a logarithmic algebraic stack. This can be seen through the fact the category of logarithmic algebraic stacks is equivalent to the category of cone stacks, but $|T(\Sigma_X)|$ cannot in general be given a proper cone structure. In order to give it a cone structure, Maulik and Ranganathan study the spaces of maps $\mathbb{X}_G$ from the graphs G associated to 1-complexes in $|T(\Sigma_X)|$ to Σ_X, and identify maps which have the same image. By taking appropriate subdivisions of the objects $\mathbb{X}_G$ and identifying them in the right way, they obtain a *moduli space of tropical expansions* T, which has the desired cone structure.

This operation results in non-uniqueness, as we are making a choice of polyhedral subdivision and there is in general no canonical choice.

Proper Deligne-Mumford stacks. In order to construct the universal family $\mathfrak{Y} \subset T \times \Sigma$, some additional choices must be made. Indeed, as mentioned above, $\operatorname{trop}(Z^\circ)$ does not in general define a blow-up, so, when fitting the expansions we constructed together into one large family over a larger base, we must modify these expansions to ensure flatness. Here this is resolved by adding distinguished vertices to the relevant complexes. These added vertices will be 2-valent vertices along edges of the 1-complexes parameterised by T and we call them *tube vertices*. Geometrically, they look like $\mathbb{P}^1$-bundles over curves in X_0 (where we took X to be a family of surfaces). Again, this operation is not canonical and results in non-uniqueness.

The addition of these tube vertices in the tropicalisation means that there are more potential components in each expansion, which interferes with the previously set up uniqueness results. Indeed, recall that $\operatorname{trop}(Z^\circ)$ gave us exactly the right number of vertices in the dual complex in order for each family of

subschemes $Z^\circ \subset X^\circ$ to have a unique limit representative. Therefore, to reflect this, *Donaldson-Thomas stability* asks for subschemes to be DT stable if and only if they are tube schemes precisely along the tube components. We say that a 1-dimensional subscheme is a *tube* if it is the schematic preimage of a zero-dimensional subscheme in D. In the case of Hilbert schemes of points, this condition will translate simply to a 0-dimensional subscheme Z being DT stable if and only if no tube component contains a point of the support of Z and every other irreducible component expanded out by our blow-ups contains at least one point of the support of Z.

Maulik and Ranganathan define a subscheme to be stable if it is strongly transverse and DT stable. For fixed numerical invariants the substack of stable subschemes in the space of expansions forms a Deligne-Mumford, proper, separated stack of finite type over C.

The space $|T(\Sigma_X)|$ for the case of Hilbert schemes of points can be thought of as given by the n-tuples of points in the cone of X_0. One can notice immediately that this does not have the proper structure of a cone.

3 The expanded construction

In this section we construct explicit expanded degenerations $X[n]$ out of a 1-parameter family $X \to C$ by expanding the base and making sequences of blow-ups on the expanded family. We construct these spaces as schemes here, and later on, in Section 5, we give a stack construction building upon these schemes. Note that in the stack construction we will impose additional equivalence relations which essentially set to be equivalent any two fibres which look identical. We will touch more upon why this is necessary later.

Let $X \to C$ be a family of surfaces over a curve isomorphic to $\mathbb{A}^1$, where the family is given in étale local coordinates by $\operatorname{Spec} \mathbb{C}[x,y,z,t]/(xyz-t)$. We denote by X_0 the special fibre and by Y_1, Y_2 and Y_3 the irreducible components of this special fibre given locally by $x=0, y=0$ and $z=0$ respectively.

3.1 The blow-ups

In the following, we construct expanded degenerations by enlarging the base C and making sequences of blow-ups in the family over this larger base. We start by taking a copy of $\mathbb{A}^{n+1}$, with elements labelled $(t_1,\dots,t_{n+1}) \in \mathbb{A}^{n+1}$. Throughout this work, we shall refer to the entries t_i as *basis directions*. Now, let $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ be the fibre product given by the map $X \to C \cong \mathbb{A}^1$ and the product

$$(t_1,\dots,t_{n+1}) \longmapsto t_1 \cdots t_{n+1}.$$

In this expanded degeneration construction, as well as in the second more general construction of Section 7, we will be blowing up schemes along Weil divisors. An unusual consequence of the way these blow-ups are defined is that the blow-up

morphisms contract only components of codimension at least 2. We may therefore refer to them as *small blow-ups*.

Blow-ups of the Y_1 component. We start by blowing up $Y_1 \times_{\mathbb{A}^1} V(t_1)$ inside $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$, where $V(t_i)$ denotes the locus where $t_i = 0$. We name the space resulting from this blow-up $X_{(1,0)}$ to signify we have blown up component Y_1 once and the component Y_2 zero times. We can describe this blow-up locally in the following way. The ideal of the blow-up is $I_1 = \langle x, t_1 \rangle$. Globally this will correspond to an ideal sheaf $\mathcal{I}_1$. Then there is a surjective map of graded rings

$$A[x_0^{(1)}, x_1^{(1)}] \longrightarrow S_1 = \bigoplus_{n \geq 0} I_1^n$$

which maps

$$x_0^{(1)} \longmapsto x \quad \text{and} \quad x_1^{(1)} \longmapsto t_1,$$

where A is the affine coordinate ring given by the quotient of $k[x, y, z, t_1, \ldots, t_{n+1}]$ by the ideal generated by the equation $xyz = t_1 \cdots t_{n+1}$. This induces an embedding

$$\operatorname{Proj}(S_1) \hookrightarrow \operatorname{Proj} A[x_0^{(1)}, x_1^{(1)}] = \mathbb{P}^1 \times \operatorname{Spec} A$$

and $\operatorname{Proj}(S_1)$, i.e. our blow-up, is cut out in $\mathbb{P}^1 \times \operatorname{Spec} A$ by the equations

$$\begin{aligned} x_0^{(1)} t_1 &= x x_1^{(1)} \\ x_0^{(1)} yz &= x_1^{(1)} t_2 \cdots t_{n+1}. \end{aligned}$$

Then the fibres above $(t_1, \ldots, t_{n+1})$ where t_1 is nonzero are still the same after the blow-up and so are the fibres where $t_1 = 0$ and all the other t_i are nonzero because the total space is still smooth at all points of these fibres. However, when $t_1 = 0$ and at least one of the other t_i is zero, then we get singularities of the total space appearing in the fibre of $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ and the blow-up causes a new component to appear around the Y_1 component. We call this new component $\Delta_1^{(1)}$.

Let $b_{(1,0)} \colon X_{(1,0)} \to X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ be the map defined by the first blow-up given above. We then proceed to blow-up $b_{(1,0)}^*(Y_1 \times_{\mathbb{A}^1} V(t_2))$ inside $X_{(1,0)}$ and name the resulting space $X_{(2,0)}$. We continue to blow up each $b_{(k-1,0)}^*(Y_1 \times_{\mathbb{A}^1} V(t_k))$ inside $X_{(k-1,0)}$ for each $k \leq n$. The resulting space is denoted $X_{(n,0)}$. Finally, we denote by

$$\beta_{(k,0)}^1 \colon X_{(k,0)} \longrightarrow X_{(k-1,0)}$$

the morphisms corresponding to each individual blow-up. We therefore have the equality

$$\beta_{(k,0)}^1 \circ \cdots \circ \beta_{(1,0)}^1 = b_{(k,0)}$$

We now fix the following terminology.

Definition 3.1.1. We say that a dimension 2 component of $X_{(k,0)} \to C \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ is a Δ_1-*component* if it is contracted by the morphism $\beta_{(i,0)}^1$ for some $i \leq k$. Moreover if a Δ_1-component in a fibre is contracted by such a map then we say it is *expanded out* in this fibre.

We label by $\Delta_1^{(k)}$ the Δ_1-component resulting from the k-th blow-up. The fibre where $t_i = 0$ for all $i \in \{1, \dots, n+1\}$ has exactly n expanded Δ_1-components. The equations of the blow-ups in local coordinates are as follows:

$$\begin{aligned} &x_0^{(1)} t_1 = x x_1^{(1)}, \\ &x_1^{(k-1)} x_0^{(k)} t_k = x_0^{(k-1)} x_1^{(k)}, \qquad \text{for } \ 2 \leq k \leq n, \\ &x_0^{(n)} yz = x_1^{(n)} t_{n+1}. \end{aligned}$$

Remark 3.1.2. If we restrict X_0 to only the components Y_1 and Y_2, i.e. restrict the original degeneration to $\operatorname{Spec} \mathbb{C}[x, y, z, t]/(xy - t)$, we get back exactly the blow-ups of Gulbrandsen, Halle and Hulek [GHH19].

In fibres of the construction where $\Delta_1^{(k)}$, for some k, is not expanded out, i.e. not contracted by some map $\beta^1_{(i,0)}$, we will want to think of it in the following way.

Definition 3.1.3. When $t_k = 0$ and all other t_i are nonzero, we consider $\Delta_1^{(k)}$ and all $\Delta_1^{(j)}$ for $j \geq k$ as being *equal to* Y_1, meaning that the projective coordinates introduced by the j-th blow-up are proportional to $1/yz$. This follows from the equality

$$x_0^{(j)} yz = x_1^{(j)} t_{j+1} \cdots t_{n+1},$$

obtained from the above equations of the blow-ups. Similarly, the components $\Delta_1^{(j)}$ with $j < k$ are considered to be *equal to* the union $Y_2 \cup Y_3$, which follows from equality

$$x_0^{(j)} t_1 \cdots t_j = x x_1^{(j)},$$

obtained from the equations of the blow-up. When $t_{n+1} = 0$ and all other t_k are nonzero, then all $\Delta_1^{(k)}$ are *equal to* the union $Y_2 \cup Y_3$.

Blow-ups of the Y_2 component. For the component Y_2 we can make similar definitions to the above. We blow up $b^*_{(n,0)} Y_2 \times_{\mathbb{A}^1} V(t_{n+1})$ in $X_{(n,0)}$ and name the resulting space $X_{(n,1)}$. Let $b_{(n,k)} \colon X_{(n,k)} \to X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ be the composition of the n blow-ups of Y_1 and the first k blow-ups of Y_2 on $X_{(n,0)}$. Similarly to the above, but with the order of the basis directions reversed, we blow up $b^*_{(n,k-1)}(Y_2 \times_{\mathbb{A}^1} V(t_{n+2-k}))$ in $X_{(n,k-1)}$ for each $k \leq n$. The resulting space is denoted $X_{(n,n)}$ and the components introduced by these new blow-ups are labelled $\Delta_2^{(k)}$. The equations of the blow-ups in local coordinates are as follows:

$$\begin{aligned} &y_0^{(1)} t_{n+1} = y y_1^{(1)}, \\ &y_1^{(k-1)} y_0^{(k)} t_{n+2-k} = y_0^{(k-1)} y_1^{(k)} \ \text{ for } \ 2 \leq k \leq n, \\ &y_0^{(n)} xz = y_1^{(n)} t_1 \\ &x_0^{(k)} y_0^{(n+1-k)} z = x_1^{(k)} y_1^{(n+1-k)}. \end{aligned}$$

Note that the order in which we make the blow-ups, i.e. expand out the Δ_1 or the Δ_2-components first, makes no difference. We can therefore express the space

$X_{(m_1,m_2)}$ as the space $X_{(m_1,0)}$ on which we perform a sequence of blow-ups of the pullback of Y_2 or as the space $X_{(0,m_2)}$ on which we perform a sequence of blow-ups of the pullback of Y_1, etc.

To simplify notation, we will denote the base $\mathbb{A}^{n+1} \times_{\mathbb{A}^1} C$ by $C[n]$, the expanded construction $X_{(n,n)}$ by $X[n]$ and the natural projection to the original family X by $\pi : X[n] \to X$. We have blow-up morphisms

$$\beta^1_{(i,j)} \colon X_{(i,j)} \longrightarrow X_{(i-1,j)},$$
$$\beta^2_{(i,j)} \colon X_{(i,j)} \longrightarrow X_{(i,j-1)},$$

corresponding to each individual blow-up of a pullback of the Y_1-component and Y_2-component respectively. The composition of all the blow-up morphisms is denoted

$$b \coloneqq \beta^2_{(n,n)} \circ \cdots \circ \beta^2_{(n,1)} \circ \cdots \circ \beta^1_{(n,0)} \circ \cdots \circ \beta^1_{(1,0)} \colon X[n] \to X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}.$$

Proposition 3.1.4. *If we take $X \to C$ to be the étale local model*

$$\operatorname{Spec} k[x,y,z,t]/(xyz - t) \longrightarrow \operatorname{Spec} k[t],$$

the corresponding scheme $X[n]$ obtained after the sequence of blow-ups b is a subvariety of $(X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}) \times (\mathbb{P}^1)^{2n}$ defined by the local equations of the blow-ups given above.

Proof. This is immediate from the local description of the blow-ups above. □

Proposition 3.1.5. *The family $X[n] \to C[n]$ thus constructed is projective.*

Proof. The morphism $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1} \to C[n]$ must be projective since $X \to C$ is projective. Then $X[n] \to X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ is just a sequence of blow-ups along Weil divisors, hence projective. This proves projectivity of the morphism $X[n] \to C[n]$. □

Remark 3.1.6. The issue with projectivity in Proposition 1.10 of [GHH19] only arises if the local descriptions of the blow-ups they use to create the family $X[n] \to C[n]$ do not glue globally to define blow-ups.

We now extend the definition of Δ_1-components to the schemes $X[n]$ and fix some additional terminology.

Definition 3.1.7. We say that a dimension 2 component of $X[n] \to C[n]$ is a Δ_i-*component* if it is contracted by the morphism $\beta^i_{(j,k)}$ for some i, j, k. Moreover if a Δ_i-component in a fibre is contracted by such a map then we say it is *expanded out in this fibre*. We say that a dimension 2 component of $X[n]$ is a Δ-*component* if it is a Δ_i-component for some i. If it is expanded out in some fibre we may alternatively refer to it as an *expanded component*. Similarly, we may extend Definition 3.1.3 to say that a Δ-component is *equal to* a component W of a fibre of $X[n]$ if the projective coordinates associated to this Δ-component are proportional to the non-vanishing coordinates of W.

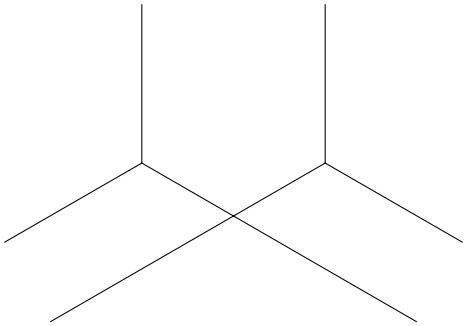


Figure 3: Local picture at $t_i = t_j = 0$.

Definition 3.1.8. We say that a Δ_i-component is of *pure type* if it is not equal to any Δ_j-component for $j \neq i$. Otherwise we say it is of *mixed type*.

In order to understand what these blow-ups look like, let us consider first a fibre of the scheme $X[n]$ over $C[n]$, where $t_i = t_j = 0$ for some $i < j$ and no other $t_k = 0$. The blow-ups of pullbacks of the Y_1-component cause exactly one Δ_1-component to be expanded in such a fibre, and this expanded component is given by $\Delta_1^{(i)} = \ldots = \Delta_1^{(j-1)}$. In this case, the singularities occurring at the intersection of Y_1 and Y_2 have already been resolved by expanding out this Δ_1-component. As the blow-ups of pullbacks of the Y_2-component also cause one Δ_2-component to be expanded in this fibre, given by $\Delta_2^{(n+2-j)} = \ldots = \Delta_2^{(n+1-i)}$, we therefore have

$$\Delta_1^{(i)} = \ldots = \Delta_1^{(j-1)} = \Delta_2^{(n+2-j)} = \ldots = \Delta_2^{(n+1-i)}$$

in the $\pi^*((Y_1 \cap Y_2)^\circ)$ locus of the fibre. This can be easily deduced from studying the equations of the blow-ups. In the $\pi^*((Y_1 \cap Y_3)^\circ)$ locus of the fibre, we see a single expanded component of pure type given by $\Delta_1^{(i)} = \ldots = \Delta_1^{(j-1)}$. Similarly, in the $\pi^*((Y_2 \cap Y_3)^\circ)$ locus of the fibre, we see a single expanded component of pure type given by $\Delta_2^{(n+1-j)} = \ldots = \Delta_2^{(n+1-i)}$. Finally, the component $\Delta_1^{(k)}$ is equal to the union $Y_2 \cup Y_3$ for $k < i$ and $\Delta_1^{(l)}$ is equal to the component Y_1 if $l > j - 1$. The situation for the Δ_2 components is similar. This can be seen in Figure 3.

Before we continue we fix some terminology which will help us describe the expanded components.

Definition 3.1.9. We refer to an irreducible component of a Δ-component as a *bubble*. The notions of two bubbles being *equal* and a bubble being *expanded out* in a certain fibre follow directly from our previous definitions.

When $t_i = t_j = t_k = 0$, where i, j, k are distinct, and all other basis directions are non-zero, then in the locus $\pi^*((Y_1 \cap Y_2)^\circ)$, we see exactly two expanded components, which are both of mixed type. Note that, more generally in any fibre of $X[n]$, all expanded components in the $\pi^*((Y_1 \cap Y_2)^\circ)$ locus are of mixed type. Moreover, in any fibre of $X[n]$ we have that $\Delta_1^{(l)} = \Delta_2^{(n+1-l)}$ in the $\pi^*((Y_1 \cap Y_2)^\circ)$ locus for all l.

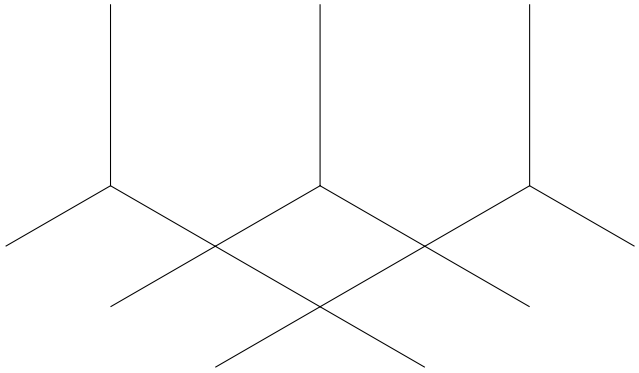


Figure 4: Local picture at $t_i = t_j = t_k = 0$.

In the example given here, the two bubbles in the $\pi^*((Y_1 \cap Y_2)^\circ)$ locus can be described as follows. The bubble which intersects Y_1 non-trivially is given by $\Delta_1^{(i)} = \ldots = \Delta_1^{(j-1)}$. By the above, each of these Δ_1-components is equivalent to a Δ_2-component in the $\pi^*((Y_1 \cap Y_2)^\circ)$ locus, so this bubble is equivalently given by $\Delta_2^{(n+1-j)} = \ldots = \Delta_2^{(n+1-i)}$. The second bubble in this locus, which intersects Y_2 non-trivially, is given by

$$\Delta_1^{(j)} = \ldots = \Delta_1^{(k-1)} = \Delta_2^{(n+2-k)} = \ldots = \Delta_2^{(n+1-j)}.$$

There is a single bubble expanded out in the $\pi^*(Y_1 \cap Y_2 \cap Y_3)$ locus. This is a $\mathbb{P}^1 \times \mathbb{P}^1$, given by the meeting of the $\Delta_1^{(i)} = \ldots = \Delta_1^{(j-1)}$ and $\Delta_2^{(n+2-k)} = \ldots = \Delta_2^{(n+1-j)}$ components. Finally, in the $\pi^*((Y_1 \cap Y_3)^\circ)$ locus we see exactly two bubbles given by the two distinct expanded Δ_1-components and in the $\pi^*((Y_2 \cap Y_3)^\circ)$ locus we see also two bubbles given by the two distinct expanded Δ_2-components. This can be seen in Figure 4.

Now, we note that there is a natural inclusion

$$\begin{aligned} C[n] &\hookrightarrow C[n+1] \\ (t_1, \ldots, t_{n+1}) &\longmapsto (t_1, \ldots, t_{n+1}, 1), \end{aligned} \tag{3.1.1}$$

which, in turn, induces a natural inclusion

$$X[n] \hookrightarrow X[n+1].$$

Under these inclusions, we may consider the space $X[n]$ as the locus of a larger space $X[n+k]$ where all $t_i \neq 0$ for $i > n+1$.

The group action. We may define a group action on $X[n]$ very similarly to [GHH19]. Let $G \subset \mathrm{SL}(n+1)$ be the maximal diagonal torus. We have $\mathbb{G}_m^n \cong G \subset \mathbb{G}_m^{n+1}$, where we can view an element of G as an $(n+1)$-tuple $(\sigma_1, \ldots, \sigma_{n+1})$ such that $\prod_i \sigma_i = 1$. This acts naturally on $\mathbb{A}^{n+1}$, which induces an action on $C[n]$. The isomorphism $\mathbb{G}_m^n \cong G$ is given by

$$(\tau_1, \ldots, \tau_n) \longrightarrow (\tau_1, \tau_1^{-1}\tau_2, \ldots, \tau_{n-1}^{-1}\tau_n, \tau_n^{-1}).$$

We shall use the notation $(\tau_1, \ldots, \tau_n)$ to describe elements of G throughout this work.

Proposition 3.1.10. *We have the following properties.*

1. *There is a unique G-action on $X[n]$ such that $X[n] \to X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ is equivariant with respect to the natural action of G on $\mathbb{A}^{n+1}$.*

2. *In the étale local model, this action is the restriction of the action on $(X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}) \times (\mathbb{P}^1)^{2n}$, which is trivial on X, acts by*

$$\begin{aligned} t_1 &\longmapsto \tau_1^{-1} t_1 \\ t_k &\longmapsto \tau_k^{-1} \tau_{k-1} t_k \\ t_{n+1} &\longmapsto \tau_n t_{n+1} \end{aligned}$$

   on the basis directions, and by

$$\begin{aligned} (x_0^{(k)} : x_1^{(k)}) &\longmapsto (\tau_k x_0^{(k)} : x_1^{(k)}) \\ (y_0^{(k)} : y_1^{(k)}) &\longmapsto (y_0^{(k)} : \tau_{n+1-k} y_1^{(k)}). \end{aligned}$$

   on the Δ-components.

Proof. This follows immediately from [GHH19]. □

Note that the group action on the $(y_0^{(k)} : y_1^{(k)})$ coordinates follows immediately from the fact that $\Delta_1^{(k)} = \Delta_2^{(n+1-k)}$ in the $\pi^*((Y_1 \cap Y_2)^\circ)$ locus. Given the equations of the blow-ups above, there is no other possible choice of action such that the map $\pi : X[n] \to X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ is G-equivariant (the equations must be invariant under group action). Note also that the natural inclusions

$$X[n] \hookrightarrow X[n+k],$$

we described in the previous section are equivariant under the group action. Moreover, up to linear independence there is a single function on $C[n]$ which is G-invariant.

Lemma 3.1.11. *We have the isomorphism*

$$H^0(C[n], \mathcal{O}_{C[n]})^G \cong k[t],$$

where $H^0(C[n], \mathcal{O}_{C[n]})^G$ denotes the space of G-invariant sections of $H^0(C[n], \mathcal{O}_{C[n]})$.

Proof. From the above description of the group action, it is clear that up to linear independence there is only one G-invariant function on $C[n]$. □

Remark that we abuse notation slightly by referring to the group acting on $X[n]$ by G, instead of $G[n]$. It should always be clear from the context what group G is meant.

