Kuga varieties, K3 surfaces and the Kuga-Satake construction

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Abstract

Kuga varieties are a natural generalisation of universal families of abelian varieties. This thesis describes the candidate's work on the geometry of some types of Kuga varieties. In Part I, by considering a special kind of Kuga varieties resulting from the Kuga-Satake construction, we construct an explicit map from a moduli space of K3 surfaces of Picard rank 14 to a moduli space of polarised abelian 8-folds with totally definite quaternion multiplication. This is a geometric interpretation of an exceptional coincidence between locally symmetric spaces of type II_4 and type IV_6. In Part II, we study the n-fold Kuga varieties associated to the moduli space of $(1,p)$-polarised abelian surfaces with canonical level structure for prime p at least 3, and compute their Kodaira dimensions for all but 27 possible combinations of (n,p).

Contents

1 Introduction

Moduli spaces, which are geometric spaces parametrising a collection of mathematical objects, have been a very active area of research. In particular, the notion of Kuga varieties arise naturally in the study of moduli spaces of abelian varieties. Given an embedding from any moduli variety to a moduli space of abelian varieties, a Kuga variety is defined as the pullback of the universal family over the moduli space of abelian varieties by this embedding. Apart from being beneficial for the study of the Hodge conjecture [HaKu], Kuga varieties are interesting varieties standalone as they generalise universal families of abelian varieties. There is work on their classification [A] and geometric properties such as unirationality of special examples [FV].

One special type of Kuga varieties can be constructed with the Kuga-Satake (KS) construction. The process associates an abelian variety called a Kuga-Satake variety [KS] to a K3 surface. A Kuga variety can be obtained by associating to each point of a moduli variety of K3 surfaces its corresponding KS variety. Moreover, the KS construction can be lifted to the level of moduli, whose geometry such as endomorphism structure and simpleness of a generic member is controlled by the family of K3 surfaces one starts with [vG1].

Another special type of Kuga varieties is the n-fold Kuga varieties, *i.e.* varieties over a Siegel modular variety such that each fibre is a product of n copies of the abelian variety or Kummer variety to which it corresponds in the base.

In Part I of this thesis, we study the Kuga-Satake construction. Our main result is Theorem 5.1.6, in which we lift the Kuga-Satake construction and construct a map F from a moduli space of K3 surfaces of Picard rank 14 to a moduli space of polarised abelian 8-folds with totally definite quaternion multiplication.

Furthermore, we realise F using MAGMA for a few specific families (Table 6) studied in [CM2], and investigate some special loci. Our main result here is Theorem 6.4.1.

In Part II of the thesis, we focus on the special n-fold Kuga varieties $\mathfrak{X}_p^n$, where the fibres are polarised abelian surfaces of polarisation type $(1, p)$, denoted as $\mathfrak{X}_p^n$, with p being an odd prime number and $n \geq 1$. Our main result is Theorem 8.2.11, which shows that the Kodaira dimension of $\mathfrak{X}_p^n$ is 3 (the maximum possible) for almost every pair (n, p). We do this using modular forms and a special toroidal compactification of $\mathfrak{X}_p^n$. This result has appeared in [Po] and is accepted for publication in the Tohoku Mathematical Journal. It complements previous works [Ve], [FV] on unirationality of certain 1-fold Kuga varieties, and also [PSMS] where we computed the Kodaira dimension of any Kuga variety over moduli spaces of principally polarised abelian varieties of dimension $g \geq 2$.

The outline for Part I is as follows. In Section 2, we explain the motivation of this work, which comes from the geometric interpretation of locally symmetric varieties. In Section 3, we give the definitions of abelian varieties and K3 surfaces, introduce their polarisation and endomorphism structures, and describe their moduli spaces. In Sections 4 and 5, we recall the classical Kuga-Satake construction, then explain how it is lifted to the maps F between moduli spaces and $\widetilde{F}$ between their Hermitian symmetric domain overspaces respectively. In Section 6, we focus on six special families of lattice polarised K3 surfaces and explain some results and observations from our MAGMA realisation of the map $\widetilde{F}$. Finally in Section 7, we discuss some possible directions for future investigation. The MAGMA code used is included in the appendix at the end of the thesis.

Part II has the following outline: Section 8 contains definitions concerning Kodaira dimension, modular forms, and the Kuga varieties $\mathfrak{X}_p^n$, their singularities and compactification. We also give the general strategy of applying S. Ma's theorem (Theorem 8.2.10), and introduce some necessary tools for our investigation. In Section 9, we show that the assumptions in Ma's theorem are satisfied, namely, for $n > 2$ and any p, the particular compactification X of $\mathfrak{X}_p^n$ constructed in [PSMS] has canonical singularities. In Section 10, we compute the Kodaira dimension of $\mathfrak{X}_p^n$ for all but 27 combinations of the indices (n, p), using some results and techniques about modular forms.

Remark on notations

We will use the following notation throughout the thesis.

- The symbol W usually means a complex vector space, and Λ means a lattice of full rank in W. The symbol V usually means a real vector space, and Λ' means a lattice of full rank in V.
- Let L/k be a finite extension of fields. Let G be an algebraic group defined over k. We denote by $G(L)$ the corresponding algebraic group defined over L.
- Let R be a ring, V be an R-module, and $\mathbb{K}$ be a field over R. Then we write $\mathbb{K}$-extension of V as $V_{\mathbb{K}} := V \otimes_R \mathbb{K}$.
- Round brackets are used in matrices, and squared brackets are used in block matrices.
- In Part I Section 3.3 and Section 6, we choose most of our notations to match with that in the main reference [Sh]. In particular, we use Z to denote an element in any Hermitian symmetric domain, including in a Siegel upper half space. In Part II, we follow the traditional notation to denote an element in the Siegel upper half space $\mathcal{S}_g$ by τ.

Note that some necessary definitions for the work in Part II have already been introduced in Part I. We hope that the index pages at the very end of the document will be helpful for navigating through the thesis.

Part I
Kuga-Satake varieties of moduli spaces of K3 surfaces of Picard rank 14

2 Locally symmetric varieties

In this section we give a classification of locally symmetric varieties, and explain how it motivates our work in Part I.

A locally symmetric space is an arithmetic quotient of a symmetric space of non-compact type. It is a locally symmetric variety if and only if the overspace of this quotient is a Hermitian symmetric domain. We will explain these notions in the following subsections.

2.1 Symmetric spaces and locally symmetric spaces

Locally symmetric varieties are locally symmetric spaces: locally, they look like symmetric spaces, which are differentiable complex manifolds with extra structure. In some texts *e.g.* [He], symmetric spaces are also referred to as Riemannian globally symmetric spaces. In this subsection, we will define symmetric space and locally symmetric space. First let us recall the definition of a Riemannian manifold.

Definition 2.1.1. [He, Chapter I, Section 9]

Let M be a differentiable complex manifold. A **Riemannian structure** *g on M is a tensor field of type* $(0,2)$, i.e. *an element in $\Gamma((T^*M)^{\otimes 2})$, such that*

(i) $g(X,Y) = g(Y,X)$ *for all* $X, Y \in TM$.

(ii) For all $p \in M$, g_p is a positive definite bilinear form on $T_pM \times T_pM$.

The pair (M,g) is called a **Riemannian manifold**.

A diffeomorphism of Riemannian manifolds $s\colon (M,g) \to (N,h)$ satisfying $s^*h = g$ is called an **isometry**. *An isometry s of M is* ***involutive*** *if s^2 is the identity morphism.*

A symmetric space and a locally symmetric space are defined as follows.

Definition 2.1.2. [He, Chapter IV, Section 3 and 5]

A Riemannian manifold M is a **symmetric space** *if every point $p \in M$ is an isolated fixed point of an involutive isometry of M.*

A Riemannian manifold M' is a **locally symmetric space** *if there exists a symmetric space M such that for any point $p' \in M'$, there is a neighbourhood $N_{p'}$ of p' and an isometry $\varphi_{p'}$ taking $N_{p'}$ to an open neighbourhood of $\varphi_{p'}(p')$ in M.*

In fact, one can characterise symmetric spaces in terms of Lie groups.

Theorem 2.1.3. [He, Theorem IV.2.5, IV.3.3]

Let M be a symmetric space and p be a point in M. Let $G := I_0(M)$ be the identity component of the group of isometries of M and K be the stabiliser subgroup with respect to p. Then G is a connected Lie group, K is a compact subgroup of G, and the quotient G/K is diffeomorphic to M.

Moreover, symmetric spaces can be divided into compact type and non-compact type depending on the Lie groups G, and K. We are interested in the symmetric spaces of non-compact type. They are homogeneous manifolds *i.e.* they admit a transitive G-action. In particular, a locally symmetric space is an arithmetic quotient of a symmetric space of non-compact type. To conclude this subsection, we give the specific characterisations of symmetric spaces of non-compact type and define a locally symmetric space using Theorem 2.1.3.

Definition 2.1.4. [BJ, III.2.1, III.2.5]

A **symmetric space of non-compact type** *is a Lie group quotient G/K, where G is a connected reductive real Lie group, and K is a maximal compact subgroup of G.*

If G is an algebraic group defined over $\mathbb{Q}$, then a subgroup $\Gamma < G(\mathbb{Q})$ is called an **arithmetic subgroup** *if it is commensurable with $G(\mathbb{Z})$* i.e. *$\Gamma \cap G(\mathbb{Z})$ has finite index in both Γ and $G(\mathbb{Z})$.*

Let G/K be a symmetric space of non-compact type. Let Γ be an arithmetic subgroup of G. Then Γ acts properly discontinuously on X and the biquotient $\Gamma\backslash G/K$ is called a **locally symmetric space***.*

Remark 2.1.5.

(i) [K, VII.2] *Any semisimple Lie group with finite centre is reductive.*

(ii) [He, Theorem VI.2.2] *All maximal compact subgroups K of a connected semi-simple Lie group G are connected, and conjugate under an inner automorphism of G. This gives the quotient G/K the structure of a homogeneous manifold.*

(iii) [BJ, III.2.5] *Γ admits a torsion-free subgroup of finite index. When Γ is torsion-free, then the resulting locally symmetric space is smooth.*

2.2 Hermitian symmetric domains

Hermitian symmetric domains form a special class of symmetric spaces of non-compact type, and are vital in the construction of locally symmetric varieties. In this subsection, we introduce the definition and a characterisation of Hermitian symmetric domains, and give a classification of these objects.

Definition 2.2.1. [He, Chapter VIII, Section 1 and 4]

Let M be a differentiable complex manifold with tangent bundle TM.

An **almost complex structure** *J on M is a tensor field of $(1,1)$ type such that $J^2 = -1$.*

Suppose M admits both an almost complex structure J and a Riemannian structure g. Then M is a **Hermitian symmetric space** *if*

(i) it admits a Hermitian structure. i.e. *$g(JX, JY) = g(X, Y)$ for all $X, Y \in TM$; and*

(ii) every point $p \in M$ is an isolated fixed point of an involutive holomorphic isometry of M. In particular, M is a symmetric space.

A Hermitian symmetric space is called a **Hermitian symmetric domain** *if it is a symmetric space of non-compact type.*

Remark 2.2.2. [He, Theorem VIII.7.1]

A Hermitian symmetric domain is a bounded domain. i.e. *a bounded open connected subset of $\mathbb{C}^n$ for some positive integer n.*

From now on, we use the abbreviation **HSD** for a Hermitian symmetric domain.

We are interested in irreducible HSDs. Their characterisation in terms of Lie groups is, of course, more restrictive than the one in Definition 2.1.4.

Theorem 2.2.3. [He, Theorem VIII.6.1]

The irreducible HSDs are exactly the manifolds G/K where G is a connected non-compact simple Lie group with centre containing only the trivial element; K has a non-discrete centre and is a maximal compact subgroup of G.

Base on this characterisation, there is a classification of HSDs which depends only on the Lie group G in the quotient G/K. This separates all HSDs into four classical types I to IV, and two more exceptional types when $G = E_6$ and E_7. See [Lo, Section 3], [He, Table X.6.V].

Type of HSD	G	K
$\mathrm{I}_{p,q}$	$\mathrm{SU}(p,q)$	$\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$
II_m	$\mathrm{SO}^*(2m)$	$\mathrm{U}(m)$
III_g	$\mathrm{Sp}(2g)$	$\mathrm{U}(g)$
IV_n	$\mathrm{SO}^+(2,n)$	$\mathrm{SO}(2)\times\mathrm{SO}(n)$

Table 1: Classical types of HSDs

We give the definitions of the Lie groups $\mathrm{U}(m)$, $\mathrm{Sp}(2g)$, $\mathrm{SO}^+(p,q)$ and $\mathrm{SO}^*(2m)$, where the notations for the last two are less standard. The definitions of all Lie groups appearing above can be found in [He, Chapter X, Section 2.1].

Definition 2.2.4. *Let $\mathbf{1}_n$ be the identity matrix of size n. Define the matrices*

$$I_{p,q} := \begin{bmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{bmatrix}, \qquad J_g := \begin{bmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{bmatrix}$$

*(i) The **indefinite unitary group** $\mathrm{U}(p,q)$ is the group of matrices*

$$\left\{ M \in \mathrm{GL}_{p+q}(\mathbb{C}) : M^t I_{p,q} \overline{M} = I_{p,q} \right\}.$$

*We call the group $\mathrm{U}(m) = \mathrm{U}(m,0) = \mathrm{U}(0,m)$ the **definite unitary group** of degree m.*

*(ii) The **symplectic group** $\mathrm{Sp}(2g)$ is the group of matrices*

$$\left\{ M \in \mathrm{GL}_{2g}(\mathbb{R}) : M^t J_g M = J_g \right\}.$$

*In fact J_g is the matrix associated to a skew-symmetric bilinear form on $\mathbb{R}^{2g}$ called the **standard symplectic form**.*

*(iii) The **indefinite special orthogonal group** $\mathrm{SO}(p,q)$ is the group of matrices*

$$\left\{ M \in \mathrm{SL}_{p+q}(\mathbb{R}) : M^t I_{p,q} M = I_{p,q} \right\}.$$

The determinant of any member is 1, i.e. *it preserves orientation of the entire $p+q$-dimensional vector space.*

*We call the group $\mathrm{SO}(n) = \mathrm{SO}(n,0) = \mathrm{SO}(0,n)$ the **definite special orthogonal group** of degree n.*

*The **reduced orthogonal group** $\mathrm{SO}^+(p,q)$ is the identity component of the group $\mathrm{SO}(p,q)$. It contains matrices that preserve orientation of the p-dimensional positive definite subspaces.*

(iv) The Lie group $\mathrm{SO}^*(2m)$ is the group of matrices

$$\left\{ M \in M_{2m}(\mathbb{C}) : M^t J_m \overline{M} = J_m,\ M^t M = \mathbf{1}_{2m} \right\}.$$

Equivalently, it is the group of matrices in $\mathrm{SO}(2m,\mathbb{C}) = \left\{ M \in \mathrm{SL}_{2m}(\mathbb{C}) : M^t M = I_{2m} \right\}$ which leaves invariant the skew Hermitian form

$$-z_1\bar{z}_{m+1} + z_{m+1}\bar{z}_1 - z_2\bar{z}_{m+2} + z_{m+2}\bar{z}_2 - \cdots - z_m\bar{z}_{2m} + z_{2m}\bar{z}_m.$$

2.3 Locally symmetric varieties

In general, locally symmetric spaces are not projective. However, a locally symmetric space that is an arithmetic quotient of a HSD is a quasi-projective variety by the **Baily-Borel Theorem** [Lo, Section 4]. We call varieties that arise in this way **locally symmetric varieties**, abbreviated as **LSVs**. In this subsection, we will study the geometric interpretation of LSVs as modular varieties.

From Definition 2.1.4, LSVs are exactly the Lie group biquotients $\Gamma\backslash G/K$ where G/K gives a HSD, and Γ is an arithmetic subgroup of G. Such a characterisation allows LSVs to inherit a classification from HSDs: the type of a LSV is the type of its overspace HSD.

It is well known that type III and certain type IV classical types of LSVs are modular varieties.

Definition 2.3.1. [Mi, Section 7]

A **moduli problem** *over a field k is a contravariant functor $\mathcal{F}$ from the category of (some class of) schemes over k to the category of sets. A variety S over k is called a* ***modular variety*** *if it is a solution to the moduli problem $\mathcal{F}$* i.e. *there is a natural isomorphism $\phi\colon \mathcal{F} \mapsto \mathrm{Hom}_k(\bullet, S)$.*

In this thesis, by a moduli space we always mean a coarse moduli space: a solution to a moduli problem $\mathcal{F}$ such that $\mathcal{F}(S)$ is the set of isomorphism classes of the structured algebraic varieties that belong to a family $f : \mathcal{X} \to S$. Some common structures shared by a family of varieties include polarisation (Sections 3.1.1 and 3.5.1) and endomorphism structures (Section 3.3.2). For those type III or type IV LSVs that are coarse moduli spaces, the HSD overspace of such a LSV behaves like a parametrisation space of the structured algebraic varieties (this space is what will be called the period domain from Section 3 onwards). The quotient of the HSD overspace by the arithmetic group Γ identifies the isomorphic varieties in the family. In fact the type II LSVs also admit a similar modular interpretation ([He, Exercise X.D.1], [BL, Section 9.5]). Table 2 gives a summary of the modular interpretations of LSVs of classical type II, III and IV. Note that the specific structures of the varieties parametrised by a LSV of one of the above types depend on the group Γ.

Type of LSV	G	**Modular interpretation**
II_m	$\mathrm{SO}^*(2m)$	Moduli spaces of polarised abelian $2m$-folds with totally definite quaternion multiplication
III_g	$\mathrm{Sp}(2g)$	Moduli spaces of polarised abelian g-folds
IV_r	$\mathrm{SO}^+(2, r)$	Moduli spaces of lattice polarised K3 surfaces of Picard rank $(20 - r)$ for $0 \leq r \leq 20$

Table 2: Some types of LSVs with modular intepertation

We are especially interested in the type IV_r series, because for r large, *i.e.* close to 20, the HSD overspace of each moduli variety of K3 surfaces of Picard rank r coincides with that of a different modular variety $\mathcal{M}$:

r	$\mathcal{M}$
20	(supersingular) points
19	modular curves
18	Hilbert modular surfaces
17	modular varieties of polarised abelian surfaces with level structure
16	modular varieties of deformation of generalised Kummer varieties
15	modular varieties of deformation of hyperkähler manifolds of type OG6
14	modular varieties of abelian 8-folds with totally definite quaternion multiplication

Table 3: Some modular varieties $\mathcal{M}$ with an analytic overspace being a type IV_r HSD for r large.

In particular, upon choosing the suitable group Γ in the biquotient $\Gamma\backslash \mathrm{SO}^+(2,20-r)/(\mathrm{SO}(2)\times\mathrm{SO}(r))$, it is possible that the resulting type IV_r LSV is isomorphic to $\mathcal{M}$ as modular varieties. We will see (Lemma 3.5.15) that as r decreases, the dimension of the LSV increases, and therefore the difficulty of finding a type IV_r LSV with two modular interpretations also increases. The case $r = 14$ is the case with smallest r where an identification of the analytic overspaces of two different modular varieties is known to the author. In fact, the cases $r = 16$ and $r = 15$ are also hard because there is no known explicit family of generalised Kummer varieties or OG6 varieties.

In fact, there is a necessary condition from classical literature for the existence of an isomorphism

$$F : \Gamma_1\backslash\mathcal{D}_1 \longrightarrow \Gamma_2\backslash\mathcal{D}_2$$

from a LSV of one of the classical types to a LSV of type II or III. Note that F lifts to a holomorphic isometry $\widetilde{F} : \mathcal{D}_1 \to \mathcal{D}_2$ that is equivariant with respect to the actions of the groups G_1 and G_2 on $\mathcal{D}_1 = G_1/K_1$ and $\mathcal{D}_2 = G_2/K_2$ respectively. Satake studied [Sa] a more general question of when does $\mathcal{D}_1$ **holomorphically imbed** into $\mathcal{D}_2$, *i.e.* when does an equivariant holomorphic isometry that embeds $\mathcal{D}_1$ into $\mathcal{D}_2$ exist. The existence of such a holomorphic imbedding is equivalent to the existence of an injective homomorphism of the Lie algebras $\mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$ and an extra analytic condition, and by checking the latter conditions, Satake has come up with a complete classification of the problem.

In particular for the case $r = 17$, there is an exceptional Lie algebra isomorphism between the associated Lie algebras of modular varieties of K3 surfaces and abelian surfaces [He, Section X.6.4(iii)]

$$\mathfrak{so}^+(2,3) \simeq \mathfrak{sp}(2).$$

In [Sa], it is proved that given any HSDs $\mathcal{D}_1$ of type IV_1 and $\mathcal{D}_2$ of type III_3, there exists a holomorphic imbedding $\widetilde{F}\colon \mathcal{D}_1 \to \mathcal{D}_2$. So by choosing an arithmetic subgroup Γ of $\mathrm{SO}^+(2,3)$, we have a mapping of LSVs

$$F : \Gamma\backslash\mathcal{D}_1 \longrightarrow \widetilde{F}(\Gamma)\backslash\mathcal{D}_2.$$

Subject to a suitable choice of Γ, it is possible that F is in fact an isomorphism.

The source [He, Section X.6.4(viii)] states another exceptional Lie algebra isomorphism which corresponds to the case $r = 14$:

$$\mathfrak{so}^+(2,6) \simeq \mathfrak{so}^*(8).$$

Moreover from [Sa], a type IV_6 HSD is always holomorphically imbedded into a type II_4 one. These results further assert the possibility for an isomorphism between a modular variety of K3 surfaces of Picard rank 14 and a modular variety of abelian 8-folds with totally definite quaternion multiplication to exist. On top of that, it is hinted in [KSTT] that such an isomorphism comes from the Kuga-Satake construction [KS] which takes a K3 surface to an abelian variety called the Kuga-Satake variety.

3 Moduli of K3 surfaces and abelian varieties

Before we give the details of the Kuga-Satake construction, let us recall some basic facts about K3 surfaces and abelian varieties, as well as their moduli spaces. Thoughout the thesis, we work over the complex numbers.

Both an abelian variety and a K3 surface are smooth complex projective varieties. In Section 3.1, we give the notions of polarisations, polarised Hodge structures and period maps related to (a family of) smooth complex projective varieties. This is essential for the treatment of moduli spaces of abelian varieties, abelian varieties with totally definite quaternion multiplication and lattice polarised K3 surfaces in Sections 3.2, 3.3 and 3.5 respectively. In Section 3.4, we recall some classical facts about lattices, as they encode a lot of information about polarised K3 surfaces.

3.1 Smooth complex projective varieties

3.1.1 Polarisation

Before giving the definition of a polarisation, we need to first recall the definition of the first Chern map [Vo, Section 7.1.3]. Consider the exponential short exact sequence for a smooth complex projective variety X.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

It induces a long exact sequence of cohomology groups. We define the **first Chern map** to be the connecting homomorphism

$$c_1 \colon H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}).$$

Remark 3.1.1. *Concerning the first Chern map, note that*

(i) [Hart, Exercise III.4.5] *The domain $H^1(X, \mathcal{O}_X^*)$ can be identified with the **Picard group** $\mathrm{Pic}(X)$, the group of isomorphism classes of line bundles on X.*

(ii) [Vo, Section 7.2.2] *We define $\mathrm{Pic}^0(X)$ to be the kernel of the first Chern map c_1. It is also the subgroup in $\mathrm{Pic}(X)$ of line bundles that are algebraically equivalent to 0. The image of c_1 is called the* **Néron-Severi group** $\mathrm{NS}(X)$ *of X.*

Definition 3.1.2. [Vo, Theorem 7.8, 7.10] *A* **polarisation** *of a projective variety X is given by $c_1(L)$, the first Chern class of a choice of an ample line bundle L on X.*

The ample line bundle L realises projectivity of the variety X: there exists an integer $m > 0$ such that $L^{\otimes m}$ is very ample [Hart, Remark 7.4.3]. The existence of a very ample line bundle is equivalent to the existence of an embedding of X into a projective space. To be specific,

Theorem 3.1.3. [Hart, Section II. 5, Theorem II.7.1(b)]
Let L be a very ample line bundle on X over $\mathbb{C}$. Let $\{s_0, \cdots, s_r\} \subset H^0(X, L)$ be global sections which generate L. Then there exists an immersion

$$\varphi \colon X \longrightarrow \mathbb{P}^r_{\mathbb{C}} := \mathrm{Proj}\ \mathbb{C}[x_0, \cdots .x_r]$$

such that $L \simeq \varphi^*(\mathcal{O}(1))$ and $s_i = \varphi^*(x_i)$.

Since the polarisation $c_1(L)$ is determined by the ample line bundle L, we also say as a shorthand that the polarisation of X is the line bundle $L \in \mathrm{Pic}(X)$.

Remark 3.1.4. *Note that the Chern map may not be injective. As long as $c_1(L_1) \simeq c_1(L_2)$ for the two ample line bundles L_1 and L_2, we do not distinguish the polarisations L_1 and L_2 in our shorthand.*

We will explain in details a few alternative definitions of a polarisation on a complex projective variety X: as the first Chern class of a positive line bundle (Remark 3.1.10); and as certain bilinear or Hermitian forms when in particular X is an abelian variety (Sections 3.2.1 and 3.2.2). We will also consider a lattice polarisation, which is a generalisation of a polarisation, for algebraic K3 surfaces (Section 3.5.1).

3.1.2 Hodge structure

We will recall the definition and some facts about Hodge structures, which are of particular importance in the study of moduli spaces of K3 surfaces and abelian varieties due to the famous Global Torelli Theorem.

Let V be a free R-module of finite rank where $R = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$.

Definition 3.1.5. [Hu, Definition 3.1.1], [Mi, Section 5]

A **Hodge structure of weight** k **on** V *is given by a direct sum decomposition of its complexified vector space*

$$V_{\mathbb{C}} := V \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$

The Hodge structure is said to be **real**, **rational** *or* **integral** *if* V *is real, rational or integral respectively.*

The dimensions $h^{p,q}$ *of the vector spaces* $V^{p,q}$ *are called the* **Hodge numbers**.

We will mainly use Definition 3.1.5 as the definition of a Hodge structure, but there is an alternative definition, which leads to the definition of the Mumford-Tate group and Hodge group of a rational Hodge structure.

Definition 3.1.6. [vG1, Proposition 1.4]

A Hodge structure of weight k *on* V *can be identified to a real representation of* $\mathbb{C}^*$*, which is the group homomorphism*

$$h\colon \mathbb{C}^* \longrightarrow \mathrm{GL}(V_{\mathbb{R}})$$

where for any $v \in V^{p,q}$, $h(z)$ *sends* v *to* $z^p \bar{z}^q v$.

Remark 3.1.7. [Hu, Section 3.1.4]

The homomorphism h *restricted to* $\mathbb{R}^*$ *is the* k*-th power map. Therefore,* h *can be recovered from its restriction to the kernel of the norm map* $\mathbb{U} := \ker(\mathrm{Nm}) = \{z \in \mathbb{C}^* : z\bar{z} = 1\} \simeq \mathbb{C}^*/\mathbb{R}_{>0}$.

Definition 3.1.8. [Hu, Section 3.3.4]

The ***Mumford-Tate group*** $\mathrm{MT}(V)$ *of a rational Hodge structure* h *is the smallest algebraic subgroup of* $\mathrm{GL}(V)$ *defined over* $\mathbb{Q}$ *satisfying* $h(\mathbb{C}^*) \subset \mathrm{MT}(V)(\mathbb{R})$. *Similarly, the* ***Hodge group*** $\mathrm{Hdg}(V)$ *of a rational Hodge structure* h *is the smallest algebraic subgroup of* $\mathrm{GL}(V)$ *over* $\mathbb{Q}$ *with* $h(\mathbb{U}) \subset \mathrm{Hdg}(V)(\mathbb{R})$. *Equivalently, the Hodge group can be defined by the surjection*

$$\begin{aligned} \mathrm{Hdg}(V) \times \mathbb{R}^* &\longrightarrow \mathrm{MT}(V) \\ (g, \mu) &\longmapsto g\mu. \end{aligned}$$

By [Vo, Section 6.1], if X is a smooth complex projective variety of dimension n, then the torsion free part of its singular cohomology group $H^k(X, \mathbb{Z})$ for $0 \leq k \leq 2n$ has a Hodge structure of weight k

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}(X)$ is given by the Dolbeault cohomology group $H^q(X, \Omega_X^p)$.

With the notion of Hodge structure, we can say more about the image of the first Chern map c_1.

Theorem 3.1.9 (Lefschetz' theorem on (1, 1) classes)**.** [Vo, Theorem 11.30]

Let X *be a smooth complex projective variety. Then*

$$\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z}).$$

Therefore, a polarisation $c_1(L)$ of X is a $(1,1)$-form. Moreover, it is a **Kähler form**: it corresponds to the Kähler metric which gives X the structure of a Kähler manifold (see [Vo, Sections 3.1, 3.3.1] and Remark 3.5.4). In particular, all complex projective varieties are Kähler manifolds.

Remark 3.1.10. *Any line bundle L on a compact complex manifold X is said to be* ***positive*** *[Vo, Section 3.3.1] if $c_1(L)$ is a Kähler form for X. In fact by the Kodaira embedding theorem [Vo, Theorem 7.3], any positive line bundle on a complex projective variety is ample. This leads to an alternative definition of a polarisation on a complex projective variety [BL] as the first Chern class of a positive line bundle.*

There is also a notion of polarisation on Hodge structures.

Definition 3.1.11. [Hu, Section 3.1]

Let $V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$ be a Hodge structure of weight k over $R = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$.

Define the **Weil operator** *C as the element in $\mathrm{GL}(V_{\mathbb{C}})$ which acts on $V^{p,q}$ by multiplication by i^{q-p}. In particular it preserves the real vector space $(V^{p,q} \oplus V^{q,p}) \cap V_{\mathbb{R}}$.*

Define the ***Tate Hodge structure*** *$R(m)$ to be the Hodge structure of the R-submodule/subvector space $(2\pi i)^m R$ of $\mathbb{C}$ of weight $-2m$, such that $R(m)^{-m,-m}$ is 1-dimensional.*

A **morphism of Hodge structures of weight** *l is an R-linear map $f\colon V_1 \to V_2$ such that its $\mathbb{C}$-linear extension $f_{\mathbb{C}}$ satisfies $f_{\mathbb{C}}(V_1^{p,q}) \subset V_2^{p+l,q+l}$.*

If V_1 and V_2 are Hodge structures of weight k and l respectively, then the tensor product vector space $V_1 \otimes V_2$ also has a natural Hodge structure of weight $k+l$ given by

$$(V_1 \otimes V_2)^{p,q} = \bigoplus (V_1^{p_1,q_1} \otimes V_2^{p_2,q_2}),$$

where the direct sum is taken over all pairs (p_1, q_1) and (p_2, q_2) such that $p_1 + p_2 = p$.

A **polarisation** *of Hodge structure V of weight k is a morphism of Hodge structures of weight 0*

$$\Psi\colon V \otimes V \longrightarrow R(-k)$$

such that its real linear extension gives a positive definite symmetric form

$$\begin{aligned} q\colon V_{\mathbb{R}} \otimes V_{\mathbb{R}} &\longrightarrow \mathbb{R} \\ (v, w) &\longmapsto (2\pi i)^k \Psi_{\mathbb{R}}(v, Cw) \in \mathbb{R}. \end{aligned}$$

We say the Hodge structure of V is ***polarised*** *if it admits a polarisation.*

From the definition of a polarsation for a Hodge structure, we may derive a set of equivalent conditions.