3.2 Embedding into product of projective bundles

In this section, we show how $X[n]$ can be embedded into a fibre product of projective bundles. We will then be able to define a G-linearised ample line bundle $\mathcal{L}$ on $X[n]$ by taking the tautological bundle of this fibre product of projective bundles. From this line bundle we will then construct a second line bundle $\mathcal{M}$ on the relative Hilbert scheme of m points $H^m_{[n]} \coloneqq \mathrm{Hilb}^m(X[n]/C[n])$ with an induced G-linearisation.

More precisely, keeping the notation as above, let $\mathcal{I}_1^{(k)}, \mathcal{I}_2^{(k)}$ be the ideal sheaves corresponding to each blow-up we perform, for example $\mathcal{I}_1^{(1)}$ is the ideal sheaf of $Y_1 \times_{\mathbb{A}^1} V(t_1)$ on $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$. Then $\mathcal{I}_2^{(1)}$ is the ideal sheaf of $b^*_{(1,0)}(Y_2 \times_{\mathbb{A}^1} V(t_{n+1}))$ on $X_{(1,0)}$, and so on for $\mathcal{I}_j^{(k)}$.

Let pr_1 and pr_2 be the projections of $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ to X and $\mathbb{A}^{n+1}$ respectively. Similarly to [GHH19], define vector bundles

$$\begin{aligned}\mathcal{F}_1^{(k)} &= \mathrm{pr}_1^* \mathcal{O}_X(-Y_1) \oplus \mathrm{pr}_2^* \mathcal{O}_{\mathbb{A}^{n+1}}(-V(t_k)) \\ \mathcal{F}_2^{(k)} &= \mathrm{pr}_1^* \mathcal{O}_X(-Y_2) \oplus \mathrm{pr}_2^* \mathcal{O}_{\mathbb{A}^{n+1}}(-V(t_{n+2-k}))\end{aligned}$$

on $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$. As we will explain below, we then have, for each of these vector bundles, the embeddings

$$\begin{aligned}X_{(k_1,k_2)} &\hookrightarrow \mathbb{P}(b^*_{(k_1-1,k_2)}\mathcal{F}_1^{(k_1)}), \\ X_{(k_1,k_2)} &\hookrightarrow \mathbb{P}(b^*_{(k_1,k_2-1)}\mathcal{F}_2^{(k_2)}),\end{aligned}$$

where $b_{(0,0)}$ is understood to be just the identity map on $X_{(0,0)} \coloneqq X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$. Indeed, the scheme $X_{(k_1,k_2)}$ embeds into the projectivisations of the ideals of these blow-ups $\mathbb{P}(\mathcal{I}_1^{(k_1)})$ and $\mathbb{P}(\mathcal{I}_2^{(k_2)})$. For a reference on projectivisations of ideals see [EH00]. There is a surjection

$$b^*_{(k_1-1,k_2)}\mathcal{F}_1^{(k_1)} \longrightarrow \mathcal{I}_1^{(k_1)} \text{ given by } \begin{pmatrix} b^*_{(k_1-1,k_2)}x \\ t_{k_1} \end{pmatrix},$$

where x is a defining equation of the locus to be blown up projected forward to X, i.e. it is the defining equation of Y_1. Similarly, there is a surjection

$$b^*_{(k_1,k_2-1)}\mathcal{F}_2^{(k_2)} \longrightarrow \mathcal{I}_2^{(k_2)}.$$

From this, we deduce that there are embeddings

$$\begin{aligned}\mathbb{P}(\mathcal{I}_1^{(k_1)}) &\hookrightarrow \mathbb{P}(b^*_{(k_1-1,k_2)}\mathcal{F}_1^{(k_1)}), \\ \mathbb{P}(\mathcal{I}_2^{(k_2)}) &\hookrightarrow \mathbb{P}(b^*_{(k_1,k_2-1)}\mathcal{F}_2^{(k_2)}).\end{aligned}$$

Hence we have embeddings

$$\begin{array}{ccc} & X_{(k_1,k_2)} & \\ \swarrow & & \searrow \\ \mathbb{P}(b^*_{(k_1-1,k_2)}\mathcal{F}_1^{(k_1)}) & & \mathbb{P}(b^*_{(k_1,k_2-1)}\mathcal{F}_2^{(k_2)}). \end{array}$$

Now, similarly to [GHH19], we can embed $X[n] = X_{(n,n)}$ into $\prod_{i,j} \mathbb{P}(\mathcal{F}_i^{(j)})$, which is to be understood as the fibre product over $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$. This can be seen by iteration on i, j in the following way. The simplest case is $X_{(1,0)} \hookrightarrow \mathbb{P}(b_{(0,0)}^* \mathcal{F}_1^{(1)}) = \mathbb{P}(\mathcal{F}_1^{(1)})$, which is obvious. Then for $X_{(1,1)}$, we have the following commutative diagram

$$\begin{array}{ccc} X_{(1,1)} \subset b_{(1,0)}^* \mathbb{P}(\mathcal{F}_2^{(1)}) & \longrightarrow & X_{(1,0)} \subset \mathbb{P}(\mathcal{F}_1^{(1)}) \\ \downarrow & & \downarrow{\scriptstyle b_{(1,0)}} \\ \mathbb{P}(\mathcal{F}_2^{(1)}) & \longrightarrow & X \times_{\mathbb{A}^1} \mathbb{A}^{n+1} \end{array}$$

(recall $b_{(1,0)}^* \mathbb{P}(\mathcal{F}_2^{(1)})$ is a vector bundle over $X_{(1,0)}$ and $\mathbb{P}(\mathcal{F}_2^{(1)})$ is a vector bundle over $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$, giving us the horizontal maps). By the universal property of fibre products, there is a unique map $X_{(1,1)} \to \mathbb{P}(\mathcal{F}_1^{(1)}) \times \mathbb{P}(\mathcal{F}_2^{(1)})$. But by universal property of the pullback there is also a unique map $\mathbb{P}(\mathcal{F}_1^{(1)}) \times \mathbb{P}(\mathcal{F}_2^{(1)}) \to b_{(1,0)}^* \mathbb{P}(\mathcal{F}_2^{(1)})$, hence the embedding $X_{(1,1)} \hookrightarrow b_{(1,0)}^* \mathbb{P}(\mathcal{F}_2^{(1)})$ factors through $\mathbb{P}(\mathcal{F}_1^{(1)}) \times \mathbb{P}(\mathcal{F}_2^{(1)})$. Since the composition of the two maps is injective, the first map, i.e. $X_{(1,1)} \to \mathbb{P}(\mathcal{F}_1^{(1)}) \times \mathbb{P}(\mathcal{F}_2^{(1)})$, must be injective and the image in $\mathbb{P}(\mathcal{F}_1^{(1)}) \times \mathbb{P}(\mathcal{F}_2^{(1)})$ is closed by properness. We can then iterate this argument until we obtain the embedding $X_{(n,n)} \hookrightarrow \prod_{i,j} \mathbb{P}(\mathcal{F}_i^{(j)})$.

The G-action is a restriction of the torus action on $\prod_{i,j} \mathbb{P}(\mathcal{F}_i^{(j)})$, described étale locally in Proposition 3.1.10.

Linearisations. The following lemma gives a method to construct all the linearised line bundles we will need to vary the GIT stability condition.

Lemma 3.2.1. *There exists a G-linearised ample line bundle $\mathcal{L}$ on $X[n]$ such that locally the lifts to this line bundle of the G-action on each $\mathbb{P}^1$ corresponding to a $\Delta_1^{(k)}$ and on each $\mathbb{P}^1$ corresponding to a $\Delta_2^{(n+1-k)}$ are given by*

$$(x_0^{(k)}; x_1^{(k)}) \longmapsto (\tau_k^{a_k} x_0^{(k)}; \tau_k^{-b_k} x_1^{(k)}) \tag{3.2.1}$$

$$(y_0^{(n+1-k)}; y_1^{(n+1-k)}) \longmapsto (\tau_k^{-c_k} y_0^{(n+1-k)}; \tau_k^{d_k} y_1^{(n+1-k)}) \tag{3.2.2}$$

for any choice of positive integers a_k, b_k, c_k, d_k.

Proof. Similarly to the proof of Lemma 1.18 in [GHH19], we see that each locally free sheaf $\mathcal{F}_i^{(k)}$ on $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ has a canonical G-linearisation. There is an induced G-action on the projective product $\prod_{i,k} \mathbb{P}(\mathcal{F}_i^{(k)})$, which is equivariant under the embedding

$$X[n] \hookrightarrow \prod_{i,k} \mathbb{P}(\mathcal{F}_i^{(k)}).$$

The G-action on each $\mathbb{P}(\mathcal{F}_i^{(k)})$ lifts to a G-action on the corresponding vector bundle, which gives us a canonical linearisation of the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F}_i^{(k)})}(1)$. Locally, the actions on $\mathcal{O}_{\mathbb{P}(\mathcal{F}_1^{(k)})}(1)$ and $\mathcal{O}_{\mathbb{P}(\mathcal{F}_2^{(k)})}(1)$ are given by

$$(x_0^{(k)}; \tau_k^{-1} x_1^{(k)}) \quad \text{and}$$
$$(y_0^{(k)}; \tau_{n+1-k} y_1^{(k)})$$

respectively. We therefore may define the lifts (3.2.1) and (3.2.2) on the line bundles $\mathcal{O}_{\mathbb{P}(\mathcal{F}_1^{(k)})}(a_k + b_k)$ and $\mathcal{O}_{\mathbb{P}(\mathcal{F}_2^{(n+1-k)})}(c_k + d_k)$ respectively. We then pull back each $\mathcal{O}_{\mathbb{P}(\mathcal{F}_1^{(k)})}(a_k + b_k)$ and $\mathcal{O}_{\mathbb{P}(\mathcal{F}_2^{(k)})}(c_{n+1-k} + d_{n+1-k})$ to $\prod_{i,k} \mathbb{P}(\mathcal{F}_i^{(k)})$ and form their tensor product to obtain a G-linearised line bundle, which we denote by $\mathcal{L}$. □

Each such line bundle $\mathcal{L}$ which can be constructed in this way will induce a G-linearised line bundle $\mathcal{M}$ on $H^m_{[n]}$. This, in turn, will yield a GIT stability condition on $H^m_{[n]}$.

4 GIT stability

In this section, we set up some results analogous to those of [GHH19] to describe various GIT stability conditions on the scheme $X[n]$ with respect to the possible choices of G-linearised line bundles described in the previous section. In particular, we show that these stability conditions does not depend on the scheme structure of the length m zero-dimensional subschemes, but instead can be reduced to a combinatorial criterion on configurations of n points.

4.1 Hilbert-Mumford criterion

In this section, we shall recall the definition of Hilbert-Mumford invariants and give a numerical criterion for stability and semi-stability in terms of these invariants.

Let H be a reductive group and S be a scheme, proper over an algebraically closed field k. Let L be a H-linearised ample line bundle. Then a 1-parameter subgroup of H (denoted 1-PS for convenience) is defined to be a homomorphism

$$\lambda \colon \mathbb{G}_m \to H.$$

Now let P be any point in S. Obviously, $\tau \in \mathbb{G}_m$ acts on P and we denote by P_0 the limit of τP as τ tends towards 0 if such a limit exists. Then let $\mu^L(\lambda, P)$ be the negative of the weight of the $\mathbb{G}_m$-action on the fibre $L(P_0)$. We call $\mu^L(\lambda, P)$ a Hilbert-Mumford invariant.

In our case we will want to think of H as being our group G, of S as being the relative Hilbert scheme of points $H^m_{[n]}$ and of L as being the line bundle $\mathcal{M}$

on $H^m_{[n]}$, which we define in the next section. A 1-parameter subgroup of G will be given by a map

$$\lambda\colon \mathbb{G}_m \to G, \quad \tau \mapsto (\tau^{s_1}, \ldots, \tau^{s_n}),$$

where $(s_1, \ldots, s_n) \in \mathbb{Z}^n$. The following result will allow us to use these invariants to determine stability and semi-stability in our GIT constructions. It is a relative version of the Hilbert-Mumford criterion (see Mumford, Fogarty and Kirwan [MFK94]) proven by Gulbrandsen, Halle and Hulek in [GHH15].

Theorem 4.1.1. *Let k be an algebraically closed field and $f\colon S \to B$ a projective morphism of k-schemes. Assume $B = \operatorname{Spec} A$ is noetherian and B is of finite type over k. Let H be an affine, linearly reductive group over k acting on S and B such that f is equivariant and let L be an ample H-linearised line bundle on S. Suppose $P \in S$ is a closed point. Then P is stable (or semistable) if and only if $\mu^L(\lambda, P) > 0$ (or ≥ 0) for every non-trivial 1-PS $\lambda\colon \mathbb{G}_m \to H$.*

4.2 Action of 1-parameter subgroup

Let P be any point in $X[n]$ and let $p_n\colon X[n] \to C[n]$ be the projection to the base. As stated in [GHH19], the limit P_0 of P under a 1-PS as defined above exists if and only if its projection onto the base, $p_n(P) \in C[n]$, has a limit. The G-action on the base is a pullback of the action on $\mathbb{A}^{n+1}$ and the corresponding action of a 1-PS is

$$\begin{aligned} t_1 &\longmapsto \tau^{-s_1} t_1, \\ t_k &\longmapsto \tau^{s_{k-1}-s_k} t_k, \quad \text{for } 1 < k \leq n, \\ t_{n+1} &\longmapsto \tau^{s_n} t_{n+1}. \end{aligned}$$

The projection $p_n(P)$ of the point P to the base has a limit as τ tends to zero if and only if each power of τ in the action is nonnegative on the nonzero basis directions t_i, i.e. if and only if

$$0 \geq s_1 \geq \ldots \geq s_{n+1} \geq 0, \tag{4.2.1}$$

where each inequality from left to right must hold if $t_1, \ldots, t_{n+1}$ is nonzero respectively. Thus we obtain boundedness conditions on the weights s_i dependent on where P lies over the base. In particular, when $t_i \neq 0$ for all i, this implies that $s_i = 0$ for all i, so the 1-PS are trivial and all points are trivially semistable.

Let $\mathcal{L}$ be a line bundle as described in Lemma 3.2.1. Assume that locally the lifts to $\mathcal{L}$ of the G-action on each $\mathbb{P}^1$ corresponding to a $\Delta_1^{(k)}$ and on each $\mathbb{P}^1$ corresponding to a $\Delta_2^{(n+1-k)}$ are given by

$$\begin{aligned} (x_0^{(k)}; x_1^{(k)}) &\longmapsto (\tau_k^{a_k} x_0^{(k)}; \tau_k^{-b_k} x_1^{(k)}) \\ (y_0^{(n+1-k)}; y_1^{(n+1-k)}) &\longmapsto (\tau_k^{-c_k} y_0^{(n+1-k)}; \tau_k^{d_k} y_1^{(n+1-k)}) \end{aligned}$$

for some choice of positive integers a_k, b_k, c_k, d_k. Then the corresponding lifts of the 1-PS action to $\mathcal{L}$ are given by

$$(x_0^{(k)}; x_1^{(k)}) \longmapsto (\tau^{a_k s_k} x_0^{(k)} : \tau^{-b_k s_k} x_1^{(k)}),$$
$$(y_0^{(n+1-k)}; y_1^{(n+1-k)}) \longmapsto (\tau^{-c_k s_{n+1-k}} y_0^{(k)} : \tau^{d_k s_{n+1-k}} y_1^{(k)}).$$

We will see in the next section that the Hilbert-Mumford invariants that interest us, which are the invariants relating to 1-PS subgroups of the induced action of G on $H^m_{[n]}$ with associated line bundle $\mathcal{M}$, can be calculated by simply adding the invariants of the points (with multiplicity) in $X[n]$ which make up the support of an element of $H^m_{[n]}$. Indeed, we will see that it is possible for this purpose to think of an element of $H^m_{[n]}$ as just a union of points with multiplicity in $X[n]$ and forget about its scheme structure.

4.3 Bounded and combinatorial weights

In this section, we explain the relation between what [GHH19] call the bounded and combinatorial weights of the Hilbert-Mumford invariants.

Keeping the notation as consistent as possible with [GHH19], let

$$Z^m_{[n]} \subset H^m_{[n]} \times_{C[n]} X[n]$$

be the universal family, with first and second projections p and q. The line bundle

$$\mathcal{M}_l \coloneqq \det p_*(q^*\mathcal{L}^{\otimes l}|_{Z^m_{[n]}})$$

is relatively ample when $l \gg 0$ and is G-linearised, exactly as in Section 2.2.1 of [GHH19].

Given a point $[Z] \in H^m_{[n]}$ and a 1-PS λ_s given by $(s_1, \ldots, s_n) \in \mathbb{Z}^n$, denote the limit of $\lambda_s(\tau) \cdot Z$ as τ tends to zero by Z_0. The Hilbert-Mumford invariant $\mu^{\mathcal{M}_l}(Z, \lambda_s)$ is given by the negative of the weight of the $\mathbb{G}_m$-action on the line bundle $\mathcal{M}_l$ at the point Z_0. At the point Z_0, the line bundle $\mathcal{M}_l$ is given by $\det(H^0(\mathcal{O}_{Z_0} \otimes \mathcal{L}^{\otimes l}))$. We can write Z_0 as a union of length m_P zero-dimensional subschemes $\bigcup_P Z_{0,P}$ supported at points P. Let $\mathcal{L}^{\otimes l}(P)$ denote the fibre of $\mathcal{L}^{\otimes l}$ at P. Following [GHH19], there is an isomorphism

$$H^0(\mathcal{O}_{Z_0} \otimes \mathcal{L}^{\otimes l}) \cong \bigoplus_P \big(H^0(\mathcal{O}_{Z_{0,P}}) \otimes \mathcal{L}^{\otimes l}(P)\big).$$

Then, by taking determinants, as in [GHH19], we get

$$\bigwedge^m H^0(\mathcal{O}_{Z_0} \otimes \mathcal{L}^{\otimes l}) \cong \Big(\bigwedge^m H^0(\mathcal{O}_{Z_0})\Big) \otimes \Big(\bigotimes_P \mathcal{L}^{\otimes l m_P}(P)\Big).$$

which allows us to write the invariant $\mu^{\mathcal{M}_l}(Z, \lambda_s)$ as a sum of what Gulbrandsen, Halle and Hulek call the *bounded weight* $\mu_b^{\mathcal{M}_l}(Z, \lambda_s)$, coming from $\bigwedge^m H^0(\mathcal{O}_{Z_0})$ in

the above, and the *combinatorial weight* $\mu_c^{\mathcal{M}_l}(Z, \lambda_s)$, coming from $\bigotimes_i \mathcal{L}^{\otimes l m_P}(P)$. It is clear also that

$$\mu_b^{\mathcal{M}_l}(Z, \lambda_s) = \mu_b^{\mathcal{M}_1}(Z, \lambda_s),$$

since the bounded weight does not depend on the value of l, and

$$\mu_c^{\mathcal{M}_l}(Z, \lambda_s) = l \cdot \mu_c^{\mathcal{M}_1}(Z, \lambda_s).$$

Hence, we have

$$\mu^{\mathcal{M}_l}(Z, \lambda_s) = \mu_b^{\mathcal{M}_1}(Z, \lambda_s) + l \cdot \mu_c^{\mathcal{M}_1}(Z, \lambda_s).$$

Note that, whereas the combinatorial weight depends on the choice of linearised line bundle, the bounded weight does not. Similarly to [GHH19], we can show that the bounded weight, as its name suggests, can be given an upper bound. The following result is based on Lemma 2.3 of [GHH19], with some slight modifications to suit our setting.

Lemma 4.3.1. *Let $\mu_b^{\mathcal{M}_1}(Z, \lambda_s)$ be the bounded weight of $[Z] \in H^m_{[n]}$ and $s \in \mathbb{Z}^n$ such that the limit of $\lambda_s(\tau) \cdot Z$ as τ goes to zero exists. Then*

$$\mu_b^{\mathcal{M}_1}(Z, \lambda_s) = \sum_{i=1}^{n} b_i s_i,$$

where $|b_i| \leq 2m^2$ for every i.

Proof. Let Z_0 be the limit point of Z with respect to some $s \in \mathbb{Z}^n$, where Z_0 is supported at points $Q_i \in X[n]$. Since Z_0 is a limit point of the action, the Q_i must be a $\mathbb{G}_m$-fixpoint. And since Z_{0,Q_i} is a finite local scheme, we can work with the local coordinates we set up earlier.

Following our previous notation, let n_{Q_i} denote the multiplicity of the scheme Z_0 at point Q_i. The coordinate ring of Z_{0,Q_i} is then generated by n_{Q_i} monomials in the variables $x, y, z, t_1, \ldots, t_{n+1}$ and $x_0^{(k)}/x_1^{(k)}$ or $x_1^{(k)}/x_0^{(k)}$ depending on which side of $\Delta_1^{(k)}$ the point Q_i lies, and $y_0^{(k)}/y_1^{(k)}$ or $y_1^{(k)}/y_0^{(k)}$ depending on which side of $\Delta_2^{(k)}$ the point Q_i lies. Note that the coordinate ring of Z_{0,Q_i} will only contain monomials in the variable $x_0^{(k)}/x_1^{(k)}$ or $x_1^{(k)}/x_0^{(k)}$ if $Q_i \in \Delta_1^{(k)}$, and similarly for the variable $y_0^{(k)}/y_1^{(k)}$ or $y_1^{(k)}/y_0^{(k)}$. Moreover, if $Q_i \in (\Delta_1^{(k)})^\circ \cup (\Delta_2^{(n+1-k)})^\circ$, then this means that $s_k = 0$ as Z_0 is the limit of the 1-PS action. So the weight of the $\mathbb{G}_m$-action on a Δ-component will be nontrivial only if Q_i lies on the boundary of this component.

The weight a_k restricted to the point Q_i is given by adding the multiplicity of $x_0^{(k)}/x_1^{(k)}$ times s_k or that of $x_1^{(k)}/x_0^{(k)}$ times $-s_k$ (depending on which side of $\Delta_1^{(k)}$ the point Q_i lies) in each monomial, plus the multiplicity of $y_0^{(n+1-k)}/y_1^{(n+1-k)}$ times $-s_k$ or that of $y_1^{(n+1-k)}/y_0^{(n+1-k)}$ times s_k (depending on which side of $\Delta_2^{(n+1-k)}$ the point Q_i lies) in each monomial. Each monomial has degree at most n_{Q_i}. The parts of a_k coming from the actions on $\Delta_1^{(k)}$ and $\Delta_2^{(n+1-k)}$ therefore both have absolute value at most m^2, so $|a_k| \leq 2m^2$. □

Let us discuss now how the bounded weight affects the overall stability condition. The following lemma is immediate from [GHH19], but we recall their proof here for convenience.

Lemma 4.3.2. *Let Z be a length m zero-dimensional subscheme in a fibre of $X[n]$. Assume that, for all $s \in \mathbb{Z}^n$ such that the limit $\lambda_s(\tau) \cdot Z$ as τ tends to zero exists, the combinatorial weight can be written as*

$$\mu_c^{\mathcal{M}_1}(Z, \lambda_s) = \sum_{i=1}^{n} c_i s_i,$$

where $c_i s_i \geq 0$ with equality if and only if $s_i = 0$. Then Z is stable (not strictly semistable) with respect to the G-linearised line bundle $\mathcal{M}_l$ on $H^m_{[n]}$ for some large enough l.