Theorem 3.1.12. [Mi, Section 5, Polarizations]

Consider an R-module V with a weight k Hodge structure, and let b be an R-bilinear form on V. Then $\Psi := (2\pi i)^{-k} b$ is a polarisation of the Hodge structure on V if and only if it satisfies the ***Hodge-Riemann relations****:*

(i) For $x \in V^{p_1,q_1}, y \in V^{p_2,q_2}$, $\Psi_{\mathbb{C}}(x, y) \neq 0$ only if $(p_1, q_1) = (q_2, p_2)$.

(ii) $\Psi(v, w) = (-1)^k \Psi(w, v)$ for all $v, w \in V$.

(iii) $(2\pi i)^k i^{p-q} \Psi_{\mathbb{C}}(x, \overline{x}) > 0$ for all $x \in V^{p,q}$.

Proof. Let us first prove the only if part.

Firstly for part (i): let $x \in V^{p_1,q_1}$ and $y \in V^{p_2,q_2}$. Then $(x, y) \in (V \otimes V)^{p_1+p_2,q_1+q_2}$. Since Ψ is a weight 0 morphism of Hodge structures, $\Psi_{\mathbb{C}}(x, y) \neq 0$ only if $\Psi_{\mathbb{C}}(x, y) \in R(-k)^{k,k}$. That is $p_1 + p_2 = k$, or $p_2 = q_1$.

For part (ii): the $\mathbb{C}$-extension of the symmetric form b is also symmetric, so we have

$$i^{p-q} \Psi_{\mathbb{C}}(x, y) = (2\pi i)^{-k} b(x, y) = (2\pi i)^{-k} b(y, x) = i^{q-p} \Psi_{\mathbb{C}}(y, x) = (-1)^k i^{p-q} \Psi_{\mathbb{C}}(y, x)$$

for any $x \in V^{p,q}$ and $y \in V^{q,p}$. Now for any general $v, w \in V_{\mathbb{R}}$, we may write $v = \sum v^{p,q}$ and $w = \sum w^{p,q}$ summing over p, q, where $v^{p,q}, w^{p,q} \in V^{p,q}$. Then by above,

$$\Psi(v,w) = \sum \Psi_{\mathbb{C}}(v^{p,q}, w^{q,p}) = \sum (-1)^k \Psi_{\mathbb{C}}(w^{q,p}, v^{p,q}) = (-1)^k \Psi(w,v).$$

Finally for part (iii): let $x \in V^{p,q}$, we have $x + \overline{x} \in V_{\mathbb{R}}$. So $q(v,v) > 0$ implies

$$0 < (2\pi i)^k \Psi_{\mathbb{C}}(x + \overline{x}, i^{q-p}x + i^{p-q}\overline{x}) = (2\pi i)^k (i^{p-q}\Psi_{\mathbb{C}}(x, \overline{x}) + i^{q-p}\Psi_{\mathbb{C}}(\overline{x}, x)) = 2 \cdot (2\pi i)^k i^{p-q} \Psi_{\mathbb{C}}(x, \overline{x})$$

where the last equality is due to part (i).

We now give the proof for the if part. The image of Ψ lies in $(2\pi i)^{-k}R$, which has a Tate Hodge structure $R(-k)$. For any $x \in V^{p_1,q_1}$ and $y \in V^{p_2,q_2}$ such that $\Psi_{\mathbb{C}}(x,y) \in R(-k)^{k,k} \neq 0$, part (i) implies that

$$(x,y) \in (V \otimes V)^{p_1+q_1,p_2+q_2} = (V \otimes V)^{k,k}.$$

Therefore $\Psi : V \times V \to R(-k)$ is a morphism of Hodge structures of weight 0. By extending to $\mathbb{C}$, parts (i) and (ii) imply that for all $v, w \in V_{\mathbb{R}}$ with $v = \sum v^{p,q}$ and $w = \sum w^{p,q}$ summing over $v^{p,q}, w^{p,q} \in V^{p,q}$,

$$q(v,w) = \sum (2\pi i)^k i^{p-q} \Psi_{\mathbb{C}}(v^{p,q}, w^{q,p}) = \sum (2\pi i)^k (-1)^k i^{p-q} \Psi_{\mathbb{C}}(w^{q,p}, v^{p,q}) = q(w,v).$$

Again consider $v \in V_{\mathbb{R}}$ with $v = \sum v^{p,q}$, then $v^{p,q} = \overline{v^{q,p}}$. So parts (i) and (iii) imply

$$q(v,v) = \sum (2\pi i)^k (i)^{p-q} \Phi_{\mathbb{C}}(v^{p,q}, v^{q,p}) > 0.$$

□

In fact, a polarisation on the rational Hodge structure of the first and second cohomology groups of a projective variety is induced by the polarisation on the projective variety. Let $R = \mathbb{Q}$. Consider a polarised variety X of dimension n with a (integral) Kähler form ω. Then for $k \leq n$, its k^{th} cohomology group $H^k(X, \mathbb{Q})$ (up to torsion) is a Hodge structure with the **Hodge-Riemann pairing** [Hu, Section 3.1, Equation (1.5)]

$$(u,v) \longmapsto (-1)^{k(k-1)/2} \int_X u \wedge v \wedge \omega^{n-k} \in \mathbb{Q}.$$

Note that when $H^k(X, \mathbb{Q})$ is the middle cohomology *i.e.* $n = k$, the Hodge-Riemann pairing is just the intersection form (up to sign), and is independent of ω.

Define the **primitive part** of the cohomology group $H^k(X, \mathbb{Q})$ to be

$$H^k(X, \mathbb{Q})_p := \ker \left(\wedge \omega^{n-k+1} \colon H^k(X, \mathbb{Q}) \longrightarrow H^{2n-k+2}(X, \mathbb{Q}) \right).$$

Then the Hodge-Riemann pairing twisted by $(2\pi i)^{-k}$ and restricted to the primitive cohomology satisfies the Hodge-Riemann relations [GrifH, Section 1.7, The Lefschetz Decomposition]. In other words, it is a polarisation form Ψ on $H^k(X, \mathbb{Q})_p$.

In particular, when $k = 1$, we have $H^1(X, \mathbb{Q}) = H^1(X, \mathbb{Q})_p$, so the twisted Hodge-Riemann pairing gives the polarisation on the first cohomology.

As for $k = 2$, we have

$$H^2(X, \mathbb{Q}) = \mathbb{Q} \cdot \omega \oplus H^2(X, \mathbb{Q})_p$$

by [GrifH, Section 1.7, Hard Lefschetz Theorem]. If we change the sign of the twisted Hodge-Riemann pairing on $\mathbb{Q} \cdot \omega$, then this gives a polarisation form Ψ on the entire cohomology group $H^2(X, \mathbb{Q})$.

The Kähler form ω also determines a polarisation on the integral Hodge structure of $H^k(X, \mathbb{Z})$ for $k = 1, 2$. By restricting the polarisation form for $H^k(X, \mathbb{Q})$ obtained from the twisted Hodge-Riemann pairing to $H^k(X, \mathbb{Z})$, we have a bilinear map

$$H^k(X, \mathbb{Z}) \times H^k(X, \mathbb{Z}) \longmapsto \mathbb{Q}(-k).$$

The image in $\mathbb{Q}(-k)$ is a $\mathbb{Z}$-module, so the denominators of the fractions in the image are bounded, and rescaling the bilinear map by a sufficiently large integer gives a polarisation form on $H^k(X, \mathbb{Z})$.

3.1.3 Period map

We will discuss the period map associated to a family of polarised complex projective varieties.

First, we explain the notion of a period domain, which is the set of polarised Hodge structures of the same Hodge numbers and polarised by the same bilinear form ([DK, Section 3], [Mi, Section 7]). This uses the definition of a Hodge filtration, which is equivalent to a polarised Hodge structure.

Definition 3.1.13. [Hu, Section 3.1.1]

A **Hodge filtration** *associated to a Hodge structure V of weight k is the flag of subspaces $(F^{\bullet})$*

$$0 \subset F^k V_{\mathbb{C}} \subset F^{k-1} V_{\mathbb{C}} \subset \cdots \subset F^0 V_{\mathbb{C}} = V_{\mathbb{C}}$$

where $F^l V_{\mathbb{C}} := \oplus_{p \geq l} V^{p,q}$.

The Hodge structure can be recovered from the Hodge filtration by

$$V^{p,k-p} = F^p V_{\mathbb{C}} \cap \overline{F^{k-p} V_{\mathbb{C}}}.$$

A Hodge filtration can be identified to a point in a Grassmann variety, which is a product of Grassmannians. To be specific, let $(F^{\bullet})$ be a Hodge filtration associated to a polarised Hodge structure (V, Ψ) of weight k. We define $\underline{f} = (f_0, \cdots, f_k)$ where $f_l = \sum_{p \geq l} h^{p,q} = \dim F^l$, which is equivalent to giving the Hodge numbers. Thus $(F^{\bullet})$ is a point in the Grassmann variety $\mathrm{Gr}_{\underline{f}}(V_{\mathbb{C}}) := \Pi_{l=0}^k \mathrm{Gr}(f_l, V_{\mathbb{C}})$.

We now restrict ourselves to consider a vector space V over $\mathbb{R}$. Let Ψ be a bilinear form on V. The set of Hodge filtrations on (V, Ψ) of dimensions $\underline{f}$ such that Ψ is the polarisation of the Hodge structure is called the **period domain of** (V, Ψ) **of type** $\underline{f}$, denoted by $\mathcal{D}_{\underline{f}}(V, \Psi)$.

Theorem 3.1.14. [Mi, Section 7, Period domains]

A flag of subspaces in $\mathrm{Gr}_{\underline{f}} V_{\mathbb{C}}$ is a polarised Hodge structure of weight k with respect to a bilinear form Ψ on V if

(i) $V_{\mathbb{C}} = F^l \oplus \overline{F^{k-l+1}}$ *for all l;*

(ii) $\Psi_{\mathbb{C}}(F^l, F^{k-l+1}) = 0$ *for all l;*

(iii) $\Psi(v, w) = (-1)^k \Psi(w, v)$ *for all $v, w \in V$;*

(iv) $(2\pi i)^k i^{2l-k} \Psi_{\mathbb{C}}(x, \overline{x}) > 0$ *for all non zero elements $x \in F^l V_{\mathbb{C}} \cap \overline{F^{k-l} V_{\mathbb{C}}}$.*

Remark 3.1.15. *The first condition is required for a flag to give a Hodge structure. The second to the fourth conditions come from the Hodge-Riemann relations (Theorem 3.1.12). The last condition is open. This [Mi, Theorem 7.2] identifies $\mathcal{D}_{\underline{f}}(V, \Psi)$ with an open submanifold of a compact complex submanifold of* $\mathrm{Gr}_{\underline{f}}(V_{\mathbb{C}})$ *.*

One can obtain a specific expression of the period domain associated to (V, Ψ) of type $\underline{f}$. Fix a standard basis in V which identifies (V, Ψ) to $(\mathbb{R}^{f_0}, \Psi_0)$. Then with respect to these bases one can express a flag of subspaces as a square matrix of size f_0. The conditions in Theorem 3.1.14 can be translated into the language of matrices. Moreover, the following generalisation of **Witt's Theorem** is known.

Theorem 3.1.16. [DK, Section 3]

Let $G_{\mathbb{R}}$ be the group $\mathrm{Aut}(\mathbb{R}^{f_0}, \Psi_0)$. Then it acts transitively on $\mathcal{D}_{\underline{f}} := \mathcal{D}_{\underline{f}}(\mathbb{R}^{f_0}, \Psi_0)$ with compact isotropy subgroup K, and

$$\mathcal{D}_{\underline{f}} \simeq G_{\mathbb{R}}/K$$

is a complex non-compact homogeneous space.

We will provide more details of this isomorphism for a few specific period domains containing Hodge structures of weight 1 or 2 in later sections. Those period domains parametrise certain structured smooth complex projective varieties. More generally, let $f\colon \mathcal{X} \to S$ be a family of smooth complex projective varieties. Let $X_s = f^{-1}(s)$ be its fibre at $s \in S$. For each $s \in S$, consider $V_s := H^k(X_s, \mathbb{R})$ a real Hodge structure of weight k with polarisation determined by a $(1,1)$-form. Let Ψ_s be the bilinear form obtained as in Definition 3.1.11. By [DK, Section 3] and [Vo, 10.1.2], the set $(V_s, \Psi_s)_{s\in S}$ is a **polarised family of real Hodge structures**: it is a vector bundle $\mathbf{V} = \{V_s\}_{s\in S}$ together with a filtration on the associated **Hodge bundle** $\mathcal{V} := \mathcal{O}_S \otimes_{\mathbb{R}} \mathbf{V}$ and a polarisation of weight k given by a bilinear pairing of vector bundles

$$\Psi : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbb{R}(-k)$$

that give a Hodge filtration and a polarisation form Ψ_s on V_s at each point $s \in S$. In simpler terms, this means that S is covered by open connected subsets U's on which $(V_s, \Psi_s)_{s\in U} \to U$ is trivial. in particular, we can fix a k^{th} **marking of** X_s for each $s \in U$, which is an isomorphism $\mathcal{P}_s\colon (V, \Psi) \to (V_s, \Psi_s)$ where Ψ is a bilinear form on V. For each $s \in U$, $\mathcal{P}_s^{-1}\left((V_s, \Psi_s)\right)$ is a polarised Hodge structure of weight k on (V, Ψ) and of the same Hodge numbers $\underline{f}$, thus corresponds to a point in the period domain $\mathcal{D}_{\underline{f}}$. This gives a map from U to $\mathcal{D}_{\underline{f}}$ called the **period map**.

Theorem 3.1.17. [Mi, Theorem 7.3]
The period map

$$\begin{aligned} \mathcal{P}_U\colon U &\longrightarrow \mathcal{D}_{\underline{f}} \\ s &\longmapsto \mathcal{P}_s^{-1}\left((V_s, \Psi_s)\right) \end{aligned}$$

is a holomorphic map.

We would like to have a global version of the period map. For a general vector bundle, there is no canonical way to patch the trivialisation mappings, or the k^{th} markings in our case. There are many choices of transition functions up to monodromy of S, which give rise to a multi-valued global mapping $\mathcal{P} : S \to \mathcal{D}_{\underline{f}}$. However [DK, Section 3], the vector bundle $\mathbf{V}$ is in fact a real local coefficient system. That is, we can fix a standard basis in each (V_s, Ψ_s) which varies holomorphically with $s \in S$, such that $\mathbf{V}$ has transition functions given by matrices with constant entries. Specifically if we fix a point $s_0 \in S$ and let $V := V_{s_0}$ and $\Psi := \Psi_{s_0}$, then we have a homomorphism of groups

$$\pi_1(S, s_0) \longrightarrow \mathrm{GL}(V)$$

called a **monodromy representation**. A monodromy representation preserves the polarisation form Ψ, and the images of the monodromy representations form a subgroup $\Gamma(f)$ in $\mathrm{Aut}(V, \Psi)$ which we call the **monodromy group**. We can deduce more information about the monodromy group and define a global period map if each X_s is a smooth complex projective variety.

Theorem 3.1.18. [DK, Equation 3.4]
Let $f\colon \mathcal{X} \to S$ be a family of smooth complex projective varieties. Let $\underline{f}$ be the dimensions of the flag of subspaces associated to the real polarised Hodge structure $(H^k(X_s, \mathbb{R}), \Psi_s)$ for any $s \in S$. Fix a k-th marking $(H^k(X_s, \mathbb{R}), \Psi_s) \to (V, \Psi)$ and let Λ be the image of $H^k(X_s, \mathbb{Z})$.

*Then the monodromy group $\Gamma(f)$ is discrete, and is the group $\mathrm{Aut}(\Lambda, \Psi|_\Lambda)$ of automorphisms of the lattice Λ preserving the bilinear form $\Psi|_\Lambda$. Furthermore, there is a holomorphic map called the **global period map***

$$S \longrightarrow \Gamma(f)\backslash\mathcal{D}_{\underline{f}}.$$

Remark 3.1.19. [Mi, Theorem 7.10]
Every HSD arises as a connected component of a period domain This aligns with the existence of a modular intepretation of some locally symmetric spaces.

Finally, let us state the Global Torelli Theorem for a family of smooth complex projective varieties.

Theorem 3.1.20. [DK, Section 3]
A family of structured smooth complex projective varieties $f\colon \mathcal{X} \to S$ satisfies the **Global Torelli Theorem** *if for any two points $s, s' \in S$ with the same image under the global period map, there is an isomorphism of the fibres $\phi\colon X_s \mapsto X_{s'}$ such that $f^*([\omega_{s'}]) = [\omega_s]$, where ω_s and ω'_s are the Kähler forms defining the polarisations for X_s and $X_{s'}$ respectively.*

If a family $f\colon \mathcal{X} \to S$ satisfies the Global Torelli Theorem, then the image of S in $\Gamma(f)\backslash\mathcal{D}_{(\underline{f})}$ under the holomorphic map in Theorem 3.1.18 is a **coarse moduli space** *i.e.* it parametrises the members in the family $\mathcal{X}$ up to isomorphsm. Moreover, if $\mathcal{D}_{\underline{f}}$ is a HSD, then the quotient is a quasi-projective variety by the Baily-Borel Theorem. In particular, the quotient is also a LSV.

Remark 3.1.21. *We will show that any family of abelian varieties or K3 surfaces to be studied in this thesis satisfies the Global Torelli Theorem, and that the associated period domain is a (union of) HSD(s). The base S is therefore a quasi-projective variety, and we say that S is a modular variety.*

Conversely if a family $f\colon \mathcal{X} \to S$ of complex projective varieties is also a smooth projective map of complex algebraic varieties, then the associated period domain $D_{\underline{f}}$, as a polarised family of Hodge structures, satisfies ***Griffiths transversality*** *[Mi, Theorem 5.2]. It is a compatibility condition for a flat connection on the Hodge bundle $\mathcal{V}$ and the filtration on $\mathcal{V}$. Furthermore, this implies that $D_{\underline{f}}$ is a HSD ([Mi, Theorem 7.9]).*

We will discuss more about polarisations, polarised Hodge structures, period maps and moduli varieties specific to abelian varieties and K3 surfaces respectively in the following subsections.

3.2 Abelian varieties

3.2.1 Polarised abelian varieties

We start with the definition of a complex abelian variety.

Definition 3.2.1. [BL, Section 4.1]
An **abelian variety** *A* **of dimension** *g is a pair $(\mathbb{T} = W/\Lambda, c_1(L))$ where $W \simeq \mathbb{C}^g$, Λ is a lattice of rank $2g$ (full rank) in W, and $c_1(L)$, the first Chern class of an ample line bundle L on the complex torus $\mathbb{T}$, is the* **polarisation** *of A.*

For a detailed definition of lattices, see Theorem 3.4.1. In particular, a lattice is a $\mathbb{Z}$-module, and can be described by a basis. The expression of a basis of Λ embedded in W depends on the choice of a basis of W.

Definition 3.2.2. [BL, Section 1.1]
Let W/Λ be a complex torus. Fix a basis $e_1, \cdots, e_g$ of W and a basis $\lambda_1, \cdots, \lambda_{2g}$ of Λ. Write each λ_i in terms of the basis $e_1, \cdots, e_g$

$$\lambda_i = \sum_{j=1}^{g} \lambda_{j,i} e_j.$$

We call the $g \times 2g$ complex matrix

$$\Pi = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,2g} \\ \vdots & & \vdots \\ \lambda_{g,1} & \cdots & \lambda_{g,2g} \end{pmatrix}$$

the **period matrix** *of the torus.*

The polarisation of an abelian variety only depends on the choice of an ample line bundle L, so we also write $A := (W/\Lambda, L)$. We can also consider a polarisation on A as an alternating form that satisfies the following conditions.

Theorem 3.2.3. [BL, Theorems 2.1.6]
Let $\mathbb{T} = W/\Lambda$ be a complex torus. Let $E\colon W \times W \to \mathbb{R}$ be an alternating form. Then E represents the first Chern class $c_1(L)$ of an ample line bundle L on $\mathbb{T}$ if and only if the following conditions are satisfied

(i) $E(\Lambda, \Lambda) \subset \mathbb{Z}$;

(ii) $E(ix, iy) = E(x, y)$ *for all* $x, y \in W$; *and*

(iii) $E(x, ix) > 0$ *for all* $x \in W$.

Remark 3.2.4. *In Lemma 3.2.12, we will show how these conditions are related to the Hodge-Riemann relations.*

This gives rise to two more equivalent expressions for a polarisation of an abelian variety; by a Hermitian symmetric form or by a symmetric form.

Theorem 3.2.5. [BL, Theorem 2.1.7], [Harv, Lemma 2.63]
Let W be a $\mathbb{C}$-vector space. Let H be an arbitrary Hermitian symmetric form on W, i.e. *a bilinear form $H\colon W \times W \to \mathbb{C}$ which is $\mathbb{C}$-linear in the first component and satisfies $H(x, y) = \overline{H(y, x)}$ for all $x, y \in W$. Then there exists a real-valued alternating form E on W satisfying $E(ix, iy) = E(x, y)$ such that*

$$\mathrm{Im}(H(x, y)) = E(x, y) \text{ and } H(x, y) = E(x, iy) + iE(x, y).$$

Moreover, if H has signature (p, q), then the symmetric form

$$Q := \mathrm{Re}(H)\colon (x, y) \longmapsto E(x, iy)$$

has signature $(2p, 2q)$.

Clearly, any one of the three forms H, $Q = \mathrm{Re}(H)$, and $E = \mathrm{Im}(H)$ recovers the other two forms. Moreover, if E corresponds to the first Chern class of an ample line bundle on a complex torus, then Theorem 3.2.3(iii) implies that both Q and H are positive definite.

By considering a polarisation of an abelian variety as an alternating form, one can define the type of polarisation.

Definition 3.2.6. [BL, Section 3.1]
Let $A = (\mathbb{C}^g/\Lambda, E)$ be an abelian variety of dimension g where its polarisation is given by the alternating form E. Then there exits a basis of the lattice Λ with respect to which E is given by the matrix

$$\begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}$$

where D is the g-by-g diagonal matrix $\mathrm{diag}(d_1, \cdots, d_g)$ with $d_i > 0$, satisfying $d_i | d_{i+1}$ for $i = 1, \cdots, g-1$. Such basis of Λ is called a **symplectic basis** *for the associated Hermitian form H. The* **polarisation type of** *A is the vector $(d_1, \cdots, d_g)$, which is uniquely determined by E. We say that A is* **principally polarised** *if A has polarisation type $(1, \cdots, 1)$.*

Remark 3.2.7. *There exists a basis of $\mathbb{R}^{2g}$ such that this alternating form E is the standard symplectic form* i.e. *given by the matrix J_g.*

Remark 3.2.8. [BL, Section 8.3.1]

For any abelian variety $A = (W/\Lambda, E)$ of type $D = (d_1, \cdots, d_g)$, there are choices of a ***canonical level structure*** *which describes the chosen symplectic basis. To be specific, we define*

$$\Lambda(E) := \{w \in W : E(w, \Lambda) \subset \mathbb{Z}\}$$
$$K(E) := \Lambda(E)/\Lambda.$$

The group $K(E)$ has a pairing that depends on E. On the other hand, we define the group

$$K(D) := \mathbb{Z}^g/D\mathbb{Z}^g \oplus \mathbb{Z}^g/D\mathbb{Z}^g \quad \text{with} \quad \mathbb{Z}^g/D\mathbb{Z}^g := \prod_{i=1}^{g} \mathbb{Z}/d_i\mathbb{Z}$$

which also has a pairing that depends on D. As groups, $K(E) \simeq K(D)$. A canonical level structure of A is a choice of group isomorphism $K(E) \to K(D)$ that preserves the respective pairings. It is possible to futher rigidify an abelian variety by imposing a ***level n structure*** *for some positive integer n (see [BL, Section 8.3.2]).*

3.2.2 Complex structure

The complex torus $\mathbb{C}^g/\Lambda$ structure of an abelian g-fold A can be seen [DK, Section 4] as a real torus V/Λ' where $V = \Lambda'_{\mathbb{R}} \simeq \mathbb{R}^{2g}$, with a complex structure J on V. A real torus is a manifold with an additive group structure. In particular, any real torus V/Λ' of dimension $2g$ is diffeomorphic to $(S^1)^{2g} = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$.

Definition 3.2.9. *A* **complex structure** J **on a real vector space** V *is a linear operator satisfying $J^2 = -1$.*

The pair (V, J) can be identified to $\mathbb{C}^g$ by

$$V \xrightarrow{\iota} V_{\mathbb{C}} \simeq W \oplus \overline{W} \xrightarrow{\pi} W \simeq \mathbb{C}^g$$

where ι is the natural inclusion map, W and $\overline{W}$ are the $+i$ and $-i$ eigenspaces of J in V respectively with i being the imaginary unit in $\mathbb{C}$, and π is the projection map onto W. In fact, the above map is a $\mathbb{R}$-linear isomorphism given by

$$\mu \colon V \longrightarrow W$$
$$v \longmapsto \frac{1}{2}(v - iJ(v)).$$

The decomposition $V_{\mathbb{C}} \simeq W \oplus \overline{W}$ is a weight one Hodge structure of V. Setting $V^{0,1} = W$, then J is the Weil operator of the Hodge structure. Furthermore, if Ψ is the polarisation of this weight one Hodge structure, then in particular we have

$$2\pi i \Psi(v, J(v)) > 0$$

for all $v \in V$, and we also call J the **positive complex structure** with respect to Ψ. Let us give a precise description of this equivalence between a polarised Hodge structure of weight one and a positive complex structure.

Lemma 3.2.10. [DK, Theorem 4.1]

There is a natural bijection between the set of Hodge structures of weight one on V with polarisation form Ψ and the set of positive complex structures on V with respect to Ψ.

Proof. Given a positive complex structure J on V, we can define the $\mathbb{R}$-linear map μ as above and obtain the decomposition $V_{\mathbb{C}} = \mu(V) \oplus \overline{\mu(V)}$. Indeed for any $x \in V^{0,1} = \mu(V)$, there exists $v \in V$ such that $\mu(v) = x$, and

$$\begin{aligned} 0 < 2\pi i \cdot (-i) \cdot \Psi_{\mathbb{C}}(x, \overline{x}) &= \frac{-i}{4} \cdot 2\pi i \cdot \Psi_{\mathbb{C}}(v - iJ(v), v + iJ(v)) \\ &= \frac{-i}{4} \cdot 2\pi i \cdot (\Psi(v,v) + \Psi(J(v), J(v)) + i\Psi(v, J(v)) - i\Psi(J(v), v)) \\ &= \frac{1}{2} \cdot 2\pi i \cdot \Psi(v, J(v)). \end{aligned}$$

The last equality is due to Theorem 3.1.12(ii).

On the other hand, given a weight one Hodge structure $V_{\mathbb{C}} \simeq V^{1,0} \oplus V^{0,1}$, then for all $v \in V$, we can write $v = x + \overline{x}$ where $x \in V^{0,1}$. We can define a complex structure J by $J(v) = ix - i\overline{x}$. This is also a positive complex structure with respect to the polarisation form Ψ of V due to the same equations above. □

Remark 3.2.11. *With respect to Definition 3.1.6, the weight one Hodge structure on V that corresponds to the complex structure J is given by the group homomorphism*

$$\begin{aligned} h\colon \mathbb{C}^* &\longrightarrow \mathrm{GL}(V_{\mathbb{R}}) \\ a + bi &\longmapsto a + bJ. \end{aligned}$$

Lemma 3.2.12. [DK, Section 5] *The polarisation form Ψ of the weight one Hodge structure on (V, J) determines a polarisation of the abelian variety A.*

Proof. Define a real bilinear form E on W such that $E(x, y) = 2\pi i\Psi(u, v)$ for all $x = \mu(u)$ and $y = \mu(v)$ in W. From Theorem 3.1.12(ii), it is clear that E is an alternating form. We will show that E satisfies all three conditions in Theorem 3.2.3.

First, $(\Lambda', \Psi|_{\Lambda'})$ is a $\mathbb{Z}$-sub Hodge structure of (V, Ψ), so $2\pi i\Psi$ (resp. E) is integral with respect to Λ' (resp. Λ), which is the statement of 3.2.3(i). Let $x, y \in W$ with $x = \mu(u)$ and $y = \mu(v)$ for some $u, v \in V$. Note that $u = x + \overline{x}$ and $v = y + \overline{y}$, and so

$$\Psi(J(u), J(v)) = \Psi_{\mathbb{C}}(ix - i\overline{x}, iy - i\overline{y}) = \Psi_{\mathbb{C}}(x, \overline{y}) + \Psi_{\mathbb{C}}(\overline{x}, y) = \Psi(u, v).$$

Since

$$\mu(J(w)) = \frac{1}{2}(J(w) + iw) = i\mu(w)$$

for any $w \in V$, we have 3.2.3(ii):

$$E(ix, iy) = 2\pi i\Psi(J(u), J(v)) = 2\pi i\Psi(u, v) = E(x, y).$$

Also,

$$E(x, ix) = 2\pi i\Psi(u, J(u)) > 0$$

by the definition of a polarisation form. This gives 3.2.3(iii). □

Remark 3.2.13. *From now on, we will interchangeably use an ample line bundle L, or one of the forms Ψ, q on $V \simeq \mathbb{R}^{2g}$, or one of the forms H, E, Q on $W \simeq \mathbb{C}^g$ to denote the polarisation of an abelian variety or that of its weight one Hodge structure.*

Remark 3.2.14.
An almost complex structure of a differentiable complex manifold M as in Definition 2.2.1 gives a complex structure on the tangent space $T_p(M)$ that varies continuously with the point $p \in M$. In fact the almost complex structure of an abelian variety A is invariant with respect to translation in the torus [DK, Section 5], so the holomorphic tangent bundle is isomorphic to the trivial bundle with fibre $(V \simeq T_0(A), J)$. In particular, any abelian variety has a translation invariant Hermitian structure, which is given by the positive definite symmetric form $q = 2\pi i\Psi(\cdot, J\ \cdot)$.

3.2.3 Homomorphisms of abelian varieties

A homomorphism of abelian varieties is a homomorphism of complex tori compatible with the respective polarisations.

Definition 3.2.15. [BL, Section 1.4, 4.1]

A **homomorphism of complex tori** $f\colon \mathbb{T}_1 \to \mathbb{T}_2$ *is a holomorphic map that preserves the respective group structure.*

A **homomorphism of abelian varieties** $f : (\mathbb{T}_1, L_1) \to (\mathbb{T}_2, L_2)$ *is a homomorphism of complex tori* $f\colon \mathbb{T}_1 \to \mathbb{T}_2$ *such that* $f^* c_1(L_2) = c_1(L_1)$.

A homomorphism of complex tori $f\colon \mathbb{T}_1 \to \mathbb{T}_2$ is an **isomorphism** if there exists another homomorphism of complex tori $g\colon \mathbb{T}_2 \to \mathbb{T}_1$ such that $f \circ g = \mathbf{1}_{\mathbb{T}_2}$ and $g \circ f = \mathbf{1}_{\mathbb{T}_1}$. Similarly, a homomorphism of abelian varieties $f\colon A_1 \to A_2$ is an **isomorphism** if there exists another homomorphism of abelian varieties $g\colon A_2 \to A_1$ such that $f \circ g = \mathbf{1}_{A_2}$ and $g \circ f = \mathbf{1}_{A_1}$.

Under addition, the set of homomorphisms of abelian varieties from A_1 to A_2 forms an abelian group $\mathrm{Hom}(A_1, A_2)$. An **endomorphism of an abelian variety** A is a homomorphism of A into itself. Denote the set of endomorphisms of an abelian variety A by $\mathrm{End}(A)$. Under addition and composition, the set of $\mathbb{Q}$-endomorphisms $\mathrm{End}_{\mathbb{Q}}(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ forms an algebra.