Proof. As we have shown that the bounded weight can be expressed as

$$\mu_b^{\mathcal{M}_1}(Z, \lambda_s) = \sum_{i=1}^{n} b_i s_i,$$

where $|b_i| \leq 2m^2$, and recalling that

$$\mu^{\mathcal{M}_l}(Z, \lambda_s) = \mu_b^{\mathcal{M}_1}(Z, \lambda_s) + l \cdot \mu_c^{\mathcal{M}_1}(Z, \lambda_s),$$

it is just a matter of choosing a big enough value of l to make the combinatorial weight overpower the bounded weight. This allows us effectively to treat the bounded weight as negligible and ignore it in our computations. □

Remark 4.3.3. The assumption of Lemma 4.3.2 does not hold in general for all possible G-linearised line bundles on $H^m_{[n]}$.

Let Z be a length m zero-dimensional subscheme in a fibre of $X[n]$. With the following lemmas, we shall establish that if there is at least one point of the support of Z in the union $(\Delta_1^{(k)})^\circ \cup (\Delta_2^{(n+1-k)})^\circ$ for every k, then there exists a GIT stability condition which makes Z stable. We start by showing that for such a subscheme Z there exists a G-linearised line bundle $\mathcal{M}$ on $H^m_{[n]}$ such that the corresponding combinatorial weight will be strictly positive. We will then use Lemma 4.3.2 to show that Z is stable in the corresponding GIT stability.

Remark 4.3.4. Note, here, that such a Z will not necessarily have smooth support, i.e. be supported in the smooth locus of the fibre in which it lies, and every point of the support of Z need not necessarily be contained in a Δ-component.

Lemma 4.3.5. *Let Z be in a fibre of $X[n]$ as above. If there is at least one point of the support of Z in the union $(\Delta_1^{(k)})^\circ \cup (\Delta_2^{(n+1-k)})^\circ$ for every k, then there exists a G-linearised line bundle on $H^m_{[n]}$ with respect to which the combinatorial weight of Z is strictly positive for every nontrivial 1-PS λ_s such that the limit of $\lambda_s(\tau) \cdot Z$ as τ tends to zero exists.*

Proof. We will construct a G-linearised line bundle $\mathcal{L}$ on $X[n]$ as in Lemma 3.2.1, by specifying lifts of the G-action on each $\mathbb{P}(\mathcal{F}_1^{(k)})$ and $\mathbb{P}(\mathcal{F}_2^{(n+1-k)})$ to line bundles $\mathcal{O}_{\mathbb{P}(\mathcal{F}_1^{(k)})}(a_k+b_k)$ and $\mathcal{O}_{\mathbb{P}(\mathcal{F}_2^{(n+1-k)})}(c_k+d_k)$ for some chosen values $a_k, b_k, c_k, d_k \in \mathbb{Z}_{\geq 0}$.

Let $k \in \{1,\dots,n\}$. If there is some point of the support of Z, denoted P, in $(\Delta_1^{(k)})^\circ \subseteq \pi^*(Y_1 \cap Y_3)$, and if m' points of the support lie on the $(1:0)$ side of $\Delta_1^{(k)}$, then we will want the lift of the G-action on $\mathbb{P}(\mathcal{F}_1^{(k)})$ to $\mathcal{O}_{\mathbb{P}(\mathcal{F}_1^{(k)})}(a_k+b_k)$ to be locally given by

$$(x_0^{(k)}; x_1^{(k)}) \longmapsto (\tau_k^{m(m-m')} x_0^{(k)}; \tau_k^{-m(m'+1)} x_1^{(k)}) \in \mathbb{A}^2.$$

This lift is therefore defined on $\mathcal{O}_{\mathbb{P}(\mathcal{F}_1^{(k)})}(m^2+m)$, i.e. we have chosen $a_k = m(m-m')$ and $b_k = n(m'+1)$. We will then choose $c_k = 0$ and $d_k = 1$, so that the action on $\mathbb{P}(\mathcal{F}_2^{(n+1-k)})$ lifts to $\mathcal{O}_{\mathbb{P}(\mathcal{F}_2^{(n+1-k)})}(1)$ and it is locally given by

$$(y_0^{(n+1-k)}; y_1^{(n+1-k)}) \longmapsto (y_0^{(n+1-k)}; \tau_k y_1^{(n+1-k)}) \in \mathbb{A}^2$$

If there is no point of the support of Z in $\Delta_1^{(k)}$, we set the lift of the G-action on $\mathbb{P}(\mathcal{F}_1^{(k)})$ to $\mathcal{O}_{\mathbb{P}(\mathcal{F}_1^{(k)})}(1)$ to be locally given by

$$(x_0^{(k)}; x_1^{(k)}) \longmapsto (\tau_k x_0^{(k)}; x_1^{(k)}) \in \mathbb{A}^2,$$

i.e. we have chosen $a_k = 1$ and $b_k = 0$. In this case there must be at least one point of the support in $(\Delta_2^{n+1-k})^\circ$. Let m'' be the number of points of the support on the $(1:0)$ side of $\Delta_2^{(n+1-k)}$. We then set $c_k = m(m-m'')$ and $d_k = m(m''+1)$, i.e. we have a lift of the G-action on $\mathbb{P}(\mathcal{F}_2^{(n+1-k)})$ to $\mathcal{O}_{\mathbb{P}(\mathcal{F}_2^{(n+1-k)})}(m^2+m)$, locally given by

$$(y_0^{(n+1-k)}; y_1^{(n+1-k)}) \longmapsto (\tau_k^{-m(m-m'')} y_0^{(n+1-k)}; \tau_k^{-m(m''+1)} y_1^{(n+1-k)}) \in \mathbb{A}^2.$$

Repeating this process over all $k \in \{1,\dots,n\}$ will give us a description of $\mathcal{L}$ and we may form the G-linearised line bundle $\mathcal{M}$ from this line bundle in the way described at the start of this section. For more details on why this yields a positive combinatorial weight, see the proof of the following lemma. Note that this is not the only GIT stability condition for which Z is stable. □

Lemma 4.3.6. *Let Z be as in the statement of Lemma 4.3.5 and let $\mathcal{M}$ be a G-linearised line bundle constructed as in the proof of Lemma 4.3.5. Then, for any $s \in \mathbb{Z}^n$, the combinatorial weight can be written*

$$\mu_c^{\mathcal{M}}(Z, \lambda_s) = \sum_{i=1}^{n} c_i s_i,$$

where $c_i s_i \geq 0$ with equality if and only if $s_i = 0$.

Proof. It is clear that the combinatorial weight may be written as a sum

$$\mu_c^{\mathcal{M}}(Z, \lambda_s) = \sum_{i=1}^{n} c_i s_i.$$

Now, let us take any $k \in \{1, \ldots, n\}$. First, let us assume that there is at least one point of the support in $(\Delta_1^{(k)})^\circ \subseteq \pi^*(Y_1 \cap Y_3)$ and denote by m' the number of points of the support on the $(1:0)$ side of $\Delta_1^{(k)}$. Then, if $s_k > 0$,

$$c_k s_k \geq (-m'm(m-m') + (m-m')m(m'+1))s_k - (m')s_k = (m^2 - m')s_k \geq 0.$$

Here, the first bracketed term on the left hand side corresponds to the weight coming from $\mathbb{P}(\mathcal{F}_1^{(k)})$ and the second bracketed term corresponds to the weight coming from $\mathbb{P}(\mathcal{F}_2^{(n+1-k)})$. The value of the second bracketed term arises from the fact that there are at most m' points of the support on the $(0:1)$ side of $\Delta_2^{(n+1-k)}$. And since m' can be at most $m-1$, we have that $m^2 - m' > 0$.

Now, if $s_k < 0$, then

$$c_k s_k \geq (-(m'+1)m(m-m') + (m-m'-1)m(m'+1))s_k + 0 = -(m'+1)ms_k \geq 0,$$

where again the first bracketed term on the left hand side corresponds to the weight coming from $\mathbb{P}(\mathcal{F}_1^{(k)})$ and the second term corresponds to the weight coming from $\mathbb{P}(\mathcal{F}_2^{(n+1-k)})$. As $(m'+1)m > 0$, this gives the desired answer.

Finally, if there is no point of the support in $(\Delta_1^{(k)})^\circ \subseteq \pi^*(Y_1 \cap Y_3)$, we can make a very similar argument, as the weight coming from $\mathbb{P}(\mathcal{F}_1^{(k)})$ is overpowered by the weight coming from $\mathbb{P}(\mathcal{F}_2^{(n+1-k)})$ in the line bundle $\mathcal{M}$ we set up. □

4.4 Semistable locus and GIT quotient

Lemma 4.4.1. *Let Z be as in the statement of Lemma 4.3.5. Then there exists a GIT stability condition on $H^m_{[n]}$ which makes Z stable.*

Proof. This follows from Lemmas 4.3.2 and 4.3.6. Indeed, by Lemma 4.3.2, if the combinatorial weight can be written in the form

$$\mu_c^{\mathcal{M}}(Z, \lambda_s) = \sum_{i=1}^{n} c_i s_i,$$

where $c_i s_i \geq 0$ with equality if and only if $s_i = 0$, then we may choose a high enough tensor power l of $\mathcal{M}$ such that Z is stable if and only if the combinatorial weight is strictly positive. But this condition is satisfied by Lemma 4.3.6. □

Lemma 4.4.2. *Let Z be a length m zero-dimensional subscheme in a fibre of $X[n]$, such that no point of the support is contained in the union $(\Delta_1^{(k)})^\circ \cup (\Delta_2^{(n+1-k)})^\circ$ for some k (these components may be expanded or not in the fibre). Then there exists no GIT stability condition on $H^m_{[n]}$ with respect to the group G which makes Z stable.*

Proof. Let us choose an arbitrary G-linearised line bundle $\mathcal{M}$, not necessarily constructed as above, with respect to which Z has Hilbert-Mumford invariant

$$\mu^{\mathcal{M}}(Z, \lambda_s) = \sum_{i=1}^{n} a_i s_i,$$

for some $s \in \mathbb{Z}^n$ such that the limit of $\lambda_s(\tau) \cdot Z$ as τ tends to zero exists. Either $a_k = 0$, in which case Z cannot be stable (it will at best be semistable) with respect to the stability condition given by the chosen linearisation, or $a_k \neq 0$.

If $\Delta_1^{(k)}$ and $\Delta_2^{(n+1-k)}$ are expanded out in the fibre, then s_k is not bounded above or below by 0 or by any weights acting nontrivially outside of these components. Moreover, as no point of the support of Z are contained in $(\Delta_1^{(k)})^\circ \cup (\Delta_2^{(n+1-k)})^\circ$, we know that τ^{s_k} acts trivially on all points of the support. The integer a_k is therefore independent of the value of s_k; different values of s_k will not change a_k. If $a_k > 0$, we may choose s_k to be negative with large enough absolute value to destabilise Z. Similarly, if $a_k < 0$, we may choose s_k to be positive and large enough to destabilise Z.

Finally, if $\Delta_1^{(k)}$ and $\Delta_2^{(n+1-k)}$ are not expanded out in the fibre, either $t_l \neq 0$ for $l \geq k$ or $t_l \neq 0$ for $l \leq k$. If $t_l \neq 0$ for $l \geq k$, then $\Delta_1^{(k)} = Y_1$ and $\Delta_2^{(n+1-k)} = Y_1 \cup Y_3$. All points of the support of Z must therefore be on the $(1:0)$ side of $\Delta_1^{(k)}$ and on the $(0:1)$ side of $\Delta_2^{(n+1-k)}$, which implies that $a_k < 0$. But by the condition (4.2.1), we have $s_k \geq 0$, and we can therefore choose s_k large enough to destabilise Z. A very similar argument can be made if instead $t_l \neq 0$ for $l \leq k$. □

Theorem 4.4.3. *Let Z be a length m zero-dimensional subscheme in a fibre of $X[n]$. Then there exists a GIT stability condition on $H^m_{[n]}$ which makes Z stable if and only if there is at least one point of the support of Z in the union $(\Delta_1^{(k)})^\circ \cup (\Delta_2^{(n+1-k)})^\circ$ for every k.*

Proof. This follows directly from Lemmas 4.4.1 and 4.4.2. □

We can now describe the GIT quotients resulting from these constructions. Let

$$A[n] \coloneqq H^0(C[n], \mathcal{O}_{C[n]}).$$

Then we recall from Lemma 3.1.11, the isomorphism

$$H^0(C[n], \mathcal{O}_{C[n]})^G \cong k[t].$$

For all choices of linearised line bundle described in the above, the GIT quotient on the base therefore behaves as follows

$$C[n]//G = \operatorname{Spec} A[n]//G = \operatorname{Spec}(A[n]^G) \cong \mathbb{A}^1.$$

Now let us denote by $H^{m,s}_{[n],\mathcal{M}}$ the locus of GIT stable subschemes in $H^m_{[n]}$ with respect to the stability condition determined by one of the choices of G-linearised line bundle $\mathcal{M}$ as constructed in Section 4.3 and let

$$I^m_{[n],\mathcal{M}} \coloneqq H^{m,s}_{[n],\mathcal{M}}//G$$

denote the corresponding GIT quotient.

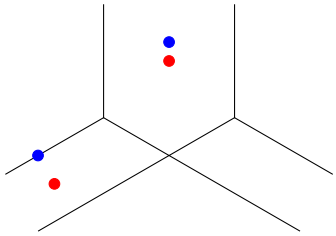


Figure 5: Non-separatedness in $t_1 = t_2 = 0$

Theorem 4.4.4. *The GIT quotients $I^m_{[n],\mathcal{M}}$ thus constructed are projective over*

$$\mathrm{Spec}(A[n]^G) \cong \mathbb{A}^1.$$

Proof. This result follows directly from the relative Hilbert-Mumford criterion of [GHH15]. □

We remark that two types of subscheme arise in the stable loci $H^{m,s}_{[n],\mathcal{M}}$. Firstly, subschemes whose support is entirely in the smooth locus of the fibre in which they lie and, secondly, subschemes whose support has nonempty intersection with the singular locus of this fibre. Since we do not necessarily require all points of the support to lie in the interior of a Δ-component, some of these singularities may be quite complicated.

Remark 4.4.5. We note that, although the GIT quotients $I^m_{[n],\mathcal{M}}$ are projective and thus proper over $\mathbb{A}^1$, their corresponding stack quotients are not necessarily proper. Indeed, the GIT quotient does not see the orbits of the group action themselves but the closures of these orbits. For example in Figure 5, the red pair of points and the blue pair of points are in the same orbit closure, so the GIT quotient considers them as equivalent, while the corresponding stack quotient regards them as belonging to separate orbits. This means that, in the stack, allowing for both pairs will break separatedness.

In the following sections, when studying quotient stacks, we will therefore want to consider the sublocus of the GIT stable stable locus containing only length m zero-dimensional subschemes which have smooth support in a given fibre of $X[n]$. Building a compactification in which limits are represented by subschemes with smooth support will also be useful for future applications as it allows us to break down the problem of a Hilbert scheme of m points on a singular surface into products of Hilbert schemes of fewer than m points on smooth components.

No single GIT quotient $I^m_{[n],\mathcal{M}}$ contains all desired limits with smooth support. Therefore in the stack construction, the stability condition we define will draw on these local quotients, but globally will not correspond to one single GIT stability condition. We now define a notion that we will use in the following sections.

Definition 4.4.6. We say that a fibre in some expanded degeneration $X[n]$ has *base codimension* k if exactly k basis directions vanish at this fibre. This is independent of the value n.

Finally, we would like to point out that if we construct a unique GIT quotient which does not realise all limit subschemes with smooth support then the limits given by orbit closures containing only subschemes with singular support will not lie in a fibre of the expected base codimension. This gives an intuition that the degeneration we have chosen is too small. That being said, it can be useful to think about this GIT quotient if what we are trying to do is simply to resolve singularities in a way that preserves some good properties of the space, e.g. in the context of constructing minimal models for type III degenerations of Hilbert schemes of points on K3 surfaces. We will touch on this again briefly in Section 8.

5 Stack perspective

In this section, we generalise the scheme construction of Section 3 and define the analogous stack of expansions and its family $\mathfrak{X} \to \mathfrak{C}$. As mentioned before, we impose additional equivalences in the stack, which have the effect of setting any two fibres with the same expanded components to be equivalent. We examine the loci of GIT stable points again on this stack and discuss their relation with stability conditions of Li and Wu and of Maulik and Ranganathan ([LW15], [MR20]). Finally, we construct a proper Deligne-Mumford stack which we will show to be isomorphic to a choice of underlying algebraic stack obtained through the Maulik-Ranganthan construction. We use the word underlying here because what is constructed in [MR20] is a logarithmic algebraic stack and we impose no logarithmic structure on our space.

5.1 Expanded construction for stacks

In this section we construct a stack of expansions $\mathfrak{C}$ and family over it $\mathfrak{X}$, keeping our notation as close as possible to that of [LW15].

The stack $\mathfrak{C}$. In the following we define the stack $\mathfrak{C}$ identically to the stack of expanded degenerations defined by Li and Wu. For convenience, we recall the details of this construction here.

Let us consider $\mathbb{A}^{n+1}$ with its natural torus action $\mathbb{G}_m^n$ as defined above. We then impose some additional relations given by a collection of isomorphisms which we describe in the following. As before, we label elements of the base as $(t_1, \dots, t_{n+1})$. We start by defining the set

$$[n+1] \coloneqq \{1, \dots, n+1\}.$$

Let $I \subseteq [n+1]$ and $I^\circ = [n+1] - I$ be the complement of I. For $|I| = r+1$, let

$$\mathrm{ind}_I \colon [r+1] \longrightarrow I \subset [n+1]$$

and

$$\mathrm{ind}_{I^\circ} \colon [n-r] \longrightarrow I^\circ \subset [n+1]$$

be the order preserving isomorphisms. Let

$$\mathbb{A}^{n+1}_I = \{(t) \in \mathbb{A}^{n+1} |\ t_i = 0,\ t_i \in I\} \subset \mathbb{A}^{n+1}$$

and

$$\mathbb{A}^{n+1}_{U(I)} = \{(t) \in \mathbb{A}^{n+1} |\ t_i \neq 0,\ t_i \in I^\circ\} \subset \mathbb{A}^{n+1}.$$

Then we have the isomorphism

$$\widetilde{\tau}_I \colon (\mathbb{A}^{r+1} \times \mathbb{G}_m^{n-r}) \longrightarrow \mathbb{A}^{n+1}_{U(I)}$$

given by

$$(a_1, \ldots, a_{r+1}, \sigma_1, \ldots, \sigma_{n-r}) \longmapsto (t_1, \ldots, t_{n+1}),$$

where

$$t_k = a_l,\ \text{if}\ \ \mathrm{ind}_I(l) = k,$$
$$t_k = \sigma_l,\ \text{if}\ \ \mathrm{ind}_{I^\circ}(l) = k.$$

Then, given $I, I' \subset [n+1]$ such that $|I| = |I'|$, we define an isomorphism

$$\widetilde{\tau}_{I,I'} = \widetilde{\tau}_I \circ \widetilde{\tau}_{I'}^{-1} \colon \mathbb{A}^{n+1}_{U(I')} \longrightarrow \mathbb{A}^{n+1}_{U(I)}.$$

Recall from Section 3.1 that we had natural inclusions (3.1.1)

$$C[n] \hookrightarrow C[n+1],$$

which Li and Wu refer to as *standard embeddings*.

Finally, we define $\mathfrak{U}^n$ to be the quotient $[\mathbb{A}^{n+1}/\!\sim]$ by the equivalences generated by the $\mathbb{G}_m^n$-action and the equivalences $\widetilde{\tau}_{I,I'}$ for pairs I, I' with $|I| = |I'|$. We can define open immersions

$$\mathfrak{U}^n \longrightarrow \mathfrak{U}^{n+1},$$

induced by the standard embeddings. Let $\mathfrak{U} \coloneqq \varinjlim \mathfrak{U}^n$ be the direct limit over n and let $\mathfrak{C} \coloneqq C \times_{\mathbb{A}^1} \mathfrak{U}$.

The stack $\mathfrak{X}$. Let $X[n] \to C[n]$ be as in Section 3 and recall that $\pi \colon X[n] \to X$ is the projection to the original family. Let

$$\overline{\tau}_I \colon C[m] \hookrightarrow C[n]$$

be the standard embedding. Then the induced family

$$(\overline{\tau}_I^* X[n], \overline{\tau}_I^* \pi)$$

is isomorphic to $(X[m], \pi)$ over $C[m]$. The equivalences on $\mathfrak{U}^n$ lift to C-isomorphisms of fibres.

We define $\mathfrak{X}^n$ to be the quotient $[X[n]/\!\sim]$ by the equivalences generated by the $\mathbb{G}_m^n$-action and equivalences lifted from $\mathfrak{U}^n$. There are natural immersions of stacks

$$\mathfrak{X}^n \longrightarrow \mathfrak{X}^{n+1},$$

induced by the immersions $\mathfrak{U}^n \to \mathfrak{U}^{n+1}$. Finally, we define $\mathfrak{X} = \varinjlim \mathfrak{X}^n$ to be the direct limit over n. It is an Artin stack.

5.2 Stability conditions.

We start by examining some stability conditions on the scheme $H^m_{[n]}$. We have defined several G-linearised line bundles $\mathcal{M}$ on this space. As before, let us denote by $H^{m,ss}_{[n],\mathcal{M}}$ and $H^{m,s}_{[n],\mathcal{M}}$ the corresponding GIT semistable and stable loci respectively. As discussed in Section 4.4, considering the GIT stable locus does not give us a separated quotient stack, among other reasons because it contains subschemes with non-smooth support in some fibres. Recall that the relative Hilbert scheme of m points on $X[n] \to C[n]$, which we denoted $H^m_{[n]}$, is the scheme which represents the functor

$$h\colon \underline{\text{k-Sch}}^{\text{op}} \longrightarrow \underline{\text{Sets}},$$

where $\underline{\text{k-Sch}}^{\text{op}}$ is the category of k-schemes. This functor associates to any k-scheme B the set of flat families over B of subschemes of fibres of $X[n]$ over $C[n]$. Restricting $X[n]$ to the smooth locus of its fibres yields a family of open subschemes $X[n]^{\text{sm}}$ over $C[n]$, and we can similarly define a Hilbert functor h_{sm} on this family. There is a morphism from the corresponding Hilbert scheme $H^m_{[n],\text{sm}} \coloneqq \text{Hilb}(X[n]^{\text{sm}}/C[n])$ to $H^m_{[n]}$ which is clearly a monomorphism and it is étale since deformations of subschemes with smooth support have smooth support. We could also note that the complement of $H^m_{[n],\text{sm}}$ in $H^m_{[n]}$ is closed by the valuative criterion since the limit of any subscheme with part of its support in the singular locus of a fibre must also have part of its support in the singular locus of a fibre.

We remark that since the smooth locus of the fibres of $X[n]$ is G-invariant, restricting the functor to this locus preserves the G-invariance. The restriction of the semistable and stable loci, denoted $H^{m,ss}_{[n],\mathcal{M}}$ and $H^{m,s}_{[n],\mathcal{M}}$, to the loci of subschemes with smooth support will therefore yield a G-invariant open subscheme. We may therefore write inclusions

$$H^{m,s}_{[n],\mathcal{M},\text{sm}} \subset H^{m,ss}_{[n],\mathcal{M},\text{sm}} \subset H^{m,ss}_{[n],\mathcal{M}},$$

where the first two terms in the above chain of inclusions denote the loci of GIT stable and semistable subschemes which have smooth support.