We would like to focus on a special kind of homomorphisms of abelian varieties.

Definition 3.2.16. [BL, Section 1.2, Section 4.1]

An **isogeny of complex tori** *is a surjective homomorphism of complex tori with finite kernel.*

An **isogeny of abelian varieties** *is an isogeny between the underlying complex tori.*

Remark 3.2.17. [BL, Corollary 3.2.7]
Isogenies of abelian varieties define an equivalence relation. We say two abelian varieties are ***isogenous*** *if there is an isogeny between them. If* A_1 *and* A_2 *are non-isogenous abelian varieties, then* $\mathrm{Hom}(A_1, A_2) = 0$.

Remark 3.2.18. *An isogeny of abelian varieties does not preserve the polarisation type. Zarhin's trick shows that for all abelian* g*-fold* A*, and* g*-tuple* $D = (d_1, \cdots, d_g)$ *such that* $d_i | d_{i+1}$*, there exists an isogeny* f *such that* $f(A)$ *has polarisation type* D.

3.2.4 Moduli of polarised abelian varieties

Let $f\colon \mathcal{X} \to S$ be a family of polarised abelian g-folds over a connected complex variety S. By Theorem 3.1.18, the set of polarised weight one Hodge structures of the first cohomology groups $H^1(f^{-1}(s), \mathbb{R}) \simeq \mathbb{R}^{2g}$ forms the associated period domain for the family f. In this section, we will give more details of this period domain which leads to the expression of a coarse moduli space for the family.

Lemma 3.2.19. *The type of polarisation of the abelian variety* $A_s = f^{-1}(s)$ *is independent of* s.

Proof. The variable s varies in S continuously, while the type of polarisation is discrete. □

Let us describe the period domain that locally parametrises a family of polarised abelian g-folds $f\colon \mathcal{X} \to S$ of polarisation type D up to isomorphism.

Since the fibre A_s at any point $s \in S$ is diffeomorphic to a product of S^1, it is clear that its first cohomology group $H^1(A_s, \mathbb{R})$ is torsion free. One can compute the weight one Hodge structure on $H^1(A_s, \mathbb{R})$ using the de Rham cohomology [BL, Section 1.1.4], and associate it to a Hodge filtration of degrees $\underline{f} = (2g, g)$. The period domain $\mathcal{D}_{\underline{f}}$ is the set of $2g \times 2g$ complex matrices which represent bilinear forms that satisfy the Hodge-Riemann relations. But since it is possible to recover the entire

Hodge decomposition from $H^{1,0}(A_s)$, one can consider elements of $\mathcal{D}_{(2g,g)}$ as $2g \times g$ complex matrices. Moreover [DK, Equation 4.9, 4.10], under the symplectic basis of Λ_s in $A_s = (\mathbb{R}^{2g}, \Lambda_s)$, the corresponding element $\Pi \in \mathcal{D}_{\underline{f}}$ is a $2g \times g$ matrix

$$\Pi_s = \begin{bmatrix} Z \\ D \end{bmatrix}$$

where Z is a square complex matrix of degree g, and D is the polarisation type of A_s independent of s by Lemma 3.2.19. The conditions in Theorem 3.1.14 are equivalent to the two conditions in terms of Z

$$Z^t = Z, \mathrm{Im}(Z) = \frac{1}{2i}(Z - \overline{Z}) > 0.$$

Therefore we have

Theorem 3.2.20. *The period domain $\mathcal{D}_{(2g,g)}$ is isomorphic to the **Siegel upper half space of degree** g*

$$\mathcal{S}_g := \left\{ Z \in M_g(\mathbb{C}) : Z^t = Z,\ \mathrm{Im}(Z) > 0 \right\}.$$

Given the above expression of the period domain $\mathcal{D}_{(2g,g)}$, we can deduce that $\mathcal{D}_{(2g,g)}$ parametrises a family of polarised abelian varieties.

Theorem 3.2.21. [BL, Proposition 8.1.2]

*There exists a **complete family** $f\colon \mathcal{X} \to S$ of abelian varieties of some polarisation type D;* i.e. *such that under the multi-valued period map*

$$\mathcal{P}\colon S \longrightarrow \mathcal{S}_g,$$

every element in $\mathcal{S}_g$ has non-empty preimage.

Proof. Let $A_s = (\mathbb{C}^g/\Lambda_s, E_s)$ be the abelian g-fold corresponding to $s \in S$. Let $\lambda_1, \cdots, \lambda_g, \mu_1, \cdots, \mu_g$ be the symplectic basis of Λ_s, and $D = \mathrm{diag}(d_1, \cdots, d_g)$ be the polarisation type of A_s. Then with respect to the basis $\lambda_1, \cdots, \lambda_g, \mu_1/d_1, \cdots, \mu_g/d_g$ of $\mathbb{R}^{2g}$, the basis of Λ_s is given by the rows of

$$\Pi_s = \begin{bmatrix} Z \\ D \end{bmatrix}$$

where $Z = \mathcal{P}(s)$. That is Π_s is a period matrix of A_s. Moreover, the matrix $\mathrm{Im}(Z)^{-1}$ gives a Hermitian form H_z with respect to the basis $\mu_1/d_1, \cdots, \mu_g/d_g$ of $\mathbb{C}^g$, which is a polarisation of A_s of type D by Theorem 3.2.5. Therefore, given any $Z \in \mathcal{S}_g$, we have an abelian variety $A_Z := \mathbb{C}^g / \left((Z, D)\mathbb{Z}^{2g}\right)$ whose polarisation is H_Z. So A_Z belongs to the family of polarised abelian g-folds of type D with symplectic basis. □

In another direction, the period domain $\mathcal{D}_{(2g,g)} = \mathcal{S}_g$ can be identified with the quotient of a Lie group by a compact subgroup as in Theorem 3.1.16.

Theorem 3.2.22.

$$\mathcal{S}_g \simeq \mathrm{Sp}(2g)/\,\mathrm{U}(g).$$

Proof. The symplectic group $\mathrm{Sp}(2g)$ acts on $\mathcal{S}_g$ by [BL, Proposition 8.2.2]

$$\begin{aligned} \mathrm{Sp}(2g) \times \mathcal{S}_g &\longrightarrow \mathcal{S}_g \\ \left(M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, Z \right) &\longmapsto (A \cdot Z + B)(C \cdot Z + D)^{-1}. \end{aligned}$$

By [BL, Proposition 8.2.3], every $Z \in \mathcal{S}_g$ lies on the $\mathrm{Sp}(2g)$-orbit of $i\mathbf{1}_g \in \mathcal{S}_g$. Moreover, the stabiliser group of the action [BL, Proposition 8.2.3] is the compact subgroup

$$\mathrm{Sp}(2g) \cap \mathrm{O}(2g) \simeq \mathrm{U}(g).$$

□

Remark 3.2.23. *We can also prove transitivity of the action by applying Theorem 3.1.16. By Remark 3.2.7, each polarised Hodge structure of weight one on $H^1(A_s, \mathbb{R})$ is equivalent to the pair $(\mathbb{R}^{2g}, J_g)$, where J_g is the matrix associated to the standard symplectic form. From Definition 2.2.4,* $\mathrm{Sp}(2g) = \mathrm{Aut}(\mathbb{R}^{2g}, J_g)$. *So* $\mathrm{Sp}(2g)$ *preserves the polarisation form of a Hodge filtration in $\mathcal{D}_{(2g,g)}$, and therefore acts on $\mathcal{S}_g$ transitively.*

Remark 3.2.24. [HKW2, Chapter I.1]

The action of $\mathrm{Sp}(2g)$ *on $\mathcal{S}_g$ is an analogue of the linear fractional transformation.*

On the other hand, note that for any integer $k \geq g$, the Grassmannian $\mathrm{Gr}(g, \mathbb{C}^k)$ *is isomorphic to the orbit space $M_{k\times g}(\mathbb{C})/\,\mathrm{GL}(g, \mathbb{C})$ of all $k \times g$ matrices modulo right multiplication by* $\mathrm{GL}(g, \mathbb{C})$. *The Siegel upper half space $\mathcal{S}_g$ can be identified with a subset of* $\mathrm{Gr}(g, \mathbb{C}^{2g})$ *by sending an element Z to the* $\mathrm{GL}(g, \mathbb{C})$*-equivalence class of block matrices:*

$$Z \mapsto \begin{bmatrix} Z \\ \mathbf{1}_g \end{bmatrix}.$$

With respect to this alternative expression of elements in $\mathcal{S}_g$, the group $\mathrm{Sp}(2g)$ *acts by left multiplication:*

$$\gamma \cdot Z = \left[\gamma \cdot \begin{bmatrix} Z \\ \mathbf{1}_g \end{bmatrix}\right], \quad \gamma \in \mathrm{Sp}(2g), Z \in \mathcal{S}_g.$$

Remark 3.2.25. *By comparing to [He, Table X.6.V], the unitary group $U(g)$ is the maximal compact subgroup of* $\mathrm{Sp}(2g)$*, and $\mathcal{S}_g$ is an irreducible HSD of type III_g.*

Moreover, we have the expression for the monodromy group $\Gamma(f) < \mathrm{Sp}(2g)$ for a complete family f which leads to the holomorphic map

$$S \longrightarrow \Gamma(f)\backslash\mathcal{D}_{(2g,g)}$$

as in Theorem 3.1.18. The group $\Gamma(f)$ is an arithmetic subgroup of the symplectic group, and is called a **modular group** .

Theorem 3.2.26. *The monodromy group $\Gamma(f)$ for a family of polarised abelian varieties $f \colon \mathcal{X} \to S$ of polarisation type D depends on D, and is given by*

$$\Gamma_D = \left\{ M \in \mathrm{Sp}(2g, \mathbb{Q}) : M^t \cdot \Lambda_D \subset \Lambda_D \right\} < \mathrm{Sp}(2g)$$

where

$$\Lambda_D = \begin{bmatrix} \mathbf{1}_g & 0 \\ 0 & D \end{bmatrix} \mathbb{Z}^{2g}.$$

Proof. By [BL, Proposition 8.1,3], two polarised abelian varieties (A_Z, H_Z) and $(A_{Z'}, H_{Z'})$ parametrised by $\mathcal{S}_g$ are isomorphic if Z and Z' lie in the same Γ_D-orbit. □

In particular, we have verified that a family of polarised abelian varieties of type D satisfies the Global Torelli Theorem.

The quotient $\mathcal{A}_D := \Gamma_D\backslash\mathcal{S}_g$ is a normal complex analytic space as $\Gamma(f)$ acts properly discontinuously on $\mathcal{S}_g$ by [BL, Proposition 8.2.5], and is quasi-projective by the Baily-Borel theorem. We call $\mathcal{A}_D$ the **moduli variety of polarised abelian varieties of type** D. It has dimension $g(g+1)/2$ as $\mathcal{S}_g$ is of the same dimension, and it is a LSV of type III.

3.3 Abelian varieties with totally definite quaternion multiplication

In Table 2, we mentioned moduli space of abelian varieties with totally definite quaternion multiplication. In fact, a totally definite quaternion multiplication is a special type of endomorphism structure admitted by an abelian variety. Before studying this special moduli space, we first consider the endomorphism structure of simple abelian varieties.

3.3.1 Simple abelian subvarieties

In this subsection we will give a characterisation of simple abelian subvarieties.

Let $A := (\mathbb{T}, L)$ be an abelian variety. The polarisation L induces an anti-involution ρ on $\mathrm{End}_{\mathbb{Q}}(A)$, *i.e.* a self-inverse anti-homomorphism on $\mathrm{End}_{\mathbb{Q}}(A)$:

$$\rho^2 = \mathbf{1}_{\mathrm{End}_{\mathbb{Q}}(A)}, \text{ and } (f \circ g)^\rho = g^\rho \circ f^\rho \text{ for all } f, g \in \mathrm{End}_{\mathbb{Q}}(A).$$

This special anti-involution is called the **Rosati involution** (see [BL, Section 5.1]).

For an element f in $\mathrm{End}_{\mathbb{Q}}(A)$, we say f is **symmetric** if it is stable under the Rosati involution of $\mathrm{End}_{\mathbb{Q}}(A)$; f is an **idempotent** if $f^2 = f$. We denote the set of symmetric idempotents in $\mathrm{End}_{\mathbb{Q}}(A)$ as $\mathrm{End}^s_{\mathbb{Q}}(A)$.

Any abelian subvariety of A corresponds to a symmetric idempotent in $\mathrm{End}_{\mathbb{Q}}(A)$.

Theorem 3.3.1. [BL, Theorem 5.3.2]

There is a bijection between the set of abelian subvarieties of an abelian variety A and the set of symmetric idempotents in $\mathrm{End}_{\mathbb{Q}}(A)$

To be specific, if ε is an element in $\mathrm{End}^s_{\mathbb{Q}}(A)$, and d is the smallest positive integer such that $d\varepsilon \in \mathrm{End}(A)$, then under the above bijection, ε corresponds to the abelian subvarity $A^\varepsilon := \mathrm{Im}(d\varepsilon)$ in A.

An abelian variety is called **simple** if it does not contain any abelian subvariety apart from itself and 0. There is a simple decomposition of any abelian variety, unique up to isogeny.

Theorem 3.3.2 (Poincaré's Complete Reducibility Theorem)**.** [BL, Theorem 5.3.7]

Given an abelian variety A, there is an isogeny

$$A \sim A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_k^{n_k}$$

where $A_1, \cdots, A_k$ are non-isogenous simple abelian varieties. Moreover, the abelian varieties A_i and the integers n_i are unique up to isogenies and permutations.

Note that any non-zero endomorphism of a simple abelian variety A is an isogeny, thus a unit in $\mathrm{End}_{\mathbb{Q}}(A)$. Therefore $\mathrm{End}_{\mathbb{Q}}(A)$ is a division ring of finite dimension over $\mathbb{Q}$. In fact, for any (non-simple) abelian variety A, its $\mathbb{Q}$-endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(A)$ is semisimple.

Corollary 3.3.3. [BL, Corollary 5.3.8]

Suppose $A \sim A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_k^{n_k}$ is an isogeny decomposing the abelian variety A into a product of simple subvarieties. Then

$$\mathrm{End}_{\mathbb{Q}}(A) \simeq M_{n_1}(F_1) \oplus M_{n_2}(F_2) \oplus \cdots \oplus M_{n_r}(F_r)$$

where $F_i = \mathrm{End}_{\mathbb{Q}}(A_i)$ are division rings of finite dimension over $\mathbb{Q}$, and $M_{n_i}(F_i)$ are the rings of $n_i \times n_i$ matrices with entries in F_i.

3.3.2 Endomorphism structure of abelian varieties

In this subsection, we will introduce the possible endomorphism structures of an abelian variety. We will first focus on simple abelian varieties and give a classification of division rings that arise as their endomorphism algebras.

Let (F, ρ) be a division ring of finite dimension over $\mathbb{Q}$ with an anti-involution ρ. Let K be the centre of F, and K_0 be the fixed part in K by ρ.

One can define a quadratic form associated to (F, ρ) which we will describe as below [BL, Section 5.5]. The degree $[F : K]$ of F over K is a square d^2. Any element f in F has **reduced characteristic**

polynomial of f over K, which is the d^{th} power of a polynomial $t^d - a_1 t^{d-1} + \cdots + (-1)^d a_0 \in K[t]$. The **reduced trace of f over K** is defined as

$$\mathrm{tr}_{F|K}(f) = a_1.$$

For any subfield $k \leq K$, the **reduced trace of f over k** is defined as

$$\mathrm{tr}_{F|k}(f) = \mathrm{tr}_{K|k}(\mathrm{tr}_{F|K}(f))$$

where $\mathrm{tr}_{K|k}$ denotes the usual trace for field extension $k \subseteq K$. Finally we define a quadratic form on F by $f \mapsto \mathrm{tr}_{F|\mathbb{Q}}(f^\rho \cdot f)$.

Before we proceed, we give a list of definitions.

Definition 3.3.4. *[BL, Section 5.5]*
Let F be a division ring of finite dimension over $\mathbb{Q}$ with an anti-involution ρ.

(i) F is a **quaternion algebra** *if its degree over its centre K is 4, and it has a canonical anti-involution*

$$f \mapsto \overline{f} := \mathrm{tr}_{F|K}(f) - f.$$

(ii) ρ is **positive** *if the quadratic form $f \mapsto \mathrm{tr}_{F|\mathbb{Q}}(f^\rho \cdot f)$ is positive definite.*

Definition 3.3.5. *[BL, Section 5.5]*
Let Q be a number field.

(i) Q is a **totally complex number field** *if there is no embedding $Q \hookrightarrow \mathbb{C}$ that factors via $\mathbb{R}$.*

(ii) Q is a **totally real number field** *if every embedding $Q \hookrightarrow \mathbb{C}$ factors via $\mathbb{R}$.*

(iii) If Q is totally real, then an element $a \in Q$ is **totally positive** *(resp.* **totally negative***) if $\sigma(a) > 0$ (resp. $\sigma(a) < 0$) for every embedding $\sigma \colon Q \hookrightarrow \mathbb{R}$.*

If (F, ρ) is the endomorphism algebra of a simple abelian variety, then more can be said about the pair.

Proposition 3.3.6. [BL, Theorem 5.1.8, Lemma 5.5.2]
Let $A = (\mathbb{T}, L)$ be a simple abelian variety. Consider $F = \mathrm{End}_{\mathbb{Q}}(A)$, which is a division ring of finite dimension over $\mathbb{Q}$. Let ρ be the Rosati involution on F induced by the polarisation L. Then

(i) ρ is positive.

(ii) The fixed part K_0 of K by ρ is a totally real number field.

There is a classification of endomorphism algebras of simple abelian varieties:

Theorem 3.3.7. [BL, Theorem 5.5.3]
Let F be the endomorphism algebra over $\mathbb{Q}$ of a simple abelian variety and ρ be its Rosati involution. Let $K, K_0 \subset F$ be as defined at the beginning of the section. Then F falls into one of the following cases:

1. *F is* **of the first kind***: $K = K_0$. In particular, F can be a*

    i. **totally real number field***: $F = K$ and $f^\rho = f$ for all $f \in F$.*

ii. **totally indefinite quaternion algebra***: F is a quaternion algebra over K and every embedding $\sigma\colon K \hookrightarrow \mathbb{R}$ satisfies*

$$\sigma(F) \otimes \mathbb{R} \simeq M_2(\mathbb{R}).$$

Moreover, there exists an element $a \in F$ with $a^2 \in K$ totally negative such that $f^\rho = a^{-1}\overline{f}a$ for all $f \in F$.

iii. **totally definite quaternion algebra***: F is a quaternion algebra over K and every embedding $\sigma\colon K \hookrightarrow \mathbb{R}$ satisfies*

$$\sigma(F) \otimes \mathbb{R} \simeq \mathbb{H}$$

where $\mathbb{H}$ is the **Hamilton quaternions** *$\langle -1, -1 \rangle_{\mathbb{R}}$. Moreover, $f^\rho = \overline{f}$ for all $f \in F$.*

2. F is **of the second kind***: $K \neq K_0$. In particular, K is a totally complex number field [BL, Lemma 5.5.4].*

More generally, for any (possibly non-simple) polarised abelian variety, we can describe its **endomorphism structure** as a division ring (F, ρ) of one of the above types together with a representation Φ of F, subject to satisfying some compatibility conditions.

Definition 3.3.8. [BL, Section 9.1]
Let (F, ρ) be a division ring of finite dimension over $\mathbb{Q}$ and ρ a positive anti-involution. Let Φ be a representation of F by g-by-g complex matrices

$$\Phi\colon F \longrightarrow M_g(\mathbb{C}).$$

Then a **polarised abelian variety with endomorphism structure** *(F, ρ, Φ) is a triplet (A, E, ι) where $A \simeq (\mathbb{C}^g/\Lambda)$ is an abelian variety, E is an alternating form on $\mathbb{C}^g$ defining a polarisation on X, and ι is an embedding*

$$\iota : F \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A) \subset M_g(\mathbb{C})$$

such that

(i) Φ and ι are equivalent representations, i.e. *there is a $\mathbb{C}$-linear map G on $\mathbb{C}_g$ such that $\iota(f) \circ G = G \circ \Phi(f)$ for all $f \in F$, and*

(ii) (Rosati condition) *the Rosati involution on* $\mathrm{End}_{\mathbb{Q}}(A)$ *extends the anti-involution ρ on F via ι.*

For the obvious reason, an abelian variety is said to admit a **real multiplication**, a **totally indefinite quaternion multiplication**, a **totally definite quaternion multiplication** or a **complex multiplication** if there is respectively an embedding of a totally real number field, a totally indefinite quaternion algebra, a totally definite quaternion algebra or a division ring of the second kind into its endomorphism algebra over $\mathbb{Q}$.

Remark 3.3.9. *The representation Φ of F realises $\mathbb{C}^g$ as a F-module.*

Remark 3.3.10. *Let (F, ρ) be a division ring with positive anti-involution ρ. Then for any integer n, by putting $M^\rho = (m_{ij}^\rho)^t$ for any $M = (m_{ij}) \in M_n(F)$, the Rosati involution ρ extends to a positive anti-involution on $M_n(F)$ which we also call ρ. Under this notation, the Rosati condition then translates to: for any $a \in F$,*

$$\iota(a^\rho) = \iota(a)^\rho.$$

Moreover [BL, Proposition 5.1.1], *if $F =$ $\mathrm{End}_{\mathbb{Q}}(A)$ for some abelian g-fold A, then the Rosati involution on* $\mathrm{End}_{\mathbb{Q}}(A)$ *is the adjoint operator with respect to the alternating form E associated to the polarisation of A: for all $x, y \in \mathbb{C}^g$ and $a \in F$,*

$$E(x, \iota(a)y) = E(\iota(a^\rho)x, y).$$

We will discuss coarse moduli spaces of polarised abelian varieties with certain endomorphism structure in the next subsection. Recall a moduli space parametrises objects in a family up to isomorphisms.

Definition 3.3.11. *Two polarised abelian varieties (A, E, ι) and (A', E', ι') with endomorphism structure (F, ρ, Φ) are **isomorphic** if there is an isomorphism of polarised abelian varieties*

$$f\colon (A, E) \longrightarrow (A', E')$$

such that for all $a \in F$,

$$f \circ \iota(a) = \iota'(a) \circ f.$$

3.3.3 Moduli of polarised abelian varieties with totally definite quaternion multiplication

Referencing [Sh, Section 2.2], we construct the moduli space for a family of polarised abelian varieties with totally definite quaternion multiplication given by (F, ρ, Φ). For simplicity, we will restrict our discussion to the case $F \simeq \mathbb{H}_{\mathbb{Q}} := \langle -1, -1 \rangle_{\mathbb{Q}}$, and in particular, $K = \mathbb{Q}$. One can refer to [Sh] for the more general argument with respect to a totally definite quaternion algebra $F \simeq (\mathbb{H}_{\mathbb{Q}})^e$ with index $e := [K : \mathbb{Q}] \geq 1$.

Similar to Section 3.2.4, we would like to obtain an expression of the period domain of the weight one Hodge structures on the first cohomology groups. We will first introduce some attributes associated with any polarised abelian g-fold $A = (\mathbb{T} = \mathbb{C}^g/\Lambda, E, \iota)$ with totally definite quaternion multiplication (F, ρ, Φ), which describe the abelian g-fold as a member in a family. These attributes depend on an explicit expression of the representation Φ. Following [Sh], we will fix the representation Φ of F to appear in a standard form Φ_{std}.

Theorem 3.3.12. [Sh, Section 2.1]

Let χ be the representation of $\mathbb{H}_{\mathbb{Q}}$ by 2-by-2 complex matrices

$$\begin{aligned} \chi\colon \mathbb{H}_{\mathbb{Q}} &\longrightarrow M_2(\mathbb{C}) \\ a + bj &\longmapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \text{ with } a, b \in \mathbb{Q}\langle 1, i \rangle, \end{aligned}$$

where $\mathbb{Q}\langle 1, i \rangle$ is the $\mathbb{Q}$-algebra generated by 1 and the imaginary unit i.

Let $m := g/2$. Then for any representation Φ of $\mathbb{H}_{\mathbb{Q}}$ by g-by-g complex matrices such that (F, ρ, Φ) is the endomorphism structure of an abelian g-fold, Φ is equivalent to a m-multiple of the representation χ. That is, there exists $G \in \mathrm{GL}_g(\mathbb{C})$ such that

$$G\Phi(x)G^{-1} = \Phi_{\mathrm{std}}(x) := \chi(x) \otimes \mathbf{1}_m$$

for any x in $\mathbb{H}_{\mathbb{Q}}$, where $\otimes$ is the **Kronecker product of matrices***: given any positive integers m, n, r, s and $\mathbb{K}$ a field, define*

$$\begin{aligned} \otimes\colon M_{m,n}(\mathbb{K}) \times M_{r,s}(\mathbb{K}) &\longrightarrow M_{mr,ns}(\mathbb{K}) \\ (A = \{a_{ij}\}_{i,j}, B = \{b_{kl}\}_{k,l}) &\longmapsto \{a_{ij}b_{kl}\}_{r(i-1)+k, s(j-1)+l} = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}. \end{aligned}$$

Remark 3.3.13. *The representation χ extends to a representation of $M_d(\mathbb{H}_{\mathbb{Q}})$, which we still denote as χ:*

$$\begin{aligned} \chi\colon M_d(\mathbb{H}_{\mathbb{Q}}) &\longrightarrow M_{2d}(\mathbb{C}) \\ A + Bj &\longmapsto \begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix}, \text{ where } A, B \in M_d(\mathbb{Q}\langle 1, i \rangle). \end{aligned}$$

In the rest of the subsection, we will fix the representation Φ to be the standard representation Φ_{std}. By picking the right coordinate system for $\mathbb{C}^g$, the matrix $\iota(x)$ is represented by $\Phi(x)$ for all $x \in F$, so $\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q}$ can be considered as a left F-module, as $F \hookrightarrow \operatorname{End}_{\mathbb{Q}}(A)$.

Remark 3.3.14. [BL, Section 9.9]
In our case ($m > 2$ and $F = \mathbb{H}_{\mathbb{Q}}$), we have $\iota(F) = \operatorname{End}_{\mathbb{Q}}(A)$ for most members A in the moduli space, i.e. *A is in the complement of the union of at most countably many proper subvarieties of the moduli space. We call such a member A a* ***very general*** *member of the family.*

The first attribute associated to A is a set of m vectors $\{x_1, \cdots, x_m\}$ in $\mathbb{C}^g \supset \Lambda$, such that

$$\Lambda_{\mathbb{Q}} = \sum_{i=1}^{m} \Phi(F) x_i. \tag{1}$$

The second attribute is a free $\mathbb{Z}$-module $\mathcal{M}$ of rank $4m$ in F^m such that when restricting the equation (1) to the lattice Λ, we have

$$\Lambda = \left\{ \sum_{i=1}^{m} \Phi(a_i) x_i \colon (a_1, \cdots, a_m) \in \mathcal{M} \right\}. \tag{2}$$

The third attribute is a non-degenerate matrix $\mathcal{T} \in M_m(F)$ which determines the alternating form E on $\mathbb{C}^g$, indexed with respect to $\{x_1, \cdots, x_m\}$. Note that for each index i and j, the mapping

$$\begin{aligned} F &\longrightarrow \mathbb{Q} \\ a &\longmapsto E(\Phi(a)x_i, x_j) \end{aligned}$$

is $\mathbb{Q}$-linear. Therefore, there exists an element $t_{ij} \in F$ such that $E(\Phi(a)x_i, x_j) = \operatorname{tr}_{F|\mathbb{Q}}(a t_{ij})$. Combining with Remark 3.3.10, for all $x, y \in \Lambda_{\mathbb{Q}}$ we have

$$E(x, y) = E\left(\sum_{i=1}^{m} \Phi(a_i) x_i, \sum_{j=1}^{m} \Phi(b_j) x_j \right) = \operatorname{tr}_{F|\mathbb{Q}} \left(\sum_{i,j=1}^{m} a_i t_{ij} b_j^{\rho} \right) \tag{3}$$

for some suitable a_i's and b_j's. Thus $\mathcal{T} := (t_{ij})$ is the matrix for $E|_{\Lambda_{\mathbb{Q}}}$, which extends $\mathbb{R}$-linearly to E. Note that $\mathcal{T}$ reflects the properties of E as in Theorem 3.2.3: $(\mathcal{T}^{\rho})^t = -\mathcal{T}$ for E is skew symmetric, and $\operatorname{tr}_{F|\mathbb{Q}}(\mathcal{M}\mathcal{T}\mathcal{M}^{\rho}) \subset \mathbb{Z}$ for E is $\mathbb{Z}$-valued on Λ.

Before we show the significance of these three attributes in the next theorem, let us derive another matrix $\mathcal{H}$ out of the present data. Extend equation (1) linearly to $\mathbb{R}$. Then any $x \in \mathbb{C}^g$ can be expressed as a sum $\sum_{i=1}^{m} \Phi(a_i) x_i$ for some $a_1, \cdots, a_m$ in $F_{\mathbb{R}} := F \otimes \mathbb{R}$. Therefore, the map sending $x \mapsto (a_1, \cdots, a_m)$ gives an isomorphism $\mathbb{C}^g \to F_{\mathbb{R}}^m$. In particular, if we denote by $\sqrt{-1}$ the push forward $\mu_*(J)$, where $\mu \colon \mathbb{R}^{2g} \to \mathbb{C}^g$ is the isomorphism in Definition 3.2.9 and J is the positive complex structure J associated to A, then there exists a unique matrix $\mathcal{H} = (h_{ij}) \in M_m(F_{\mathbb{R}})$ such that

$$\sqrt{-1} x_i = \sum_{i=1}^{m} \Phi(h_{ij}) x_j \tag{4}$$

Remark 3.3.15. *In Section 3.2.2, we used the imaginary unit $i \in \mathbb{C}$ to represent the action of J on the $\mathbb{C}$-vector space $\mathbb{C}^g$. But here, we deliberately choose a different symbol $\sqrt{-1}$ for the action of J (or equivalently $h(i)$ if we use the alternative definition for Hodge structure in Remark 3.2.11). This is to avoid confusion with the purely imaginary elements i, j and k in the endomorphism algebra* $\operatorname{End}_{\mathbb{Q}}(A) \supset \mathbb{H}_{\mathbb{Q}}$ *which also act on $\mathbb{C}^g$: we now deem $\mathbb{C}^g$ as an $\mathbb{H}_{\mathbb{Q}}$-module by Remark 3.3.9 .*

Indeed, by definition any endomorphism $f \in \mathrm{End}_{\mathbb{Q}}(A)$ preserves the complex structure J of A. In other words, the action of J commutes with the action of f. This further implies that for any simple abelian g-fold A with a division ring $F := \mathrm{End}_{\mathbb{Q}}(A)$, the action of any element $f \in F$ is on the right. To see this, let $\sqrt{-1}$ be a general complex structure of A. It acts on the left as a complex structure is a linear operator by definition. By Poincaré's Complete Reducibility Theorem, we have $\mathrm{End}_{\mathbb{Q}}(A \times A) \simeq M_2(F)$. In order for $g \in \mathrm{End}_{\mathbb{Q}}(A \times A)$ to commute with the action of $\sqrt{-1} \oplus \sqrt{-1}$, the matrix $g \in M_2(F)$ must act by right multiplication. In particular, an endomorphism $f \in \mathrm{End}_{\mathbb{Q}}(A)$ induces an element $f \oplus f \in \mathrm{End}_{\mathbb{Q}}(A \times A)$ which is equivalent to a scalar matrix in $M_2(F)$. Projection from $M_2(F)$ onto F in the $(1,1)^{th}$-entry implies F also acts from the right.

However, under the representation Φ (equivalent to ι) which identifies $f \in F$ to an element in $M_g(\mathbb{C})$ as in Definition 3.3.8, the matrix $\Phi(f) \in M_g(\mathbb{C}) = \mathrm{End}(\mathbb{C}_g)$ has the usual action on $\mathbb{C}_g$ by left multiplication.

Upon satisfying some conditions, the above three attributes uniquely determine an abelian variety with endomorphism structure.

Theorem 3.3.16. *Let (F, ρ, Φ) as defined in Definition 3.3.8. Let $\mathcal{M}$ be a free $\mathbb{Z}$-module in F^m of rank $4m$, and $\mathcal{T}$ a non-degenerate matrix in $M_m(F)$ such that $\mathcal{T}^\rho = -\mathcal{T}$ and $\mathrm{tr}_{F|\mathbb{Q}}(\mathcal{M}\mathcal{T}\mathcal{M}^\rho) \subset \mathbb{Z}$. Then $\mathcal{M}$ and $\mathcal{T}$, together with a set of $\mathbb{C}$-linearly independent vectors $\{x_1, \cdots, x_m\} \subset \mathbb{C}^g$, completely determine a polarised abelian g-fold $(\mathbb{T}, E, \iota)$ with endomorphism structure (F, ρ, Φ) if and only if both of the following conditions are satisfied:*

(a) $\Lambda_{\mathbb{R}} = \sum_{i=1}^m \Phi(F_{\mathbb{R}})x_i$; *and*

(b) If $\mathcal{H} \in M_m(F_{\mathbb{R}})$ is derived from (a) and satisfies Equation (4), then $\mathcal{T}\mathcal{H}^\rho$ is ρ-symmetric and ρ-positive, i.e. $(\mathcal{T}\mathcal{H}^\rho)^\rho = \mathcal{T}\mathcal{H}^\rho$ *and* $\mathrm{tr}_{F|\mathbb{Q}}(x(\mathcal{T}\mathcal{H}^\rho)x^\rho) > 0$ *for all* $x \in F^m_{\mathbb{R}} \setminus \{0\}$.

Proof. ($\Rightarrow$) Condition (a) was shown in the derivation of Equation (4). Condition (b) is immediate because of our construction of the alternating form E from the polarisation form Ψ on $H^1(A, \mathbb{R})$..

($\Leftarrow$) It is clear that $\mathbb{T}$ is determined and ι is equivalent to Φ as representations. We can define E by the $\mathbb{R}$-linear extension of equation (3). It remains to show that E defines a polarisation on $\mathbb{T}$ as in Theorem 3.2.3, and the Rosati condition in Definition 3.3.8.

Theorem 3.2.3(i) is a result of the assumption $\mathrm{tr}_{F|\mathbb{Q}}(\mathcal{M}\mathcal{T}\mathcal{M}^\rho) \subset \mathbb{Z}$.

Combining equations (3) and (4), then we can represent the form $(x, y) \mapsto E(x, \sqrt{-1}y)$ by the matrix $\mathcal{T}\mathcal{H}^\rho$. Then Theorem 3.2.3(iii) follows from condition (b). So by Remark 3.3.10, the form $(x, y) \mapsto E(\sqrt{-1}x, \sqrt{-1}y)$ is given by the matrix $\mathcal{H}\mathcal{T}\mathcal{H}^\rho$. Condition (b) and $\mathcal{T}^\rho = -\mathcal{T}$ give us

$$\mathcal{T}\mathcal{H}^\rho = (\mathcal{T}\mathcal{H}^\rho)^\rho = -\mathcal{H}\mathcal{T}$$

so the matrix $\mathcal{H}\mathcal{T}\mathcal{H}^\rho$ is in fact $-\mathcal{H}^2\mathcal{T}$. Moreover, by multiplying equation (4) by $\sqrt{-1}$, we know $\mathcal{H}^2 = -1$. So Theorem 3.2.3(ii) is also satisfied.

Finally as in [BL, Proposition 9.5.3], for all $c \in F$ and $x, y \in \mathbb{C}^g$ such that $x = \sum_{i=1}^m \Phi(a_i)x_i$ and $y = \sum_{j=1}^m \Phi(b_j)x_j$, we have

$$E(\Phi(c)x, y) = \mathrm{tr}_{F|\mathbb{Q}} \left(\sum_{i,j=1}^m c a_i t_{ij} b_j^\rho \right) = \mathrm{tr}_{F|\mathbb{Q}} \left(\sum_{i,j=1}^m a_i t_{ij} (c^\rho b_j)^\rho \right) = E(x, \Phi(c^\rho)y).$$

Again by Remark 3.3.10, the extension of the anti-involution ρ on F via Φ is the Rosati involution as it is the adjoint operator with respect to E. □

Given a pair $(\mathcal{M}, \mathcal{T})$ satisfying $\mathcal{T}^\rho = -\mathcal{T}$ and $\mathrm{tr}_{F|\mathbb{Q}}(\mathcal{M}\mathcal{T}\mathcal{M}^\rho) \subset \mathbb{Z}$, the abelian g-folds with the attributes $(\mathcal{M}, \mathcal{T})$ then form a family $f_{\mathcal{M},\mathcal{T}}$ which we call a **family of polarised abelian g-folds with endomorphism structure** (F, ρ, Φ) **associated to** $(\mathcal{M}, \mathcal{T})$. Clearly the remaining attribute $\{x_1, \cdots, x_m\} \subset \mathbb{C}^g$ distinguishes isomorphic abelian varieties. It is therefore natural to consider the set of $\{x_1, \cdots, x_m\}$ as the period domain associated to $f_{\mathcal{M},\mathcal{T}}$.

Remark 3.3.17. *In fact the only important information that the attribute $\{x_1, \cdots, x_m\} \subset \mathbb{C}^g$ associated to a member A of the family $f_{\mathcal{M},\mathcal{T}}$ encode, is the complex structure J of A. We will later see in the explicit calculations in Section 6.2.3 that we may choose a set of real vectors $\{(x_\mathbb{R})_1, \cdots, (x_\mathbb{R})_m\} \subset \mathbb{R}^{16}$ shared by all members in the $f_{\mathcal{M},\mathcal{T}}$, such that the attribute $\{x_1, \cdots, x_m\}$ of a member A can be recovered from this set and the complex structure J of A. Moreover, there are "real" versions of Equations (2) and (3) that only depend on the set of real vectors. Therefore, the parameter of the family $f_{\mathcal{M},\mathcal{T}}$ is effectively the complex structure of the members.*

One can in fact standardise the attribute $\{x_1, \cdots, x_m\}$ by associating it to a period matrix $X \in M_g(\mathbb{C})$. Write each vector x_i in the form

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix}$$

where $u_i, v_i \in M_{m\times 1}(\mathbb{C})$ and put $U = (u_1, \cdots, u_m)$, $V = (v_1, \cdots, v_m)$. Define a matrix

$$X := \begin{bmatrix} U & V \\ \overline{V} & -\overline{U} \end{bmatrix}.$$

Upon choosing a suitable basis of $F_\mathbb{R}^m$ such that $\mathcal{T}^{-1}$ is given by $\sqrt{-1} \cdot \mathbf{1}_m$ with respect to $\mathcal{M}$, or equivalently the complex matrix $\sqrt{-1}\chi(\mathcal{T})^{-1}$ is in the form $\mathrm{diag}(-\mathbf{1}_m, \mathbf{1}_m)$, then the m-by-m complex matrix $Z := -V^{-1}U$ satisfies $Z^t = -Z$ and $1 - Z\overline{Z}^t > 0$. Furthermore by change of basis of $\mathbb{C}^g$, that is by the left multiplication action of $\mathrm{GL}_g(\mathbb{C})$, we can assume that $V = \mathbf{1}_m$, and the period matrix X is in the standardised normalised form

$$\begin{bmatrix} -Z & \mathbf{1}_m \\ \mathbf{1}_m & \bar{Z} \end{bmatrix}$$

which is unique to the attribute $\{x_1, \cdots, x_m\}$. Therefore, we have the following theorem analogous to Theorem 3.2.21.

Theorem 3.3.18. *There exists a complete family $f : \mathcal{X} \to S$ of polarised abelian g folds with endomorphism structure (F, ρ, Φ) associated to $(\mathcal{M}, \mathcal{T})$, that is, the association of a normalised period matrix to a member in the family which is determined by the attribute $\{x_1, \cdots, x_m\} \subset \mathbb{C}^g$ gives a multi-valued map*

$$S \longrightarrow \mathcal{H}_m := \left\{ Z \in M_m(\mathbb{C}) : -Z = Z^t, 1 - Z\overline{Z}^t > 0 \right\}$$

such that the preimage of any $Z \in \mathcal{H}_m$ is non-empty.

As in Section 3.2.4, $\mathcal{H}_m$ is a period domain for some polarised weight 1 Hodge structures.

Corollary 3.3.19. *In a family $f_{\mathcal{M},\mathcal{T}}$ of polarised abelian g-folds with endomorphism structure (F, ρ, Φ) associated to a pair $(\mathcal{M}, \mathcal{T})$, the analytic manifold $\mathcal{H}_m$ is the period domain of the weight 1 Hodge structures on $\mathbb{R}^{2g} = \Lambda'_\mathbb{R}$ of the abelian varieties as real tori $\mathbb{R}^{2g}/\Lambda'$.*

Proof. It can be seen that the standardised normalised period matrix X of any abelian variety A in $f_{\mathcal{M},\mathcal{T}}$ depends only on $\sqrt{-1}$, which is the action of the positive complex structure J of A: it gives the generators of $\Lambda = \mu(\Lambda')$ in terms of a suitable basis of $\mathbb{C}^g$ such that $\mathcal{T}^{-1} = \sqrt{-1} \cdot \mathbf{1}_m$.

In particular, if we replace Φ by $\Phi_\mathbb{R}$, a representation of F by $2g$-by-$2g$ real matrices, then there exists a set of m vectors $\{(x_\mathbb{R})_1, \cdots, (x_\mathbb{R})_m\} \subset \mathbb{R}^{2g}$ such that $\Lambda' = \sum_{i=1}^m \Phi_\mathbb{R}(F)(x_\mathbb{R})_i$. This set of real vectors of length $2g$ is invariant in the family. So $\mathcal{H}_m$ parametrises the positive complex structures on $\mathbb{R}^{2g}/\Lambda'$ whose action commutes with that of F, which are equivalent to the weight one Hodge structures of the first cohomology of the members in $f_{\mathcal{M},\mathcal{T}}$. □

With this interpretation of $\mathcal{H}_m$, we can derive another expression of $\mathcal{H}_m$ as a quotient of Lie groups similar to Theorem 3.2.22.

Theorem 3.3.20.

$$\mathcal{H}_m \simeq \mathrm{SO}^*(2m)/\,\mathrm{U}(m).$$

To prove the statement, first we derive another expression of the Lie group $\mathrm{SO}^*(2m)$. Similar to χ in Remark 3.3.13, there is natural embedding

$$\begin{aligned} \chi_{\mathbb{R}} := M_{2m}(\mathbb{C}) &\hookrightarrow M_{4m}(\mathbb{R}) \\ A + Bi &\mapsto \begin{bmatrix} A & B \\ -B & A \end{bmatrix}, \text{ where } A, B \in M_{2m}(\mathbb{R}). \end{aligned}$$

We will show that

Theorem 3.3.21.

$$\chi_{\mathbb{R}}\left(\mathrm{SO}^*(2m)\right) = \chi_{\mathbb{R}}\left(M_{2m}(\mathbb{C})\right) \cap \mathrm{Aut}_{\mathbb{H}}(\mathbb{R}^{4m}) \cap \mathrm{Sp}(\widetilde{J}).$$

where

$$\mathrm{Aut}_{\mathbb{H}}(\mathbb{R}^{4m}) := \{M \in \mathrm{GL}_{4m}(\mathbb{R}) : Mh = hM \text{ for all } h \in \mathbb{H}\}$$

is the automorphism group of $\mathbb{R}^{4m}$ *as a* $\mathbb{H}$*-module, and*

$$\mathrm{Sp}(\widetilde{J}) := \left\{M \in M_{4m}(\mathbb{R}) : M^t \widetilde{J} M = \widetilde{J}\right\} \text{ with } \widetilde{J} = \begin{bmatrix} J_m & 0 \\ 0 & J_m \end{bmatrix}.$$

Proof. Recall from Definition 2.2.4(iv)

$$\mathrm{SO}^*(2m) = \left\{M \in M_{2m}(\mathbb{C}) : M^t J_m \overline{M} = J_m,\ M^t M = \mathbf{1}_{2m}\right\}$$

Write $M \in \mathrm{SO}^*(2m) = A + Bi$, where $A, B \in M_{2m}(\mathbb{C})$. Then the condition $M^t M = \mathbf{1}_{2m}$ translates to

$$\begin{cases} A^t A - B^t B = \mathbf{1}_{2m} \\ A^t B + B^t A = 0_{2m}. \end{cases} \tag{*}$$

Moreover, $M^t M = \mathbf{1}_{2m}$ is equivalent to $M^t = M^{-1}$. So the condition $M^t J_m \overline{M} = J_m$ is equivalent to saying $J_m \overline{M} = M J_m$, which translates to

$$\begin{cases} J_m A = A J_m \\ J_m B = -B J_m. \end{cases} \tag{**}$$

On the other hand, to give an explicit expression of the group $\mathrm{Aut}_{\mathbb{H}}(\mathbb{R}^{4m})$, let us realise $\mathbb{R}^{4m}$ as a $\mathbb{H}$-module using a specific representation $\hat{\Phi} \colon \mathbb{H} \to M_{4m}(\mathbb{R})$ determined by

$$\begin{aligned} i &\longmapsto \hat{\imath} := J_{2m} \\ j &\longmapsto \hat{\jmath} := \begin{bmatrix} J_m & 0 \\ 0 & -J_m \end{bmatrix}. \end{aligned}$$

One can check that indeed $\hat{\imath}^2 = \hat{\jmath}^2 = -1$ and $\hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath}$. This gives

$$\mathrm{Aut}_{\mathbb{H}}(\mathbb{R}^{2m}) = \{M \in \mathrm{GL}_{4m}(\mathbb{R}) : M\hat{\imath} = \hat{\imath}M \text{ and } M\hat{\jmath} = \hat{\jmath}M\}.$$

So for $M = \chi_{\mathbb{R}}(A + Bi)$ where $A, B \in M_{2m}(\mathbb{C})$, then $M \in \mathrm{Aut}_{\mathbb{H}}(\mathbb{R}^{4m})$ if and only if A and B satisfy (**). Moreover, $M^t \widetilde{J} M = \widetilde{J}$ if and only if

$$\begin{bmatrix} A^t J_m A + B^t J_m B & A^t J_m B - B^t J_m A \\ B^t J_m A - A^t J_m B & B^t J_m B + A^t J_m A \end{bmatrix} \underset{\text{by } (**)}{=} \begin{bmatrix} A^t A - B^t B & -A^t B - B^t A \\ B^t A + A^t B & -B^t B + A^t A \end{bmatrix} \widetilde{J} = \widetilde{J},$$

i.e. if and only if (*) is satisfied. □

Remark 3.3.22. *By Corollary 3.3.19, the group* $\mathrm{SO}^*(8)$ *is the automorphism group of the period domain for weight one Hodge structures on* $\Lambda'_{\mathbb{R}}$ *associated to the family* $f_{\mathcal{M},\mathcal{T}}$.

We are ready to prove Theorem 3.3.20.

Proof. As in [BL, Chapter 9.7], consider the group

$$\mathrm{U}_m(\mathbb{H}) := \left\{ M \in M_m(\mathbb{H}) : M^t (i\mathbf{1}_m) \overline{M} = i\mathbf{1}_m \right\}$$

where $\overline{M}$ is the canonical involution of the quaternion algebra on M as defined in Definition 3.3.4. The representation $\hat{\Phi}$ in the proof of Theorem 3.3.21 identifies $\chi(\mathrm{U}_m(\mathbb{H})) < \mathrm{GL}_{2m}(\mathbb{C})$ to a subgroup of $\mathrm{Aut}_{\mathbb{H}}(\mathbb{R}^{4m})$. Moreover, if we denote by ϵ the standard $\mathbb{H}$-Hermitian skew form represented by the matrix $i\mathbf{1}_m$

$$\epsilon(x, y) = \overline{x}_1 i y_1 + \cdots + \overline{x}_m i y_m,$$

then $i\epsilon$ is a $\mathbb{H}$-Hermitian symmetric form. Therefore $\chi(\mathrm{U}_m(\mathbb{H}))$ also leaves the Hermitian form $\chi_*(i\epsilon)$ invariant.

On the other hand, note that the symplectic form represented by $\widetilde{J}$ is equivalent to the standard symplectic form represented by J_{2m} by reordering the chosen basis of $\mathbb{R}^{4m}$. By the same argument as in the proof of Theorem 3.2.22, preserving J_{2m} is equivalent to preserving an alternating form E associated to the polarisation of a weight 1 Hodge structure on $\mathbb{R}^{4m}$. Note that the Rosati condition for E is automatically satisfied. Since E uniquely determines a Hermitian symmetric form by Theorem 3.2.5, the group $\chi(\mathrm{U}_m(\mathbb{H}))$ is isomorphic to $\mathrm{SO}^*(2m)$.

By [BL, Proposition 9.7.2], the group $\chi(\mathrm{U}_m(\mathbb{H}))$ acts on $\mathcal{H}_m$ by

$$\begin{aligned} \chi(\mathrm{U}_m(\mathbb{H})) \times \mathcal{H}_m &\longrightarrow \mathcal{H}_m \\ \left(\begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix}, Z \right) &\longmapsto (A \cdot Z + B)(-\overline{B} \cdot Z + \overline{A})^{-1}. \end{aligned}$$

Furthermore, [BL, Proposition 9.7.2] says this action is transitive, and the stabliser subgroup

$$\chi(\mathrm{U}_m(\mathbb{H})) \cap \mathrm{U}(2m) = \left\{ \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} \in M_{2m}(\mathbb{C}) : A^t \overline{A} = \mathbf{1}_m \right\} \simeq \mathrm{U}(m)$$

is compact. □

Remark 3.3.23. *Again, we can apply Theorem 3.1.16 to show transitivity of the action of* $\mathrm{SO}^*(2m)$ *on* $\mathcal{H}_m$*: let* $g = 2m$*, then by Theorem 3.3.21,* $\chi_{\mathbb{R}}(\mathrm{SO}^*(2m))$ *is the automorphism group of* $\mathbb{R}^{2g}$*, preserving its* $\mathbb{H}$*-module structure and the polarisation* E *of its weight* 1 *Hodge structure.*

Remark 3.3.24. *When working with Shimura's construction of the family* $f_{\mathcal{M},\mathcal{T}}$ *which we have just introduced above, the preferred definition for the group* $\mathrm{SO}^*(8)$ *is* $\chi(\mathrm{U}_m(\mathbb{H}))$. i.e. *we may consider* $\chi_{\mathbb{R}}(\chi(\mathrm{U}_m(\mathbb{H}))) \subset \mathrm{Aut}(\Lambda'_{\mathbb{R}})$ *which preserves the real torus structure* $\mathbb{R}^{2g}/\Lambda'$ *shared by every abelian g-fold in the family. Indeed, left multiplication by* $\chi_{\mathbb{R}}(\chi(\mathrm{U}_m(\mathbb{H})))$ *preserves the* $\mathbb{H}$*-module structure of* $\Lambda_{\mathbb{R}}$ *(given by* $\mathcal{M}$*), as well as the matrix* $i\mathbf{1}_m$ *(which is exactly the matrix* $\mathcal{T}^{-1}$ *when* X *is in the standardised normalised form).*

As in Remark 3.2.24, we may identify any element Z in $\mathcal{H}_m$ with a $\mathrm{GL}(g, \mathbb{C})$-equivalence class of block matrices

$$\begin{bmatrix} Z \\ \mathbf{1}_g \end{bmatrix},$$

and the action of $\chi(\mathrm{U}_m(\mathbb{H}))$ on $\mathcal{H}_m$ in [BL, Proposition 9.7.2] can be thought of as left multiplication on these classes.

Remark 3.3.25. *Referencing [He, Table X.6.V], the unitary group $\mathrm{U}(m)$ is the maximal compact subgroup of $\mathrm{SO}^*(2m)$, and $\mathcal{H}_m$ is an irreducible HSD of type II_m.*

We also have a general expression for the monodromy group of $f_{\mathcal{M},\mathcal{T}}$.

Theorem 3.3.26. *The monodromy group $\Gamma(\mathcal{M}, \mathcal{T}) < \mathrm{SO}^*(2m)$ of a family $f_{\mathcal{M},\mathcal{T}}$ is given by*

$$\Gamma(\mathcal{M}, \mathcal{T}) = \{N \in M_m(\mathbb{H}_{\mathbb{Q}}) : \mathcal{M}N = \mathcal{M}, N\mathcal{T}N^{\rho} = \mathcal{T}.\}$$

Proof. See [Sh, Theorem 2]: two members in the family represented by Z and Z' in $\mathcal{H}_m$ are isomorphic if and only if Z and Z' are in the same $\Gamma(\mathcal{M}, \mathcal{T})$-orbit. □

Therefore the Global Torelli Theorem is true for a family of polarised abelian g-folds with a totally definite quaternion multiplication. Moreover, any discrete subgroup of $\mathrm{SO}^*(2m)$ acts properly discontinuously on $\mathcal{M}$ by [BL, Proposition 9.7.4]. So

$$\mathcal{A}_{(\mathcal{M},\mathcal{T})} := \Gamma(\mathcal{M}, \mathcal{T}) \backslash \mathcal{H}_m$$

is a quasi-projective variety by the Baily-Borel Theorem. We call $\mathcal{A}_{(\mathcal{M},\mathcal{T})}$ the **moduli variety of polarised abelian g-folds with totally definite quaternion multiplication associated to the pair** $(\mathcal{M}, \mathcal{T})$, which is a moduli variety of PEL type. It has the same dimension as $\mathcal{H}_m$, which is $m(m-1)/2$, and is a LSV of type II_m.

3.4 Lattice theory

We have seen lattices in the definition of an abelian variety. Let us recall some lattice theory which will be useful for defining K3 surfaces and their lattices of polarisation.

3.4.1 K3 Lattice

First we will give the basic definitions and examples of lattices in order to define a special lattice called the K3 lattice. The followings are mainly taken from [Hu, Section 14.0].

Definition 3.4.1. *A* ***lattice*** *is a pair (Λ, b) where Λ is a free $\mathbb{Z}$-module, and b is an integral symmetric bilinear form on Λ which we will always assume to be non-degenerate. That is, the pairing b is of full rank.*

By choosing a basis of $\Lambda_{\mathbb{R}} \supset \Lambda$, the lattice Λ is often characterised by the matrix M of its symmetric bilinear form b. We define the following invariants of Λ with respect to any change of bases.

(i) The **rank** of Λ, $\mathrm{rk}(\Lambda)$, the rank of b or equivalently $\mathrm{rk}(M)$.

(ii) The pairing b on Λ induces a pairing on $\Lambda_{\mathbb{R}}$, which can be diagonalised with only ± 1 on the diagonal. The **signature** of Λ is the pair (n_+, n_-) where $n_\pm$ is the number of ± 1 on the diagonal, and they add up to $\mathrm{rk}(\Lambda)$.

(iii) The **discriminant** of Λ, $\mathrm{disc}(\Lambda)$, is the determinant of M.

Here we give some examples of lattices:

Example 3.4.2.

(1) The ***standard hyperbolic plane*** *U is the rank 2 lattice*

$$M_U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly, it has signature $(1,1)$ and discriminant -1.

(2) For $n = 6, 7$ or 8, the E_n lattice is the rank n lattice given by the root lattice of the E_n root system. It has signature $(n, 0)$. In particular, the E_8 lattice has discriminant 1.

(3) Similarly for any $n \geq 4$, the lattice D_n is the rank n lattice given by the root lattice of the D_n root system. It has signature $(n, 0)$. In particular, the D_4 lattice is given by the rank 4 matrix

$$M_{D_4} := \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}.$$

(4) For any $d \in \mathbb{Z}\backslash\{0\}$, the lattice denoted by $\langle d \rangle$ is the lattice given by the rank 1 matrix $\begin{pmatrix} d \end{pmatrix}$. In particular, $\langle 2 \rangle$ is the A_1 lattice given by the root lattice of the A_1 root system.

(5) The Nikulin lattice N defined in [Mo1, Definition 5.3] is a rank 8 lattice of signature $(0, 8)$.

(6) Let (Λ, b) be a lattice. Then for any $d \in \mathbb{Z}\backslash\{0, 1\}$, we define $\Lambda(d)$, the ***twist*** *of Λ by d, to be the lattice $(\Lambda, d \cdot b)$. Clearly, $\mathrm{rk}(\Lambda) = \mathrm{rk}(\Lambda(d))$ and $\mathrm{disc}(\Lambda(d)) = d^{\mathrm{rk}(\Lambda)} \,\mathrm{disc}(\Lambda)$.*

(7) Let (Λ_1, b_1) and (Λ_2, b_2) be two lattices of signature (n_+, n_-) and (m_+, m_-) respectively. We define the ***orthogonal direct sum*** *of the two lattices to be $\Lambda = (\Lambda_1 \oplus \Lambda_2, b)$, where*

$$b((x_1, y_1), (x_2, y_2)) := b_1(x_1, x_2) + b_2(y_1, y_2).$$

Clearly, we have $\mathrm{rk}(\Lambda) = \mathrm{rk}(\Lambda_1) + \mathrm{rk}(\Lambda_2)$, signature of Λ is $(n_+ + m_+, n_- + m_-)$ and $\mathrm{disc}(\Lambda) = \mathrm{disc}(\Lambda_1) \cdot \mathrm{disc}(\Lambda_2)$. Examples (1)–(6) are ***indecomposible*** *lattices,* i.e. *none of them can be expressed as an orthogonal direct sum of two lattices of strictly lower ranks.*

Let us define morphisms of lattices. Let (Λ, b) and (Λ', b') be two lattices.

Definition 3.4.3. *A* ***morphism of lattices*** *$\phi \colon \Lambda \to \Lambda'$ is a linear map that respects the symmetric bilinear forms b and b',* i.e. *for all $x, y \in \Lambda$,*

$$b'(\phi(x), \phi(y)) = b(x, y).$$

An ***embedding*** *is an injective morphism. An embedding is a* ***primitive embedding*** *if its cokernel is torsion free. An* ***isometry*** *is a bijective morphism.*

One significant structure associated to a lattice Λ is its discriminant group.

Definition 3.4.4. *The* ***dual*** *of the lattice Λ is the subset of $\Lambda_{\mathbb{Q}}$*

$$\Lambda^\vee := \{l \in \Lambda_{\mathbb{Q}} : b(l, m) \in \mathbb{Z} \text{ for all } m \in \Lambda\}.$$

By definition, there is a natural embedding

$$\iota_\Lambda \colon \Lambda \hookrightarrow \Lambda^\vee.$$

Define the ***discriminant group*** *of the lattice* Λ *to be the cokernel of* ι_Λ,

$$A_\Lambda = \Lambda^\vee / \Lambda$$

which is a finite group of order $|\operatorname{disc}(\Lambda)|$.

We define the ***length*** *of the lattice* Λ*, denoted* $l(\Lambda)$*, to be the minimal number of generators of its discriminant group* A_Λ.

We also define the following predicates of a lattice Λ.

Definition 3.4.5.

(i) ***Odd or even****. A lattice* Λ *is called even if for any* $x \in \Lambda$, $x^2 := b(x,x) \in 2\mathbb{Z}$*; otherwise it is called odd.*

(ii) ***Definite or indefinite****. Let the signature of* Λ *be* (n_+, n_-)*. The lattice* Λ *is called definite if* n_+ *or* n_- *is* 0*; otherwise it is called indefinite.*

(iii) ***Unimodular,*** d***-elementary****. The lattice* Λ *is unimodular if the discriminant group* A_Λ *is trivial, or equivalently* $|\operatorname{disc}(A)| = 1$*, or* $l(\Lambda) = 0$*. If* $A_\Lambda \simeq (\mathbb{Z}/d\mathbb{Z})^l$*, we say* Λ *is d-elementary.*

Remark 3.4.6. *In Example 3.4.2,*

(i) The hyperbolic plane U*, the root lattices* E_7, E_8, D_{2n} *for* $n \geq 2$*,* A_1 *and the Nikulin lattice* N *are even lattices. Orthogonal direct sums of even lattices are even.*

(ii) Clearly, the root lattices and N *are definite, but* U *is indefinite. A twist of a definite (resp. indefinite) lattice is definite (resp. indefinite).*

(iii) The lattices U *and* E_8 *are unimodular.* $U(2), E_7, E_8(2), D_{2n}$ *for* $n \geq 2$*,* A_1 *and* N*, as well as their twists by* -1*, are* 2*-elementary. Orthogonal direct sums of unimodular (resp. d-elementary) lattices are unimodular (resp. d-elementary).*

The following rank 22 lattice called the **K3 lattice** is significant in the discussion of K3 surfaces:

$$\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

It is clear that Λ_{K3} is even, 2-elementary and indefinite of signature $(3, 19)$. In fact, it is the unique lattice with these properties [Mo1, Theorem 1.3] up to isometry.

3.4.2 Primitive embeddings of even, indefinite, 2-elementary lattices

We will discuss primitive embeddings of even, indefinite, 2-elementary lattices into the K3 lattices later when we define a lattice polarisation of K3 surfaces. Let us specifically study the even, 2-elementary lattices and define more attributes.

Definition 3.4.7. *Let* (Λ, b) *be an even lattice. The pairing* b *on* Λ *induces a* $\mathbb{Q}$*-valued pairing on* $\Lambda^\vee$*, thus a pairing*

$$b_{A_\Lambda} \colon A_\Lambda \times A_\Lambda \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

There is a unique associated quadratic form

$$q_{A_\Lambda} \colon A_\Lambda \longrightarrow \mathbb{Q}/2\mathbb{Z}$$

such that for any $\sigma, \tau \in A_\Lambda$, it satisfies the ***polarisation formula***

$$q_{A_\Lambda}(\sigma + \tau) = q_{A_\Lambda}(\sigma) + 2b_{A_\Lambda}(\sigma, \tau) + q_{A_\Lambda}(\tau).$$

We call q_{A_Λ} the ***discriminant form*** *of Λ.*

Moreover [CM1, Section 2]*, if Λ is 2-elementary, then the discriminant form q_{A_Λ} takes values in $\mathbb{Z}/2\mathbb{Z}$. We define the* ***parity*** *of Λ to be*

$$\delta(\Lambda) = \begin{cases} 0, & \text{if } q_{A_\Lambda}(\tau) \in \mathbb{Z} \text{ for all } \tau \in A_\Lambda; \\ 1, & \text{otherwise.} \end{cases}$$

The even, indefinite, 2-elementary lattices are classified by the following.

Theorem 3.4.8. [Ni, Theorem 3.6.2]
An even, indefinite, 2-elementary lattice Λ is uniquely determined by the triple $((n_+, n_-), l(\Lambda), \delta(\Lambda))$ up to isometry. Moreover, such a lattice exists only if $l + n_+ + n_- \equiv 0 \mod 2$.