Now recall that the Li-Wu stability states that a subscheme Z in $\mathfrak{X}$ is stable if and only if it is admissible and has finite automorphism group. Let $X[n]_0$ be a fibre of $X[n]$ over a closed point and let D denote the singular locus of $X[n]_0$. A subscheme Z in $X[n]_0$ is said to be admissible if the morphism

$$\mathcal{I}_Z \otimes \mathcal{O}_D \to \mathcal{O}_D$$

is injective, where $\mathcal{I}_Z$ is the ideal sheaf of Z, i.e. when Z is normal to D. A family Z in $X[n]$ is admissible if it is admissible in every fibre over a closed point. For a length m zero-dimensional scheme, the admissible condition will mean that none of the points in the support of Z lie in the singular locus of the given fibre. The second condition means that the stabiliser of Z with respect to the torus action we defined on the blow-ups must be finite. Denote the Li-Wu stable locus by

$H^m_{[n],\mathrm{LW}}$. From now on, we will abbreviate to LW stability for convenience. Note that we have an inclusion

$$H^{m,s}_{[n],\mathcal{M},\mathrm{sm}} \subset H^m_{[n],\mathrm{LW}}$$

for all G-linearised line bundles $\mathcal{M}$ on $H^m_{[n]}$ since, if points are GIT stable, they must have finite stabilisers. This inclusion no longer holds for the GIT semistable locus. This is a strict inclusion as the LW stability is clearly a weaker condition than the GIT strict stability with smooth support. We now make the following definition on schemes.

Definition 5.2.1. Let $[Z] \in H^m_{[n]}$. We say that Z is *weakly stable* (we will abbreviate to WS stable) if there exists a G-linearised ample line bundle on $H^m_{[n]}$ with respect to which Z is stable. We denote the WS stable locus in $H^m_{[n]}$ by $H^m_{[n],\mathrm{WS}}$.

We shall denote by $H^m_{[n],\mathrm{SWS}}$ the locus of WS stable subschemes with smooth support. From the above, it is then obvious that we have an inclusion $H^m_{[n],\mathrm{SWS}} \subset H^m_{[n],\mathrm{LW}}$. We will now want to compare these stability conditions on the stack $\mathfrak{X}$, so we will need to extend our definition of WS stability to this stack.

Given a C-scheme S, an object of $\mathfrak{X}(S)$ is a pullback family $\xi^*X[n]$ for a morphism

$$\xi\colon S \to C[n].$$

Now we describe WS stability on the stack $\mathfrak{X}$.

Definition 5.2.2. A pair $(Z, \mathcal{X}) \subset \mathfrak{X}(S)$ is said to be WS stable if and only if $\mathcal{X} \coloneqq \xi^*X[n]$ for some morphism $\xi\colon S \to C[n]$ and there exists some G-linearised ample line bundle on $H^m_{[n]}$ which makes Z be GIT stable. We will say that Z is SWS stable if it has smooth support and is WS stable.

Remark 5.2.3. Note that we are slightly abusing notation in the above definition, by asking for Z to be GIT stable in $H^m_{[n]}$, when Z is defined in $\mathcal{X}$, and it is in fact ξ_*Z which must be GIT stable in $H^m_{[n]}$. This is a harmless a simplification as it will always be clear from context what we mean. We continue to use it throughout the work for convenience, especially where the map ξ has not been specified.

Let us denote by $\mathfrak{M}^m_{\mathrm{SWS}}$ and $\mathfrak{M}^m_{\mathrm{LW}}$ the stacks of SWS and LW stable length m zero-dimensional subschemes in $\mathfrak{X}$ respectively. Let S be a C-scheme. An object of $\mathfrak{M}^m_{\mathrm{SWS}}(S)$ is defined to be a pair $(Z, \mathcal{X})$, where $\mathcal{X} \in \mathfrak{X}(S)$ and Z is an S-flat SWS stable family in $\mathcal{X}$. Similarly, an object of $\mathfrak{M}^n_{\mathrm{LW}}(S)$ is a pair $(Z, \mathcal{X})$, where $\mathcal{X} \in \mathfrak{X}(S)$ and Z is an S-flat LW stable family in $\mathcal{X}$.

Remark 5.2.4. Note that it does not make sense in general to speak of Maulik-Ranganathan stability (MR stability) without defining an appropriate notion of tube components on our stacks as in Section 2.5. In this specific setting, however, we will see that there is no need to specify tube components as the stacks $\mathfrak{M}^m_{\mathrm{SWS}}$ and $\mathfrak{M}^m_{\mathrm{LW}}$ are already proper. The LW stability which we extended to our situation will therefore be equivalent to MR stability on $\mathfrak{X}$. In this setting we may therefore use both terminologies interchangeably.

We will see in Section 7 that we can also make constructions, equivalent to some of the constructions of Maulik-Ranganathan, which require choices of representatives of limit subschemes and labelling of components as tube components. As the construction we make here requires the minimal amount of choice (the only choice was in choosing to blow up Y_1 and Y_2 but not Y_3 at the very start) we shall refer to it as the canonical construction.

6 The canonical moduli stack

In this section we show that the stacks $\mathfrak{M}^m_{\mathrm{SWS}}$ and $\mathfrak{M}^m_{\mathrm{LW}}$ are proper and Deligne-Mumford and that they are in fact isomorphic.

6.1 Properness and Deligne-Mumford property

In this section, we show that the stacks $\mathfrak{M}^m_{\mathrm{SWS}}$ and $\mathfrak{M}^m_{\mathrm{LW}}$ are universally closed, separated and have finite automorphisms. Before we give these proofs, we make the following definition.

Definition 6.1.1. Let $S \coloneqq \operatorname{Spec} R \to C$, where R is some discrete valuation ring and let η denote the generic point of S. Now, let $(Z, \mathcal{X})$ be a pair where $\mathcal{X} \in \mathfrak{X}(S)$ and Z is an S-flat family of length m zero-dimensional subschemes in $\mathcal{X}$. Let $S' \to S$ be some finite base change and denote the generic and closed points of S' by η' and η'_0 respectively. We say that a pair $(Z'_{\eta'_0}, \mathcal{X}'_{\eta'_0})$ is an *extension* of $(Z_\eta, \mathcal{X}_\eta)$ if there exists such a base change and $(Z'_{\eta'_0}, \mathcal{X}'_{\eta'_0})$ is the restriction to η'_0 of some S-flat family $(Z', \mathcal{X}')$ with $\mathcal{X}' \in \mathfrak{X}'(S')$ such that $Z_\eta \times_\eta \eta' \cong Z'_{\eta'}$ and $\mathcal{X}_\eta \times_\eta \eta' \cong \mathcal{X}'_{\eta'}$.

Proposition 6.1.2. *The stack $\mathfrak{M}^m_{\mathrm{SWS}}$ is universally closed.*

Proof. Let $S \coloneqq \operatorname{Spec} R \to C$, where R is some discrete valuation ring with uniformising parameter w and quotient field k. We denote by η and η_0 the generic and closed points of S respectively. Let $(Z, \mathcal{X})$ be an S-flat family of length m zero-dimensional subschemes such that $\mathcal{X} \coloneqq \zeta^* X[r] \in \mathfrak{X}(S)$ for some morphism $\zeta \colon S \to C[r]$ and $(Z_\eta, \mathcal{X}_\eta) \in \mathfrak{M}^m_{\mathrm{SWS}}(\eta)$. We show that there exists a finite base change $S' \coloneqq \operatorname{Spec} R' \to S$, for some discrete valuation ring R' and a pair $(Z', \mathcal{X}') \in \mathfrak{M}^m_{\mathrm{SWS}}(S')$ satisfying the following condition. We denote by η' and η'_0 the generic and closed points of S' respectively. Then S' and $(Z', \mathcal{X}')$ are chosen such that we have an equivalence $\mathcal{X}'_{\eta'} \cong \mathcal{X}_\eta \times_\eta \eta'$ which induces an equivalence $Z'_{\eta'} \cong Z_\eta \times_\eta \eta'$.

Let $\mathfrak{X}(S)$ be defined by the equation $xyz = cw^h$, where w is the uniformising parameter of R as above and $c \in R^\times$ (the notation $R^\times$ is used to denote the invertible elements of R). The subscheme Z is a union of irreducible components Z_i whose defining equations we will want to express in terms of the uniformising parameter. We therefore start by taking an appropriate base change $S' \coloneqq \operatorname{Spec} R' \to S$, which maps $u^k \to w^h$, where u is the uniformising parameter

of R' and where u is chosen such that each Z_i can be written locally in terms of its x, y and z coordinates as

$$\{(c_{i,1}u^{e_{i,1}}, c_{i,2}u^{e_{i,2}}, c_{i,3}u^{e_{i,3}})\}, \tag{6.1.1}$$

for some $e_{i,j} \in \mathbb{Z}$ and $c_{i,j} \in (R')^\times$. Note that $\mathfrak{X}(S')$ is defined by the equation $xyz = cu^k$ and we therefore have the equality

$$c_{i,1}c_{i,2}c_{i,3}u^{e_{i,1}+e_{i,2}+e_{i,3}} = cu^k$$

for all i.

We now form an ordered list $(d_1u^{e_1}, \dots, d_{2m}u^{e_{2m}})$, where we arrange all values $c_{i,1}u^{e_{i,1}}$ and $(c_{i,2})^{-1}cu^{k-e_{i,2}}$ from smallest to largest power of u. If two terms have the same power of u, we may place them in any order. We shall now inductively construct an element $(t_1, \dots, t_{n+1})$ of $\mathbb{A}^{n+1}$ determining a morphism $\xi\colon S' \to C[n]$ such that the pullback $\xi^*X[n]$ defines the family $\mathcal{X}'$. We start by setting

$$(t_1, t_2) = (d_1u^{e_1}, (d_1u^{e_1})^{-1}cu^k).$$

If $e_1 = e_2$, then we do not include $d_2u^{e_2}$ and move on to e_3. If $e_1 \neq e_2$, however, we set

$$(t_1, t_2, t_3) = (d_1u^{e_1}, (d_1u^{e_1})^{-1}d_2u^{e_2}, (d_2u^{e_2})^{-1}cu^k).$$

We continue to iterate this process in the following way. Assume we have $(t_1, \dots, t_j)$, where $t_j = (d_lu^{e_l})^{-1}cu^k$. Then, if $e_{l+1} \neq e_l$, we write

$$(t_1, \dots, t_j, t_{j+1}) = (d_1u^{e_1}, \dots, (d_lu^{e_l})^{-1}d_{l+1}u^{e_{l+1}}, (d_{l+1}u^{e_{l+1}})^{-1}cu^k),$$

and if $e_{l+1} = e_l$, then we move on to $l+2$ without including $d_{l+1}u^{e_{l+1}}$ in the expression. We iterate this until we find

$$(t_1, \dots, t_{n+1}) = (f_1u^{g_1}, \dots, f_{n+1}u^{g_{n+1}}) \tag{6.1.2}$$

which has exactly one entry for each different power of u contained in the list $(d_1u^{e_1}, \dots, d_{2m}u^{e_{2m}})$.

We now denote by $\pi_n\colon C[n] \to \mathbb{A}^{n+1}$ the natural projection. The morphism $\xi\colon S' \to C[n]$ is defined by the condition that

$$\pi_n \circ \xi = (f_1u^{g_1}, \dots, f_{n+1}u^{g_{n+1}}). \tag{6.1.3}$$

We may then define $\mathcal{X}' \coloneqq \xi^*X[n]$ and let $Z' \coloneqq Z \times_{\mathcal{X}} \mathcal{X}'$. We show now that this satisfies all the necessary conditions.

Since $\mathcal{X} \in \mathfrak{X}(S)$ is a pullback $\mathcal{X} = \zeta^*X[r]$ for some r, where $\zeta\colon S \to C[r]$ is given by a similar expression to (6.1.3), then we have that $\mathcal{X}'_{\eta'} \cong \mathcal{X}_\eta \times_\eta \eta'$. Indeed, over the generic point, the uniformising parameter is invertible and any two expressions $(t_1, \dots, t_l)$ and $(t_1, \dots, t_{l'})$ are equivalent in $\mathfrak{C}$ up to the equivalences of this stack if they have the same product $t_1 \cdots t_l = t_1 \cdots t_{l'}$. But in our case this product was chosen to be identical up to the base change factor.

Moreover, the expression (6.1.3) is chosen precisely to give an S'-flat extension of $Z \times_\eta \eta'$ to the closed point η'_0, where all points of the support of this extension lie in the smooth locus of the fibre $\mathcal{X}_{\eta_0}$. Finally, the expression (6.1.3) ensures that we have expanded out the Δ-components in the fibre $\mathcal{X}'_{\eta_0}$ in such a way that every expanded Δ-component in this fibre contains some point of the support of Z'. By Theorem 4.4.3, such a configuration will be stable with respect to some GIT stability condition on $H^m_{[n]}$.

The above discussion shows that if $(Z_\eta, \mathcal{X}_\eta)$ is pulled back from a fibre above a point $(t_1, \ldots, t_{n+1})$ in some $C[n]$ whose entries are all invertible, then $(Z_\eta, \mathcal{X}_\eta)$ has an SWS stable extension. See Corollary 7.5.11 for a proof that there exists an extension if $\mathcal{X}_\eta$ is a modified special fibre, i.e. if some of the entries of $(t_1, \ldots, t_{n+1})$ are not invertible. □

Corollary 6.1.3. *The stack $\mathfrak{M}^m_{\mathrm{LW}}$ is universally closed.*

Proof. As every SWS stable subscheme must be LW stable, the existence of limits in $\mathfrak{M}^m_{\mathrm{LW}}$ follows from the existence of limits in $\mathfrak{M}^m_{\mathrm{SWS}}$. □

Proposition 6.1.4. *The stacks $\mathfrak{M}^m_{\mathrm{SWS}}$ and $\mathfrak{M}^m_{\mathrm{LW}}$ are separated.*

Proof. Let $S \coloneqq \operatorname{Spec} R \to C$, where R is a discrete valuation ring with uniformising parameter u. Let η denote the generic point of S and η_0 its closed point. Now, assume that there are two pairs $(Z, \mathcal{X})$ and $(Z', \mathcal{X}')$ in $\mathfrak{M}^m_{\mathrm{SWS}}(S)$ such that $(Z_\eta, \mathcal{X}_\eta) \cong (Z'_\eta, \mathcal{X}'_\eta)$. We will show that it must follow that $(Z_{\eta_0}, \mathcal{X}_{\eta_0}) \cong (Z'_{\eta_0}, \mathcal{X}'_{\eta_0})$.

We may assume that S is chosen so that the i-th irreducible component of Z is given in terms of its local coordinates x, y and z by

$$\{(c_{i,1}u^{e_{i,1}}, c_{i,2}u^{e_{i,2}}, c_{i,3}u^{e_{i,3}})\}, \tag{6.1.4}$$

and the i-th irreducible component of Z' is given in terms of its local coordinates x, y and z by

$$\{(d_{i,1}u^{f_{i,1}}, d_{i,2}u^{f_{i,2}}, d_{i,3}u^{f_{i,3}})\}. \tag{6.1.5}$$

Since the equivalences of the stack fix x, y and z and we know that $(Z_\eta, \mathcal{X}_\eta) \cong (Z'_\eta, \mathcal{X}'_\eta)$, it must therefore follow that Z and Z' have the same number of irreducible components. Moreover, if these components are labelled in a compatible way, then $c_{i,1} = d_{i,1}$ and $e_{i,1} = f_{i,1}$ for all i. But now, by flatness, each Z_i and Z'_i component must satisfy the equations

$$x = c_{i,1}u^{e_{i,1}}, \tag{6.1.6}$$

$$y = c_{i,2}u^{e_{i,2}}, \tag{6.1.7}$$

$$z = c_{i,3}u^{e_{i,3}}, \tag{6.1.8}$$

also above the closed point. If more than one element of the set $\{e_{i,1}, e_{i,2}, e_{i,3}\}$ is nonzero, then this implies that either Z_i and Z'_i must either have non-smooth support or be supported in a component blown-up along the vanishings of both sides of the above components. The stability condition forces Z_i and Z'_i to have

smooth support, so the latter must be true. Moreover, since in our construction we have chosen to do our blow-ups along the vanishing of x and the vanishing of y, this implies that Z_i and Z_i' must be supported in a component blown up along $\langle x, cu^{e_{i,1}}\rangle$ and $\langle y, c'u^{e_{i,2}}\rangle$ over the closed point η_0, for some $c, c' \in R^\times$.

Note that different values of c and c' will cause the relevant points of the support of Z_i and Z_i' to take on different values in the interior of the $\mathbb{P}^1$ introduced by each blow-up. Since the $\mathbb{G}_m$-action imposed on the $\mathbb{P}^1$ identifies all points within the interior of a $\mathbb{P}^1$, this choice makes no difference.

Notice also that blowing up along $\langle x, cu^{e_{i,1}}\rangle$ is the same as blowing up along $\langle yz, (cu^{e_{i,1}})^{-1}du^k\rangle$, where $\mathfrak{X}(S)$ is defined by the equation $xyz = du^k$. This allows us to obtain the equation (6.1.8).

We have established that both Z_i and Z_i' must be supported in the blown-up components described above for all i such that more than one element of the set $\{e_{i,1}, e_{i,2}, e_{i,3}\}$ is nonzero. We also know that by the stability conditions the pairs $(Z_{\eta_0}, \mathcal{X}_{\eta_0})$ and $(Z'_{\eta_0}, \mathcal{X}'_{\eta_0})$ cannot have an expanded component containing no point of the support. Let $\pi_n \colon C[n] \to \mathbb{A}^{n+1}$ denote the natural projection, as above. It follows that the morphism

$$\pi_n \circ \xi = (h_1 u^{g_1}, \ldots, h_n u^{g_n}) \colon S \to C[n] \to \mathbb{A}^{n+1} \tag{6.1.9}$$

defining the family $\mathcal{X} = \xi^* X[n]$ is uniquely determined up to the choices of values $h_i \in R^\times$ and embeddings by the standard embeddings. If the family $\mathcal{X}'$ is defined by a morphism as in (6.1.9) but with different nonzero g_i values, then the pair $(Z'_{\eta_0}, \mathcal{X}'_{\eta_0})$ will necessarily have either non-smooth support or an expanded component containing no point of the support. This shows uniqueness of limits. □

Proposition 6.1.5. *The stacks $\mathfrak{M}^m_{\mathrm{LW}}$ and $\mathfrak{M}^m_{\mathrm{SWS}}$ have finite automorphisms.*

Proof. On the stack $\mathfrak{M}^m_{\mathrm{LW}}$ this is immediate from the definition of LW stability. Since the SWS stable locus is a subset of the LW stable locus, it follows that $\mathfrak{M}^m_{\mathrm{SWS}}$ must also have finite automorphisms. Alternatively, one can recall that a GIT stable point must have finite stabiliser with respect to the relevant G-action.

Note that any equivalence on $\mathfrak{X}$ lifted from an isomorphism $\widetilde{\tau_{I,I'}}$ does not fix any object unless $\widetilde{\tau_{I,I'}}$ is the identity map. This is clear from the fact that $\widetilde{\tau_{I,I'}}$ acts on a tuple in $\mathbb{A}^{n+1}$ by changing the position of its zero entries while preserving the relative order of its nonzero entries. The only way to fix a tuple is to leave its zero entries in their original position, but any map $\widetilde{\tau_{I,I'}}$ which does this is just the identity map. □

Corollary 6.1.6. *The stacks $\mathfrak{M}^m_{\mathrm{LW}}$ and $\mathfrak{M}^m_{\mathrm{SWS}}$ are Deligne-Mumford and proper.*

Proof. This follows directly from the results of this section. □

6.2 An isomorphism of stacks

We shall now show that the stacks $\mathfrak{M}^m_{\mathrm{SWS}}$ and $\mathfrak{M}^m_{\mathrm{LW}}$ are isomorphic. The following lemma is a standard result, quoted from [GHH19].

Lemma 6.2.1. *Let $\mathfrak{W}$ and $\mathfrak{Y}$ be Deligne-Mumford stacks of finite type over an algebraically closed field k, and let*

$$f\colon \mathfrak{W} \to \mathfrak{Y}$$

be a representable étale morphism of finite type. Let $|\mathfrak{W}(k)|$ denote the set of equivalence classes of objects in $\mathfrak{W}(k)$ and similarly for $|\mathfrak{Y}(k)|$. Assume

- *$|f|\colon |\mathfrak{W}(k)| \to |\mathfrak{Y}(k)|$ is bijective.*
- *For every $x \in \mathfrak{W}(k)$, f induces an isomorphism*

$$\mathrm{Aut}_{\mathfrak{W}}(x) \to \mathrm{Aut}_{\mathfrak{Y}}(f(x)).$$

Then f is an isomorphism of stacks.

We may construct such a map $f\colon \mathfrak{M}^m_{\mathrm{SWS}} \to \mathfrak{M}^m_{\mathrm{LW}}$, which we will show to have the required properties, in the following way. First recall from above that we have an inclusion $H^m_{[n],\mathrm{SWS}} \subset H^m_{[n],\mathrm{LW}}$. Therefore the natural morphism $H^m_{[n],\mathrm{LW}} \to \mathfrak{M}^m_{\mathrm{LW}}$ restricts to give a morphism $H^m_{[n],\mathrm{SWS}} \to \mathfrak{M}^m_{\mathrm{LW}}$. This morphism is equivariant under the group action so must factor through the morphism

$$f\colon \mathfrak{M}^m_{\mathrm{SWS}} \longrightarrow \mathfrak{M}^m_{\mathrm{LW}}.$$

Lemma 6.2.2. *The function $|f|\colon |\mathfrak{M}^m_{\mathrm{SWS}}(k)| \to |\mathfrak{M}^m_{\mathrm{LW}}(k)|$ induced by f is a bijection.*

Proof. As we have an inclusion of the SWS stable locus into the LW stable locus, we know that this map must be injective. It remains to show that it is surjective. Let us take any point in $|\mathfrak{M}^m_{\mathrm{LW}}(k)|$. This is given by the equivalence class of a pair $(Z_k, \mathcal{X}_k)$, where Z_k is a length m zero-dimensional subscheme in a fibre $\mathcal{X}_k$ over the point $\operatorname{Spec} k$. Either the pair $(Z_k, \mathcal{X}_k)$ is already SWS stable, in which case there is nothing left to prove, or we are in the following case. If $(Z_k, \mathcal{X}_k)$ is LW stable but not SWS stable then it must mean that there is at least a point of the support in each expanded Δ-component, but there is at least one Δ-component which is not expanded out which contains no point of the support. Let us say this Δ-component is equal to Y_i. But by the equivalences of the stack $\mathfrak{X}$ such a fibre is equivalent to a fibre where Y_i is not equal to any Δ-component. It will therefore be equivalent to a fibre in which every Δ-component contains at least one point of the support, which gives us an SWS stable fibre-subscheme pair. □

We will need also the following result from Alper and Kresch [AK16].

Lemma 6.2.3. *Let $\mathfrak{W}$ be a Deligne-Mumford stack with finite inertia, let $\mathfrak{Y}$ be an algebraic stack with separated diagonal and let $f\colon \mathfrak{W} \to \mathfrak{Y}$ be a morphism. Then the largest open substack $\mathfrak{U}$ of $\mathfrak{W}$ on which the restriction of f is a representable morphism enjoys the following characterisation: the geometric points of $\mathfrak{U}$ are precisely those at which f induces an injective homomorphism of stabiliser group schemes.*

Now we are in a position to prove the following theorem:

Theorem 6.2.4. *The map $f\colon \mathfrak{M}^m_{\mathrm{SWS}} \to \mathfrak{M}^m_{\mathrm{LW}}$ is an isomorphism of stacks.*

Proof. This can be seen by applying Lemma 6.2.1 to the map f. In order to do this we must show that this morphism is representable, with the help of Lemma 6.2.3. It follows directly from the fact that $\mathfrak{M}^m_{\mathrm{SWS}}$ is a separated Deligne-Mumford stack that it has finite inertia. By Lemma 6.2.2, the first condition of Lemma 6.2.1 is satisfied. And the map f defined above must also induce a bijective homomorphism of stabilisers since the only elements which can stabilise a family $(Z, \mathcal{X})$ in $\mathfrak{M}^m_{\mathrm{SWS}}$ or $\mathfrak{M}^m_{\mathrm{LW}}$ are elements of $\mathbb{G}^n_m$ (the other equivalences on $\mathfrak{X}$ do not stabilise any families as explained in Proposition 6.1.5) and, by construction, if a family $(Z, \mathcal{X})$ in $\mathfrak{M}^m_{\mathrm{SWS}}$ has stabiliser $\mathrm{Stab}_{(Z,\mathcal{X})} \subset \mathbb{G}^n_m$ then $f((Z, \mathcal{X}))$ must have the same stabiliser in $\mathbb{G}^n_m$. Lemma 6.2.1 therefore holds and f is an isomorphism of stacks.

□

7 A second expanded construction

In this section we present a second construction of a stack of expansions and its family, which we will denote by $\mathfrak{X}' \to \mathfrak{C}'$ and which will contain more choices of expansions of X. The SWS and LW stability conditions described in the previous sections may be extended to this setting and we will see that, because of these additional expansions, the stacks of stable SWS and LW objects will not separated here. Indeed, there will be too many stable limits for a single object. We will then explore how a separated stack may be constructed, either by identifying all possible limits of a same object in a non-algebraic stack, similar to ideas of Kennedy-Hunt in [Ken23], or by including an additional stability condition which cuts out a separated substack. These choices of proper substacks will give us more (though not all) examples of proper Deligne-Mumford stacks arising from the methods of Maulik and Ranganathan in [MR20].

7.1 Scheme construction

The blow-ups. Let $X \to C \cong \mathbb{A}^1$ be as before, a projective family of surfaces locally given by $\operatorname{Spec} k[x, y, z, t]/(xyz - t)$. We form a fibre product $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ in the same way as in Section 3.1.

Similarly to the first construction, we now make a sequence of blow-ups of Y_1 and Y_2 in $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ along the vanishing of certain basis directions, but we do not require here that both Y_1 and Y_2 should be blown up along all basis directions.

Definition 7.1.1. Let A and B be subsets of $[n+1] \coloneqq \{1, \dots, n+1\}$. We say that the triple (A, B, n) is *unbroken* if there are real intervals $[1, a)$ and $(b, n+1]$ with $1 < b < a < n+1$ such that $A = \mathbb{N} \cap [1, a)$ and $B = \mathbb{N} \cap (b, n+1]$.

In other words, if (A, B, n) is an unbroken triple, then elements of $[n+1]$ in order from 1 to $n+1$ are contained first in $A \setminus B$, then in $A \cap B$ and then in $B \setminus A$, where $A \cap B$ may be empty but the other two are not. In particular, A is forced to contain 1 but not $n+1$ and, similarly, B must contain $n+1$ but not 1.

Let (A, B, n) be an unbroken triple. We blow up $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ in the ideals

$$\langle x, t_1 \rangle, \langle x, t_1 t_2 \rangle, \dots, \langle x, t_1 t_2 \cdots t_{\lfloor a \rfloor} \rangle$$

and the ideals

$$\langle y, t_{n+1} \rangle, \langle y, t_{n+1} t_n \rangle, \dots, \langle y, t_{n+1} t_n \cdots t_{\lceil b \rceil} \rangle.$$

We denote the resulting space by $X[A, B]$. Note that if $A = \{1, \dots, n\}$ and $B = \{2, \dots, n+1\}$, then we get back exactly the space $X[n]$ of Section 3.1. We extend the terminology of *expanded components*, Δ-*components*, *bubbles*, etc. from Section 3 in the obvious way. We abuse notation slightly and again denote by $\pi \colon X[A, B] \to X$ the projection to the original family (it will always be clear from context to which map π we are referring). We will denote by $\Delta_1^{(i)}$ the component introduced by the blow-up of the ideal $\langle x, t_1 t_2 \cdots t_i \rangle$. Similarly we will denote by $\Delta_2^{(n+2-j)}$ the component introduced by the blow-up of the ideal $\langle y, t_{n+1} t_n \cdots t_j \rangle$. Finally, we set the notation $C[A, B]$ to refer to the base $C \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ together with the data of the sets A and B. This is the same scheme as $C[n]$ from Section 3, but here for each basis direction t_i we retain the extra information of whether i belongs to the set A or B. Before we continue describing the schemes $X[A, B]$, we make one additional definition.

Definition 7.1.2. Let $X[A, B]$ be a scheme as defined above. A Δ-component in a fibre of $X[A, B]$ is said to have Δ_i-*multiplicity* l if this component is equal to $\Delta_i^{(j)} = \cdots = \Delta_i^{(j+l)}$ for some j and it is not equal to any other Δ_i-components.

Note that when restricting to the setting of the construction $X[n]$ of Section 3, then the Δ_1- and Δ_2-multiplicity of a component will always be identical.

Now, let $(t) \in C[A, B]$ and let t_i and t_j be two consecutive zero entries of (t), i.e. $t_i = t_j = 0$ and $t_{i+1}, \dots, t_{j-1} \neq 0$. If both i and j are in A and $j \notin B$, then the component $\Delta_1^{(i)} = \cdots = \Delta_1^{(j-1)}$ is expanded out in the fibre of $X[A, B]$ over (t) and it is of pure type; there is no component Δ_2^{n+2-k} in $X[A, B]$ for $k \leq j$. Similarly, if both i and j are in B and $i \notin A$, then the component $\Delta_2^{(n+3-i)} = \cdots = \Delta_1^{(n+2-j)}$ is expanded out in the fibre of $X[A, B]$ over (t) and of pure type.

If $i \in A$ and $j \in B$, then $\Delta_1^{(i)} = \Delta_1^{(n+2-j)}$ is expanded out in the fibre of $X[A, B]$ over (t). The Δ_1-multiplicity of this component will be equal to the number of elements of the set $\{i, \dots, j-1\}$ which are contained in A. Similarly, the Δ_2-multiplicity of this component is given by the number of elements of the set $\{i+1, \dots, j\}$ contained in B.

Remark 7.1.3. The condition in Definition 7.1.1 is there to ensure that all modified special fibres have the expected base codimension. Indeed, if we allowed for example $i \in B \setminus A$ and $j \in A \setminus B$ with $i < j$ and the blow-ups as given above, then when $t_i = t_j = 0$ and all other basis directions are non-vanishing, we would see a copy of X_0. Then we would need to identify such a fibre with other copies of X_0 in our stack $\mathfrak{X}'$, but this would imply identifying fibres of different base codimension, which cannot be done through isomorphisms on the base. To avoid this unpleasantness, we therefore fix an ordering of the elements of $A \setminus B$, $A \cap B$ and $B \setminus A$.

Now, let $n' > n$, let (A', B', n') be an unbroken triple, and suppose that $|A| \leq |A'|$ and $|B| \leq |B'|$, where $|A \setminus B| = i$, $|A \cap B| = j - i$ and $|B \setminus A| = n - j$. In this case, we may define a natural inclusion

$$\iota \colon C[A, B] \hookrightarrow C[A', B'], \tag{7.1.1}$$

given by

$$\begin{array}{c} (t_1, \ldots, t_i, t_{i+1}, \ldots, t_j, t_{j+1} \ldots, t_n) \\ \big\downarrow \\ (t_1, \ldots, t_i, 1, \ldots, 1, t_{i+1}, \ldots, t_j, 1, \ldots, 1, t_{j+1} \ldots, t_n, 1, \ldots, 1). \end{array}$$

These, in turn, determine embeddings

$$X[A, B] \hookrightarrow X[A', B'].$$

The group action. We may define a natural torus action on $X[A, B]$ similarly to the first construction. Let $G \coloneqq \mathbb{G}_m^n$, where again we denote an element of this group $(\tau_1, \ldots, \tau_n)$. This group acts trivially on the original family X. If the component $\Delta_1^{(k)}$ exists in $X[A, B]$, then G acts on the coordinates of this component by

$$(x_0^{(k)} : x_1^{(k)}) \longmapsto (\tau_k x_0^{(k)} : x_1^{(k)}).$$

If the component $\Delta_2^{(k)}$ exists in $X[A, B]$, then G acts on it by

$$(y_0^{(k)} : y_1^{(k)}) \longmapsto (y_0^{(k)} : \tau_{n+1-k} y_1^{(k)}).$$

As before, this action induces a natural action on the base given by

$$\begin{aligned} t_1 &\longmapsto \tau_1^{-1} t_1, \\ t_k &\longmapsto \tau_k^{-1} \tau_{k-1} t_k, \\ t_{n+1} &\longmapsto \tau_n t_{n+1}. \end{aligned}$$

The morphisms $\pi \colon X[A, B] \to X$ and $X[A, B] \hookrightarrow X[A', B']$ are equivariant under the group action. Again, up to linear independence there is a single function on $C[A, B]$ which is G-invariant.

The G-linearisation. Similarly to Section 3.2, we may embed $X[A,B]$ into a product of projective bundles. This is done again by defining vector bundles

$$\mathcal{F}_1^{(i)} = \mathrm{pr}_1^* \mathcal{O}_X(-Y_1) \oplus \mathrm{pr}_2^* \mathcal{O}_{\mathbb{A}^{n+1}}(-V(t_i))$$
$$\mathcal{F}_2^{(j)} = \mathrm{pr}_1^* \mathcal{O}_X(-Y_2) \oplus \mathrm{pr}_2^* \mathcal{O}_{\mathbb{A}^{n+1}}(-V(t_{n+2-j}))$$

on $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ for $i \in A$ and $j \in B$ and repeating the steps of Section 3.2. We may then construct the following G-linearised ample line bundles on $X[A,B]$. Then we may extend Lemma 3.2.1 of Section 3.2 to this situation as follows.

Lemma 7.1.4. *There exists a G-linearised ample line bundle $\mathcal{L}$ on $X[A,B]$ such that locally the lifts to this line bundle of the G-action on each $\mathbb{P}^1$ corresponding to a $\Delta_1^{(i)}$ and on each $\mathbb{P}^1$ corresponding to a $\Delta_2^{(n+2-j)}$ are given by*

$$(x_0^{(i)}; x_1^{(i)}) \longmapsto (\tau_i^{a_i} x_0^{(i)}; \tau_i^{-b_i} x_1^{(i)}) \tag{7.1.2}$$
$$(y_0^{(n+2-j)}; y_1^{(n+2-j)}) \longmapsto (\tau_{j+1}^{-c_j} y_0^{(n+2-j)}; \tau_{j+1}^{d_j} y_1^{(n+2-j)}) \tag{7.1.3}$$

for any $i \in A$ and $j \in B$ and any choice of positive integers a_i, b_i, c_j, d_j.

Proof. This follows directly from Lemma 3.2.1. □

As before, we will apply the relative Hilbert-Mumford criterion of [GHH15] to define GIT stability conditions with respect to the G-linearised line bundles we defined. We leave this discussion for Section 7.3.

7.2 The stack of expansions and its family

Here we define a second version of the stack of expansions and its family, building on the ideas of Section 5.

The stack $\mathfrak{C}'$. In the following we define the stack of expansions $\mathfrak{C}'$. This will differ slightly from the construction of Li and Wu in that we will want to remember what components of the special fibre X_0 are blown up along which basis directions.

Let us consider $\mathbb{A}^{n+1}$ with its natural torus action $\mathbb{G}_m^n$ as before and denote by $(\mathbb{A}^{n+1})_{[A,B]}$ this space enhanced by a choice of sets A and B such that (A,B,n) is unbroken. We label elements of $(\mathbb{A}^{n+1})_{[A,B]}$ by $(t_1,\dots,t_{n+1})$ as before and, for any k, we define the set

$$[k] \coloneqq \{1,\dots,k\}.$$

Let $J \subseteq [n+1]$ and let $J^\circ = [n+1] \setminus J$ be its complement. We denote the cardinality of J by $r \coloneqq |J|$ and define

$$\mathrm{ind}_J \colon [r] \longrightarrow J \quad \text{and}$$
$$\mathrm{ind}_{J^\circ} \colon \{r+1,\dots,n+1\} \longrightarrow J^\circ$$

to be the order preserving isomorphisms.

Remark that though the map ind_J is defined identically to the map of the same name in Section 5, the map ind_{J° is given a different definition here. This modification is introduced so that we may keep track of which indices of the entries of $(t_1, \ldots, t_{n+1})$ lie in A or B in the following isomorphisms.

Let

$$(\mathbb{A}^{n+1}_J)_{[A,B]} = \{(t) \in (\mathbb{A}^{n+1})_{[A,B]} |\ t_i = 0,\ i \in J\} \subset (\mathbb{A}^{n+1})_{[A,B]}$$

and

$$(\mathbb{A}^{n+1}_{U(J)})_{[A,B]} = \{(t) \in (\mathbb{A}^{n+1})_{[A,B]} |\ t_i \neq 0,\ i \in J^\circ\} \subset (\mathbb{A}^{n+1})_{[A,B]}.$$

Now, given $(\mathbb{A}^{n+1}_{U(J)})_{[A,B]}$, we define a corresponding space $(\mathbb{A}^r \times \mathbb{G}^{n+1-r}_m)_{[\epsilon(A),\epsilon(B)]}$ by specifying the sets $\epsilon(A)$ and $\epsilon(B)$ in the following way. If $\mathrm{ind}_J(1) \in A \cap B$, then let $1 \in \epsilon(A)$. Then, for $i > 1$, if $\mathrm{ind}_J(i) \in A \cap B$ then $i \in \epsilon(A) \cap \epsilon(B)$. For $i \geq 1$, if $\mathrm{ind}_J(i) \in A \setminus B$ then $i \in \epsilon(A) \setminus \epsilon(B)$ and if $\mathrm{ind}_J(i) \in B \setminus A$ then $i \in \epsilon(B) \setminus \epsilon(A)$. Finally, let $i \in \epsilon(B) \setminus \epsilon(A)$ for all $i > r$. We note that $(\epsilon(A), \epsilon(B), n)$ is unbroken. We may then define the isomorphism

$$\tau^{[A,B]}_J \colon (\mathbb{A}^r \times \mathbb{G}^{n+1-r}_m)_{[\epsilon(A),\epsilon(B)]} \longrightarrow (\mathbb{A}^{n+1}_{U(J)})_{[A,B]}$$

given by

$$(a_1, \ldots, a_r, \sigma_{r+1}, \ldots, \sigma_{n+1}) \longrightarrow (t_1, \ldots, t_{n+1}),$$

where

$$t_i = a_j,\ \text{if}\ \ \mathrm{ind}_J(j) = i,$$
$$t_i = \sigma_j,\ \text{if}\ \ \mathrm{ind}_{J^\circ}(j) = i.$$

Now, given two triples (A, B, J) and (A', B', J') such that $|J| = |J'|$ and such that $\epsilon(A) = \epsilon(A')$ and $\epsilon(B) = \epsilon(B')$, we define an isomorphism

$$\tau^{J',[A',B']}_{J,[A,B]} = \tau^{[A,B]}_J \circ (\tau^{[A',B']}_{J'})^{-1} \colon (\mathbb{A}^{n+1}_{U(J')})_{[A',B']} \longrightarrow (\mathbb{A}^{n+1}_{U(J)})_{[A,B]}.$$

Now, we define a second set of isomorphisms similarly to the above. Given any $(\mathbb{A}^{n+1}_{U(J)})_{[A,B]}$, we may define an alternative space $(\mathbb{A}^r \times \mathbb{G}^{n+1-r}_m)_{[\delta(A),\delta(B)]}$ by describing sets $\delta(A)$ and $\delta(B)$ in the following way. If $\mathrm{ind}_J(r) \in A \cap B$, then let $r \in \delta(B)$. For $i < r$, if $\mathrm{ind}_J(i) \in A \cap B$, then let $i \in \delta(A) \cap \delta(B)$. For $i \leq r$, if $\mathrm{ind}_J(i) \in A \setminus B$, then let $i \in \delta(A) \setminus \delta(B)$ and if $\mathrm{ind}_J(i) \in B \setminus A$ let $i \in \delta(B) \setminus \delta(A)$. Again, the triple $(\delta(A), \delta(B), n)$ is unbroken. We then define an isomorphism

$$\rho^{[A,B]}_J \colon (\mathbb{A}^r \times \mathbb{G}^{n+1-r}_m)_{[\delta(A),\delta(B)]} \longrightarrow (\mathbb{A}^{n+1}_{U(J)})_{[A,B]}$$

given by

$$(a_1, \ldots, a_r, \sigma_{r+1}, \ldots, \sigma_{n+1}) \longrightarrow (t_1, \ldots, t_{n+1}),$$

where

$$t_i = a_j,\ \text{if}\ \ \mathrm{ind}_J(j) = i,$$
$$t_i = \sigma_j,\ \text{if}\ \ \mathrm{ind}_{J^\circ}(j) = i.$$

As above, given two triples (A, B, J) and (A', B', J') such that $|J| = |J'|$ and such that $\delta(A) = \delta(A')$ and $\delta(B) = \delta(B')$, we may then specify an isomorphism

$$\rho^{J',[A',B']}_{J,[A,B]} = \rho^{[A,B]}_{J} \circ (\rho^{[A',B']}_{J'})^{-1} \colon (\mathbb{A}^{n+1}_{U(J')})_{[A',B']} \longrightarrow (\mathbb{A}^{n+1}_{U(J)})_{[A,B]}.$$

Finally, given (A, B, n), we define $\mathfrak{U}^{A,B}$ to be the quotient $[(\mathbb{A}^{n+1})_{[A,B]}/\sim]$ by the equivalences generated by the $\mathbb{G}^n_m$-action and the equivalences $\tau^{J',[A',B']}_{J,[A,B]}$ and $\rho^{J',[A',B']}_{J,[A,B]}$ for compatible triples (A, B, J) and (A', B', J'). Recall from Section 7.1 that we had natural inclusions (7.1.1)

$$C[A, B] \longhookrightarrow C[A'', B''],$$

for all A'' and B'' such that (A'', B'', n'') is unbroken for some $n'' > n$ and such that $|A| \leq |A''|$ and $|B| \leq |B''|$. These induce open immersions of stacks

$$\mathfrak{U}^{A,B} \longrightarrow \mathfrak{U}^{A',B'}.$$

Let $\mathfrak{U}' \coloneqq \varinjlim \mathfrak{U}^{A,B}$ be the direct limit over unbroken (A, B, n) and let $\mathfrak{C}' \coloneqq C \times_{\mathbb{A}^1} \mathfrak{U}'$.

We now make a few remarks concerning the isomorphisms described in this section. These isomorphisms are effectively a reordering of the basis directions, where the t_i's whose indices lie in J preserve their order relative to each other and the corresponding indices preserve their A, B labelling, apart in certain cases for the first and last index in J. As we will see in the proof of Proposition 7.2.1, these are exactly the isomorphisms on the base which correspond to isomorphisms of the corresponding fibres above these elements.

Finally, we note that if $(\mathbb{A}^{n+1}_{U(J)})_{[A,B]}$ is such that $\mathrm{ind}_J(1) \in A \cap B$, then as well as being isomorphic to the space $(\mathbb{A}^r \times \mathbb{G}^{n+1-r}_m)_{[\epsilon(A),\epsilon(B)]}$ defined above, it is also isomorphic to a spaces $(\mathbb{A}^{n+1}_{U(J')})_{[A',B']}$ where $\mathrm{ind}_{J'}(1) \in A' \cap B'$ and the equalities $\epsilon(A) = \epsilon(A')$ and $\epsilon(B) = \epsilon(B')$ hold.

The stack $\mathfrak{X}'$. Let $X[A, B] \to C[A, B]$ be a construction as described in Section 7.1 and recall that $\pi \colon X[A, B] \to X$ is the projection to the original family. Let $n' > n$, let (A', B', n') be unbroken, and let $|A| \leq |A'|$ and $|B| \leq |B'|$. In such a case, we have the natural inclusion (7.1.1)

$$\iota \colon C[A, B] \longhookrightarrow C[A', B']$$

Then the induced family $(\iota^* X[A', B'], \iota^* \pi)$ is isomorphic to $(X[A, B], \pi)$ over $C[A, B]$. The following proposition shows that the equivalences on $\mathfrak{U}^{A,B}$ lift to C-isomorphisms of fibres. Let $X[A, B]_{U(J)}$ and $X[A, B]_J$ be the restrictions of $X[A, B]$ to $(\mathbb{A}^{n+1}_{U(J)})_{[A,B]}$ and $(\mathbb{A}^{n+1}_{J})_{[A,B]}$ respectively.

Proposition 7.2.1. *The schemes $X[A, B]_{U(J)}$ and $X[A', B']_{U(J')}$ are isomorphic if and only if $(\mathbb{A}^{n+1}_{U(J)})_{[A,B]}$ and $(\mathbb{A}^{n+1}_{U(J')})_{[A',B']}$ are related by the isomorphisms $\tau^{J',[A',B']}_{J,[A,B]}$ and $\rho^{J',[A',B']}_{J,[A,B]}$.*

Proof. It is clear that $X[A,B]_{U(J)}$ and $X[A',B']_{U(J')}$ are isomorphic on $\pi^{-1}_{[A,B]}(X^\circ) \cong \pi^{-1}_{[A',B']}(X^\circ)$, i.e. on the locus where all basis directions are nonzero. We must show that for each modified special fibre in $X[A,B]_{U(J)}$ there is an isomorphic fibre in $X[A',B']_{U(J')}$ if and only if the corresponding bases are related by $\tau^{J',[A',B']}_{J,[A,B]}$ and $\rho^{J',[A',B']}_{J,[A,B]}$.

First, assume there is a sequence of $\tau^{J',[A',B']}_{J,[A,B]}$ and $\rho^{J',[A',B']}_{J,[A,B]}$ isomorphisms such that $(\mathbb{A}^{n+1}_{U(J)})_{[A,B]} \cong (\mathbb{A}^{n+1}_{U(J')})_{[A',B']}$. Since $|J| = |J'| = r$, this implies that, for $1 < i < r$, the element $\mathrm{ind}_J(i)$ belongs to $A \setminus B$ if and only if $\mathrm{ind}_{J'}(i)$ belongs to $A' \setminus B'$. The same goes for the sets $A \cap B$ with $A' \cap B'$ and $B \setminus A$ with $B' \setminus A'$.

Next, we note that the placement of the nonzero entries in $(t_1, \ldots, t_{n+1})$ influences the Δ-multiplicity of the components but has no effect on how many components are expanded out in the fibre above this point or whether these expanded components are of type Δ_1 or type Δ_2. Moreover, changing the first zero entry of $(t_1, \ldots, t_{n+1})$ from an element of $A \setminus B$ to an element of $A \cap B$ or vice versa also does not affect which components are expanded out in this fibre. The same is true of changing the last zero entry of $(t_1, \ldots, t_{n+1})$ from $B \setminus A$ to $A \cap B$ or vice versa. This can be easily seen by studying the equations of the blow-ups. It therefore must follow that a sequence of isomorphisms $\tau^{J',[A',B']}_{J,[A,B]}$ and $\rho^{J',[A',B']}_{J,[A,B]}$, as they only modify the base in ways that preserve the fibres, induces an isomorphism $X[A,B]_{U(J)} \cong X[A',B']_{U(J')}$.