Suppose an even, indefinite, 2-elementary lattice Λ given by the triple $((n_+, n_-), l, \delta)$ primitively embeds into an even lattice $(\widetilde{\Lambda}, \widetilde{b})$ of signature $(\widetilde{n}_+, \widetilde{n}_-)$. Consider its orthogonal complement $\Lambda^\perp$ in $\widetilde{\Lambda}$ with respect to $\widetilde{b}$. We have the following general theorem.

Theorem 3.4.9. [Mo1, Lemma 2.4]
Let $\Lambda \hookrightarrow \widetilde{\Lambda}$ be a primitive embedding of even lattices. If $\widetilde{\Lambda}$ is unimodular, then $q_{A_\Lambda} = q_{A_{(\Lambda^\perp)}}$.

If $\Lambda^\perp$ is also indefinite, then $\Lambda^\perp$ is also fully determined by the triple $((\widetilde{n}_+ - n_+,\ \widetilde{n}_- - n_-), l, \delta)$ applying Theorem 3.4.8.

For convenience, here we list the values of the triples $((n_+, n_-), l(\Lambda), \delta(\Lambda))$ for each unimodular or 2-elementary indecomposible lattice Λ in Example 3.4.2. These values can be checked using MAGMA.

Λ	(n_+, n_-)	$l(\Lambda)$	$\delta(\Lambda)$
U	$(1, 1)$	0	0
E_8	$(8, 0)$	0	0
$U(2)$	$(1, 1)$	2	0
$E_8(2)$	$(8, 0)$	8	0
$D_n,\ n \equiv 0 \pmod 4$	$(n, 0)$	2	0
$D_n,\ n \equiv 2 \pmod 4$	$(n, 0)$	2	1
$\langle 2 \rangle$	$(1, 0)$	1	1
E_7	$(7, 0)$	1	1
N	$(0, 8)$	6	0

Table 4: Lattices and their invariants

3.5 K3 surfaces

3.5.1 Polarised K3 surfaces

We continue to work over the complex numbers. Furthermore, we only consider algebraic K3 surfaces.

Definition 3.5.1. [Hu, Definition 1.1]
An ***algebraic K3 surface*** *X over $\mathbb{C}$ is a complete non-singular variety of dimension 2 such that $\Omega^2_{X/k} \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.*

Here we give a few explicit examples of K3 surfaces.

Example 3.5.2. [SS, Example 12.2]

(i) ***Quartic surfaces****. A smooth quartic in* $\mathbb{P}^3$ *is a K3 surface. For any quartic surface with isolated rational double points (*i.e. *canonical surface singularities), its minimal (crepant) desingularisation is also a K3 surface.*

(ii) ***Double sextics****. A double sextic is a double cover of* $\mathbb{P}^2$ *ramified along a smooth sextic curve.*

(iii) ***Kummer surfaces****. Let* $A \simeq \mathbb{C}^2/\Lambda$ *be an abelian surface. Blowing up at the* 16 *ordinary double points in the quotient of* A *by the action of* $-1 \in \mathbb{C}$ *yields a K3 surface* $\mathrm{Kum}(A)$ *called a Kummer surface. In particular, there is a rational double cover* $(A \to A/\langle -1 \rangle \dashrightarrow \mathrm{Kum}(A))$.

Much of the geometry of a K3 surface X can be extracted from its second integral cohomology. Here we present some properties of $H^2(X,\mathbb{Z})$.

Theorem 3.5.3. [SS, Section 12.2]

For any K3 surface X*,* $H^2(X,\mathbb{Z})$ *is torsion-free. Moreover,* $H^2(X,\mathbb{Z})$ *with its intersection form is an even unimodular indefinite lattice of rank* 22 *isomorphic to the K3 lattice* Λ_{K3}*. An explicit isomorphism*

$$\phi\colon H^2(X,\mathbb{Z}) \xrightarrow{\simeq} \Lambda_{K3}$$

is called a ***marking*** *of the K3 surface* X*.*

The lattice $H^2(X,\mathbb{Z})$ has a polarised weight two Hodge structure. By [Hu, Remark 1.2], any complete smooth algebraic surface, in particular a K3 surface, is projective. That is, for a K3 surface X, there exists an ample line bundle L in its Picard group $\mathrm{Pic}(X)$. The image under the first Chern map $c_1(L) \in H^2(X,\mathbb{Z})$ defines a **polarisation** of the K3 surface. As described in Section 3.1.2, a Kähler form ω given by $c_1(L)$ also determines a polarisation form on the weight two Hodge structure on $H^2(X,\mathbb{Z})$.

Remark 3.5.4. [Hu, Theorem 7.3.6]

Choosing a Kähler form ω *is also equivalent to choosing the complex structure* J *of a complex K3 surface and a Kähler metric* g*: we may define*

$$\omega := g(J(\cdot),\cdot).$$

Given Theorem 3.5.3, let us state some facts about the first Chern map specific to K3 surfaces. Since $H^1(X,\mathcal{O}_X)$ is trivial, the first Chern map c_1 as part of the long exact sequence in Section 3.1.1

$$\cdots \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow \cdots$$

is injective. Since $H^2(X,\mathbb{Z})$ is torsion-free, the Picard group $\mathrm{Pic}(X)$, as the source of the first Chern map, has no torsion. If we denote the intersection form on $\mathrm{Pic}(X)$ by b, then $\mathrm{Pic}(X)$ is a lattice which we refer to as the **Picard lattice**. We call the rank of $\mathrm{Pic}(X)$ the **Picard rank** and denote by $\rho(X)$. The Hodge index theorem (see [Hu, Section 1.2.2]) further says that $\mathrm{Pic}(X)$ is a lattice of signature $(1, \rho(X)-1)$.

Remark 3.5.5. *Since* c_1 *is injective, we identify the Picard lattice* $\mathrm{Pic}(X)$ *with the Néron-Severi group* $\mathrm{NS}(X)$ *which we also call the* ***Néron-Severi lattice****. We will use these notions interchangeably.*

Fix an ample line bundle $L \in \mathrm{Pic}(X)$. Since $H^2(X,\mathbb{Z})$ is an even lattice, $c_1(L)^2 := b(c_1(L), c_1(L)) = 2d$ for some positive integer d. We say (X,L) is a **polarised K3 surface of degree** d.

However, we are interested in K3 surfaces with a more general notion of polarisation called a lattice polarisation. Without a lattice polarisation, it is not possible to give a moduli space of K3 surfaces that is Hausdorff [Hu, Section 6.3.1]. First we need some definitions concerning **$\mathbb{R}$-line bundles** on a K3 surface X, which are vectors in $\mathrm{Pic}(X) \otimes \mathbb{R}$.

Note that [GrifH, Chapter 1.1, Line Bundles] for a complex variety X, there is a one-to-one correspondence between $\mathrm{Pic}(X)$, the group of isomorphism classes of line bundles on X, and $\mathrm{CaCl}(X)$, the group of Cartier divisors of X modulo linear equivalence. We may extend this correspondence for the $\mathbb{R}$-line bundles by defining a **$\mathbb{R}$-Cartier divisor** to be a Weil divisor D such that $\mathcal{O}(D)$ is a $\mathbb{R}$-line bundle.

Suppose L is a $\mathbb{R}$-line bundle on X, and let the global sections $\{s_0, \cdots, s_r\}$ be a basis of $\Gamma(X, L)$ (which do not necessarily generate L). Then there exists a rational map $\varphi_L \colon X \dashrightarrow \mathbb{P}^r_{\mathbb{C}}$ given by $\varphi_L(x) = (s_0(x), \cdots, s_r(x))$ defined away from the base locus of L. Recall that if L is a line bundle, then L (or its corresponding Cartier divisor) is big if there exists an integer m_0 such that the rational map $\phi_{L^{\otimes m}}$ is birational for all $m \geq m_0$.

Definition 3.5.6. *A $\mathbb{R}$-Cartier divisor D of X is **big** if D can be expressed as a positive $\mathbb{R}$-linear combination of big Cartier divisors.*

*A $\mathbb{R}$-Cartier divisor is **nef** if it has non-negative intersection with any curve in X.*

We also need the notion of a very irrational vector in the $\mathbb{R}$-span of a lattice.

Definition 3.5.7. [AE, Section 2.2]

*Let Λ be a lattice. A vector $h \in \Lambda \otimes \mathbb{R}$ is **very irrational** if $h \notin \Lambda' \otimes \mathbb{R}$ for any proper sublattice $\Lambda' \subsetneq \Lambda$.*

We may now define a lattice polarisation for a K3 surface X. Let P be a lattice of signature $(1, r-1)$ where $1 \leq r \leq \rho(X) \leq 20$, and fix a very irrational vector h in $P \otimes \mathbb{R}$.

Definition 3.5.8. [AE, Definition 2.6]

*A **lattice polarisation** of X is a primitive embedding*

$$j \colon P \hookrightarrow \mathrm{Pic}(X)$$

*such that $j(h)$ is big and nef. We say (X, j) is a **P-polarised K3 surface**.*

Remark 3.5.9. *In [AE], a lattice polarisation (P, j) in the sense of Definition 3.5.8 is called a P-quasipolarisation, and is distiguished from a (strictly ample) lattice polarisation. The latter requires a stronger condition: the class $j(h)$ must be ample.*

Remark 3.5.10. *A primitive ample line bundle on a K3 surface X spans a primitive sub-lattice of rank 1 in $\mathrm{Pic}(X)$. Therefore, a polarised K3 surface is also a lattice polarised K3 surface.*

Remark 3.5.11. *Let P' be a sublattice of P of signature $(1, r'-1)$ for some $r \in \mathbb{Z}_{\geq 1}$. If P' primitively embeds into Λ_{K3}, then a P-polarised K3 surface is also a P'-polarised K3 surface.*

Let us define an isomorphism of lattice polarised K3 surfaces before discussing their moduli spaces.

Definition 3.5.12. [AE, Definition 2.6]

*Two P-polarised K3 surfaces (X, j) and (X', j') are **isomorphic** if there exists an isomorphism of varieties $f \colon X \to X'$ such that $j = f^* \circ j'$ and $f^*(j'(h)) = j(h)$.*

3.5.2 Moduli of lattice polarised K3 surfaces

We consider moduli spaces of K3 surfaces polarised by the same lattice P. We would like to to obtain an expression for the moduli space of P-polarised K3 surfaces. The followings are mainly taken from [AE, Section 2.2], [D, Section 3] and [DK, Section 9, 10].

Let P be a lattice of signature $(1, r-1)$ that can be primitively embedded into the K3 lattice Λ_{K3}. Let $\iota\colon P \hookrightarrow \Lambda_{K3}$ be such a primitive embedding and identify P with its image $\iota(P)$. Let T be the orthogonal complement of P in Λ_{K3}. It is a lattice of signature $(2, 20-r)$. We denote their associated symmetric bilinear forms by b, and the quadratic forms that satisfy the polarisation formula by q.

Remark 3.5.13.

1. *If the lattice P has rank r, then the family f_P of P-polarised K3 surfaces is also said to have Picard rank r. For any member in f_P, its Picard rank is at least r by Remark 3.5.11.*

2. *We say a member in f_P is **very general** if it corresponds to a very general point in the family of deformations preserving the Picard group. That is, the lattice embedding $P \hookrightarrow \mathrm{Pic}(X)$ is surjective. Therefore we abuse notation and say that P is the **Néron-Severi lattice** associated to the the family f_P. We also call $T = P^{\perp}_{\Lambda_{K3}}$ the **transcendental lattice** associated to f_P.*

We first rigidify our family by considering the family of marked P-polarised K3 surfaces instead.

Definition 3.5.14. *A **marked P-polarised K3 surface** is a pair (X, ϕ), where X is a K3 surface and ϕ is a marking of X such that $\phi^{-1}(P) \subset \mathrm{Pic}(X)$. In particular, $j_P := \phi^{-1}|_P\colon P \to \mathrm{Pic}(X)$ gives the lattice polarisation of X.*

*Two marked P-polarised K3 surfaces (X, ϕ), (X', ϕ') are **isomorphic** if there exists an isomorphism of varieties $f\colon X \to X'$ such that $\phi' = \phi \circ f^*$.*

Let $f_P^M\colon \mathcal{X}^M \to S^M$ be the family of marked P-polarised K3 surface. As in Section 3.1.3, the base S^M is covered by opens where each admits a holomorphic period map into $\mathcal{D}_T$, the period domain of weight two Hodge structures on T. Indeed, the lattice T carries a Hodge structure: by the Lefschetz theorem on $(1,1)$ classes, $H^{2,0}(X)$ which determines the weight two Hodge structure on $H^2(X, \mathbb{Z})$ is sent to the $\mathbb{C}$-extension of the transcendental lattice $T_{\mathbb{C}}$ under ϕ. Therefore $\phi(H^{2,0}(X)) \subset T_{\mathbb{C}}$ determines a weight two Hodge structure on T.

Before having further discussion on the period maps of the family f_P^M, let us derive two explicit expressions for the period domain $\mathcal{D}_T$: as an analytic subspace of $\mathbb{C}^{22-r}$, and as the quotient of Lie groups. Note that $\phi(H^{2,0}(X)) \subset T_{\mathbb{C}}$ is completely determined by the conditions in Theorem 3.1.14, which translates to the following: for all $l \in \phi(H^{2,0}(X))$,

(i) $l^2 \in \phi(H^{4,0}(X)) = 0$, and

(ii) $l \cdot \bar{l} > 0$.

Therefore the period domain of the weight two Hodge structures on T is

$$\mathcal{D}_T = \left\{ [l] \in \mathbb{P}(T_{\mathbb{C}}) : l^2 = 0,\ l \cdot \bar{l} > 0 \right\}.$$

Lemma 3.5.15. *The space $\mathcal{D}_T$ is $(20-r)$-dimensional.*

Proof. The $\mathbb{C}$-vector space $T_{\mathbb{C}}$ has dimension $22-r$. Its projectivisation has dimension $21-r$. The closed condition defining $\mathcal{D}_T$ as a subset in $\mathbb{P}(T_{\mathbb{C}})$ further subtracts one from the dimension. □

Again, we can express the period domain as a quotient of Lie groups.

Theorem 3.5.16.

$$\mathcal{D}_T \simeq \mathrm{O}(2, 20-r)/(\mathrm{SO}(2) \times \mathrm{O}(20-r)).$$

Proof. For any $[l] \in \mathbb{P}(T_{\mathbb{C}})$, write $l = x + iy$ where x, y are vectors in the $(22-r)$-dimensional real vector space $T_{\mathbb{R}}$. Then x, y span a positive definite plane Π in $T_{\mathbb{R}}$ as $l \cdot \bar{l} > 0$.

Conversely, given any positive definite plane $\Pi \subset T_{\mathbb{R}}$, choose any basis $\{x, y\}$ such that $x \cdot x = y \cdot y = 1$ and $x \cdot y = 0$, and consider $[l] = [x + iy] \in \mathbb{P}(T_{\mathbb{C}})$. Suppose another basis that satisfies the same conditions is given by

$$\left\{x' = ax + by, y' = cx + dy\right\}, \text{ where } \begin{cases} ad - bc = \Delta \neq 0 \\ a^2 + b^2 = c^2 + d^2 = 1 \\ ac + bd = 0 \end{cases}.$$

The system of equations gives $a = \Delta d$, $b = -\Delta c$ and $\Delta = \pm 1$. The two values of Δ correspond to the the choice of orientation of the plane Π. Fixing the orientation of Π is the same as fixing $\Delta = 1$, in which case $[l'] := [x' + iy'] = [(a - bi)l] = [l]$. On the other hand, if $\Delta = -1$, then $[l'] = [\bar{l}]$. Therefore, $\mathcal{D}_T$ is the space of oriented positive definite planes in $T_{\mathbb{R}}$.

It is clear that $\mathrm{Aut}(T_{\mathbb{R}}, q) = \mathrm{O}(2, 20-r)$. By Theorem 3.1.16, it acts transitively on $\mathcal{D}_T$. Moreover, the isotropy group of this action is $\mathrm{SO}(2) \times \mathrm{O}(20-r)$, where $\mathrm{SO}(2)$ gives the condition $\Delta = 1$. $\square$

With this expression, $\mathcal{D}_T$ has two connected components which correspond to either choosing $H^{2,0}(X)$ to be generated by l or $\bar{l}$. Each connected component is given by the orbit space

$$\mathrm{SO}^+(2, 20-r)/(\mathrm{SO}(2) \times \mathrm{SO}(20-r)).$$

Remark 3.5.17. *By comparing to [He, Table X.6.V],* $\mathrm{SO}(2) \times \mathrm{SO}(20-r)$ *is the maximal compact subgroup of* $\mathrm{SO}^+(2, 20-r)$*, and the quotient is an irreducible HSD of type* IV_{20-r}*.*

Let us return to considering the period maps for the family f_P^M. Recall from 3.1.3 that by patching all the period mappings together, one obtains the global multi-valued mapping $\mathcal{P}^M : S^M \to \mathcal{D}_T$. In fact:

Theorem 3.5.18. [AE, Theorem 2.8]

The multi-valued map $\mathcal{P}^M : S^M \to \mathcal{D}_T$ *is surjective. Moreover, for each point* $\Pi \in \mathcal{D}_T$ *as a plane in* $T_{\mathbb{R}}$ *(as described in the proof of Theorem 3.5.16), there is a natural bijection between the fibre* $\mathcal{P}^{-1}(\Pi)$ *and the group* $W_\Pi(T)$*, which is the set of isometries on* Λ_{K3} *generated from reflections in vectors from*

$$\left\{e \in \Pi^\perp \cap T : e^2 = -2\right\}.$$

Remark 3.5.19. *Suppose* $\Pi \in \mathcal{D}_T$ *is the period point of a marked K3 surface with a strictly ample polarisation (Remark 3.5.9). Then [D, Corollary 3.2] the group* $W_\Pi(T)$ *is trivial. Theorem 3.5.18 implies that* $\mathcal{P}^M$ *restricted to the set of marked K3 surfaces with a strictly ample polarisation is a bijection onto its image. That is, the Global Torelli Theorem is satisfied for a family of marked K3 surfaces with a strictly ample polarisation [Hu, Theorem 7.5.3]*

Finally, we remove the marking of f_P^M to obtain the global period map for the family of (unmarked) P-polarised K3 surfaces $f : \mathcal{X} \to S$. Consider the **orthogonal group** of Λ_{K3}, which is the group of isometries $\mathrm{O}(\Lambda_{K3}) = \mathrm{Aut}((\Lambda_{K3})_{\mathbb{R}}, q)$. It has a subgroup

$$\widetilde{\Gamma}(P) := \{\sigma \in \mathrm{O}(\Lambda_{K3}) : \sigma(p) = p \text{ for all } p \in P\}.$$

Note that if a marked P-polarised K3 surface (X, ϕ) is isomorphic to another marked P-polarised (X', ϕ') in the sense of Definition 3.5.12, then it is also isomorphic to $(X', \phi' \circ \sigma)$ for any $\sigma \in \widetilde{\Gamma}(P)$. Denote the image of $\widetilde{\Gamma}(P)$ under the natural injective homomorphism from $\widetilde{\Gamma}(P)$ to $\mathrm{O}(T)$ by $\Gamma(P)$. Then $\Gamma(P)$ acts on $\mathcal{D}_T$, and by Theorem 3.1.18, the global period map for the family f is given by the descent of $\mathcal{P}^M$:

$$\widetilde{\Gamma}(P)\backslash S^M \simeq S \longrightarrow \Gamma(P)\backslash\mathcal{D}_T.$$

Since for any $\Pi \in \mathcal{D}_{\mathcal{T}}$ the group $W_\Pi(T)$ is a subgroup of $\widetilde{\Gamma}(P)$, Theorem 3.5.18 implies that the global period map is a bijection. In particular, the Global Torelli Theorem for a family of P-polarised K3 surfaces is satisfied.

Moreover by [D, Proposition 3.3], $\Gamma(P)$ is a subgroup of finite index in $\mathrm{O}(T)$, where the latter is an arithmetic subgroup of $\mathrm{O}(2, 20-r)$. So by the Baily-Borel Theorem,

$$\mathcal{K}_P := \Gamma(P)\backslash\mathcal{D}_T$$

is a quasi-projective variety. We call $\mathcal{K}_P$ the **moduli variety of P-polarised K3 surfaces**. It has at most two irreducible components; each is a LSV of type IV_{20-r}, and has the same dimension as $\mathcal{D}_T$, which is $20-r$. In fact,

Theorem 3.5.20. [D, Lemma 5.4, 5.6]
Let $\mathcal{K}_P$ be the moduli space of P-polarised K3 surfaces. Let T be the associated transcendental lattice. If T admits the lattice U or $U(2)$ as an orthogonal summand, then the moduli space $\mathcal{K}_P$ is irreducible.

3.5.3 Shioda-Inose structure

In this subsection, we introduce the Shioda-Inose structure of a lattice polarised K3 surface associated to an abelian surface. The main reference is [Mo1].

Definition 3.5.21. [Mo1, Defintion 5.1, Lemma 5.2]
An involution ι on a K3 surface X is a ***Nikulin involution*** *if $\iota^*(l) = l$ for any $l \in H^{2,0}(X)$. Each Nikulin involution has eight isolated fixed points.*

Definition 3.5.22. [Mo1, Definition 6.1]
A K3 surface X is said to admit a ***Shioda-Inose structure*** *associated to an abelian surface A if there is a Nikulin involution ι on X such that the Kummer surface $Y = \mathrm{Kum}(A)$ is the minimal resolution of $X/\langle\iota\rangle$, and if the associated rational double cover $\pi_X \colon X \dashrightarrow Y$ induces a Hodge isometry $(\pi_X)_* \colon T_X(2) \to T_Y$, where T_X and T_Y are the transcendental lattices of X and Y respectively.*

If X has a Shioda-Inose structure, then there are rational double covers as in Diagram 1, where

(i) the map $\pi_A \colon A \dashrightarrow Y$ is the rational map induced by the blow-up $Y \to A/\langle -1\rangle$ at the 16 isolated double points in the quotient of an abelian surface A by the group $\langle -1\rangle$; and

(ii) the map $\pi_X \colon X \dashrightarrow Y$ is the rational map induced by the blow up $Y \to X/\langle\iota\rangle$ at the 8 isolated double points in the quotient of X by the group generated by the Nikulin involution ι.

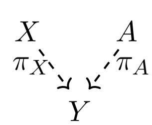


Diagram 1: A K3 surface X with a Shioda-Inose structure associated to an abelian surface A.

One can in fact define a transcendental lattice T_A of the abelian surface A, and relate it to T_X and T_Y. For an abelian surface A, the cohomology group $H^2(A, \mathbb{Z}) \simeq U^{\oplus 3}$ is a lattice. Therefore, the Néron-Severi group $\mathrm{NS}(A)$ in $H^{1,1}(A) \cap H^2(A, \mathbb{Z})$ is torsion-free. We call $\mathrm{NS}(A)$ the **Néron-Severi**

lattice of A and its rank $\rho(A)$ the **Picard rank**. We call its orthogonal complement T_A in $H^2(A,\mathbb{Z})$ the **transcendental lattice** of A. The Néron-Severi lattice and the transcendental lattice are of signatures $(1,\rho(A)-1)$ and $(2,4-\rho(A))$ respectively. The transcendental lattice T_A carries a natural weight two sub-Hodge structure of $H^2(A,\mathbb{Z})$ equipped with the intersection form (up to a sign on the $T_A^{2,0}$ component). Moreover, we have the following result that says a sublattice in $U^{\oplus 3}$ of signature $(1,\cdot)$ or $(2,\cdot)$ determines an abelian surface.

Theorem 3.5.23. [Mo1, Corollary 1.9]
Suppose $P \hookrightarrow U^{\oplus 3}$ (resp. $T \hookrightarrow U^{\oplus 3}$) is a primitive sublattice of signature $(1,\rho-1)$ (resp. $(2,4-\rho)$). Then there exists an abelian surface A and a Hodge isometry $\mathrm{NS}(A) \simeq P$ (resp. $T_A \simeq T$).

Of course by our discussion in Section 3.5.2, Theorem 3.5.23 is also true for K3 surfaces when replacing $U^{\oplus 3}$ by the K3 lattice and using the appropriate signatures.

The Shioda-Inose structure of X described in Diagram 1 induces Hodge homotheties, *i.e.* Hodge isometries up to twisting, of the transcendental lattices T_X, T_Y and T_A of X, Y and A respectively.

Theorem 3.5.24. [Mo1, Remark 6.2]
Let $\pi_A\colon A \dashrightarrow Y$ be the rational double cover associated to the Kummer surface $Y = \mathrm{Kum}(A)$. Then π_A induces the Hodge isometry

$$(\pi_A)_*\colon T_A(2) \xrightarrow{\simeq} T_Y.$$

If the K3 surface X has a Shioda-Inose structure associated to A, then we have the Hodge isometry

$$T_X \simeq T_A.$$

These Hodge isometries give us the exact criteria for a lattice T of signature $(2,20-\rho)$ to be the transcendental lattice of a K3 surface with a Shioda-Inose structure.

Theorem 3.5.25. [Mo1, Corollary 6.3]
A K3 surface X of Picard rank ρ with transcendental lattice T_X admits a Shioda-Inose structure if and only if

(i) $\rho = 19$ *or* 20*; or*

(ii) $\rho = 18$*, and* $T_X = U \oplus T'$ *for some lattice* T'*; or*

(iii) $\rho = 17$*,* $T_X = U^{\oplus 2} \oplus T'$ *for some lattice* T'*.*

Proof. From [Mo1, Corollary 2.6], we know exactly when a lattice T of signature $(2,k)$ admits a primitive embedding into $T \hookrightarrow U^{\oplus 3}$ for each value of k between 0 and 3. By Theorem 3.5.24, this translates to the criteria for a lattice of signature $(2, k = 20-\rho)$ to be the transcendental lattice of a Kummer surface of Picard rank ρ satisfying $17 \leq \rho \leq 20$. In fact, any Kummer surface has Picard rank at least 17: it has 16 exceptional curves from blowing up the 16 double points in $A/\langle -1\rangle$, as well as a Kähler class. So applying Theorem 3.5.24 again gives us the complete set of conditions for a lattice of signature $(2,20-\rho)$ to be the transcendental lattice of a K3 surface admitting a Shioda-Inose structure. □

4 Kuga-Satake construction

The Kuga-Satake construction produces an abelian variety called a Kuga-Satake variety from the Clifford algebra of a weight two Hodge structure of K3 type. To prepare ourselves for the classical Kuga-Satake construction, we will first recall the definition of a Clifford algebra and its remarkable subgroup called the spin group. From now on, we replace the phrase “Kuga-Satake” by the abbreviation “KS”.

4.1 Clifford algebra

The main references for Clifford algebras over a field and for spin groups are [Harv, Part II] (over $\mathbb{R}$), [LM, Chapter I], [La, Chapter V] and [Hu, Section 4.1]. We will extend their definitions of a Clifford algebra to one over $\mathbb{Z}$.

Let R be $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Let V be a R-module of rank n equipped with a non-degenerate symmetric bilinear form $b\colon V \times V \to R$ (thus a quadratic form q via the polarisation formula). Suppose V is of signature (n_+, n_-) and $\mathrm{O}(V)$ is the orthogonal group of V.

Definition 4.1.1. [Hu, Section 4.1.1]
*The **Clifford algebra** $\mathrm{Cl}(V)$ over (V, q) is defined as*

$$\mathrm{Cl}(V) := \mathrm{T}(V)/I(V)$$

*where $\mathrm{T}(V) := \sum_{k=0}^{\infty} \bigotimes^k V$ is the **tensor algebra** over V, and $I(V)$ is the ideal generated by elements in the form*

$$v \otimes v - q(v).$$

*Any element $v_1 v_2 \cdots v_d$ in $\mathrm{Cl}(V)$ which is a product of d non-scalar elements in V is called a **simple element of degree** d*

Remark 4.1.2. [Harv, Equation 9.6] *The ideal $I(V)$ determines the rules of **Clifford multiplication**, which is the multiplication operation of $\mathrm{Cl}(V)$. From the polarisation formula, $v \otimes v - q(v) \in I(V)$ gives the relation $u \otimes v + v \otimes u - 2b(u, v) \in I(V)$. So considering $u, v \in \mathrm{Cl}(V)$, we have $u \cdot v + v \cdot u = 2b(u, v)$.*

Note that the tensor algebra $T(V)$ has a natural $\mathbb{Z}$-grading: for any integers $k, l \geq 0$,

$$T^k(V) \otimes T^l(V) \simeq T^{k+l}(V),$$

where $T^k(V) := \bigotimes^k V$ is the degree k subspace of $T(V)$. Moreover, since any element in the ideal $I(V) < T(V)$ has even degree, the Clifford multiplication respects parity of degree in $T(V)$. Therefore the Clifford algebra has a $\mathbb{Z}_2$-grading

$$\mathrm{Cl}(V) = \mathrm{Cl}^+(V) \oplus \mathrm{Cl}^-(V),$$

where $\mathrm{Cl}^+(V)$ is the **even part** of $\mathrm{Cl}(V)$ spanned by the classes of the even degree elements in $\mathrm{T}(V)$, and $\mathrm{Cl}^-(V)$ is the **odd part** spanned by the classes of the odd degree elements in $\mathrm{T}(V)$.

Sometimes we forget the Clifford multiplication and view $\mathrm{Cl}(V)$ as a vector space/module. By the following theorem, we may identify the Clifford algebra $\mathrm{Cl}(V)$ over V of dimension/rank n with the **exterior algebra** $\bigwedge V := \sum_{k=0}^{n} \bigwedge^k V$ as graded R-vector spaces/modules.

Theorem 4.1.3. [LM, Proposition 1.3]
There is a canonical isomorphism $\bigwedge V \to \mathrm{Cl}(V)$ of vector spaces/modules such that on each simple element in $\bigwedge V$ of degree d, it is given by

$$v_1 \wedge \cdots \wedge v_d \longmapsto \sum_{\sigma \in S_d} \mathrm{sign}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(d)},$$

where the sum is over S_d, the symmetric group of degree d.

This isomorphism preserves (the parity of) the degrees of the simple elements, so it respects the $\mathbb{Z}_2$-grading of the Clifford algebra. Therefore, it is clear that

$$\dim(\mathrm{Cl}(V)) = \dim(\bigwedge V) = 2^n.$$

There is a universal property for Clifford algebras.

Lemma 4.1.4 (The Fundamental Lemma for Clifford algebras)**.** [Harv, Lemma 9.7]
Let A be an associative algebra with unit over R. Let $\varphi\colon V \longrightarrow A$ be a linear map from V into A. If for all $v \in V$ we have

$$\varphi(v)\varphi(v) - q(v) \cdot \mathbf{1}_A = 0,$$

then φ has a unique extension to an algebra homomorphism of $\mathrm{Cl}(V)$ into A.

Proof. There is also a universal property for tensor algebra that extends the linear map φ on V to an algebra homomorphism $\mathrm{T}(V) \to A$ which we also denote by φ. In particular, for any $u, v \in V$, then $u \otimes v \in \mathrm{T}(V)$, and set

$$\varphi(u \otimes v) := \varphi(u) \otimes \varphi(v).$$

By the hypothesis, elements in the form $v \otimes v \in \mathrm{T}(V)$ lie in the kernel of φ, thus φ descends to an algebra homomorphism $\mathrm{Cl}(V) \to A$. □

The Fundamental Lemma for Clifford algebras implies that any linear map between two R-modules that preserves their associated symmetric bilinear forms extends uniquely to an algebra homomorphism between the respective Clifford algebras. Moreover [Harv, Theorem 9.20], automorphisms of $\mathrm{Cl}(V)$ are exactly those extended from the isometries of V. In this way, a Clifford algebra admits a distinguished automorphism, the canonical automorphism.