Conversely, let us assume that all fibres in $X[A,B]_{U(J)}$ are isomorphic to fibres in $X[A',B']_{U(J')}$. This means that for every fibre of $X[A,B]_{U(J)}$ over a point $(t_1, \ldots, t_{n+1})$ of the base $(\mathbb{A}^{n+1}_{U(J)})_{[A,B]}$, there exists a fibre of $X[A',B']_{U(J')}$ over a point $(t'_1, \ldots, t'_{n+1})$ of the base $(\mathbb{A}^{n+1}_{U(J')})_{[A',B']}$ such that the exact same number of components are expanded out in the same loci of both fibres and these components must have the same type. Clearly, there must be the same amount of zero entries in $(t_1, \ldots, t_{n+1})$ and $(t'_1, \ldots, t'_{n+1})$; call this number k. Moreover, for $k > 1$, by studying the equations of the blow-ups we can see that for the above statement to be true, the first $k-1$ zeroes of $(t_1, \ldots, t_{n+1})$ must have indices belonging to A if and only if the first $k-1$ zeroes of $(t'_1, \ldots, t'_{n+1})$ have indices belonging to A'; and the last $k-1$ zeroes of $(t_1, \ldots, t_{n+1})$ must have indices belonging to B if and only if the last $k-1$ zeroes of $(t'_1, \ldots, t'_{n+1})$ have indices belonging to B'.

If the index of the first zero in $(t_1, \ldots, t_{n+1})$ is not contained in A, then it must be contained in $B \setminus A$ and all expanded components in the fibre above this point are of type Δ_2. This can only be isomorphic to the fibre above the point $(t'_1, \ldots, t'_{n+1})$ if the indices of all zero entries in $(t'_1, \ldots, t'_{n+1})$ are contained in $B' \setminus A'$. If the index of the first zero in $(t_1, \ldots, t_{n+1})$ is contained in A and the indices of all subsequent zeroes are contained in B, however, then whether or not this first index is contained in B has no effect on the components expanded out in this fibre. In conclusion, if the fibres above points $(t_1, \ldots, t_{n+1})$ and $(t'_1, \ldots, t'_{n+1})$ are isomorphic, then the index of the first zero entry of $(t_1, \ldots, t_{n+1})$ is in $B \setminus A$ if and only if the index of the first zero entry of $(t'_1, \ldots, t'_{n+1})$ is in $B' \setminus A'$; and

if the index of the first zero entry of $(t_1, \ldots, t_{n+1})$ is in $A \cap B$ then the index of the first zero entry of $(t'_1, \ldots, t'_{n+1})$ is in either of $A' \setminus B'$ or $A' \cap B'$. A similar reasoning holds with respect to the index of the last zero entry being contained in $B \setminus A$ or $A \cap B$. Indeed, if this index is contained in $A \cap B$, then $\mathrm{ind}_{J'}(k)$ may be contained in $A' \cap B'$ or $B' \setminus A'$. But these are exactly the elements of the base which are related by the isomorphisms $\tau^{J',[A',B']}_{J,[A,B]}$ and $\rho^{J',[A',B']}_{J,[A,B]}$. □

We define $\mathfrak{X}^{A,B}$ to be the quotient $[X[A,B]/\!\sim]$ by the equivalences generated by the $\mathbb{G}_m^n$-action and equivalences lifted from $\mathfrak{U}^{A,B}$. There are natural immersions of stacks

$$\mathfrak{X}^{A,B} \hookrightarrow \mathfrak{X}^{A',B'},$$

induced by the immersions $\mathfrak{U}^{A,B} \hookrightarrow \mathfrak{U}^{A',B'}$. Finally, we define $\mathfrak{X} = \varinjlim \mathfrak{X}^{A,B}$ to be the direct limit over all unbroken triples (A, B, n). It is an Artin stack.

7.3 Stability conditions

We now discuss stability conditions on the stack $\mathfrak{X}'$. As in Section 5.2, we will give two perspectives, one coming from the GIT stability conditions arising from the G-linearised line bundles we constructed in Section 7.1 and one which relates to the stability conditions presented by Li and Wu in [LW15] and Maulik and Ranganathan in [MR20].

Let (A, B, n) be an unbroken triple and denote by

$$H^m_{[A,B]} \coloneqq \mathrm{Hilb}^m(X[A,B]/C[A,B])$$

the relative Hilbert scheme of m points. Recall that in Section 7.1, we discussed possible choices for a G-linearised ample line bundle $\mathcal{L}$ on $X[A,B]$. As before, this will induce a G-linearised ample line bundle on $H^m_{[A,B]}$ in the following way. Let

$$Z^m_{[A,B]} \subset H^m_{[A,B]} \times_{C[A,B]} X[A,B]$$

be the universal family, with first and second projections p and q. The line bundle

$$\mathcal{M}_l \coloneqq \det p_*(q^*\mathcal{L}^{\otimes l}|_{Z^m_{[A,B]}})$$

is relatively ample when $l \gg 0$ and is G-linearised, by the same argument as in [GHH19].

Now, take a point $[Z] \in H^m_{[A,B]}$ and a 1-PS λ_s for $s \in \mathbb{Z}^n$ such that the limit of $\lambda_s(\tau) \cdot Z$ as τ tends to zero exists; we denote it by Z_0. We can write Z_0 as a union of length m_P zero-dimensional subschemes $\bigcup_P Z_{0,P}$ supported at points P. Let $\mathcal{L}^{\otimes l}(P)$ denote the fibre of $\mathcal{L}^{\otimes l}$ at P. Then, as before, at the point Z_0, the line bundle $\mathcal{M}_l$ is given by the left hand side of the following isomorphism

$$\bigwedge^m H^0(\mathcal{O}_{Z_0} \otimes \mathcal{L}^{\otimes l}) \cong \Big(\bigwedge^m H^0(\mathcal{O}_{Z_0})\Big) \otimes \Big(\bigotimes_P \mathcal{L}^{\otimes l m_P}(P)\Big)$$

and the Hilbert-Mumford invariant $\mu^{\mathcal{M}_l}(Z, \lambda_s)$ is given by the negative of the weight of the $\mathbb{G}_m$-action on the line bundle $\mathcal{M}_l$ at this point Z_0. There are again bounded and combinatorial weights in the sense of [GHH19], which we denote by $\mu_b^{\mathcal{M}_l}(Z, \lambda_s)$ and $\mu_c^{\mathcal{M}_l}(Z, \lambda_s)$ respectively. As before, we may deduce the following equality

$$\mu^{\mathcal{M}_l}(Z, \lambda_s) = \mu_b^{\mathcal{M}_1}(Z, \lambda_s) + l \cdot \mu_c^{\mathcal{M}_1}(Z, \lambda_s).$$

Again, the bounded weight has an upper bound by the same reasoning as the proof of Lemma 4.3.1.

Lemma 7.3.1. *Let Z be a length m zero-dimensional subscheme in a fibre of $X[A, B]$ such that no point of the support of Z lies in $(\Delta_1^{(k)} \cup \Delta_2^{(n+1-k)})^\circ$ (if either of these Δ-components does not exist in $X[A, B]$ replace it by $\emptyset$). There is no GIT stability condition on $H^m_{[A,B]}$ for which Z is stable.*

Proof. This follows directly from the proof of Lemma 4.4.2. □

We extend the concept of LW stability to our current setting similarly to before.

Definition 7.3.2. Let Z be a length m zero-dimensional subscheme in a fibre of $X[A, B]$. We say that Z is *LW stable* if Z has smooth support and finite automorphisms. Denote the LW stable locus by $H^m_{[A,B],\mathrm{LW}}$.

Remark 7.3.3. In the case of Hilbert schemes of points it is sufficient to describe LW stability in this simplified way, but if we want to generalise these results to Hilbert schemes with non-constant polynomials we will need to take the original definition of Li-Wu stability which we described in Section 5.2.

We also extend the definition of SWS stability to this setting in the obvious way as follows.

Definition 7.3.4. Let Z be a length m zero-dimensional subscheme in a fibre of $X[A, B]$. We say that Z is *weakly strictly stable* if there exists any G-linearised ample line bundle $\mathcal{M}'$ on $H^m_{[A,B]}$ with respect to which Z is GIT stable. The term *strictly* is used here to emphasize that we exclude strictly semistable points. If Z also has smooth support in the fibre, then we say it is *smoothly weakly strictly stable* (abbreviated SWS stable as before) and we denote the SWS stable locus by $H^m_{[A,B],\mathrm{SWS}}$.

We then have the inclusion

$$H^m_{[A,B],\mathrm{SWS}} \subset H^m_{[A,B],\mathrm{LW}}$$

since, if points are GIT stable, they must have finite stabilisers. We now extend our definition of SWS stability to the stack $\mathfrak{X}'$.

Given any C-scheme S, an object of $\mathfrak{X}'(S)$ is a pullback family $\xi^* X[A, B]$ for a morphism

$$\xi \colon S \to C[A, B].$$

Definition 7.3.5. A pair $(Z, \mathcal{X}) \subset \mathfrak{X}'(S)$ is said to be *SWS stable* if and only if $\mathcal{X} \coloneqq \xi^* X[A,B]$ for some morphism $\xi\colon S \to C[A,B]$, where Z has smooth support in $\mathcal{X}$ and there exists some G-linearised ample line bundle on $H^m_{[A,B]}$ which makes Z be GIT stable.

Lemma 7.3.6. *Let Z be a length m zero-dimensional subscheme in a fibre $\mathfrak{X}'(k)$ as above. If there is at least one point of the support of Z in the union $(\Delta_1^{(i)})^\circ \cup (\Delta_2^{(n+1-i)})^\circ$ for every i for which either of these components is expanded out, then $(Z, \mathfrak{X}'(k))$ is SWS stable.*

Proof. This follows directly from Lemma 4.4.1. □

Let us denote by $\mathfrak{N}^m_{\mathrm{SWS}}$ and $\mathfrak{N}^m_{\mathrm{LW}}$ the stacks of SWS and LW stable length m zero-dimensional subschemes in $\mathfrak{X}'$ respectively. Let S be a C-scheme. An object of $\mathfrak{N}^n_{\mathrm{SWS}}(S)$ is defined to be a pair $(Z, \mathcal{X})$, where $\mathcal{X} \in \mathfrak{X}'(S)$ and Z is an S-flat SWS stable family in $\mathcal{X}$. Similarly, an object of $\mathfrak{N}^n_{\mathrm{LW}}(S)$ is a pair $(Z, \mathcal{X})$, where $\mathcal{X} \in \mathfrak{X}'(S)$ and Z is an S-flat LW stable family in $\mathcal{X}$.

Remark 7.3.7. It does not make sense here to speak of Maulik-Ranganathan stability on the stack $\mathfrak{X}'$ as we have not yet defined a notion of tube component and choices of limit representatives. Unlike the stacks $\mathfrak{M}^m_{\mathrm{SWS}}$ and $\mathfrak{M}^m_{\mathrm{LW}}$, the stacks $\mathfrak{N}^m_{\mathrm{SWS}}$ and $\mathfrak{N}^m_{\mathrm{LW}}$ are universally closed but not separated, so here Li-Wu stability and Maulik-Ranganathan stability do not coincide. Indeed, as we will see, in order to construct separated stacks we will need to select one of two options. The first is to identify all choices of representatives for a limit, which will yield a stack which is no longer algebraic. This parallels work of Kennedy-Hunt [Ken23]. The second is to make a choice of substack which picks out exactly one representative for each limit and these substacks will be isomorphic to the underlying stacks produced by the methods of Maulik and Ranganathan in [MR20] (the word underlying here is used to signify that we do not speak of logarithmic structures).

7.4 Universal closure

Let $S \coloneqq \operatorname{Spec} R \to C$, where R is some discrete valuation ring and η denotes the generic point of S. Let $(Z_\eta, \mathcal{X}_\eta)$ be a pair such that $\mathcal{X}_\eta \in \mathfrak{X}'(\eta)$ and Z_η is a length m zero-dimensional subscheme in $\mathcal{X}_\eta$. We adjust the notion of an *extension* of the pair $(Z_\eta, \mathcal{X}_\eta)$ to the closed point of S from Definition 6.1.1 to the stack $\mathfrak{X}'$ in the obvious way.

We will now show that the stacks $\mathfrak{N}^m_{\mathrm{SWS}}$ and $\mathfrak{N}^m_{\mathrm{LW}}$ are universally closed. The proof of this proposition will illustrate also the fact that neither stack is separated, and where these different choices of limit representatives for a family of length m zero-dimensional subschemes occur.

Proposition 7.4.1. *The stack $\mathfrak{N}^m_{\mathrm{SWS}}$ is universally closed.*

Proof. Let $S \coloneqq \operatorname{Spec} R \to C$, where R is some discrete valuation ring with uniformising parameter w and quotient field k. We denote by η and η_0 the generic

and closed points of S respectively. Let $(Z, \mathcal{X})$ be an S-flat family of length m zero-dimensional subschemes such that $\mathcal{X} \in \mathfrak{X}'(S)$ and $(Z_\eta, \mathcal{X}_\eta) \in \mathfrak{N}^m_{\mathrm{SWS}}(\eta)$. As in the proof of Proposition 6.1.2, we show that there exists a finite base change $S' \coloneqq \operatorname{Spec} R' \to S$, for some discrete valuation ring R' and a pair $(Z', \mathcal{X}') \in \mathfrak{N}^m_{\mathrm{SWS}}(S')$ satisfying the following condition. We denote by η' and η'_0 the generic and closed points of S' respectively. Then S' and $(Z', \mathcal{X}')$ are chosen such that we have an equivalence $\mathcal{X}'_{\eta'} \cong \mathcal{X}_\eta \times_\eta \eta'$ which induces an equivalence $Z'_{\eta'} \cong Z_\eta \times_\eta \eta'$.

We start by repeating exactly the procedure of Proposition 6.1.2 in order to find an expression (6.1.2). Note that this part of the proof can be adapted easily to our current situation as all results up to the expression (6.1.2) still hold if we replace $\mathfrak{X}$ by $\mathfrak{X}'$ and $\mathfrak{M}^m_{\mathrm{SWS}}$ by $\mathfrak{N}^m_{\mathrm{SWS}}$. We thus construct

$$(t_1, \dots, t_{n+1}) = (f_1 u^{g_1}, \dots, f_{n+1} u^{g_{n+1}}) \in \mathbb{A}^{n+1}, \tag{7.4.1}$$

where u is the uniformising parameter of R' and $f_i \in (R')^\times$. It has been constructed, as in the proof of Proposition 6.1.2, such that the associated pullback of some scheme $X[A, B]$ to S' defines the family $\mathcal{X}'$. In this case, however, there is a choice to be made in how A and B are defined, as we do not necessarily blow up both x and y along every basis directions in this construction. We will see that this choice is non-unique.

We will now describe the possible choices of A and B for each Z' and study the corresponding stable loci. Recall from the proof of Proposition 6.1.2 that the subscheme Z' is a union of irreducible components Z'_i and that u was chosen such that each Z'_i can be written locally in terms of its x, y and z coordinates as

$$\{(c_{i,1} u^{e_{i,1}}, c_{i,2} u^{e_{i,2}}, c_{i,3} u^{e_{i,3}})\}, \tag{7.4.2}$$

for some $e_{i,j} \in \mathbb{Z}$ and $c_{i,j} \in (R')^\times$. As Z' is a flat family given by the above expression, it must satisfy the equations (6.1.6), (6.1.7) and (6.1.8). Part of our stability conditions is to require smooth support, so, as in the proof of Proposition 6.1.2, if more than one element of the set $\{e_{i,1}, e_{i,2}, e_{i,3}\}$ is nonzero, then $Z'_{\eta'_0}$ must be supported in a component blown up along the vanishing of both sides of the relevant equations (6.1.6), (6.1.7) or (6.1.8).

In the first construction there was no choice in how these blow-ups were made, but here there is some choice.

First, we discuss limits where the choice is unique. If $e_{i,1}$ and $e_{i,3}$ are nonzero but $e_{i,2} = 0$, then the support of $(Z'_i)_{\eta'_0}$ lies in the $\pi^*(Y_1 \cap Y_3)^\circ$ locus. In our construction there is only one way of expanding a Δ-component in this locus which will contain the support of $(Z'_i)_{\eta'_0}$ in its interior, namely that which in the localisation is given by blowing up the ideal $\langle x, u^{e_{i,1}} \rangle$. There is exactly one j such that $t_1 \cdots t_j = c u^{e_{i,1}}$, for some $c \in (R')^\times$, where t_k are as in the expression (7.4.1). It follows that $j \in A$.

Similarly, if $e_{i,2}$ and $e_{i,3}$ are nonzero but $e_{i,1} = 0$, then the support of $(Z'_i)_{\eta'_0}$ lies in the $\pi^*(Y_2 \cap Y_3)^\circ$ locus. Again, there is only one way of expanding a Δ-component in this locus such that the flat limit of $(Z'_i)_{\eta'}$ is contained in the

interior of this Δ-component. In the localisation, this is given by the blow-up of the ideal $\langle y, u^{e_{i,2}}\rangle$. Now, as in the previous example, there is exactly one j such that $t_j \cdots t_{n+1} = cu^{e_{i,2}}$, for some $c \in (R')^\times$, where t_k are as in the expression (7.4.1) (the values of c and j may, of course, differ from those in the first example). It follows that $j \in B$.

Finally, if $e_{i,1}$, $e_{i,2}$ and $e_{i,3}$ are all nonzero then the support of $(Z_i')_{\eta_0'}$ lies in the $\pi^*(Y_1 \cap Y_2 \cap Y_3)$ locus which is pulled back from a codimension 2 locus in X_0. This implies that in order for $(Z_i')_{\eta_0'}$ to have smooth support, it must be contained in a $\mathbb{P}^1 \times \mathbb{P}^1$ bubble, given by blowing up both ideals $\langle x, u^{e_{i,1}}\rangle$ and $\langle y, u^{e_{i,2}}\rangle$. If j is such that $t_1 \cdots t_j = cu^{e_{i,1}}$ and k is such that $t_k \cdots t_{n+1} = cu^{e_{i,2}}$, then we must include both $j \in A$ and $k \in B$.

Now, on the other hand, if $e_{i,1}$ and $e_{i,2}$ are nonzero but $e_{i,3} = 0$, then the support of $(Z_i')_{\eta_0'}$ lies in the $\pi^*(Y_1 \cap Y_2)^\circ$ locus and there are several possible ways to expand out a component in this locus which would contain $(Z_i')_{\eta_0'}$ in its interior. Indeed, $(Z_i')_{\eta_0'}$ may lie in a Δ_1- or Δ_2-component of pure type or in a component of mixed type $\Delta_1 = \Delta_2$. In other words, if j is such that $t_1 \cdots t_j = cu^{e_{i,1}}$, for some $c \in (R')^\times$ and therefore $t_{j+1} \cdots t_{n+1} = du^{e_{i,2}}$, for some $d \in (R')^\times$, then we may pick any of the following three options

$$\begin{aligned} j \in A \quad &\text{and} \quad j+1 \notin B, \\ j \notin A \quad &\text{and} \quad j+1 \in B, \\ j \in A \quad &\text{and} \quad j+1 \in B. \end{aligned}$$

Of course, whether or not j and $j+1$ are contained in A and B will depend also on the other Z_i and will this choices is constrained by the need for (A, B, n) to be unbroken, so for certain Z, some of these choices are removed.

For any of the above compatible choices, the limit pair $(Z'_{\eta_0}, \mathcal{X}'_{\eta_0})$ is such that at least one point of the support of Z'_{η_0} is in the union $(\Delta_1^{(i)})^\circ \cup (\Delta_2^{(n+1-i)})^\circ$ for every i for which either of these components exists in $\mathcal{X}'_{\eta_0}$. By Lemma 7.3.6, this limit is SWS stable.

This shows that if $(Z_\eta, \mathcal{X}_\eta)$ is pulled back from a fibre above a point $(t_1, \ldots, t_{n+1})$ in some $C[A, B]$ whose entries are all invertible, then $(Z_\eta, \mathcal{X}_\eta)$ has an SWS stable extension. See Corollary 7.5.11 for a proof that there exists an extension if $\mathcal{X}_\eta$ is a modified special fibre. □

Corollary 7.4.2. *The stack $\mathfrak{M}^m_{\mathrm{LW}}$ is universally closed.*

Proof. As SWS stable pairs must be LW stable, this follows immediately. □

Remark 7.4.3. In the proof of Proposition 7.4.1, we saw how the stacks $\mathfrak{M}^m_{\mathrm{SWS}}$ and $\mathfrak{M}^m_{\mathrm{LW}}$ fail to be separated by highlighting several possible choices of limit representatives. We should note also that it is possible for such a pair $(Z'_{\eta_0}, \mathcal{X}'_{\eta_0})$ to have finite automorphisms while having expanded Δ-components in $\mathcal{X}'_{\eta_0}$ which contain no point of the support of Z'_{η_0}. This may happen for example if, in the expression (7.4.2), both $e_{i,1}$ and $e_{i,3}$ are nonzero but $e_{i,2} = 0$. Then, in order for

Z'_{η_0} to have smooth support in $\mathcal{X}'_{\eta_0}$, we must let $j \in A$ (using the notation of the proof) so as to expand out the component $\Delta_1^{(j)}$; but we may also let $j+1 \in B$ so as to have the component $\Delta_2^{(n+1-j)}$ expanded out in $\mathcal{X}'_{\eta_0}$, even if the latter contains no point of the support. As the same $\mathbb{G}_m$ acts on $\Delta_1^{(j)}$ and $\Delta_2^{(n+1-j)}$, this will not add any automorphisms. In fact, this is exactly the type of blow-up we made in Section 3 for the first construction. Clearly, such a pair is LW stable, and it is also SWS stable by arguments of Lemmas 4.3.2, 4.3.6 and 4.4.1.

7.5 Constructing separated stacks.

We are now in a position to construct separated stacks. As mentioned earlier, there are two ways of doing this. Firstly, one can avoid making any choices and instead identify all possible limits of a given family in the stacks $\mathfrak{N}^m_{\mathrm{SWS}}$ and $\mathfrak{N}^m_{\mathrm{LW}}$; the resulting stacks will be in a sense minimal. We denote these by $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}$ and $\overline{\mathfrak{N}^m_{\mathrm{LW}}}$ respectively. As we will see, this comes at the cost of these stacks no longer being algebraic. The second way to obtain separated stacks is to make a systematic choice of representative for a given limit. This will be done by adding an additional stability condition.

The stacks $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}$ and $\overline{\mathfrak{N}^m_{\mathrm{LW}}}$. Let $S \coloneqq \operatorname{Spec} R \to C$ for some valuation ring R as before and let η and η_0 denote the generic and closed points of S. We construct the separated stacks $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}$ and $\overline{\mathfrak{N}^m_{\mathrm{LW}}}$ from the stacks $\mathfrak{N}^m_{\mathrm{SWS}}$ and $\mathfrak{N}^m_{\mathrm{LW}}$ by introducing additional equivalences on these stacks. It is sufficient to define these equivalences over the scheme S as we already have a well-defined notion of equivalence up to base change on these stacks.