Definition 4.1.5. [LM, Equation 1.7]
*The **canonical automorphism** $(\cdot)^-\colon \mathrm{Cl}(V) \longrightarrow \mathrm{Cl}(V)$ is an involution defined by extending the isometry $v \mapsto -v$ on V to an automorphism on $\mathrm{Cl}(V)$.*

Note that $(\cdot)^-$ acts trivially on the even part $\mathrm{Cl}^+(V)$. On the odd part $\mathrm{Cl}^-(V)$, $(\cdot)^-$ acts by multiplication by $-\mathbf{1}_R$.

A Clifford algebra also admits a special anti-automorphism.

Definition 4.1.6. [LM, Equation 1.15]
Consider the involution $(\cdot)^t\colon \mathrm{T}(V) \to \mathrm{T}(V)$ such that on any simple element $v_1 \otimes v_2 \otimes \cdots \otimes v_d$,

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_d)^t = v_d \otimes \cdots \otimes v_2 \otimes v_1.$$

*Since $(\cdot)^t$ sends the ideal $I(V)$ to itself, it descends to an involution on $\mathrm{Cl}(V)$ called the **transpose**, which we still denote by $(\cdot)^t$.*

Clearly $(\cdot)^t$ is an anti-automorphism: for all $x, y \in \mathrm{Cl}^+(V)$, we have $(x \cdot y)^t = y^t \cdot x^t$. Also, given a homogeneous element $x \in \mathrm{Cl}(V)$ of degree d, that is x is the sum of finitely many simple elements of the same degree d, then

$$x^t := \begin{cases} x, & \text{if } d \equiv 0, 1 \mod 4 \\ -x, & \text{if } d \equiv 2, 3 \mod 4 \end{cases}$$

Note that the above two involutions commute. We denote their composition by

$$(\cdot)^* := ((\cdot)^t)^- = ((\cdot)^-)^t. \tag{5}$$

Like any $\mathbb{Z}_2$-graded algebra, a Clifford algebra is equipped with a **graded tensor product**. For any $\mathbb{Z}_2$-graded algebra A, we denote $A = A_0 \oplus A_1$, where A_i is the component of elements of degree i for $i = 0, 1$. We define the **degree function** ∂ for an element $a \in A$ such that $\partial(a) = i$ if $a \in A_i$. We give the definition of a graded tensor product as follows.

Definition 4.1.7. [La, Section IV.2]

The graded tensor product of two $\mathbb{Z}_2$-graded algebras A and B denoted by $A \widehat{\otimes} B$ is also a $\mathbb{Z}_2$-graded algebra, with the component of degree i elements, where $i = 0, 1$ given by

$$(A \widehat{\otimes} B)_i := \sum_{j+k \equiv i \mod 2} (A_j \otimes B_k).$$

The multiplication on $A \widehat{\otimes} B$ is defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\partial(b)\partial(a')} aa' \otimes bb'$$

for any homogeneous elements a, a' and b, b' in A and B respectively.

In particular, the graded tensor product of two Clifford algebras is also a Clifford algebra.

Theorem 4.1.8 (Gluing of Clifford algebras)**.** [La, Lemma 1.7, Theorem 1.8]

Let (V, q) and (V', q') be two R-vector spaces/modules equipped with a quadratic form q and q' respectively. Then by the Fundamental lemma for Clifford algebra, the linear map

$$\begin{aligned} V \oplus V' &\longrightarrow \mathrm{Cl}(V) \widehat{\otimes} \mathrm{Cl}(V') \\ (v, v') &\longmapsto v \otimes \mathbf{1} + \mathbf{1} \otimes v' \end{aligned}$$

extends to a morphism of $\mathbb{Z}_2$-graded algebras

$$f \colon \mathrm{Cl}(V \oplus V') \xrightarrow{\simeq} \mathrm{Cl}(V) \widehat{\otimes} \mathrm{Cl}(V'),$$

which is in fact an isomorphism.

A Clifford algebra with Clifford multiplication can be identified with (a sum of) matrix algebras with the usual matrix multiplication as a $\mathbb{Z}_2$-graded algebra. The corresponding $\mathbb{Z}_2$-grading for the matrix algebras is called the **checkerboard grading**.

Definition 4.1.9. [La, Section IV.2]

The algebra of k-by-k matrices over a $\mathbb{Z}_2$-graded algebra A, denoted by $\widehat{M}_d(A)$, is a $\mathbb{Z}_2$ graded algebra with respect to the checkerboard grading. Its degree 0 part and its degree 1 part are respectively given by

$$(\widehat{M}_d(A))_0 = \begin{bmatrix} A_0 & A_1 & A_0 & \\ A_1 & A_0 & A_1 & \\ A_0 & A_1 & A_0 & \\ & & & \ddots \end{bmatrix}, \quad (\widehat{M}_d(A))_1 = \begin{bmatrix} A_1 & A_0 & A_1 & \\ A_0 & A_1 & A_0 & \\ A_1 & A_0 & A_1 & \\ & & & \ddots \end{bmatrix}.$$

Remark 4.1.10. *When $A = R$ is a ring, then A is concentrated at degree 0,* i.e. $A_1 = 0$.

Remark 4.1.11. *A matrix algebra $M_d(A)$ over a $\mathbb{Z}_2$-graded algebra A admits a graded tensor product $\widehat{\otimes}$ similar to what is described in Definition 4.1.7. In this case, the tensor product $\otimes$ in the definition is replaced by the Kronecker product introduced in Theorem 3.3.12.*

Denote by $R^{(n_+, n_-)}$ the R-module of rank n equipped with the standard quadratic form of signature (n_+, n_-) given by

$$v_1^2 + \cdots + v_{n_+}^2 - v_{n_++1}^2 - \cdots v_n^2.$$

First we restrict ourselves to $R = \mathbb{R}$. Denote $\mathrm{Cl}(\mathbb{R}^{(n_+, n_-)})$ by $\mathrm{Cl}(n_+, n_-)$. Then for any vector space $V \simeq \mathbb{R}^{(n_+, n_-)}$, we have $\mathrm{Cl}(V) \simeq \mathrm{Cl}(n_+, n_-)$. For $0 \leq n \leq 2$, one can determine an explicit isomorphism between $\mathrm{Cl}(n_+, n_-)$ and a matrix algebra respecting their respective $\mathbb{Z}_2$-gradings for each pair (n_+, n_-) by applying [Harv, Exercise 9.3, 9.4]. For larger n, the corresponding isomorphism of graded algebras for $\mathrm{Cl}(n_+, n_-)$ can be derived by repeatedly gluing up the Clifford algebras of smaller dimensions, applying Theorem 4.1.8.

Moreover, the following theorem allows the even part of a Clifford algebra to be identified with a (sum of) matrix algebra.

Theorem 4.1.12. [Harv, Theorem 9.38, 9.43]

$$\mathrm{Cl}^+(n_+ + 1, n_-) \simeq \mathrm{Cl}(n_+, n_-)$$
$$\mathrm{Cl}(n_+, n_- + 1) \simeq \mathrm{Cl}(n_-, n_+ + 1)$$

The isomorphisms in Theorem 4.1.12 between Clifford algebras and matrix algebras only respect the involution $(\cdot)^-$, which is just multiplication by $-\mathbf{1}$ in terms of matrices. They do not respect the other two involutions introduced above.

For convenience, in Table 5 we list the (sums of) matrix algebras that are isomorphic to $\mathrm{Cl}(n_+, n_-)$ for $0 \le n_+, n_- \le 6$. In fact, there is a pattern for these matrix algebras isomorphic to $\mathrm{Cl}(n_+, n_-)$ depending on $(n_+ - n_-) \mod 8$: for details, see [Harv, Theorem 11.3, Table 11.5].

n_+ \ n_-	0	1	2	3	4	5	6
0	0	$\mathbb{R}^{\oplus 2}$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$	$M_2(\mathbb{H})$	$M_2(\mathbb{H})^{\oplus 2}$	$M_4(\mathbb{H})$
1	$\mathbb{C}$	$M_2(\mathbb{R})$	$M_2(\mathbb{R})^{\oplus 2}$	$M_4(\mathbb{R})$	$M_4(\mathbb{C})$	$M_4(\mathbb{H})$	$M_4(\mathbb{H})^{\oplus 2}$
2	$\mathbb{H}$	$M_2(\mathbb{C})$	$M_4(\mathbb{R})$	$M_4(\mathbb{R})^{\oplus 2}$	$M_8(\mathbb{R})$	$M_8(\mathbb{C})$	$M_8(\mathbb{H})$
3	$\mathbb{H}^{\oplus 2}$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R})^{\oplus 2}$	$M_8(\mathbb{R})$	$M_{16}(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})^{\oplus 2}$	$M_4(\mathbb{H})$	$M_8(\mathbb{C})$	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})^{\oplus 2}$	$M_{32}(\mathbb{R})$
5	$M_4(\mathbb{C})$	$M_4(\mathbb{H})$	$M_4(\mathbb{H})^{\oplus 2}$	$M_8(\mathbb{H})$	$M_{16}(\mathbb{C})$	$M_{32}(\mathbb{R})$	$M_{32}(\mathbb{R})^{\oplus 2}$
6	$M_8(\mathbb{R})$	$M_8(\mathbb{C})$	$M_8\mathbb{H}$	$M_8(\mathbb{H})^{\oplus 2}$	$M_{16}(\mathbb{H})$	$M_{32}(\mathbb{C})$	$M_{64}(\mathbb{R})$

Table 5: Matrix algebras isomorphic to $\mathrm{Cl}(n_+, n_-)$ for $0 \le n_+, n_- \le 6$

By restricting to $R = \mathbb{Q}$, we have a similar isomorphism from $\mathrm{Cl}_{\mathbb{Q}}(n_+, n_-) := \mathrm{Cl}(\mathbb{Q}^{(n_+, n_-)})$ to the corresponding matrix algebra A in Table 5 but with entries in $\mathbb{Q}, \mathbb{Q}[i]$ or $\mathbb{H}_{\mathbb{Q}}$. When we further restrict to $R = \mathbb{Z}$, the image of $\mathrm{Cl}_{\mathbb{Z}}(n_+, n_-) := \mathrm{Cl}(\mathbb{Z}^{(n_+ . n_-)})$ under φ is contained in a maximal order in the $\mathbb{Q}$-algebra A.

Definition 4.1.13. [Rein, Section 2.8]

*An **order** in the $\mathbb{Q}$-algebra A is a subring Λ of A with the same multiplicative identity as A, such that Λ is a lattice in A and Λ spans the vector space A over $\mathbb{Q}$.*

*An order in A is **maximal** if it is not properly contained in another order in A.*

The notion of maximal orders arises [Rein, Section 4a] as the integral structure in a number field A in the work of Dedekind: the integral closure of $\mathbb{Z}$ in A is the unique maximal order in A, and its elements are called algebraic integers. An important example [CS, Section 5.1] of a maximal order in the $\mathbb{Q}$-algebra $\mathbb{H}_{\mathbb{Q}}$ is the **Hurwitz integers**

$$\mathfrak{o} := \mathbb{Z}\left\langle h := \frac{1+i+j+k}{2}, i, j, k \right\rangle$$

with a quadratic form q given by the norm function $z \mapsto z\bar{z}$. The matrix of the associated symmetric bilinear form b with respect to the generators $\{h, i, j, k\}$ is

$$\frac{1}{2}\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}.$$

Moreover [Rein, Theorem 8.7], $M_n(\mathfrak{o})$ is a maximal order in $M_n(\mathbb{H}_{\mathbb{Q}})$ for all integer $n > 0$.

Lastly, when $R = \mathbb{C}$, any non-degenerate quadratic forms are equivalent. See [LM, Chapter I, Table I] for the corresponding identifications of $\mathrm{Cl}((\mathbb{C}^n, q))$ with matrix algebras.

4.2 Spin group and representations

In this subsection, we consider $R = \mathbb{R}$. Let V be a n-dimensional $\mathbb{R}$-vector space equipped with a non-degenerate quardratic form q. We define a few groups, including the spin group, contained in the Clifford algebra $\mathrm{Cl}(V)$ with representations given by the groups mentioned in Definition 2.2.4(iii–iv). In particular, if (V, q) is a quadratic space of signature $(2, 6)$, then the Lie groups $\mathrm{SO}^+(2, 6)$ and $\mathrm{SO}^*(8)$ associated to LSVs of type IV_6 and II_4 are involved as two inequivalent representations of the identity component of the spin group $\mathrm{Spin}^+(2, 6) \subset \mathrm{Cl}^+(2, 6)$. Later in Section 5, we will construct a map between LSVs of type IV_6 and of type II_4, and show that the map is locally equivalent to switching from one representation of $\mathrm{Spin}^+(2, 6)$ to another.

Definition 4.2.1. [LM, Section I.2], [Harv, Section 10], [Hu, Section 4.1.2]

Let $\mathrm{Cl}^*(V)$ be the multiplicative group of units of the Clifford algebra $\mathrm{Cl}(V)$. The ***twisted adjoint representation*** *of $\mathrm{Cl}^*(V)$ is given by*

$$\begin{aligned} \widetilde{\mathrm{Ad}}\colon \mathrm{Cl}^*(V) &\longrightarrow \mathrm{GL}(\mathrm{Cl}(V)) \\ x &\longmapsto \widetilde{\mathrm{Ad}}_x(\cdot) := \left[y \mapsto \left(x^- \cdot y \cdot x^{-1}\right)\right]. \end{aligned}$$

The ***Clifford group*** *associated to (V, q) is defined to be*

$$\mathrm{CPin}(V) := \left\{x \in \mathrm{Cl}^*(V) : \widetilde{\mathrm{Ad}}_x(v) \in V \text{ for all } v \in V\right\}.$$

The ***special Clifford group*** *(or the classical companion group) is defined to be*

$$\mathrm{CSpin}(V) := \mathrm{CPin}(V) \cap \mathrm{Cl}^+(V).$$

The ***pin group*** *is a subgroup of the Clifford group*

$$\begin{aligned} \mathrm{Pin}(V) &:= \{x \in \mathrm{CPin}(V) : x = v_1 \cdots v_d \text{ such that } q(v_i) = \pm 1 \text{ for all } i = 1, \cdots, d\} \\ &= \{x \in \mathrm{CPin}(V) : xx^* = 1\} \end{aligned}$$

The ***spin group*** *(or the reduced Clifford group) is a subgroup of the special Clifford group*

$$\mathrm{Spin}(V) := \mathrm{Pin}(V) \cap \mathrm{Cl}^+(V).$$

Equivalently, the spin group is defined by the short exact sequence

$$1 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{CSpin}(V) \longrightarrow \mathbb{R}^* \longrightarrow 1.$$

If $\mathrm{CSpin}(V)$ (resp. $\mathrm{Spin}(V)$) is not connected, then its identity component is denoted by $\mathrm{CSpin}^+(V)$ (resp. $\mathrm{Spin}^+(V)$).

Remark 4.2.2. *To avoid any confusion, we would like to emphasise that the $+$ decoration in $\mathrm{CSpin}^+(V)$ and $\mathrm{Spin}^+(V)$ is used in similar sense as the $+$ decoration in $\mathrm{SO}^+(V)$, rather than as in $\mathrm{Cl}^+(V)$.*

By the definition of the Clifford group, it is natural to consider the restriction of the twisted adjoint representation to the Clifford group and its subgroups. In fact [LM, Equation I.2.22], there is a nice geometric interpretation of the image $\widetilde{\mathrm{Ad}}(\mathrm{Pin}(V))$: for all $v \in V$, $\widetilde{\mathrm{Ad}}(v)$ is the element in $\mathrm{O}(V)$ that corresponds to the reflection along $v^\perp$. Since any element of $\mathrm{O}(V)$ is a composition of reflections, we have $\widetilde{\mathrm{Ad}}(\mathrm{Pin}(V)) = \mathrm{O}(V)$. We also know the kernels and the images of $\widetilde{\mathrm{Ad}}$ restricted to all three groups.

Theorem 4.2.3. [LM, Proposition 2.4, Theorem 2.9, 2.10], [vG1, Section 6.2]
There are short exact sequences

$$1 \longrightarrow \mathbb{R}^* \longrightarrow \mathrm{CPin}(V) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{GL}(V) \longrightarrow 1$$
$$1 \longrightarrow \mathbb{R}^* \longrightarrow \mathrm{CSpin}^+(V) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}^+(V) \longrightarrow 1$$
$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}(V) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(V) \longrightarrow 1$$
$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}^+(V) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}^+(V) \longrightarrow 1,$$

where $\mathrm{GL}(V)$ *is the group of invertible linear maps on* V *(not necessarily preserving the quadratic form* q*);* $\mathrm{O}(V)$ *and* $\mathrm{SO}^+(V)$ *are the orthogonal group and the identity component of the special orthogonal group of* (V, q) *respectively.*

In particular, the twisted adjoint representation restricted to the pin group (resp. spin group) is a double cover map over the orthogonal group (resp. special orthogonal group). Note that the twisted adjoint representation restricted to the spin group is simply the adjoint action.

In another direction, notice that the construction of the isomorphisms from Clifford algebras $\mathrm{Cl}(V) \simeq \mathrm{Cl}(n_+, n_-)$ to (sum of) matrix algebras in Section 4.1 gives a different matrix representation of the Clifford algebras [LM, Section I.5]. With reference to the pattern shown in Table 5 (which extends to all cases of (n_+, n_-)), we have isomorphisms of algebras

$$\begin{cases} \varphi : \mathrm{Cl}(n_+, n_-) \xrightarrow{\simeq} M_d(W) & \text{if } (n_+ - n_-) \not\equiv 3 \mod 4 \\ \varphi : \mathrm{Cl}(n_+, n_-) \xrightarrow{\simeq} M_d(W_+) \oplus M_d(W_-) & \text{otherwise,} \end{cases}$$

respecting the usual left multiplications [vG1, (6.2)].

In the first case, if $(n_+ - n_-) \not\equiv 3 \mod 4$, then φ itself is a simple representation *i.e.* there is no non-trivial sub-representation. The representation φ is called the **spin representation** of $\mathrm{Cl}(n_+, n_-)$. The vector space W is called the **space of spinors** of the spin representation [Harv, Definition 11.10]. There is only one simple representation of a simple Clifford algebra up to equivalence [LM, Theorem 5.6]: *i.e.* if (φ', W') is another simple representation, then there is an $\mathbb{R}$-linear map between W and W' that respects the representations.

In the second case, when $(n_+ - n_-) \equiv 3 \mod 4$, let π_+ and π_- be the projection maps from $M_d(W_+) \oplus M_d(W_-)$ to the first and the second component respectively. Then $\varphi_+ := \pi_+ \circ \varphi$ and $\varphi_- := \pi_- \circ \varphi$ are two simple representations of the Clifford algebra called the **half-spin representations**. Again, the vector spaces W_+ and W_- are called the **spaces of half-spinors** of the two representations respectively. In fact, the two half-spin representations are inequivalent [LM, Proposition 5.9], and are sometimes distinguished as the positive and the negative half-spin representation respectively.

A (half-)spin representation restricts to a simple matrix representation on (the identity component of) the spin group. We also know their kernels.

Theorem 4.2.4. [Harv, Theorem 13.8]
Consider the spin group $\mathrm{Spin}(V) \subset \mathrm{Cl}^+(V)$*, where the vector space* $V \simeq \mathbb{R}^n$ *is of signature* (n_+, n_-)*.*

If $(n_+ - n_-) \not\equiv 0 \mod 4$*, then (by Theorem 4.1.12) up to equivalence there is only one simple representation of the even part of the Clifford algebra* $\mathrm{Cl}^+(V)$*, which is the spin representation* φ*. Restricting* φ *to* $\mathrm{Spin}^+(V)$*, we have the left exact sequence*

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}^+(V) \xrightarrow{\varphi} \mathrm{GL}(W).$$

Otherwise, if $(n_+ - n_-) \equiv 0 \mod 4$*, then there are two inequivalent simple representations of* $\mathrm{Cl}^+(V)$*, which are the two half-spin representations* φ_+ *and* φ_-*. Restricting to* $\mathrm{Spin}^+(V)$*, we have the* *following left exact sequences*

$$\begin{aligned} 1 &\longrightarrow \mathbb{Z}_2 \simeq \{1, \lambda\} \longrightarrow \mathrm{Spin}^+(V) \xrightarrow{\varphi_+} \mathrm{GL}(W_+) \\ 1 &\longrightarrow \mathbb{Z}_2 \simeq \{1, -\lambda\} \longrightarrow \mathrm{Spin}^+(V) \xrightarrow{\varphi_-} \mathrm{GL}(W_-). \end{aligned}$$

Again the (half-)spin representations are degree two maps.

Remark 4.2.5. *By Theorem 4.1.12,*

$$\mathrm{Spin}(V) \subset \begin{cases} \mathrm{Cl}^+(n_+, n_-) \subset \mathrm{Cl}(n_+ - 1, n_-), & \text{if } n_+ > 0; \\ \mathrm{Cl}^+(n_-, n_+) \subset \mathrm{Cl}(n_- - 1, n_+), & \text{if } n_+ = 0. \end{cases}$$

It is clear that $(n_+ - 1) - n_- \equiv 3 \mod 4$ *if and only if* $n_+ - n_- \equiv 0 \mod 4$.

Remark 4.2.6. [FH, Exercise 20.36]

The Klein four-group $\{\pm 1, \pm\lambda\}$ *is the centre of* $\mathrm{Spin}^+(V)$*. Therefore, under the identification*

$$\mathrm{Cl}^+(V) \xrightarrow[\simeq]{\varphi} M_d(W_+) \oplus M_d(W_-),$$

it is clear that $\varphi(\lambda) = (D_1, D_2)$ *where* D_1 *and* D_2 *are two diagonal matrices. Let* $D_2 = \mathrm{diag}(x_1, \cdots, x_d)$*. By definition of the spin group, we have* D_2 *squares to* $\mathbf{1}_d$*, so* $x_k = \pm 1$ *for all* $k = 1, \cdots, d$*. In fact, each* x_k *acts on a copy of* W_- *(see proof of Theorem 5.1.6), so their actions have to be the same and* D_2 *is either* $\mathbf{1}_d$ *or* $-\mathbf{1}_d$*. Similarly* $D_1 = \pm\mathbf{1}_d$*. Since the positive half-spin representation is just the projection* π_+ *on* $M_d(W_+) \oplus M_d(W_-)$*, we have* $D_1 = \pm\mathbf{1}_d$ *and* $D_2 = \mp\mathbf{1}_d$*.*

Finally, let us consider the special case when $(V, q) \simeq \mathbb{R}^{(2,6)}$. We write $\mathrm{CPin}(2,6)$ for $\mathrm{CPin}(V)$, and similarly for all subgroups of $\mathrm{CPin}(V)$ defined in Definition 4.2.1. Consulting [Harv, Theorem 13.8], the image of $\mathrm{Spin}^+(2,6)$ under each half-spin representation is $\mathrm{SO}^*(8)$. Therefore we have the exact sequences in Diagram 2.

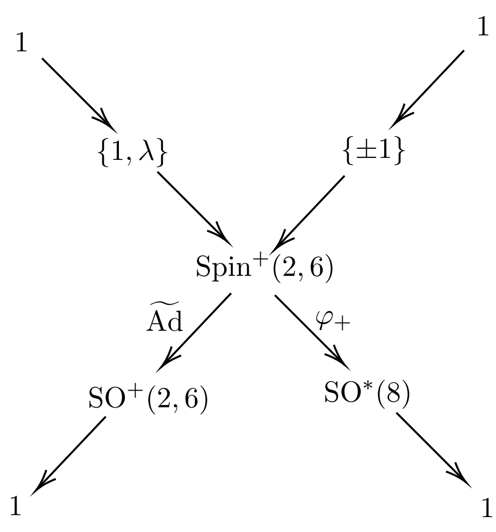


Diagram 2: Two inequivalent representations of the spin group.

Since both $\widetilde{\mathrm{Ad}}$ and φ_+ are degree two maps, the diagram gives local isomorphisms between the groups $\mathrm{SO}^+(2,6)$, $\mathrm{Spin}^+(2,6)$ and $\mathrm{SO}^*(8)$, thus the Lie algebra isomorphism

$$\mathfrak{so}^+(2,6) \simeq \mathfrak{so}^*(8)$$

mentioned at the end of Section 2.3.

4.3 KS construction

In this subsection, we explicitly construct a KS variety from a lattice polarised K3 surface. The main references are [Hu, Section 4.2] and [vG1], where the KS construction is given in greater generality: see Remark 4.3.3.

Let $(X, j\colon P \hookrightarrow \mathrm{Pic}(X))$ be a K3 surface polarised by the lattice P of signature $(1, r-1)$. Let $T := P^{\perp}_{\Lambda_{K3}}$ be the transcendental lattice of X, which is of signature $(2, 20-r)$. The Clifford algebra $\mathrm{Cl}(T)$ over T is a lattice of rank 2^{22-r}. The quotient $\mathbb{T} := \mathrm{Cl}^+(T_\mathbb{R})/\mathrm{Cl}^+(T)$ is therefore a torus of real dimension 2^{21-r}. Moreover, it has a complex structure and a polarisation, so in fact $\mathbb{T}$ is an abelian variety.

We first construct a complex structure for $\mathbb{T}$ as in [Hu, Section 4.2.1]. Recall the transcendental lattice (T, q) inherits a weight two Hodge structure from $H^2(X, \mathbb{Z})$ with the intersection form. Pick a generator $\sigma = e_1 + ie_2$ of $T^{2,0}$ such that $e_1, e_2 \in T_\mathbb{R}$ and $q(e_1) = 1$. Since $q(e_1 + ie_2) = 0$, the vectors e_1 and e_2 are orthonormal. Set $J = e_1e_2$.

Lemma 4.3.1. [vG1, Lemma 5.5, Proposition 6.3.1]
J is an element in the spin group $\mathrm{Spin}^+(T_\mathbb{R})$ satisfying $J = e_1e_2 = -e_2e_1$ and $J^2 = -1$. Moreover, J is independent of the choice of the orthonormal basis e_1, e_2.

Under the spin representation, $J \in \mathrm{Spin}^+(T_\mathbb{R})$ then gives a complex structure on $\mathrm{Cl}^+(T_\mathbb{R})$ by left multiplication, which is in accordance with Remark 3.3.15. This gives a weight one Hodge structure on $\mathrm{Cl}^+(T_\mathbb{R})$ as the decomposition into the $\pm i$ eigenspaces of J.

Next, we give a construction of a polarisation on the complex torus $(\mathbb{T}, J)$. Choose two orthogonal vectors $f_1, f_2 \in T$ with $q(f_i) > 0$, and let $\alpha = f_1f_2$. Consider the pairing E with

$$\begin{aligned} E\colon \mathrm{Cl}^+(T) \times \mathrm{Cl}^+(T) &\longrightarrow \mathbb{Z} \\ (v, w) &\longmapsto \mathrm{tr}(\alpha v^* w), \end{aligned}$$

where tr is the trace function for linear maps, and $v \mapsto v^*$ is the involution on $\mathrm{Cl}^+(T)$ defined in Section 4.1 Equation (5). One can check [vG1, Proposition 5.9] that $E_\mathbb{R}$, the real extension of E given in Theorem 3.1.11, is an alternating form. The symmetric form $E_\mathbb{R}(\cdot, J\ \cdot)$ is either positive or negative definite depending on the sign of α.

Therefore, $(\mathrm{Cl}^+(T_\mathbb{R})/\mathrm{Cl}^+(T), J, E)$ is an abelian variety of complex dimension 2^{20-r}. We call this abelian variety a **Kuga-Satake variety**, and denote it by $\mathrm{KS}(X, \alpha)$. We will suppress α in the notation, especially if it is clear what α is or if the choice of polarisation class is unimportant.

Remark 4.3.2. *Note that the vectors e_1 and e_2 defining the complex structure J in general do not belong to T. Still, we can choose f_1 and f_2 in T to be scalar multiples of e_1 and e_2 respectively.*

Remark 4.3.3. *In most literature, the starting ingredient of the KS construction is a Hodge structure of K3 type, rather than a K3 surface. A weight two Hodge structure on V of dimension n is said to be a* ***Hodge structure of K3 type*** *if $\dim V^{2,0} = 1$ and V is equipped with a quadratic form q of signature $(n-2, 2)$ which is positive definite on $V^{1,1}$. In particular, the second cohomology of a K3 surface together with the intersection form has a Hodge structure of K3 type. From a Hodge structure of K3 type V one arrives at a KS variety replacing T in the above steps by V and negating the quadratic form q. In fact, it does not matter whether we choose the associated quadratic form to be q or $-q$, because $\mathrm{Cl}^+(n_+, n_-) \simeq \mathrm{Cl}^+(n_-, n_+)$ by Theorem 4.1.12.*

If the Hodge structure of K3 type on V is only defined on $\mathbb{Q}$ but not $\mathbb{Z}$, then a KS variety is only defined up to isogenies. On the other hand, our more restrictive approach of starting from the $\mathbb{Z}$-Hodge structure of the transcendental lattice makes the KS variety an abelian variety, instead of just an isogeny class of abelian varieties.

Remark 4.3.4. *In fact, one can also define the KS variety from the odd part of the Clifford algebra* $\mathrm{Cl}^-(V_\mathbb{R})$ *instead of the even part. Concerning this, [Hu, Remark 4.2.3] says that for any lattice* V*, fixing a vector* w *in* V *gives an isomorphism of* $\mathbb{R}$*-vector space*

$$\begin{aligned} \mathrm{Cl}^+(V_\mathbb{R}) &\xrightarrow{\simeq} \mathrm{Cl}^-(V_\mathbb{R}) \\ v &\longmapsto v \cdot w \end{aligned}$$

This isomorphism induces an isogeny from the KS variety defined from $\mathrm{Cl}^+(V)$ *to the one defined from* $\mathrm{Cl}^-(V)$*.*

By Remark 4.3.3, given a K3 surface X, we may construct a KS variety from any sublattice T of $H^2(X,\mathbb{Z})$ of signature $(2, n-2)$ where $2 \leq n \leq 22$. We denote such a KS variety as $\mathrm{KS}(T)$ (suppressing the notion of α). Before we end the section, let us give a lemma concerning these more general KS varieties.

Lemma 4.3.5. [Mo2, Sections 4.4 and 4.7]

(i) Let X *be a K3 surface with transcendental lattice* T*. Let* T', T'' *be lattices such that* $T \subset T' \subset T'' \subset H^2(X,\mathbb{Z})$*, and let* $d = \dim_\mathbb{Q}((T''/T') \otimes \mathbb{Q})$*. Then*

$$\mathrm{KS}(T'') \sim \mathrm{KS}(T')^{2^d}.$$

(ii) Let X *be a K3 surface with a Shioda-Inose structure associated to an abelian surface* A*. Then*

$$\mathrm{KS}(H^2(X,\mathbb{Z})) \sim A^{2^{19}}.$$

The proof/explanation of Lemma 4.3.5(ii) in [Mo2, Section 4.7] depends on the statement in part (i), which is explained in [VV, Remark 2.4]. Let us repeat the proof here.