Let $(Z_\eta, \mathcal{X}_\eta) \in \mathfrak{N}^m_{\mathrm{SWS}}(\eta)$ (or $\mathfrak{N}^m_{\mathrm{LW}}(\eta)$), where all basis directions are invertible at the point η. Then $(Z_\eta, \mathcal{X}_\eta)$ will also be an element of $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}(\eta)$ (or $\overline{\mathfrak{N}^m_{\mathrm{LW}}}(\eta)$). We then define to be equivalent in $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}$ (or $\overline{\mathfrak{N}^m_{\mathrm{LW}}}$) all stable (for the respective stability conditions) extensions of this pair.

Another way to understand this, closer to the language of Maulik and Ranganathan, is the following. Take $(Z, \mathcal{X}) \in \mathfrak{N}^m_{\mathrm{SWS}}(S)$, where now η is not necessarily pulled back from a point with only invertible basis directions (i.e. $\mathcal{X}_\eta$ may be a modified special fibre). Then let $\mathrm{trop}(\mathcal{X})$ denote the tropicalisation of $\mathcal{X}$ as defined in Section 2.5 and let $\mathrm{trop}(Z_\eta)$ be the image of the restriction of the tropicalisation map to Z_η. The latter may be seen as the space $\mathrm{trop}(X_0)$ where certain vertices have been added. More precisely, in the notation of the proof of Proposition 7.4.1, if the irreducible components of Z_η are given in local coordinates x, y and z by

$$\{(c_{i,1}u^{e_{i,1}}, c_{i,2}u^{e_{i,2}}, c_{i,3}u^{e_{i,3}})\},$$

as in (7.4.2), then $(e_{i,1}, e_{i,2}, e_{i,3})$ are the vertices added to $\mathrm{trop}(X_0)$ to make $\mathrm{trop}(Z_\eta)$. In order to construct an extension $(Z'_{\eta'_0}, \mathcal{X}'_{\eta'_0})$ of $(Z_\eta, \mathcal{X}_\eta)$ such that $Z'_{\eta'_0}$ has smooth support in $\mathcal{X}'_{\eta'_0}$, each added vertex of $\mathrm{trop}(Z_\eta)$ must correspond

to a nonempty bubble in $\mathcal{X}'_{\eta'_0}$. This effectively determines all nonempty bubbles which must exist in $(Z'_{\eta'_0}, \mathcal{X}'_{\eta'_0})$, but there may be different blow-ups which produce such bubbles, i.e. different fibres $\mathcal{X}'_{\eta'_0}$ in which the closure of Z_η after base change is stable. For separatedness to hold, we must therefore identify all these different options. In other words, any two pairs in $\mathfrak{N}^m_{\mathrm{SWS}}(k)$ (or $\mathfrak{N}^m_{\mathrm{LW}}(k)$) whose nonempty bubbles correspond to the same vertices in $\mathrm{trop}(X_0)$ must belong to the same equivalence class in $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}(k)$ (or $\overline{\mathfrak{N}^m_{\mathrm{LW}}}(k)$).

Definition 7.5.1. In the notation of the above paragraph, we will say that a pair $(Z_{\eta_0}, \mathcal{X}_{\eta_0})$ is *associated* to a configuration of points in $\mathrm{trop}(X_0)$ if these points correspond exactly to the non-empty bubbles in $(Z_{\eta_0}, \mathcal{X}_{\eta_0})$ in the manner described above.

This stack construction will be, in a sense, more canonical, as the only choices come from the construction of the stack $\mathfrak{X}'$, but all subsequent choices are avoided. This, however, implies that the objects of the stack are equivalence classes of pairs $(Z, \mathcal{X})$, where for two representatives $(Z, \mathcal{X})$ and $(Z', \mathcal{X}')$ of a given equivalence class, there may exist no isomorphism of schemes $\mathcal{X} \cong \mathcal{X}'$. Defining this more canonical stack therefore comes at the cost of our stack being no longer algebraic. This parallels the canonical choice of underlying stack in Kennedy-Hunt's construction of a logarithmic Quot scheme [Ken23].

Theorem 7.5.2. *The stacks $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}$ and $\overline{\mathfrak{N}^m_{\mathrm{LW}}}$ have finite automorphisms and are proper.*

Proof. They have finite automorphisms because the stack $\mathfrak{N}^m_{\mathrm{SWS}}$ has finite automorphisms by the same argument as Lemma 6.1.5 and $\mathfrak{N}^m_{\mathrm{LW}}$ has finite automorphisms by construction. The universal closure follows directly from the universal closure of $\mathfrak{N}^m_{\mathrm{SWS}}$ and $\mathfrak{N}^m_{\mathrm{LW}}$ in Proposition 7.4.1 and Corollary 7.4.2, and from Corollary 7.5.11. It remains to prove that the stacks are separated. This part of the proof will build upon the proof of Proposition 6.1.4.

Let $S \coloneqq \operatorname{Spec} R \to C$, where R is a discrete valuation ring with uniformising parameter u. Let η denote the generic point of S and η_0 its closed point. Now, assume that there are two pairs $[(Z, \mathcal{X})]$ and $[(Z', \mathcal{X}')]$ in $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}$ (or $\overline{\mathfrak{N}^m_{\mathrm{LW}}}$) such that $[(Z_\eta, \mathcal{X}_\eta)] \cong [(Z'_\eta, \mathcal{X}'_\eta)]$. We will show that it must follow that $[(Z_{\eta_0}, \mathcal{X}_{\eta_0})] \cong [(Z'_{\eta_0}, \mathcal{X}'_{\eta_0})]$.

$(Z, \mathcal{X})$ and $(Z', \mathcal{X}')$ are some representatives of the above equivalence classes. As in the the proof of Proposition 6.1.4, we may assume that S is chosen so that the i-th irreducible component of Z is given in terms of its local coordinates x, y and z by

$$\{(c_{i,1}u^{e_{i,1}}, c_{i,2}u^{e_{i,2}}, c_{i,3}u^{e_{i,3}})\}, \tag{7.5.1}$$

and the i-th irreducible component of Z' is given in terms of its local coordinates x, y and z by

$$\{(d_{i,1}u^{f_{i,1}}, d_{i,2}u^{f_{i,2}}, d_{i,3}u^{f_{i,3}})\}. \tag{7.5.2}$$

Since the equivalences of the stack act trivially on x, y and z and we know that $(Z_\eta, \mathcal{X}_\eta) \cong (Z'_\eta, \mathcal{X}'_\eta)$, it must therefore follow that Z and Z' have the same number of irreducible components. Moreover, if these components are labelled in a compatible way, then $c_{i,1} = d_{i,1}$ and $e_{i,1} = f_{i,1}$ for all i. This is independent of the choices of representative Z and Z', as the valuation vectors $(e_{i,1}, e_{i,2}, e_{i,3})$ and $(f_{i,1}, f_{i,2}, f_{i,3})$ are precisely the data of the tropicalisation of Z and Z' and this is constant across the equivalence classes.

Now, as before, the fact that Z and Z' must be S-flat families in $\mathcal{X}$ and $\mathcal{X}'$ respectively means that each Z_i and Z'_i component must satisfy the equations

$$\begin{aligned} x &= c_{i,1}u^{e_{i,1}}, \\ y &= c_{i,2}u^{e_{i,2}}, \\ z &= c_{i,3}u^{e_{i,3}}, \end{aligned}$$

also above the closed point. If more than one element of the set $\{e_{i,1}, e_{i,2}, e_{i,3}\}$ is nonzero, then Z_i and Z'_i must be supported in a component blown up along $\langle x, cu^{e_{i,1}} \rangle$ and $\langle y, c'u^{e_{i,2}} \rangle$ over the closed point η_0, for some $c, c' \in R^\times$. But this tells us exactly that the nonempty bubbles of the pairs $(Z_{\eta_0}, \mathcal{X}_{\eta_0})$ and $(Z'_{\eta_0}, \mathcal{X}'_{\eta_0})$ must correspond to the same points in the tropicalisation and therefore these two pairs must belong to the same equivalence class. □

Remark 7.5.3. As mentioned above, the stacks $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}$ and $\overline{\mathfrak{N}^m_{\mathrm{LW}}}$ mirror Kennedy-Hunt's underlying stack construction of a logarithmic Quot scheme. We should note, however, that restricting the construction of [Ken23] to the case of Hilbert schemes of points, though similar, would not yield either of the stacks $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}$ and $\overline{\mathfrak{N}^m_{\mathrm{LW}}}$. This is because some choices were made in the way we constructed $X[A, B]$, and $\mathfrak{X}'$ does not contain every possible expansion of X. Kennedy-Hunt's construction precludes any such choices. Let us denote for now by $\mathfrak{Y} \to C$ the underlying stack of the restriction of the logarithmic Quot scheme construction to the case of Hilbert schemes of points. For any algebraically closed field k, an object of $\mathfrak{Y}(k)$ is an equivalence class. While there is a bijection between $|\mathfrak{Y}(k)|$ and $|\overline{\mathfrak{N}^m_{\mathrm{SWS}}}(k)|$ or $|\overline{\mathfrak{N}^m_{\mathrm{LW}}}(k)|$, some representatives of the equivalence class defining an object of $\mathfrak{Y}(k)$ do not exist in the equivalence class defining the corresponding object in $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}(k)$ or $\overline{\mathfrak{N}^m_{\mathrm{LW}}}(k)$.

Finally, we would like to comment on the fact that we do not assert an isomorphism between the stacks $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}$ and $\overline{\mathfrak{N}^m_{\mathrm{LW}}}$ here. This is because the results we used to establish equivalence of stacks require the stacks to be algebraic. We do, in fact, expect these stacks to be isomorphic but some further work is needed to show this.

Choices of proper algebraic stacks. As mentioned before, the second way we have of making separated stacks is to make a systematic choice among all pairs $(Z, \mathcal{X}) \in \mathfrak{N}^m_{\mathrm{SWS}}(k)$ (or $\mathfrak{N}^m_{\mathrm{LW}}$) associated to the same configuration of vertices in $\mathrm{trop}(X_0)$. These choices effectively cut out substacks of $\mathfrak{N}^m_{\mathrm{SWS}}$ and $\mathfrak{N}^m_{\mathrm{LW}}$.

In order to make these choices of limit representatives, we may declare that certain fibres $\mathcal{X} \in \mathfrak{X}'$ contain no stable subschemes and we may also decide that certain bubbles in given fibres contain no points of the support. The Maulik-Ranganathan stability condition is close to this description. The starting point for their construction is to look at the image of a given subscheme Z under the tropicalisation map, which is a collection of vertices in the tropicalisation of X_0, and then construct an expansion around the bubbles introduced by these new vertices. Implicitly, choices are made at this stage to exclude certain superfluous fibres. We must therefore emulate this with our stability condition. Moreover, in the modified special fibres they do construct, it is necessary to allow certain configurations of points and not others, hence labelling certain bubbles as tube components and introducing the notion of Donaldson-Thomas stability with respect to these labellings. We describe suitable additions to the stability conditions previously discussed and extend Maulik-Ranganathan stability to our situation with the following definitions.

Definition 7.5.4. Let $\mathcal{X} \in \mathfrak{X}'(k)$ and let Z be a length m zero-dimensional subscheme in $\mathcal{X}$. A *tube labelling* of $\mathcal{X}$ is a choice of a collection of bubbles in $\mathcal{X}$ which we call *tube components*. Given such a labelling, the pair $(Z, \mathcal{X})$ is *Donaldson-Thomas stable* (abbreviated DT stable) if each bubble in $\mathcal{X}$ contains no point of the support of Z if and only if it is a tube component.

Definition 7.5.5. Let α denote a collection of equivalence classes of fibres in $\mathfrak{X}'(k)$ and let β denote a collection of tube labellings on the equivalence classes of fibres remaining after the fibres α have been removed. Let $\mathcal{X} \in \mathfrak{X}'(k)$. We say that the pair $(Z, \mathcal{X})$, where Z is a length m zero-dimensional subscheme in $\mathcal{X}$, is (α, β)-*stable* if $\mathcal{X}$ is not one of the fibres in α and the pair $(Z, \mathcal{X})$ is DT stable.

We will only want to consider (α, β)-stability as a restriction of SWS or LW stability. We denote by $\mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}$ and $\mathfrak{N}^m_{\mathrm{LW},(\alpha,\beta)}$ the stacks $\mathfrak{N}^m_{\mathrm{SWS}}$ and $\mathfrak{N}^m_{\mathrm{LW}}$ restricted to their (α, β)-stable loci. As we will see with the next two results, an appropriate choice of (α, β)-stability condition will give us unique limits and allow us to build proper stacks. First, let us define some useful terminology.

Definition 7.5.6. We say that a pair (α, β) is an *almost proper SWS stability condition* if it is chosen so that for any closed point η_0' there exists a unique equivalence class $(Z'_{\eta_0'}, \mathcal{X}'_{\eta_0'}) \in \mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}(\eta_0')$ associated to each configuration of points in $\mathrm{trop}(X_0)$. The corresponding definition of *almost proper LW stability condition* can be made in a similar way.

Proposition 7.5.7. *Let (α, β) be an almost proper SWS stability condition and let $\mathcal{X} \coloneqq \xi^* X[A, B]$ for some unbroken (A, B, n), where $\xi \colon S \to C[A, B]$ and $S \coloneqq \operatorname{Spec} R$ for some discrete valuation ring R. Let η be the generic point of S. If $(Z_\eta \mathcal{X}_\eta) \in \mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}(\eta)$, then $(Z_\eta, \mathcal{X}_\eta)$ has an extension in $\mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}$.*

Proof. This follows directly from the proof of Proposition 7.4.1. Indeed, given such a pair $(Z_\eta, \mathcal{X}_\eta)$, we recall that, up to some finite base change, each irreducible

component of Z_η can be written locally in terms of its x, y and z coordinates as

$$\{(c_{i,1}u^{e_{i,1}}, c_{i,2}u^{e_{i,2}}, c_{i,3}u^{e_{i,3}})\},$$

for some $e_{i,j} \in \mathbb{Z}$ and $c_{i,j} \in R^\times$. Moreover, we showed that if more than one element of the set $\{e_{i,1}, e_{i,2}, e_{i,3}\}$ is nonzero, the closure of Z_η in any stable extension of $(Z_\eta, \mathcal{X}_\eta)$ must be supported in a component blown up along the vanishing of both sides of the relevant equations (6.1.6), (6.1.7) or (6.1.8). Now, we note that each of these triples $(e_{i,1}, e_{i,2}, e_{i,3})$ corresponds to a point in $\mathrm{trop}(X_0)$, and for such a configuration of points in the tropicalisation we know that we have an associated pair $(Z'_{\eta_0}, \mathcal{X}'_{\eta_0})$ over the closed point η_0 of S which is SWS and (α, β)-stable, since (α, β) is an almost proper SWS stability condition. But, again from the proof of Proposition 7.4.1, all choices of such a fibre and subscheme pair appear as choices of limits for $(Z_\eta, \mathcal{X}_\eta)$. □

We have therefore proven that, if (α, β) is an almost proper SWS (or LW) stability condition, families whose generic fibre is pulled back from a fibre in $X[A, B]$ over a point $(t_1, \ldots, t_{n+1})$ whose entries are all nonzero have a unique (α, β)-stable limit.

In order to ensure that the stacks $\mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}$ and $\mathfrak{N}^m_{\mathrm{LW},(\alpha,\beta)}$ are proper, we need one more compatibility condition to hold. We give here an example of how the stack $\mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}$ may fail to be proper if we do not add this extra condition. Let the equivalence class in $\mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}$ associated to $\mathrm{trop}(X_0)$ with one vertex added to the $Y_1 \cap Y_2$ edge be given by expanding one Δ_1-component of pure type. Now, we decide that the equivalence class associated to $\mathrm{trop}(X_0)$ with two vertices added to the $Y_1 \cap Y_2$ edge will be given by expanding two Δ_2-components of pure type. See Figure 7.5. Now let us take a length 2 zero-dimensional subscheme in the first fibre whose support consists of a point in $\Delta_1^\circ \cap \pi^*(Y_1 \cap Y_2)^\circ$ and a point in Y_2°. This is SWS and (α, β)-stable for our choice of (α, β). We may now study the limit of this subscheme as the point in Y_2° tends towards $\pi^*(Y_1 \cap Y_2)$, i.e. as its x coordinate tends to zero. If the stack $\mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}$ were proper it would contain this limit, but it does not. Indeed, our second fibre has the same associated tropical configuration as this limit, but is clearly not a limit for this pair. This breaks universal closure. We therefore need the following additional condition.

Definition 7.5.8. Let (α, β) define an almost proper SWS stability condition and let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two fibres of $\mathfrak{X}'$ over a closed point. Assume that $\mathcal{X}_1$ is pulled back from some $X[A, B]_{J_1}$ and $\mathcal{X}_2$ is pulled back from some $X[A', B']_{J_2}$, where $|J_1| \leq |J_2|$. We denote by $W_1 \subseteq \mathcal{X}_1$ and $W_2 \subseteq \mathcal{X}_2$ the union of all irreducible components in each fibre which are not tubes. We say that (α, β) defines a *proper SWS stability condition* if the following compatibility condition holds. If $\pi_*(W_2) \subset \overline{\pi_*(W_1)}$, then there exists a fibre $\mathcal{X}_3$ equivalent to $\mathcal{X}_2$ in $\mathfrak{X}'$ such that $\mathcal{X}_3$ is pulled back from $X[A, B]_{J_3}$ for some J_3 with $J \subseteq J_3$. A similar definition can be made for LW stability.

If (α, β) is a proper LW stability condition and the pair $(Z, \mathcal{X})$ is LW and (α, β)-stable, then we say it is *Maulik-Ranganathan stable* (abbreviated MR stable) for the given choice (α, β).

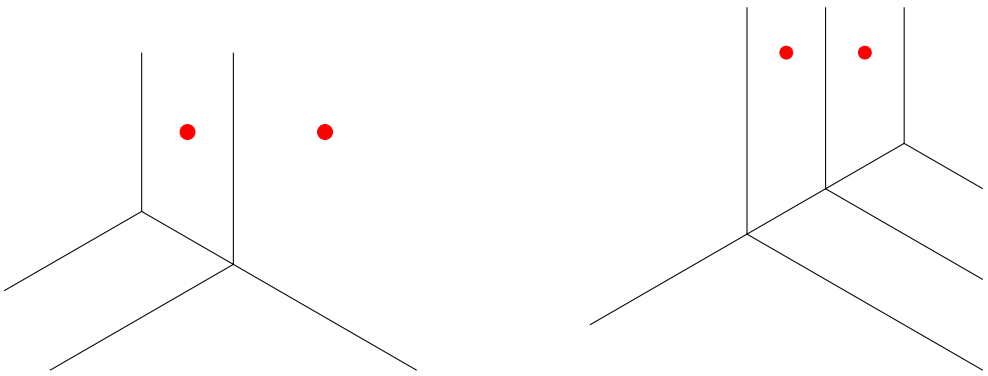


Figure 6: The picture on the right is not a limit of the one on the left.

Proposition 7.5.9. *Let (α, β) be some choice of stability condition. The corresponding stack $\mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}$ is proper if and only if (α, β) is a proper SWS stability condition.*

Proof. Once again we apply the valuative criterion. Let $S \coloneqq \operatorname{Spec} R \to C$, where R is a discrete valuation ring and let η and η_0 be denote the generic and closed points of S. Now, we take any $(Z, \mathcal{X}) \in \mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}(S)$, where (α, β) is a proper SWS stability condition. From Proposition 7.5.7, we know already that that if the restriction to the generic fibre $\mathcal{X}_\eta$ is pulled back from a fibre in some $X[A, B]$ over a point $(t_1, \ldots, t_{n+1})$ whose entries are all nonzero, then $(Z_\eta, \mathcal{X}_\eta)$ has a stable extension in $\mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}$. We may therefore assume that $\mathcal{X} \coloneqq \xi^* X[A, B]_I$ for some morphism $\xi \colon S \to C[A, B]_I$ and some nonempty set I. The generic fibre $\mathcal{X}_\eta$ is therefore a modified special fibre.

We split the proof into the following two cases. The first case is where a point P of the support of Z_η tends towards a codimension greater or equal to one stratum in $\mathcal{X}_\eta$, i.e. one or more of its local coordinates x, y or z tends to zero. The second case is where a point P of the support of Z_η has fixed x, y and z values but one or more of its $(x_0^{(i)} : x_1^{(i)})$ or $(y_0^{(i)} : y_1^{(i)})$ coordinates tends towards $(1 : 0)$ or $(0 : 1)$.

We start by proving existence and uniqueness of limits in the first case using the valuative criterion. Let V denote the irreducible component of $\mathcal{X}_\eta$ in the interior of which P lies. Notice that since P tends towards a codimension greater or equal to one stratum of $\mathcal{X}$, then in order for its limit to have smooth support in an extension of $(Z_\eta, \mathcal{X}_\eta)$, it will be necessary to expand out at least one Δ-component in this extension. Moreover, there exists a smoothing from the interior of V in the fibre over the generic point to the interior of this expanded Δ-component in such an extension of $(Z_\eta, \mathcal{X}_\eta)$ if and only if this Δ-component is equal to V in the fibre over the generic point.

Since (α, β) is an almost proper SWS stability condition, we know that there exists an associated pair $(Z'_{\eta'_0}, \mathcal{X}'_{\eta'_0})$ for the image $\operatorname{trop}(Z_\eta)$ of Z_η in $\operatorname{trop}(X_0)$. Precisely, this means that there exists some base change $S' \to S$ and some SWS

(α,β)-stable pair $(Z'_{\eta'_0},\mathcal{X}'_{\eta'_0})$ over the closed point η'_0 of S' such that the non-tube bubbles of $\mathcal{X}'_{\eta'_0}$ correspond exactly to the vertices in $\mathrm{trop}(Z_\eta)$. Moreover, as (α,β) is a proper SWS stability condition, we know that every vertex in $\mathrm{trop}(\mathcal{X}_\eta)$ is a vertex in $\mathrm{trop}(\mathcal{X}'_{\eta'_0})$, i.e. the two fibres are compatible in the sense that $\mathcal{X}'_{\eta'_0}$ can be seen as a blow-up of $\mathcal{X}_\eta$. This tells us that $(Z'_{\eta'_0},\mathcal{X}'_{\eta'_0})$ is the restriction to the closed point η'_0 of an S-flat family $(Z',\mathcal{X}')\in\mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}(S')$ such that $\mathcal{X}'_{\eta'}\cong\mathcal{X}_\eta\times_\eta\eta'$. In other words, $\mathcal{X}'_{\eta'}$ is the choice of representative of the equivalence class of $(Z_\eta,\mathcal{X}_\eta)$, possibly after some base change, such that all newly expanded components in $\mathcal{X}'_{\eta'_0}$ containing the limit of P are equal to V in $\mathcal{X}'_{\eta'}$. Note that we are abusing notation slightly here: as $\mathcal{X}_\eta$ and $\mathcal{X}'_{\eta'}$ are isomorphic, we refer to the irreducible component in $\mathcal{X}'_{\eta'}$ corresponding to V in $\mathcal{X}_\eta$ as V also for simplicity. By the above, we therefore get the required smoothing to the SWS (α,β)-stable pair $(Z'_{\eta'_0},\mathcal{X}'_{\eta'_0})$.