Proof. The isogeny of KS varieties is induced by the following isometries of lattices

$$\begin{aligned} \mathrm{Cl}^+(T'') &\simeq \Big((\mathrm{Cl}^+(T') \otimes \mathrm{Cl}^+((T')^\perp) \oplus (\mathrm{Cl}^-(T') \otimes \mathrm{Cl}^-((T')^\perp) \Big) \\ &\simeq 2^{d-1}(\mathrm{Cl}^+(T') \oplus \mathrm{Cl}^-(T')) \\ &\simeq 2^d \cdot \mathrm{Cl}^+(T'). \end{aligned}$$

The second isometry is because $(\mathrm{Cl}^+(T') \otimes \mathrm{Cl}^+((T')^\perp)$ is isomorphic to the direct sum of $\mathrm{rk}(\mathrm{Cl}^+((T')^\perp)$ copies of $\mathrm{Cl}^+(T')$; and the third isometry is due to Remark 4.3.4. □

5 KS varieties associated to families of K3 surfaces of Picard rank 14

In Section 5.1, we will modify the KS construction to give a map F from a modular variety of K3 surfaces of Picard rank 14 to a modular variety of abelian 8-folds with totally definite quaternion multiplication. Following this in Section 5.2, we will further lift the map F to an isomorphism $\widetilde{F}$ between the corresponding HSD overspaces as mentioned in Section 2.3.

5.1 KS construction on the level of families

5.1.1 General idea

In Section 4.3, we have constucted a KS variety $\mathrm{KS}(X,\alpha) = (\mathbb{T}, J, E)$ from a lattice polarised K3 surface X. We first lift the construction to a non-canonical map from a family of K3 surface to a family of KS varieties, and from there derive the map F by making some choices.

Although both $\mathbb{T}$ and J depend on the weight two Hodge structure on $H^2(X,\mathbb{Z})$, there are a lot of choices for the polarisation form E. In fact, the number of polarisation classes (unique up to scalar) depends on the size of $\mathrm{End}^s_\mathbb{Q}(A)$, the set of symmetric idempotents in $\mathrm{End}_\mathbb{Q}(A)$.

Theorem 5.1.1. [BL, Proposition 5.2.1]
Let A be an abelian variety. Then there is an isomorphism of $\mathbb{Q}$-vector spaces

$$\mathrm{NS}_{\mathbb{Q}}(A) \simeq \mathrm{End}^{s}_{\mathbb{Q}}(A).$$

To make a consistent choice of a KS variety associated to each K3 surface in a family, we may choose the same $\alpha \in \mathrm{Cl}^{+}(T)$ for every member X to give the same alternating form E. Although the corresponding polarisation class in $\mathrm{NS}_{\mathbb{Q}}(\mathrm{KS}(X))$ also depends on the Weil operator which is the positive complex structure J of X, the polarisation type remains constant as J varies in the family by the same argument in Lemma 3.2.19. So by fixing α (up to sign) in the KS construction, the KS varieties associated to the family of K3 surfaces share the same polarisation type, thus they form a family of KS varieties which lies in a family of polarised abelian varieties.

As explained in Section 2.3, we hope to construct a map from a LSV of type IV_6 to a LSV of type II_4. That is, a map from a modular variety of K3 surfaces of Picard rank 14 to a modular variety of polarised abelian 8-folds with totally definite quaternion multiplication. We will show that such a map can be obtained by modifying the above KS construction.

Let $\mathcal{K}_P$ be a modular variety of K3 surfaces with a rank 14 polarisation lattice P, and let T be the transcendental lattice of the family which has signature $(2, 6)$. We fix a choice of $\alpha \in \mathrm{Cl}^{+}(T)$, and associate to each K3 surface X in $\mathcal{K}_P$ the KS variety $\mathrm{KS}(X)$ with polarisation determined by α. The following theorem shows how we derive an abelian 8-fold with totally definite quaternion multiplication from the KS variety associated to a very general K3 surface in the family.

Theorem 5.1.2. *For a very general K3 surface X in the family $\mathcal{K}_P$, there is a simple decomposition of $\mathrm{KS}(X)$ given by*

$$\mathrm{KS}(X) \sim A_{+}^{4} \times A_{-}^{4},$$

where A_{+} and A_{-} are non-isogenous simple abelian 8-folds. Moreover,

$$\mathrm{End}_{\mathbb{Q}}(A_{+}) \simeq \mathbb{H}_{\mathbb{Q}} \simeq \mathrm{End}_{\mathbb{Q}}(A_{-}).$$

Proof. Recall in Corollary 3.3.3 that a simple decomposition of an abelian variety A and the endomorphism algebra of each simple factor can be read off from $\mathrm{End}_{\mathbb{Q}}(A)$ as a sum of some simple matrix algebras over a division ring. Denote by $\mathrm{End}_{\mathrm{Hodge}}(V)$ the $\mathbb{Q}$-vector space of endomorphisms of Hodge structures on V. An endomorphism of an abelian variety is an endomorphism of its Hodge structure, *i.e.* we have

$$\mathrm{End}_{\mathbb{Q}}(\mathrm{KS}(X)) \simeq \mathrm{End}_{\mathrm{Hodge}}(\mathrm{Cl}^{+}(T_{\mathbb{Q}})).$$

From [vG1, Corollary 3.6], we have

$$\mathrm{End}_{\mathrm{Hodge}}(\mathrm{Cl}^{+}(T_{\mathbb{Q}})) \simeq \mathrm{End}_{\mathrm{MT}}(\mathrm{Cl}^{+}(T_{\mathbb{Q}})),$$

where for any rational Hodge structure on V, $\mathrm{End}_{\mathrm{MT}}(V)$ are the vector space endomorphisms that commute with the action of the Mumford-Tate group $\mathrm{MT}(V)$:

$$\mathrm{End}_{\mathrm{MT}}(V) = \{M \in \mathrm{End}(V) : Mg = gM \text{ for all } g \in \mathrm{MT}(V)\}.$$

On the other hand, [vG1, Lemma 6.5] we have

$$\mathrm{Cl}^{+}(T_{\mathbb{Q}}) \simeq \mathrm{End}_{\mathrm{CSpin}^{+}}(\mathrm{Cl}^{+}(T_{\mathbb{Q}})),$$

where

$$\mathrm{End}_{\mathrm{CSpin}^{+}}(\mathrm{Cl}^{+}(T_{\mathbb{Q}})) = \left\{M \in \mathrm{End}(T_{\mathbb{Q}}) : Mg = gM \text{ for all } g \in \mathrm{CSpin}^{+}(T_{\mathbb{Q}})\right\}.$$

If $\mathrm{MT}(\mathrm{Cl}^+(T_{\mathbb{Q}})) = \mathrm{CSpin}^+(T_{\mathbb{Q}})$, then by considering Table 5 which says

$$\mathrm{Cl}^+(T_{\mathbb{Q}}) \simeq M_4(\mathbb{H}_{\mathbb{Q}}) \oplus M_4(\mathbb{H}_{\mathbb{Q}}),$$

so we are done: $\mathrm{KS}(X) \sim A_+^4 \times A_-^4$ with $\dim_{\mathbb{C}}(A_+) = \dim_{\mathbb{C}}(A_-) = 8$. Indeed by a result of Zarhin [Hu, Theorem 3.3.9, 6.4.9], for a very general K3 surface X, we have

$$\mathrm{MT}(T_{\mathbb{Q}}) = \mathrm{MT}(H^2(X, \mathbb{Q})) = \mathrm{O}(T_{\mathbb{Q}}).$$

Therefore by [vG1, Proposition 6.3], we have $\mathrm{MT}(\mathrm{Cl}^+(T_{\mathbb{Q}})) = \mathrm{CSpin}^+(T_{\mathbb{Q}})$. □

Remark 5.1.3. [vG1, Proposition 6.3]

In fact, $\mathrm{MT}(T_{\mathbb{Q}}) \subseteq \mathrm{O}(T_{\mathbb{Q}})$ *and* $\mathrm{MT}(\mathrm{Cl}^+(T_{\mathbb{Q}})) \subseteq \mathrm{CSpin}^+(T_{\mathbb{Q}})$ *for all K3 surfaces in the family. Therefore the corresponding Hodge groups satisfy* $\mathrm{Hdg}(T_{\mathbb{Q}}) \subseteq \mathrm{SO}^+(T_{\mathbb{Q}})$ *and* $\mathrm{Hdg}(\mathrm{Cl}^+(T_{\mathbb{Q}})) \subseteq \mathrm{Spin}^+(T_{\mathbb{Q}})$.

We can fix a representation Φ of $\mathbb{H}_{\mathbb{Q}}$ to be Φ_{std} for all $x \in \mathbb{H}_{\mathbb{Q}}$ as in Theorem 3.3.12 such that the Rosati condition is satisfied. By doing so, we give the abelian 8-folds A_+ and A_- a unique endomorphism structure.

Therefore, to associate an abelian 8-fold with definite quaternion multiplication to a very general K3 surface X, we may choose an isogeny $f\colon \mathrm{KS}(X) \sim A_1 \times \cdots \times A_8$ which gives a simple decomposition of $\mathrm{KS}(X)$, and then project to down to one of the simple factors A_k. This is again not a canonical construction, but we can make the choices of f and k consistently for all very general X in $\mathcal{K}_P$ as described in the following section.

5.1.2 Explicit construction

We start with exploring all possibilities of the isogeny f for X very general. By [vG1, Lemma 6.5], the action of $\mathrm{Cl}^+(\mathrm{T}_{\mathbb{Q}}) \simeq M_4(\mathbb{H}_{\mathbb{Q}}) \oplus M_4(\mathbb{H}_{\mathbb{Q}})$ on itself is by term-wise right multiplication, which is in accordance with Remark 3.3.15. Therefore we have the following isomorphism of vector spaces,

$$\mathrm{Cl}^+(T_{\mathbb{Q}}) \simeq W_1 \oplus \cdots \oplus W_8,$$

where each W_i is a 16-dimensional $\mathbb{Q}$-vector space spanned by the i-th column in $M_4(\mathbb{H}_{\mathbb{Q}}) \oplus M_4(\mathbb{H}_{\mathbb{Q}})$. Note that each W_i for $i = 1, \cdots, 4$ (resp. for $i = 5, \cdots 8$) is the space of positive (resp. negative) half-spinors. By restricting the above isomorphism of vector spaces to the lattice $\mathrm{Cl}^+(T)$, we have

$$\mathrm{Cl}^+(T) \simeq (W_1^{\mathbb{Z}}) \oplus \cdots \oplus (W_8^{\mathbb{Z}}),$$

where each $W_i^{\mathbb{Z}}$ is a rank 16 lattice. For each $i = 1, \cdots, 8$, this gives the complex torus $\mathbb{T}_i := \big((W_i)_{\mathbb{R}}/W_i^{\mathbb{Z}}, J_i\big)$, where J_i is the complex structure on the real torus $(W_i)_{\mathbb{R}}/W_i^{\mathbb{Z}}$ obtained by restricting the complex structure J on $\mathrm{KS}(X)$. Therefore, by Theorem 3.3.2, if $\mathrm{KS}(X) \simeq A_1 \times \cdots \times A_8$ is a simple decomposition of $\mathrm{KS}(X)$, then we have A_i isogenous to $\mathbb{T}_i$ as complex tori (up to re-ordering). In particular, knowing the lattices $W_i^{\mathbb{Z}}$ is enough to recover the complex torus structures for the A_i. Based on this fact, we have the following recipe to obtain a simple decomposition of $\mathrm{KS}(X)$ up to isogenies. It is enough to

(1) fix an explicit algebra isomorphism $\varphi\colon \mathrm{Cl}^+(T_{\mathbb{Q}}) \simeq M_4(\mathbb{H}_{\mathbb{Q}}) \oplus M_4(\mathbb{H}_{\mathbb{Q}})$;

(2) find eight symmetric idempotents $\varepsilon_1, \cdots, \varepsilon_8$ of $\mathrm{Cl}^+(T_{\mathbb{Q}})$ such that for the integers $d_1, \cdots, d_8$ as described in Theorem 3.3.1, the images $\Lambda_i' := d_i\varepsilon_i(\mathrm{Cl}^+(T))$ are rank 16 lattices, and together these lattices span $\mathrm{Cl}^+(T_{\mathbb{Q}})$ over $\mathbb{Q}$.

After making the above choices, then for each real torus $(\Lambda'_i)_{\mathbb{R}}/\Lambda'_i$ for $i = 1, \cdots, 8$, we can define a complex structure J_i as the restriction of J on it. This gives a complex torus $\mathbb{T}_i := ((\Lambda'_i)_{\mathbb{R}}/\Lambda'_i, J_i)$. The polarisation E of $\mathrm{KS}(X)$ also restricts to a polarisation E_i for $\mathbb{T}_i$, and it has to be the unique one up to scalar multiples by Theorem 5.1.1 and Theorem 3.3.1. Therefore we have the abelian 8-fold $A_i \sim (\mathbb{T}_i, J_i, E_i)$. We also have an isogeny f that gives the simple decomposition of $\mathrm{KS}(X)$, which as an isogeny of complex tori is given by

$$\begin{aligned} f\colon \mathrm{KS}(X) &\longrightarrow \mathbb{T}_1 \times \cdots \times \mathbb{T}_8 \\ [p] &\longmapsto ([\varepsilon_1(p)], \cdots, [\varepsilon_8(p)]) . \end{aligned}$$

Moreover, we know for which i's the complex torus $\mathbb{T}_i$ is isogenous to A_+. This is because $\varphi(\Lambda'_i)$ can only be non-zero in exactly one copy of $M_4(\mathbb{H})$ but not both, otherwise $\mathbb{T}_i$ is a non-simple torus.

Remark 5.1.4. *If $A_i \sim A_+$, and if we consider J as the image of the imaginary unit i under h where $h\colon \mathbb{C}^* \to \mathrm{CSpin}^+(T_{\mathbb{R}}) \subset \mathrm{GL}(\mathrm{Cl}^+(T_{\mathbb{R}}))$, then the complex structure J_i as a linear operator on the positive half-spin representation $(W_+)_{\mathbb{R}}$ is given by $(\varphi_+ \circ h)(i)$.*

Remark 5.1.5. *Since there is a unique polarisation class up to scalars for each simple factor A_i of $\mathrm{KS}(X) = \mathrm{KS}(X, \alpha)$ by Theorem 5.1.1, the choice of α we made is unimportant: for any polarisation form Q of the KS variety we started with, it has to restrict to the same polarisation form Q_i of A_i up to scalars.*

In fact, the only substantial choice to be made in the recipe is the isomorphism φ. There is an obvious choice of the symmetric idempotents ε_i's given φ. Let $E_{i,j} \in M_4(\mathbb{H}_{\mathbb{Q}})$ be the matrix with 1 at the (i, j)-th entry as the only non-zero entry. The elements in $M_4(\mathbb{H}_{\mathbb{Q}}) \oplus M_4(\mathbb{H}_{\mathbb{Q}})$ in the form of

$$(E_{1,1}, 0), \cdots, (E_{4,4}, 0), (0, E_{1,1}), \cdots, (0, E_{4,4})$$

are symmetric idempotents. Pulling back to $\mathrm{Cl}^+(T_{\mathbb{Q}})$ via φ gives a set of ε_i's satisfying (2). Moreover, φ only depends on the transcendental lattice T, but not the particular member in $\mathcal{K}_P$ we started from. Therefore, fixing the isomorphism φ alone gives a uniform choice of isogeny f across all very general members in the family $\mathcal{K}$ as in Diagram 3. Furthermore, the order of the simple factors A_i's in the simple decomposition of $\mathrm{KS}(X)$ is also fixed by the order of ε_i's, thus the choice of φ. So we may always choose the first simple factor A_1 in the decomposition $f(\mathrm{KS}(X))$ to be assigned to each very general K3 surface X in $\mathcal{K}_P$. All such A_1's, as abelian 8-folds with totally definite quaternion multiplication, have the same attributes $\mathcal{M}$ and $\mathcal{T}$ as in Section 3.3.3. This is because both attributes only depend on the representation Φ_{std}, the real torus $(\Lambda'_1)_{\mathbb{R}}/\Lambda'_1$ and the polarisation form E_1, which are the same for every A_1 obtained from our modified KS construction.

Away from the very general members, the same choice of φ still leads us to the same choice of the ε's, and therefore an isogeny from $\mathrm{KS}(X)$ to a product of eight abelian 8-folds. However, these abelian 8-folds are not very general, and they show exceptional behaviours. For example they may be no longer simple, or all of them belong to the same isogeny class. This completes our proof for the following theorem.

Theorem 5.1.6. *An isomorphism of algebras*

$$\varphi\colon \mathrm{Cl}^+(T_{\mathbb{Q}}) \simeq M_4(\mathbb{H}_{\mathbb{Q}}) \oplus M_4(\mathbb{H}_{\mathbb{Q}})$$

induces a map F from $\mathcal{K}_P$ to a modular variety $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ of polarised abelian 8-folds with totally definite quaternion multiplication (Diagram 3).

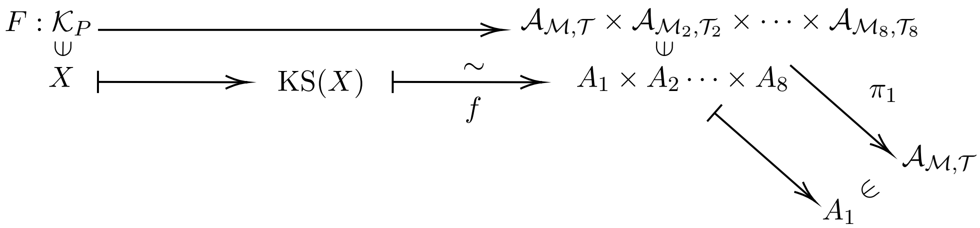


Diagram 3: A modification of the KS construction inducing a map F from a modular variety $\mathcal{K}_P$ of P-polarised K3 surfaces to a modular variety $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ of abelian 8 folds with totally definite quaternion multiplication.

5.2 Lie groups and KS construction

Recall that Diagram 2 induces a Lie algebra isomorphism $\mathfrak{so}^+(2,6) \longrightarrow \mathfrak{so}^*(8)$, which corresponds to an isomorphism between a type IV_6 HSD and a type II_4 HSD [He, X.6.4(viii)]. We will show that our map F constructed in Section 5.1 lifts to this isomorphism $\widetilde{F}$ of HSDs.

Let $\mathcal{K}_P$ be the modular variety of K3 surfaces with a rank 14 polarisation lattice P and let T be the associated transcendental lattice of signature (2,6). Let $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ be the modular variety of polarised abelian 8-folds with totally definite quaternion multiplication. Recall

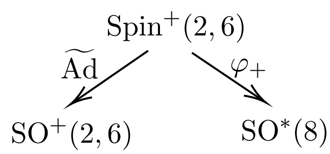
$$\mathcal{K}_P \simeq \Gamma_T \backslash \mathcal{D}_T \quad \text{and} \quad \mathcal{A}_{\mathcal{M},\mathcal{T}} \simeq \Gamma_{\mathcal{M},\mathcal{T}} \backslash \mathcal{D}_{\mathcal{M},\mathcal{T}},$$

where $\mathcal{D}_T \simeq \mathrm{O}^+(2,6)/\left(\mathrm{SO}(2) \times \mathrm{O}(6)\right)$ and $\mathcal{D}_{\mathcal{M},\mathcal{T}} \simeq \mathrm{SO}^*(8)/\,\mathrm{U}(4)$ are the HSD overspaces of the two moduli varieties respectively. Let $\mathcal{D}_T^+ \simeq \mathrm{SO}^+(2,6)/\left(\mathrm{SO}(2) \times \mathrm{SO}(6)\right)$ be a connected component of $\mathcal{D}_T$.

Theorem 5.2.1. *The map $F \colon \mathcal{K}_P \to \mathcal{A}_{\mathcal{M},\mathcal{T}}$ lifts to a map*

$$\widetilde{F} \colon \mathcal{D}_T^+ \longrightarrow \mathcal{D}_{\mathcal{M},\mathcal{T}}.$$

With reference to the diagram



the map $\widetilde{F}$ applied to an element $g \in \mathrm{SO}^+(2,6)$ corresponds to choosing an element $\widetilde{g}$ inside the preimage of the twisted adjoint representation $\widetilde{\mathrm{Ad}}^{-1}(g)$, and then mapping $\widetilde{g}$ to $\mathrm{SO}^*(8)$ via the positive half-spin representation φ_+.

Proof. Recall that $\mathcal{D}_T$ is the period domain for polarised weight two integral Hodge structures of K3 type on the transcendental lattice T. By Remarks 3.1.7 and 5.1.3, the identity component $\mathcal{D}_T^+$ of $\mathcal{D}_T$ is a set containing representations h of $\mathbb{U}$ that factor through $\mathrm{SO}^+(T_\mathbb{R})$ described in Definition 3.1.8.

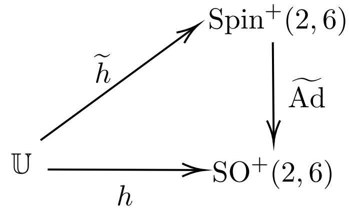
On the other hand, $\mathcal{D}_{\mathcal{M},\mathcal{T}}$ is the period domain for polarised integral weight one Hodge structures on $\Lambda'_\mathbb{R}$, where Λ' is a rank 16 lattice and a $\mathbb{H}$-module (see Corollary 3.3.19 and Remark 3.3.23). It can be identified to a set containing representations $h \colon \mathbb{U} \to \mathrm{GL}(\Lambda_\mathbb{R})$ such that $h(\pm 1)$ act by multiplication on $\Lambda_\mathbb{R} = \mu(\Lambda')_\mathbb{R}$. Each representation is uniquely determined by the Weil operator $J := h(i)$.

Let us also denote by $\mathcal{D}_{\mathrm{Cl}^+(T)}$ the period domain of weight one Hodge structures on $\mathrm{Cl}^+(T)$. Again by Remark 5.1.3, it is the set of all representations $h \colon \mathbb{U} \to \mathrm{Spin}^+(T_\mathbb{R})$ such that $h(\pm 1)$ act by multiplication, or equivalently the set of Weil operators $J = h(i) \in \mathrm{Spin}^+(T_\mathbb{R}) \subset \mathrm{GL}(\mathrm{Cl}^+(T_\mathbb{R}))$.

Our map F takes any K3 surface X in $\mathcal{K}_P$ first to a KS variety $\mathrm{KS}(X, \alpha)$ then to an abelian 8-fold A_1, solely by mapping the underlying Hodge structures of the objects up to isomorphisms. The lifted map $\widetilde{F}$ therefore should factor through $\mathcal{D}_{\mathrm{Cl}^+(T)}$:

$$\widetilde{F} \colon \mathcal{D}_T^+ \longrightarrow \mathcal{D}_{\mathrm{Cl}^+(T)} \longrightarrow \mathcal{D}_{\mathcal{M},\mathcal{T}}$$

The first arrow, which is just the KS construction, corresponds [Hu, Remark 4.2.1] to the lift of representations of $\mathbb{U}$ with respect to the twisted adjoint representation $\widetilde{\mathrm{Ad}}$, requiring $\widetilde{h}(\pm 1)$ to act by multiplication.



This lift of representations is not unique. The only substantial information encoded by any representation $\widetilde{h}$ as a weight one Hodge structure on $\mathrm{Cl}^+(T_{\mathbb{R}})$ is the Weil operator $\widetilde{h}(i)$ in $\mathrm{Spin}^+(2, 6)$. Suppose h lifts to $\widetilde{h}$ and let $\widetilde{J} := \widetilde{h}(i) \in \mathrm{Spin}^+(2, 6)$. The preimage of $h(i)$ under $\widetilde{\mathrm{Ad}}$ is exactly $\{\pm \widetilde{J}\}$, and $\widetilde{h}'$ determined by $h'(i) = -\widetilde{J}$ also descends to h by $\widetilde{\mathrm{Ad}}$. However, only one of $\widetilde{J}$ and $-\widetilde{J}$ can satisfy the Hodge-Riemann relations as in Theorem 3.2.3: if the polarisation of the KS variety is given by the alternating form E, then either $E(\cdot, \widetilde{J}\, \cdot) > 0$ or $E(\cdot, -\widetilde{J}\, \cdot) > 0$. Therefore, there is a unique choice for the lift by further requiring $\widetilde{h}$ to be the complex structure of a polarised abelian variety with polarisation given by α (see Section 5.1), and the first arrow is injective.

By Remark 5.1.4, the second arrow in $\widetilde{F}$ is the positive half-spin representation. It sends any $\widetilde{J} := \widetilde{h}(i) \in \mathrm{Spin}^+(2, 6)$ to $\varphi_+(\widetilde{J}) \in \mathrm{SO}^*(8)$. ☐

In [He, Exercise X.D.1, X.D.2(b)], an explicit holomorphic diffeomorphism between a HSD of type IV_6 and a HSD of type II_4 is given. Without showing $\widetilde{F}$ is equivalent to this explicit map, we will show that

Theorem 5.2.2. *The map $\widetilde{F} \colon \mathcal{D}_T^+ \longrightarrow \mathcal{D}_{\mathcal{M},\mathcal{T}}$ is a differentiable bijection between the two HSDs.*

Proof. Let us denote by $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}} \subset \mathcal{D}_{\mathrm{Cl}^+(T)}$ the set of weight one Hodge structures on $\mathrm{Cl}^+(T)$ obtained from the lift of representations described in the proof of Theorem 5.2.1. Furthermore, we denote by $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}}_+$ the subset of $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}}$ requiring the members to be the positive complex structure of a KS variety $\mathrm{KS}(T, \alpha)$ with respect to its polarisation. Note that $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}}_+ \simeq \left(\mathcal{D}_T^+\right)^{\mathrm{KS}} / \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\pm 1\}$. The map $\widetilde{F}$ is therefore given by

$$\widetilde{F} \colon \mathcal{D}_T^+ \xrightarrow{\ \simeq\ } \left(\mathcal{D}_T^+\right)^{\mathrm{KS}}_+ \xrightarrow{\ \varphi_+\ } \mathcal{D}_{\mathcal{M},\mathcal{T}}, \tag{6}$$

where the first arrow is now by definition a bijection.

It is clear that $\widetilde{F}$ is differentiable: the second arrow is effectively a projection map and the choice of the lift in the first arrow is continuous. This choice is equivalent to the choice of the sign of α described in Section 5.1. Since the period point in the connected component $\mathcal{D}_T^+$ varies continuously, the sign of α, which is discrete, is fixed. In fact, locally $\widetilde{F}$ is just the Lie algebra isomorphism $\mathfrak{so}^+(2, 6) \to \mathfrak{so}^*(8)$ induced by Diagram 2.

To prove the second arrow is bijective, we will need to show that the representations $\widetilde{\mathrm{Ad}}$ and φ_+ are equivariant with respect to the suitable actions of the groups $\mathrm{SO}^+(2, 6)$, $\mathrm{Spin}^+(2, 6)$ and $\mathrm{SO}^*(8)$ on $\mathcal{D}_T^+$, $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}}_+$ and $\mathcal{D}_{\mathcal{M},\mathcal{T}}$ respectively.

We may identify $\mathcal{D}_T^+$ with a set of representations of $\mathbb{U}$, or with the set of positively oriented positive definite plane $P \subset T_\mathbb{R}$ (see proof of Theorem 3.5.16). By [vG1, Remark 4.6], $\mathrm{SO}^+(2,6) \simeq \mathrm{SO}^+(T_\mathbb{R})$ acts naturally on both sets but in different ways:

- On $\mathcal{D}_T^+$ as the set of representations $h : \mathbb{U} \to \mathrm{SO}^+(T_\mathbb{R})$, the group $\mathrm{SO}^+(T_\mathbb{R})$ acts by conjugation. *i.e.* for any $g \in \mathrm{SO}^+(T_\mathbb{R})$,
$$g \colon h \longmapsto h^g := ghg^{-1}.$$

- On $\mathcal{D}_T^+$ as the set of planes, the group $\mathrm{SO}^+(T_\mathbb{R})$ acts by left multiplication. *i.e.* for any $g \in \mathrm{SO}^+(T_\mathbb{R})$,
$$g \colon P \longmapsto gP.$$

Moreover, the two actions are compatible under the identification of the two sets given in [vG1, proof of Lemma 4.4]. By definition, the left multiplication action of $\mathrm{SO}^+(2,6)$ on $\mathcal{D}_T$ is transitive, so the conjugation action of $\mathrm{SO}^+(2,6)$ is also transitive with $\mathrm{SO}(2) \times \mathrm{SO}(6)$ being the stabliser group. The period domain $\mathcal{D}_T^+$ can therefore be identified to the $\mathrm{SO}^+(2,6)$-orbit $\mathrm{SO}^+(2,6)/\left(\mathrm{SO}(2) \times \mathrm{SO}(6)\right)$ with respect to the conjugation action.

Similarly, the period domain $\mathcal{D}_{\mathcal{M},\mathcal{T}}$ as a set of representations of $\mathbb{U}$ can be identified with the set of normalised period matrices
$$\left\{ X = \begin{bmatrix} -Z & \mathbf{1}_4 \\ \mathbf{1}_4 & \bar{Z} \end{bmatrix} : Z \in \mathcal{H}_4 \right\}$$
which parametrises the lattice $\Lambda \subset \mathbb{C}^g$ of the abelian varieties $\mathbb{C}^g/\Lambda$ in $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ (see Theorem 3.3.18). The group $\mathrm{SO}^*(8) \simeq \chi(\mathrm{U}_4(\mathbb{H}))$ acts naturally on the two sets (see Remark 3.3.24):

- On $\mathcal{D}_{\mathcal{M},\mathcal{T}}$ which is a set of representations $h \colon \mathbb{U} \to \mathrm{GL}(\Lambda_\mathbb{R})$, any subgroup of $\mathrm{O}(\Lambda_\mathbb{R})$, and in particular $\chi_\mathbb{R}(\chi(\mathrm{U}_4(\mathbb{H}))$, acts by conjugation. Equivalently by Remark 3.2.11, $\mathcal{D}_{\mathcal{M},\mathcal{T}}$ is the set of complex structures $J = h(i)$, and $\chi_\mathbb{R}(\chi(\mathrm{U}_4(\mathbb{H}))$ also acts by conjugation.

- On $\mathcal{D}_{\mathcal{M},\mathcal{T}}$ as a set of normalised period matrices, the group $\chi(\mathrm{U}_4(\mathbb{H}))$ acts by left multiplication.

A bijection of the two sets is given in Section 3.2.2: any complex structure J on $\Lambda'_\mathbb{R}$ corresponds to a $\mathbb{R}$-linear isomorphism μ that sends any $v \in \Lambda'_\mathbb{R}$ to $1/2(v - iJ(v))$. For any $g \in \chi_\mathbb{R}(\chi(\mathrm{U}_4(\mathbb{H})))$, the complex structure gJg^{-1} corresponds to the isomorphism $g(1/2(\mathbf{1} - iJ))g^{-1}$, which is equivalent to a change of basis in $\mathbb{R}^{2g}$ by left multiplication of g. So the two actions are compatible. Together with Remark 3.3.23, we know that the conjugation action of $\mathrm{SO}^*(8)$ on $\mathcal{D}_{\mathcal{M},\mathcal{T}}$ is also transitive with $\mathrm{U}(4)$ being the stabiliser group. So $\mathcal{D}_{\mathcal{M},\mathcal{T}}$ can be identified to the $\mathrm{SO}^*(8)$-orbit $\mathrm{SO}^*(8)/\mathrm{U}(4)$ with respect to the conjugation action.

By the same argument, any subgroup of $\mathrm{Cl}^+(T_\mathbb{R}) \simeq \mathrm{GL}(\mathrm{Cl}^+(T_\mathbb{R}))$ acts on $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}} \subset \mathcal{D}_{\mathrm{Cl}^+(T)}$ by conjugation when $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}}$ is identified to a set of representations of $\mathbb{U}$; and by left multiplication when $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}}$ is identified to a set of period matrices.

With the above set up, we will now give an explicit expression of the space $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}}_+$. Note that for any $g, g' \in \mathrm{Spin}^+(2,6)$, $\widetilde{\mathrm{Ad}}(gg') = (\widetilde{\mathrm{Ad}}(g'))^g$. So the KS construction is equivariant with respect to the conjugation action of $\mathrm{SO}^+(2,6)$ on $\mathcal{D}_T^+$ and the left multiplication action of $\mathrm{Spin}^+(2,6)$ on $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}}$.