Now, let us dicuss the second case. Denote again by V the irreducible component of $\mathcal{X}_\eta$ in the interior of which P lies. Firstly, let us assume that any other point of the support of Z_η lying in the interior of V shares the same equations as P up to multiplication by a constant. In particular, for all these points the same coordinates $(x_0^{(i)}:x_1^{(i)})$ or $(y_0^{(i)}:y_1^{(i)})$ will be fixed and the same will vary (recall that in this case we are assuming already that all x,y and z coordinates are fixed). By flatness, it therefore follows that all points of the support of Z_η which lie in the interior of V will tend to the interior of the same irreducible component in any extension of $(Z_\eta,\mathcal{X}_\eta)$. Any extension in which an additional Δ-component is expanded out, i.e. in which an additional basis direction is set to zero, cannot be stable, since it would necessarily have an empty expanded component which would destabilise the pair.

Notice that any $(x_0^{(i)}:x_1^{(i)})$ or $(y_0^{(i)}:y_1^{(i)})$ which are fixed for one of these points contained in the interior of $V\subset\mathcal{X}_\eta$ must be fixed for all of them. We may therefore choose a representative $(Z'_\eta,\mathcal{X}'_\eta)$ in the same equivalence class as $(Z_\eta,\mathcal{X}_\eta)$ such that the coordinates $x_0^{(i)}/x_1^{(i)}$, $x_1^{(i)}/x_0^{(i)}$, $y_0^{(i)}/y_1^{(i)}$ or $y_1^{(i)}/y_0^{(i)}$ which are allowed to vary in Z are invertible only outside of $V\subset\mathcal{X}'_\eta$. Then the limit of $(Z'_\eta,\mathcal{X}'_\eta)$ is $(Z'_\eta,\mathcal{X}'_\eta)$ itself. And by the above this equivalence class gives us the only stable limit for such a family.

Now, if two points P_0 and P_1 of the support of Z_η which lie in $V\subset\mathcal{X}_\eta$ have different defining equations up to multiplication by a constant, it means that there are some coordinates $(x_0^{(i)}:x_1^{(i)})$ or $(y_0^{(i)}:y_1^{(i)})$ which are fixed for P_0 but not P_1 and vice versa. This means that, as one or the other of these coordinates tends toward $(1:0)$ or $(0:1)$, the points P_0 and P_1 must tend towards different irreducible components. Similarly to the first case, as (α,β) is an almost proper SWS stability condition, we know that there exists an associated pair $(Z'_{\eta'_0},\mathcal{X}'_{\eta'_0})$ for $\mathrm{trop}(Z_\eta)$ and since (α,β) is, in fact, a proper SWS stability condition, we know that every vertex in $\mathrm{trop}(\mathcal{X}_\eta)$ is a vertex in $\mathrm{trop}(\mathcal{X}'_{\eta'_0})$. As before this shows that $(Z'_{\eta'_0},\mathcal{X}'_{\eta'_0})$ is the unique SWS (α,β)-stable extension of $(Z_\eta,\mathcal{X}_\eta)$.

If $V\subset\mathcal{X}_\eta$ contains more than two points of the support which have different equations up to multiplication by a constant, we can just repeat the steps of the

previous paragraph until we find a stable extension. It will be unique as it will be the unique pair associated to $\mathrm{trop}(Z_\eta)$.

Finally, the converse is straightforward. Indeed if the stability condition (α, β) is not a proper SWS stability condition, it either means that for some $(Z_\eta, \mathcal{X}_\eta)$ there does not exist a unique stable pair in $\mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}$ associated to $\mathrm{trop}(Z_\eta)$ which will break universal closure or separatedness, or that this associated pair exists and is unique but not compatible with $\mathcal{X}_\eta$. In the second case, universal closure will not be satisfied because the pair associated to $\mathrm{trop}(Z_\eta)$ is the only possible limit but it is not an extension of $(Z_\eta, \mathcal{X}_\eta)$, similarly to the example preceding this proposition. □

Corollary 7.5.10. *The stack $\mathfrak{N}^m_{\mathrm{LW},(\alpha,\beta)}$ is proper if and only if (α, β) is a proper LW stability condition.*

Proof. This follows immediately from the proof of Proposition 7.5.9. □

In the following we shall denote by $\mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)}$ a stack $\mathfrak{N}^m_{\mathrm{SWS},(\alpha,\beta)}$ where (α, β) is a proper SWS stability condition, and by $\mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}$ a stack $\mathfrak{N}^m_{\mathrm{LW},(\alpha,\beta)}$ where (α, β) is a proper LW stability condition.

Corollary 7.5.11. *Let S and η as before and let $(Z_\eta, \mathcal{X}_\eta)$ be an element of $\mathfrak{M}^m_{\mathrm{SWS}}(\eta)$, $\mathfrak{M}^m_{\mathrm{LW}}(\eta)$, $\mathfrak{N}^m_{\mathrm{SWS}}(\eta)$, $\mathfrak{N}^m_{\mathrm{LW}}(\eta)$, $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}(\eta)$ or $\overline{\mathfrak{N}^m_{\mathrm{LW}}}(\eta)$ such that $\mathcal{X}_\eta$ is a modified special fibre. Then $(Z_\eta, \mathcal{X}_\eta)$ has a stable extension with respect to the relevant stability condition in the relevant stack.*

Proof. The stacks $\mathfrak{M}^m_{\mathrm{SWS}}$ and $\mathfrak{M}^m_{\mathrm{LW}}$ are examples of stacks $\mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)}$ and $\mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}$. In other words the choice (α, β) which defines these stacks clearly satisfies the conditions of a proper SWS (or LW) stability condition, therefore $\mathfrak{M}^m_{\mathrm{SWS}}$ and $\mathfrak{M}^m_{\mathrm{LW}}$ are proper stacks.

In the non-separated stacks $\mathfrak{N}^m_{\mathrm{SWS}}$ and $\mathfrak{N}^m_{\mathrm{LW}}$, we allow for all limits. In particular, each stack will contain all the associated pairs for $\mathrm{trop}(Z_\eta)$ which are compatible with $\mathcal{X}_\eta$ and stable for their respective stability conditions.

In the case of $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}(\eta)$ and $\overline{\mathfrak{N}^m_{\mathrm{LW}}}(\eta)$, we do not make any choices, so for any such $(Z_\eta, \mathcal{X}_\eta)$ all possible associated pairs of $\mathrm{trop}(Z_\eta)$ form an equivalence class in $\overline{\mathfrak{N}^m_{\mathrm{SWS}}}(\eta)$ or $\overline{\mathfrak{N}^m_{\mathrm{LW}}}(\eta)$. In particular, there will be a representative of this equivalence class which is compatible with $\mathcal{X}_\eta$ in the sense of Definition 7.5.8. □

Theorem 7.5.12. *The stacks $\mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)}$ and $\mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}$ are Deligne-Mumford and proper.*

Proof. Properness follows from Proposition 7.5.9 and Corollary 7.5.10. The Deligne-Mumford condition holds because SWS and LW stability ensure finite automorphisms. □

Remark 7.5.13. Let $(Z_\eta, \mathcal{X}_\eta)$ be a pair consisting of a length m zero-dimensional subscheme Z_η in a modified special fibre $\mathcal{X}_\eta$. Assume that the pair is either SWS or LW stable. Let P be a point of the support which lies in some irreducible

component $W \subset \mathcal{X}_\eta$ and let V be some other irreducible component in $\mathcal{X}_\eta$ which intersects W nontrivially. If there exists no representative in the equivalence class of $(Z_\eta, \mathcal{X}_\eta)$ such that there is a smoothing from the interior of W in $\mathcal{X}_\eta$ to the interior of some Δ-components expanded out in the $W \cap V$ locus, i.e. such that these Δ-components are equal to W in $\mathcal{X}_\eta$, then there exists no flat family of length m zero-dimensional subschemes such that $(Z_\eta, \mathcal{X}_\eta)$ is the generic fibre and P tends towards $W \cap V$ over the closed point. This can be seen by studying the equations of the blow-ups.

For example, let $\mathcal{X}_\eta$ be a fibre with one expanded Δ_1-component of pure type and no other expanded components. Let Z_η be a length 2 zero-dimensional subscheme with one point of its support, P_0, lying in the interior of this Δ_1-component in the $\pi^*(Y_1 \cap Y_2)^\circ$ locus and the other point of the support, P_1, lying in Y_2°, as in the picture on the left of Figure 7.5. In this fibre, there cannot be a Δ_2-component which is equal to Y_2 and thus there can be no smoothing from the interior of Y_2 to an expanded component in the $Y_2 \cap Y_3$ locus. But there also cannot be any flat family such that P_1 tends towards $z = 0$, as any equation for P_1 containing z must also contain y, which is already zero.

Instead of (α, β) stability, it is also possible to cut out a proper substack of $\mathfrak{N}^m_{\mathrm{SWS}}$ by restricting the G-linearised line bundles that we use for the stability condition. By allowing subschemes to be stable only if they are GIT stable with respect to this chosen collection of G-linearised line bundles, we may recover some of the stacks $\mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)}$. This type of stability will behave well with respect to limits of special objects, as GIT stability has good properties for smoothings. More precisely, this means that if the (α, β)-stability condition recovered by such a choice is almost proper, it is necessarily proper as well.

Relating the proper stacks. We may now show how the stacks $\mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)}$ and $\mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}$ relate to each other. Note that the stacks $\mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}$ constructed in the previous section give us some choices of compactifications arising through the methods of Maulik and Ranganathan, but they do not give us every possible choice that their constructions would yield, because of the choices we made in the definition of the stack $\mathfrak{X}'$.

Theorem 7.5.14. *Let $\mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)}$ and $\mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}$ be two choices given by the same (α, β). There then exists an isomorphism of stacks*

$$\mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)} \cong \mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}.$$

Proof. Similarly to the arguments of Section 6.2, it is clear that on the scheme level we have an inclusion

$$H^{m,[A,B]}_{\mathrm{LW},(\alpha,\beta)} \hookrightarrow H^{m,[A,B]}_{\mathrm{SWS},(\alpha,\beta)},$$

where $H^{m,[A,B]}_{\mathrm{LW},(\alpha,\beta)}$ and $H^{m,[A,B]}_{\mathrm{SWS},(\alpha,\beta)}$ are the restrictions of $H^m_{[A,B],\mathrm{SWS}}$ and $H^m_{[A,B],\mathrm{LW}}$ to their respective (α, β)-stable loci. As before, this gives rise to a morphism of stacks

$$f \colon \mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)} \longrightarrow \mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}.$$

By Lemma 7.3.6, for every MR stable pair $(Z, \mathcal{X}) \in \mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}(k)$, where k is an algebraically closed field as before, there exists a representative of the equivalence class of $(Z, \mathcal{X})$ which is SWS stable. We therefore have an induced bijection of sets

$$|f|\colon |\mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)}(k)| \longrightarrow |\mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}(k)|.$$

Moreover, f is representable by Theorem 7.5.12 as $\mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)}$ is separated and Deligne-Mumford and therefore has finite inertia. The morphism f induces a bijective homomorphism of stabilisers by the same argument as in the proof of Theorem 6.2, so f is an isomorphism of stacks by Lemma 6.2.1. □

As shown in [MR20], all choices of proper Deligne-Mumford stacks obtained for different (α, β), are birational to each other.

8 An example and discussion of minimality

Up until now, we have only been considering the property that $X \to C$ is semistable. In this section, we will discuss what properties our constructions have if we start with some added assumptions. We will want to show that the proper stacks we construct are semistable themselves, in the sense of Definition 2.3.6, and that moreover, if $K_X + (X_0)_{\mathrm{red}}$ is semi-ample, i.e. $X \to C$ is a good semistable model, then the proper stacks constructed in previous sections also satisfy this property.

In particular, we will want to consider the case where $X \to C$ is a good type III degeneration of K3 surfaces. In practice, this means that $X \to C$ is a family of simply connected surfaces together with a relative holomorphic symplectic differential 2-form, which is nowhere degenerate. Note this implies the relative canonical bundle is trivial. The general fibres of the family are smooth K3 surfaces and the dual complex of its central fibre X_0 is a triangulated sphere. In this case, we will show that such a 2-form on $X \to C$ induces a nowhere degenerate logarithmic 2-form on the proper stacks we constructed.

Most of the results in this section follow easily from the results of Gulbrandsen, Halle, Hulek and Zhang in [GHHZ21]. For clarity, however, we give certain of the relevant proofs again here, where the details are slightly different for our situation.

8.1 Good semistable and dlt minimal models

Proposition 8.1.1. *The stacks $\mathfrak{M}^m_{\mathrm{SWS}}$, $\mathfrak{M}^m_{\mathrm{LW}}$, $\mathfrak{N}^m_{\mathrm{PSWS},(\alpha,\beta)}$ and $\mathfrak{N}^m_{\mathrm{MR},(\alpha,\beta)}$ are semistable degenerations over C. Moreover, they are normal and $\mathbb{Q}$-factorial.*

Proof. Semistability follows directly from Lemma 7.3 of [GHHZ21]. The stacks are normal as they are semistable. This also applies to any constructions arising from the methods of Maulik and Ranganthan [MR20]. Finally, the stacks are $\mathbb{Q}$-factorial, as all object of the stacks have finite stabilisers. □

We will now assume that $K_X + (X_0)_{\mathrm{red}}$ is semi-ample and prove the following result.

Theorem 8.1.2. *The stacks listed in Proposition 8.1.1 are good minimal models.*

Proof. Let $S \to C$ be a scheme over C and let $\mathcal{X} \in \mathfrak{X}'(S)$ (note that $\mathcal{X} \in \mathfrak{X}(S)$ is a special case), where $\mathcal{X} \coloneqq \xi^* X[A,B]$ for some étale morphism $\xi\colon S \to C[A,B]$. We denote by $\phi\colon \xi^* X[A,B] \to X[A,B]$ the relevant strongly cartesian morphism. Now let $P_1, \dots, P_m$ be a collection of points in $\mathcal{X}$. Since ϕ is a base change morphism and $\pi\colon X[A,B] \to X$ is the G-equivariant projection, similarly to Lemma 5.12 of [GHHZ21], we have that $K_{\mathcal{X}} \cong (\pi \circ \phi)^* K_X$ (note that all blow-ups we make in our construction are also small).

Now, let $P_1, \dots, P_m$ be a finite collection of points in $\mathcal{X}$. Similarly to Lemma 5.13 of [GHHZ21], we may find a G-invariant section of $K_{\mathcal{X}}^{\otimes r}$, not vanishing at any P_i. Indeed, as we have assumed that $X \to C$ is a good semistable minimal model, for each i, we know that there exists a section $\widetilde{s_i}$ of $K_X^{\otimes r}$ that does not vanish at $(\pi \circ \phi)(P_i)$. It follows by the above that $s_i \coloneqq (\pi \circ \phi)^* \widetilde{s_i}$ is a G-invariant section of $K_{\mathcal{X}}^{\otimes r}$ which does not vanish at P_i. Therefore, as in Lemma 5.13 of [GHHZ21], if we take a k-linear combination $s = \sum_i \lambda_i s_i$, for sufficiently general $\lambda_i \in k$, the section s does not vanish at any P_i.

Now, let $Z = (P_1, \dots, P_m) \in \mathcal{X}^m$, where $\mathcal{X}^m$ denotes the product $\mathcal{X} \times_S \cdots \times_S \mathcal{X}$. Then, similarly to Lemma 5.14 of [GHHZ21], there exists a G- and S_m-invariant section σ of $(K_{\mathcal{X}^m})^{\otimes r}$ that does not vanish at Z, where S_m denotes the m-th symmetric group. This is given by the restriction to the stable locus of the tensor product $\mathrm{pr}_1^*(s) \otimes \cdots \otimes \mathrm{pr}_m^*(s)$ of the pullbacks of s along each projection $\mathrm{pr}_i\colon \mathcal{X}^m \to \mathcal{X}$. Clearly, this is G- and S_m-invariant and does not vanish at Z.

We may now apply a generalised version of Beauville's argument from [Bea83]. Let W denote the inverse image of $X \setminus \mathrm{Sing}(X_0)$ by the composition of morphisms $\pi \circ \phi$. (Note that this is not defined quite in the same way as in [GHHZ21] but the overall idea is the same.) We then denote by W^m and $W^{[m]}$ the relative product and relative Hilbert scheme of $W \to S$. Additionally, we denote by W_*^m and $W_*^{[m]}$ the restrictions of the above schemes to the open loci whose points are length m zero-dimensional subschemes which have at most one double point. The complements of W_*^m in W^m and $W_*^{[m]}$ in $W^{[m]}$ are both of codimension 2.

Now, the Hilbert scheme $W^{[m]}$ is obtained from W^m by blowing up the diagonal in W^m and quotienting by the action of S_m. Denote by $b_{\mathrm{diag}}\colon \widetilde{W_*^m} \to W_*^m$ and $q_m\colon \widetilde{W_*^m} \to W_*^{[m]}$ the appropriate restrictions of the blow-up and quotient maps and by $q_G\colon W_*^{[m]} \to W_{*,G}^{[m]}$ the quotient by the group G. Let E be the exceptional divisor of b_{diag}. Then, following [Bea83] and [GHHZ21], E is precisely the ramification locus of q_m, which gives rise to the isomorphisms

$$K_{\widetilde{W_*^m}} \cong q_m^*(K_{W_*^{[m]}})(E)$$
$$K_{\widetilde{W_*^m}} \cong b_{\mathrm{diag}}^*(K_{W_*^m})(E).$$

Note that if the blow-ups made in the construction of $X[A,B]$ are small, then q_G contracts no divisors and we have again an isomorphism

$$K_{W_*^{[m]}} \cong q_G^*(K_{W_{*,G}^{[m]}}).$$

This, in turn, will yield an isomorphism

$$(q_G \circ q_m) * (K_{W_{*,G}^{[m]}})(E) \cong b_{\mathrm{diag}}^*(K_{W_*^m})(E).$$

Finally, let $W_{*,G}^{[m],s}$ be the restriction of $W_{*,G}^{[m]}$ to some stable locus, with respect to any of the stability conditions defined in previous sections. For any point in $W_{*,G}^{[m],s}$, the section $(q_G \circ q_m)_* b_{\mathrm{diag}}^* \sigma$, where σ is defined as above gives an everywhere nonvanishing section of $(K_{W_{*,G}^{[m],s}})^{\otimes r}$. But as $W_{*,G}^{[m],s}$ has codimension 2 in $W_G^{[m],s}$, the section σ extends to a nonvanishing section of $(K_{W_G^{[m],s}})^{\otimes r}$. Then, again, as $W_G^{[m],s}$ has codimension 2 in $\mathcal{X}_G^{[m],s}$, where $\mathcal{X}_G^{[m],s}$ is the stable locus of $\mathcal{X}^{[m]}$ quotiented by G, this section extends to an everywhere nonvanishing section on $K_{\mathcal{X}_G^{[m],s}}^{\otimes r}$.

Let $\mathfrak{Y}$ be any of the stacks listed in Proposition 8.1.1. Any stable object $(Z, \mathcal{X})$ of $\mathfrak{Y}$ is such that $Z \in \mathcal{X}_G^{[m],s}$ for some stability condition. And we have shown that for any such Z, there is a section of $K_{\mathcal{X}_G^{[m],s}}^{\otimes r}$ which does not vanish at Z. Now, for any such stack $f\colon \mathfrak{Y} \to C$ that we constructed, $\mathfrak{Y}_0 \coloneqq f^*(0)$ is a reduced principal divisor in the following sense. Let S and ξ be as above and write $\mathcal{Y} \coloneqq \xi^* H_{[A,B]}^{m,s}$ in $\mathfrak{Y}$, where $H_{[A,B]}^{m,s}$ is the stable locus of length m zero-dimensional subschemes on $X[A,B]$. Let $\phi\colon \mathcal{Y} \to H_{[A,B]}^{m,s}$ be the corresponding strongly cartesian morphism. Then we have that $\mathcal{Y}_0 \coloneqq (f \circ \phi)^*(0)$ is a reduced principal divisor in $\mathcal{Y}$. We therefore have that

$$K_{\mathcal{Y}} + \mathcal{Y}_0$$

is semi-ample and $f\colon \mathfrak{Y} \to C$ is a good minimal model. □

Remark 8.1.3. In the above proof, we used the assumption that all blow-ups made in the construction of $X[A,B]$ were small, i.e. that the map $b\colon X[A,B] \to X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ contracts no divisors. This assumption happens to be true in our case and is convenient, but not strictly necessary. Indeed, keeping with the notation of the above proof, we have a diagram

$$\begin{array}{ccccc}
\widetilde{W_*^m} & \xrightarrow{b_{\mathrm{diag}}} & W_*^m & \xrightarrow{b^m} & (X \times_{\mathbb{A}^1} \mathbb{A}^{n+1})^m \\
\downarrow{\scriptstyle q_m} & & & & \downarrow \\
W_*^{[m]} & & \longrightarrow & & C \times_{\mathbb{A}^1} \mathbb{A}^{n+1} \\
\downarrow{\scriptstyle q_G} & & & & \downarrow{\scriptstyle /G} \\
W_{*,G}^{[m]} & & \longrightarrow & & C,
\end{array}$$

where the morphisms in the diagram denote the restrictions of the previous morphisms of the same names to the W_* locus. As G is given as the natural torus action on the projective coordinates introduced by the blow-up map b, the morphism q_G must contract exactly the divisors coming from the exceptional divisors of b^m and no other divisors. By applying a generalised version of the Beauville argument once more, we may obtain the desired result.

Corollary 8.1.4. *Suppose $X \to C$ has trivial relative canonical bundle. Then the relative canonical bundle $K_{\mathfrak{Y}/C}$ is trivial, for any of the stacks $\mathfrak{Y} \to C$ listed in Proposition 8.1.1.*

Remark 8.1.5. Finally, we would like to remark that the fact that we are working with stacks and not schemes here allows us to produce semistable minimal models, as we do not see any quotient singularities. It is possible to construct good minimal models in the world of schemes using these expanded degeneration techniques, but these will no longer be semistable; they will be dlt as they must contain quotient singularities, by nature of the expanded degeneration machinery. This can be seen to happen already in [GHH19] and [GHHZ21].

8.2 Symplectic structure

We will now assume that $X \to C$ is a type III good degeneration of K3 surfaces. In particular, this means that $K_{X/C}$ is trivial. By Corollary 8.1.4, this implies that $K_{\mathfrak{Y}/C}$ is trivial, for any of the stacks $\mathfrak{Y} \to C$ listed in Proposition 8.1.1. This can be seen alternatively as following from the fact that the relative holomorphic symplectic 2-form on $X \to C$ induces a relative holomorphic symplectic logarithmic 2-form on $\mathfrak{Y} \to C$.

As the dual complex of X_0 is a triangulated sphere, we may choose a labelling of the vertices of this triangulation, such that each vertex is labelled by either Y_1, Y_2 or Y_3 and each triangle has exactly one of each vertices. All the constructions we made on the local family $\operatorname{Spec} k[x,y,z,t] \to \operatorname{Spec} k[t]$ can then be extended to a family $X \to C$ of K3 surfaces by making the following modification. Everywhere we blew up the pullback of $Y_i \times_{\mathbb{A}^1} V(t_j)$ inside some scheme, we now blow up instead the pullback of $Y_{(i)} \times_{\mathbb{A}^1} V(t_j)$, where $Y_{(i)}$ denotes the union of all Y_i components in X.

Proposition 8.2.1. *The stacks listed in Proposition 8.1.1 are symplectic, in the sense that they carry a nowhere degenerate logarithmic 2-form.*

Proof. Let $f\colon \mathfrak{Y} \to C$ be one such construction. Since f is semistable, this follows from Section 7.2 of [GHHZ21]. The logarithmic 2-form is defined with respect to the divisorial logarithmic structure given by the divisors 0 in C and $\mathfrak{Y}_0 = f^*(0)$ in $\mathfrak{Y}$. □

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