Under the twisted adjoint representation, $\mathcal{D}_T^+ \simeq \mathrm{SO}^+(2,6)/\left(\mathrm{SO}(2) \times \mathrm{SO}(6)\right)$ as a quotient group pulls back to the group $\mathrm{Spin}^+(2,6)/\mathrm{K}_{\mathrm{mult}}$, where
$$\mathrm{K}_{\mathrm{mult}} := \left(\mathrm{Spin}(2) \times \mathrm{Spin}(6)\right) / \left\{\pm(1,1)\right\} < \mathrm{Spin}^+(2,6).$$
To see this, let $(V, q) = V_2 \oplus V_6$, where $V_2 \simeq \mathbb{R}^2$ and $V_6 \simeq \mathbb{R}^6$ are two orthogonal real vector spaces. As in Section 4.2, we may consider $\mathrm{SO}(V_2) \times \mathrm{SO}(V_6) \subset \mathrm{SO}^+(V)$ as the group containing the composition

of even number of reflections along $v^\perp$ for v's either all in V_2 or all in V_6. Since $uv = -vu$ for any $u \in V_2 \subset \mathrm{Cl}^+(V)$ and $v \in V_6 \subset \mathrm{Cl}^+(V)$, we have

$$\mathrm{SO}(V_2) \times \mathrm{SO}(V_6) = \{u_1 \cdots u_{2k} v_1 \cdots v_{2l} : u_i \in V_2,\ v_j \in V_6,\ q(u_i) = q(v_j) = \pm 1\}\,.$$

Its preimage under $\widetilde{\mathrm{Ad}}$ is

$$\begin{aligned}&\{\pm u_1 \cdots u_{2k} v_1 \cdots v_{2l} : u_i \in V_2,\ v_j \in V_6,\ q(u_i) = q(v_j) = \pm 1\}\\ =&\{(\pm u_1 \cdots u_{2k})(\pm v_1 \cdots v_{2l}) : u_i \in V_2,\ v_j \in V_6,\ q(u_i) = q(v_j) = \pm 1\} \,/\, \{(1,1),(-1,-1)\}\end{aligned}$$

which is the group $\mathrm{K}_{\mathrm{mult}}$. This group is the maximal compact subgroup of $\mathrm{Spin}^+(2,6)$, and is connected: fix $e_1, e_2 \in V_2$ and $e_3 \in V_6$ with $q(e_1) = q(e_2) = q(e_3) = 1$. Then any element $w := u_1 \cdots u_{2k} v_1 \cdots v_{2l}$ with $q(w) = 1$ in $\mathrm{K}_{\mathrm{mult}}$ is path connected to 1 as each u_i is path connected to e_1 and each v_j is path connected to e_3. On the other hand, 1 and -1 in $\mathrm{K}_{\mathrm{mult}}$ are also path connected by

$$\gamma\colon t \longmapsto \gamma(t) := \cos(\pi t) - \sin(\pi t) e_1 e_2 \text{ for } t \in [0,1].$$

Indeed, one can check that $-e_1e_2$ is path connected to -1, and the image of γ is contained in $\mathrm{Spin}(2) < \mathrm{K}_{\mathrm{mult}}$.

Since $-1 \in \mathrm{Spin}^+(2,6)$, the element $-\widetilde{h}$ is contained in the $\mathrm{Spin}^+(2,6)$-orbit of $\widetilde{h} \in \left(\mathcal{D}_T^+\right)^{\mathrm{KS}}$, so the left multiplication action of $\mathrm{Spin}^+(2,6)$ on $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}}$ is also transitive, and the connected maximal compact subgroup $\mathrm{K}_{\mathrm{mult}}$ is the stabiliser subgroup. So we have

$$\begin{aligned}&\left(\mathcal{D}_T^+\right)^{\mathrm{KS}} \simeq \mathrm{Spin}^+(2,6)/\mathrm{K}_{\mathrm{mult}},\\ &\left(\mathcal{D}_T^+\right)^{\mathrm{KS}}_+ \simeq \left(\mathrm{Spin}^+(2,6)/\{\pm 1\}\right)/K_{\mathrm{mult}} \simeq \mathrm{Spin}^+(2,6)/\left(\mathrm{Spin}(2) \times \mathrm{Spin}(6)\right).\end{aligned}$$

In the other direction, the positive half-spin representation $\varphi_+ = \pi_+ \circ \varphi$ is equivariant with respect to the conjugation actions of $\mathrm{Spin}^+(2,6)$ and $\mathrm{SO}^*(8)$ on $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}}$ and $\mathcal{D}_{\mathcal{M},\mathcal{T}}$ respectively. Since the two actions of $\mathrm{Spin}^+(2,6)$ on $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}}$ by conjugation and by left multiplication are equivalent, both actions are transitive, and the stabiliser subgroup $\mathrm{K}_{\mathrm{conj}} < \mathrm{Spin}^+(2,6)$ of the conjugation action is isomorphic to $\mathrm{K}_{\mathrm{mult}}$. It is clear that

$$(\varphi_+)_*\colon \mathrm{Spin}^+(2,6)/\mathrm{K}_{\mathrm{conj}} \longrightarrow \mathrm{SO}^*(8)/\,\mathrm{U}(4)$$

is surjective with kernel $\{1, \lambda\} = \ker(\varphi_+)$. In fact, only one of $\widetilde{J}$ and $\lambda\widetilde{J}$ belongs to $\left(\mathcal{D}_T^+\right)^{\mathrm{KS}}_+$. Assume otherwise, and let J be $\varphi_-(\widetilde{J})$ the image of $\widetilde{J}$ under the negative half-spin representation φ_-. Then by Remark 4.2.6, $\varphi_-(\lambda\widetilde{J}) = -J$. This is a contradiction to our construction because only one of J and $-J$ can define the positive complex structure of a member in $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ by the same reasoning as in the proof of Theorem 5.2.1. Therefore $\varphi_+\colon \left(\mathcal{D}_T^+\right)^{\mathrm{KS}}_+ \to \mathcal{D}_{\mathcal{M},\mathcal{T}}$ is bijective. □

Remark 5.2.3. *It is clear that the stabiliser* $\mathrm{K}_{\mathrm{conj}}$ *is in fact the group*

$$\left(\mathrm{Spin}(2) \times \mathrm{Spin}(6)\right)/\{1, \lambda\} < \mathrm{Spin}^+(2,6).$$

The argument in the proof above is also consistent with the fact that $\mathrm{K}_{\mathrm{conj}} \simeq \mathrm{U}(4)$ *as in [Harv, Equation 14.44].*

Remark 5.2.4. *By the Inverse Function Theorem, Theorem 5.2.2 implies that* F *is a diffeomorphism.*

Surjectivity of the second arrow of $\widetilde{F}$ implies that the Hodge group of W_+ is $\mathrm{SO}^*(8)$ for a very general member in $\mathcal{A}_{\mathcal{M},\mathcal{T}}$. Note that the dimensions of $\mathcal{K}_P$ and $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ are both 6. So if $\mathcal{K}_P$ is irreducible, for example in the case of Theorem 3.5.20, then our map F is dominant on to an irreducible component of $\mathcal{A}_{\mathcal{M},\mathcal{T}}$. However, it is not known to us whether $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ is irreducible or not.

6 Application to examples

We have shown that the map F defined in Section 5.1 indeed lifts to an isomorphism $\widetilde{F}$ between a type IV_6 HSD and a type II_4 HSD induced by Diagram 2. In this section, we will apply the construction on a few special modular varieties $\mathcal{K}_{P_1}, \cdots, \mathcal{K}_{P_6}$ of K3 surfaces of Picard rank 14, and work out the explicit properties of the resulting modular varieties of polarised abelian 8-folds with totally definite quaternion multiplication. In Section 6.1, we discuss the six special families of K3 surfaces polarised by an even, indefinite, 2-elementary lattice of rank 14; In Section 6.2.2 and 6.2.3, we will compute the attributes introduced in Section 3.3.3 associated to the image of F for each special family; in Section 6.3 and 6.4, we will perform computations of the map $\widetilde{F}$ for these families in MAGMA, and study a special locus on which $\widetilde{F}$ exhibits exceptional behaviour.

6.1 Special families of K3 surfaces of Picard rank 14

We consider families of K3 surfaces with even, indefinite, 2-elementary polarisation lattices of rank 14. The polarisation lattices and their complements in the K3 lattice Λ_{K3} can be classified by the triple (rk, l, δ) as in Theorem 3.4.8. We can exhaust all such lattices: let X be a P-polarised K3 surface where P is 2-elementary with rank 14, and let T be its transcendental lattice. Then T has signature $(2, 6)$, and is 2-elementary. *i.e.* $A_T \simeq (\mathbb{Z}/2\mathbb{Z})^l$, where $l = 2, 4, 6$ or 8 by Theorem 3.4.8. With reference to Table 4, we list in Table 6 all such transcendental lattices for each possible pair of (l, δ), as well as their corresponding polarisation lattices with the same attributes.

l	δ	T	P
2	0	$T_1 := U \oplus U \oplus D_4(-1)$	$P_1 := U \oplus D_{12}(-1)$
4	1	$T_2 := U \oplus U \oplus \langle -2 \rangle^{\oplus 4}$	$P_2 := U \oplus E_8(-1) \oplus \langle -2 \rangle^{\oplus 4}$
4	0	$T_3 := U \oplus U(2) \oplus D_4(-1)$	$P_3 := U \oplus D_8(-1) \oplus D_4(-1)$
6	1	$T_4 := U \oplus U(2) \oplus \langle -2 \rangle^{\oplus 4}$	$P_4 := U \oplus D_8(-1) \oplus \langle -2 \rangle^{\oplus 4}$
6	0	$T_5 := U(2) \oplus U(2) \oplus D_4(-1)$	$P_5 := U \oplus D_4(-1)^{\oplus 3}$
8	1	$T_6 := U(2) \oplus U(2) \oplus \langle -2 \rangle^{\oplus 4}$	$P_6 := U \oplus D_4(-1)^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 4}$

Table 6: All even, indefinite, 2-elementary transcendental lattices (T) and polarisation lattices (P) for a family of K3 surfaces of Picard rank 14 by their length (l) and parity (δ).

The modular varieties $\mathcal{K}_{P_i}$ of K3 surfaces polarised by the lattices P_i for $i = 1, \cdots, 6$ are families of Jacobian elliptic K3 surfaces: each K3 surface X in the family admits an elliptic fibration, which is a projection $X \to \mathbb{P}^1$ whose fibres are elliptic curves, as well as a section. The **Jacobian elliptic fibrations** with finite Mordell-Weil group admitted by these families are classified in [CM1]. If the elliptic fibration of X has a 2-torsion section S, then there is a canonical involution on X called the **van Geemen-Sarti involution** ι given by the fibre-wise translation that identifies the zero-section with S. By resolving the 8 singular points in X/ι, we obtain another K3 surface Y and the induced rational double cover $X \dashrightarrow Y$. In [CM2] by the same authors, it is shown that the family $\mathcal{K}_{P_5}$ is a van Geemen-Sarti dual of the family $\mathcal{K}_{P_2}$, *i.e.* there is a van Geemen-Sarti involution on any $X_5 \in \mathcal{K}_{P_5}$ which induces rational double cover from X_5 to some X_2 in $\mathcal{K}_{P_2}$, and a van Geemen-Sarti involution on X_2 which induces a rational double cover from X_2 back to X_5. In the same sense, the family $\mathcal{K}_{P_5}$ is self-dual. Moreover, the family $\mathcal{K}_{P_6}$ of double sextics is studied in [KSTT].

Let P and T be one of the above pairs P_i and T_i. We will give an explicit construction of the map $F\colon \mathcal{K}_P \to \mathcal{A}_{\mathcal{M},\mathcal{T}}$ sending a K3 surface X to an abelian 8-fold $A_1 = (\mathbb{T}_1, J_1, Q_1)$ as in Diagram 3. Note that by Theorem 3.5.20, $\mathcal{K}_P$ is irreducible, and therefore F is dominant to an irreducible component of $\mathcal{A}_{\mathcal{M},\mathcal{T}}$. In the following subsections, we will explain how one may obtain the attributes $\mathcal{M}$ and $\mathcal{T}$ which determine the modular variety $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ of abelian 8-folds of PEL type starting from each of the six $\mathcal{K}_P$'s listed above, as well as how the image of the lift $\widetilde{F}$ changes as we vary our input in the HSD $\mathcal{D}_T^+$. For this purpose, we use the computation system MAGMA which allows computation over integers as well as number fields. Specifically, we will work out the details for the family $\mathcal{K}_{P_3}$. One may refer to the variables and functions in the two MAGMA files **pre-T3.m** and **T3.m** in the appendix.

6.2 Realise map between modular varieties

6.2.1 Compute simple decomposition of a generic KS variety

Fixing any α as in Section 4.3, the original KS construction gives us a family of KS varieties, from which we derive the family of the abelian subvarieties A_1. Recall from Section 5.1 that given a KS variety KS(X) in the family, both the complex structure J_1 and the polarisation form Q_1 of the abelian subvariety A_1 depend on the 16-dimensional real torus $(\Lambda_1')_{\mathbb{R}}/\Lambda_1'$, which can be obtained by fixing an isomorphism of algebras

$$\varphi\colon \mathrm{Cl}^+(T_{\mathbb{Q}}) \simeq M_4(\mathbb{H}_{\mathbb{Q}}) \oplus M_4(\mathbb{H}_{\mathbb{Q}}).$$

We would like to obtain the abelian 8-fold A_1. Since we may glue up Clifford algebras by applying Theorem 4.1.8, and since P and T are orthogonal direct sums of the 2-elementary indecomposible lattices, it suffices to fix a map φ from the Clifford algebra over each indecomposible lattice component to the corresponding matrix algebra listed in Table 5, which extends $\mathbb{Q}$-linearly to an isomorphism, and then put them together. In fact from Table 6, it is enough to consider U, $U(2)$, $D_4(-1)$ and $\langle -2\rangle^{\oplus 4}$.

Let us first consider the lattice $U(n)$ for $n = 1, 2$. Let $\{f_1, f_2\}$ be generators of the lattice $U(n)$ such that the associated symmetric bilinear form b is given by the matrix

$$M_{U(n)} := \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$$

Consider $U(n) < \mathrm{Cl}(U(n))$, where $\mathrm{Cl}(U(n)_{\mathbb{Q}}) \simeq \mathrm{Cl}_{\mathbb{Q}}(1,1) \simeq M_2(\mathbb{Q})$. Then the Clifford multiplication is determined by the relations

- $f_1^2 = 0$;
- $f_2^2 = 0$;
- $f_1 f_2 = 2n - f_2 f_1$, so $(f_1 f_2)^2 = 2n \cdot f_1 f_2$.

Thus a choice of the map $\varphi\colon \mathrm{Cl}(U(n)) \to M_2(\mathbb{Q})$ which preserves the Clifford multiplication is given by

$$\varphi(1) = \mathbf{1}, \quad \varphi(f_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \varphi(f_2) = \begin{pmatrix} 0 & 0 \\ 2n & 0 \end{pmatrix}.$$

Observe that $\varphi(f_1), \varphi(f_2)$ span the $\mathbb{Z}$-algebra

$$\left\{ \begin{pmatrix} d + 2n\alpha & b \\ 2n\beta & d \end{pmatrix} : a, b, c, d, \alpha, \beta \in \mathbb{Z} \right\}$$

This is well defined as one can check that the integral structure is preserved: any element in the above set is the image of

$$d \cdot 1 + \alpha \cdot f_1 f_2 + b \cdot f_1 + \beta \cdot f_2 \in \mathrm{Cl}^+(U(n)).$$

So φ is a $\mathbb{Z}$-algebra isomorphism onto its image.

Secondly, let us consider the case of $D_4(-1)$. Let $\{h_1, h_2, h_3, h_4\}$ be the generators of the lattice $D_4(-1)$ such that the associated symmetric bilinear form b is given by the matrix

$$-M_{D_4} = \begin{pmatrix} -2 & -1 & -1 & -1 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ -1 & 0 & 0 & -2 \end{pmatrix}.$$

We will obtain a map $\varphi\colon \mathrm{Cl}(D_4(-1)) \to M_2(\mathbb{H}_\mathbb{Q})$ by applying the Fundamental Lemma for Clifford algebras (Lemma 4.1.4). From our discussion towards the end of Section 4.1, the lattice $D_4(-1)$ is in fact the order $\mathfrak{o}(-2) < \mathbb{H}_\mathbb{Q}$, where $\mathfrak{o} = \mathbb{Z}\langle h, i, j, k\rangle$ is the Hurwitz integers. An explicit isometry between the two lattices is given by

$$h_1 \mapsto -2h, \quad h_2 \mapsto -2i, \quad h_3 \mapsto -2j, \quad h_4 \mapsto -2k.$$

One can construct a $\mathbb{Z}$-module homomorphism

$$\begin{aligned} \varphi\colon \mathfrak{o}(-2) &\longrightarrow M_2(\mathfrak{o}) \\ -2z &\longmapsto \begin{pmatrix} 0 & z \\ -2\bar{z} & 0 \end{pmatrix} \end{aligned}$$

with $\varphi(-2z)^2 = -2q(z) \cdot \mathbf{1}$. Therefore φ extends uniquely to an algebra homomorphism

$$\varphi\colon \mathrm{Cl}(D_4(-1)) \longrightarrow M_2(\mathfrak{o}).$$

Lastly, let us consider the lattice $\langle -2\rangle^{\oplus 4}$. Let $\langle -2\rangle^{\oplus 4}$ be generated by $h_1, \cdots, h_4$ such that the associated symmetric bilinear form b is given by the matrix $-2 \cdot \mathbf{1}_4$. Again we apply the Fundamental Lemma for Clifford algebras: define a $\mathbb{Z}$-module homomorphism $\varphi\colon \langle -2\rangle^{\oplus 4} \to M_2(\mathbb{H}_\mathbb{Q})$ by defining

$$\varphi(h_1) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \ \varphi(h_2) = \begin{pmatrix} 0 & i \\ 2i & 0 \end{pmatrix}, \ \varphi(h_3) = \begin{pmatrix} 0 & j \\ 2j & 0 \end{pmatrix}, \ \varphi(h_4) = \begin{pmatrix} 0 & k \\ 2k & 0 \end{pmatrix}$$

and extend $\mathbb{Z}$-linearly. It is easy to check that $\varphi(v)^2 = q(v)\cdot\mathbf{1}$, so φ extends to an algebra homomorphism $\varphi\colon \mathrm{Cl}(\langle -2\rangle^{\oplus 4}) \to M_2(\mathbb{H}_\mathbb{Q})$ as desired.

By applying Remark 4.1.11, we may put together any two of the homomorphisms of graded algebras φ in the following way:

$$\begin{aligned} \mathrm{Cl}(L_1) \otimes \mathrm{Cl}(L_2) \cong M_{n_1}(\mathbb{F}) \otimes M_{n_2}(\mathbb{F}) &\longrightarrow M_{n_1 \cdot n_2}((\mathbb{F}) \cong \mathrm{Cl}(L_1 \oplus L_2) \\ (\{a_{ij}\}_{i,j}, \{b_{kl}\}_{k,l}) &\longmapsto \{a_{ij}b_{kl}\}_{n_2(i-1)+k, n_2(j-1)+l}. \end{aligned}$$

This gives us a homomorphism (which we still call φ) from the lattice $\mathrm{Cl}(T)$ to $M_8(\mathbb{H}_\mathbb{Q})$ for the family of K3 surfaces with transcendental lattice T. With reference to Remark 4.1.10, the image of any element x in the even part $\mathrm{Cl}^+(T)$ under φ is in the form

$$\varphi(x) = \begin{pmatrix} m_{11} & 0 & 0 & m_{14} & 0 & m_{16} & m_{17} & 0 \\ 0 & m_{22} & m_{23} & 0 & m_{25} & 0 & 0 & m_{28} \\ 0 & m_{32} & m_{33} & 0 & m_{35} & 0 & 0 & m_{38} \\ m_{41} & 0 & 0 & m_{44} & 0 & m_{46} & m_{47} & 0 \\ 0 & m_{52} & m_{53} & 0 & m_{55} & 0 & 0 & m_{58} \\ m_{61} & 0 & 0 & m_{64} & 0 & m_{66} & m_{67} & 0 \\ m_{71} & 0 & 0 & m_{74} & 0 & m_{76} & m_{77} & 0 \\ 0 & m_{82} & m_{83} & 0 & m_{85} & 0 & 0 & m_{88} \end{pmatrix} \in M_8(\mathbb{H}_\mathbb{Q}),$$

which can be identified to an element in $M_4(\mathbb{H}_\mathbb{Q}) \oplus M_4(\mathbb{H}_\mathbb{Q})$ by extracting the two obvious 4-by-4 blocks:

$$\varphi(x) = \left(\begin{pmatrix} m_{11} & m_{14} & m_{16} & m_{17} \\ m_{41} & m_{44} & m_{46} & m_{47} \\ m_{61} & m_{64} & m_{66} & m_{67} \\ m_{71} & m_{74} & m_{76} & m_{77} \end{pmatrix}, \begin{pmatrix} m_{22} & m_{23} & m_{25} & m_{28} \\ m_{32} & m_{33} & m_{35} & m_{38} \\ m_{52} & m_{53} & m_{55} & m_{58} \\ m_{82} & m_{83} & m_{85} & m_{88} \end{pmatrix} \right).$$

Extending linearly by $\mathbb{Q}$, this gives us the isomorphism φ identifying $\mathrm{Cl}^+(T_\mathbb{Q})$ with $M_4(\mathbb{H}_\mathbb{Q}) \oplus M_4(\mathbb{H}_\mathbb{Q})$ as required.

From the discussion following Theorem 5.1.6, the rank 16 lattice Λ'_1 is the image of $\mathrm{Cl}^+(T)$ under $d \cdot \varepsilon_1 \in \mathrm{Cl}^+(T)$, where $\varepsilon_1 \mathrm{Cl}^+(T_\mathbb{Q})$ is the preimage of the matrix $(E_{1,1}, 0)$ under φ. By Remark 4.1.10, it is clear that the matrices $(E_{j,j}, 0)$ and $(0, E_{j,j})$ for $j = 1, \cdots, 4$ belong to the even part of the Clifford algebra over $T_\mathbb{Q}$. By patching the two homomorphisms φ together, each ε_i can be written as a product of x_j and y_k, where x_j is the preimage of some $E_{j,j}$ under $\varphi\colon \mathrm{Cl}(U \oplus U(n)) \to M_4(\mathbb{Q})$; and y_k is the preimage of $\mathrm{diag}(1, 0)$ or $\mathrm{diag}(0, 1)$ under $\varphi\colon \mathrm{Cl}(T') \to M_2(\mathfrak{o})$. In the following example we give the idempotents ε_i's for $i = 1, \cdots, 8$ explicitly for the family $\mathcal{K}_{P_3}$, and obtain the lattices $\Lambda'_i \subset \mathbb{R}^{16}$ using MAGMA.

Example 6.2.1. ***(See file [pre-T3.m])*** *Consider $T_3 = U \oplus U(2) \oplus D_4(-1)$. Let $\{f_1, f_2\}, \{f_3, f_4\}$ and $\{h_1, h_2, h_3, h_4\}$ be the sets of generators of the indecomposible sublattices U, $U(2)$ and $D_4(-1)$ such that the matrices associated to the symmetric bilinear forms with respect to those generators are $M_U, M_{U(2)}$ and $-M_{D_4}$ respectively.*

Notice that we have the following ***pseudo-idempotents*** *in* $\mathrm{Cl}(U \oplus U(2))$, i.e. *primitive elements in* $\mathrm{Cl}(U \oplus U(2))$ *that are integral multiples of idempotents in* $\mathrm{Cl}\left((U \oplus U(2))_\mathbb{Q}\right)$*:*

$$\begin{aligned} x_1 &:= f_3 f_1 f_2 f_4 \\ x_2 &:= 4 f_1 f_2 - x_1 \\ x_3 &:= 2 f_3 f_4 - x_1 \\ x_4 &:= 8 \cdot \mathbf{1} - x_1 - x_2 - x_3. \end{aligned}$$

Their images under $\varphi\colon \mathrm{Cl}(U \oplus U(2)) \to M_4(\mathbb{Q})$ are $8E_{1,1}, \cdots, 8E_{4,4}$. By considering the element

$$H := h_1 h_2 h_3 h_4 + h_2 h_3 + h_3 h_4 + h_4 h_2 \in \mathrm{Cl}(D_4(-1)),$$

we also have pseudo-idempotents in $\mathrm{Cl}(D_4(-1))$

$$\begin{aligned} y_1 &:= 2 - H \\ y_2 &:= 2 + H \end{aligned}$$

whose images under $\varphi\colon \mathrm{Cl}(D_4(-1)) \to M_2(\mathfrak{o})$ are the diagonal matrices $\mathrm{diag}(4, 0)$ *and* $\mathrm{diag}(0, 4)$ *respectively.*

Therefore in $\mathrm{Cl}(T_3)$ *we have eight pseudo-idempotents*

$$[32\varepsilon_1, \cdots, 32\varepsilon_8] = [x_1 y_1,\ x_2 y_2,\ x_3 y_2,\ x_4 y_1,\ x_1 y_2,\ x_2 y_1,\ x_3 y_1,\ x_4 y_2]$$

whose respective images under $\varphi\colon \mathrm{Cl}(T_3) \to M_4(\mathbb{H}_\mathbb{Q}) \oplus M_4(\mathbb{H}_\mathbb{Q})$ are

$$[(32E_{1,1}, 0), \cdots, (32E_{4,4}, 0), (0, 32E_{1,1}), \cdots, (0, 32E_{4,4})].$$

Since the sub-sublattices $U \oplus U(2)$ and $D_4(-1)$ are orthogonal to each other, the actions of the x_j's commute with that of the y_k's. Therefore, for the pseudo-idempotent $32\varepsilon_i = x_j y_k$, the lattice Λ'_i is given by the set

$$\mathrm{Cl}^+(T) \cdot 32\varepsilon_i = \left\{ L \cdot K \in \mathrm{Cl}^+(T) : L \in (\mathrm{Cl}(U \oplus U(2)) \cdot x_j),\ K \in (\mathrm{Cl}(D_4(-1)) \cdot y_k) \right\}.$$

Note that the image of the multiplication by a pseudo-idempotent x_j in $\mathrm{Cl}(U \oplus U(-2))$ is the kernel of the multiplication by $8 \cdot \mathbf{1} - x_j$ in $\mathrm{Cl}(U \oplus U(-2))$. By writing out the matrix associated to the map of multiplication by $8 \cdot \mathbf{1} - x_j$, one can obtain primitive generators of the lattice $\mathrm{Cl}(U \oplus U(2)) \cdot x_j$ by applying the MAGMA built-in function ***KernelMatrix****. There are four such generators $L_1, \cdots, L_4$,* *two of them are in the even degree part* $\mathrm{Cl}^+(U \oplus U(2))$, *and the other two are in the odd degree part* $\mathrm{Cl}^-(U \oplus U(2))$ ***(See function "L_CUp")****. Similarly, one can obtain eight generators* $K_1, \cdots, K_8$ *for the lattice* $\mathrm{Cl}(D_4(-1)) \cdot y_k$ *where four of them are in the even degree part* $\mathrm{Cl}^+(D_4(-1))$, *and the other four are in the odd degree part* $\mathrm{Cl}^-(D_4(-1))$ ***(See function "L_CDp")****. There are only* 16 *combinations of the* L_s*'s and the* K_w*'s such that their product lies in* $\mathrm{Cl}^+(T)$*. These* 16 *vectors form the* 16 *generators of the lattice* $\Lambda_i' \subset \mathbb{R}^{16}$.

Finally, the complex structure J_1 and the polarisation E_1 of A_1 are obtained by restricting J and E of $\mathrm{KS}(X)$ to $(\Lambda_1')_{\mathbb{R}}$.

Remark 6.2.2. *By Theorem 5.1.2, for a very general* $X \in \mathcal{K}_P$*, the* $A_i := ((\Lambda_i')_{\mathbb{R}}/\Lambda_i', J_i, E_i)$ *obtained from above satisfy* $A_1 \sim \cdots \sim A_4$ *and* $A_5 \sim \cdots \sim A_8$*, but* $A_1 \not\sim A_5$*.*

6.2.2 Compute representation of endomorphism algebra

Recall the attributes $\{x_1, \cdots, x_4\}, \mathcal{M}, \mathcal{T}$ and $\mathcal{H}$ associated to the abelian 8-fold A_1 as a member of the target family $\mathcal{A}_{\mathcal{M},\mathcal{T}}$, which were introduced in Section 3.3.3 assuming that the representation homomorphism Φ of $\mathbb{H}_{\mathbb{Q}} = \mathrm{End}_{\mathbb{Q}}(A_1)$ is the standard one Φ_{std}. However our expression of Λ_1' obtained from Section 6.2.1 already determines a basis of the real ambient space $\mathbb{R}^{16}$. Therefore we will first compute a real representation $\Phi_{\mathbb{R}}$ out of J_1 and Λ_1' with respect to the current basis of $\mathbb{R}^{16}$, and then transform to a complex representation Φ, and finally to the standard representation Φ_{std}.

Each of the six families in Table 6 has a transcendental lattice in the form $T = U \oplus U(n) \oplus T'$, where T' is either $D_4(-1) \simeq \mathfrak{o}(-2)$ or $\langle -2 \rangle^{\oplus 4}$. For any very general KS variety associated to one of the families, and for A_1 the simple abelian subvariety in the KS variety as defined in Section 5.1, let $F = \mathrm{End}_{\mathbb{Q}}(A_1) \simeq \mathbb{H}_{\mathbb{Q}}$, and let $R = \mathrm{End}(A_1)$ which is an order in F. As a $\mathbb{Z}$-submodule of F, R is of rank 4. On the other hand, recall from [vG1, Lemma 6.5] we have

$$\mathrm{Cl}^+(T_{\mathbb{Q}}) \simeq \mathrm{End}_{\mathrm{CSpin}^+}(\mathrm{Cl}^+(T_{\mathbb{Q}})).$$

Since Λ_1' is a primitive sublattice in $\mathrm{Cl}^+(T)$, the algebra R is generated by the action of some elements in the integral part $\mathrm{Cl}^+(T)$ on Λ_1'. However, R is not a free subalgebra of $\mathrm{Cl}^+(T)$: two elements in $\mathrm{Cl}^+(T)$ may act differently on $\mathrm{Cl}^+(T)$ even though they have the same action restricted to Λ_1'.

In the following we will show how to obtain the representation Φ from the algebra isomorphism

$$\varphi\colon \mathrm{Cl}(T') \longrightarrow M_2(\mathbb{H}_{\mathbb{Q}})$$

given in Section 6.2.1. It is enough to define Φ on a set of four generators $\{r_1, \cdots, r_4\} \subset R < \mathbb{H}_{\mathbb{Q}}$. The strategy is to first obtain a set of elements $\{\widetilde{h}_1, \cdots, \widetilde{h}_4\} \subset \mathrm{Cl}^+(T)$ whose actions on $\Lambda_1' < \mathrm{Cl}^+(T)$ generate $R < \mathbb{H}_{\mathbb{Q}}$. Then we can compute matrices N_i's in $M_{16}(\mathbb{Z})$ (with usual action on $\mathbb{R}^{16} \simeq (\Lambda_1')_{\mathbb{R}}$ by left multiplication) which represent the right actions of the $\widetilde{h}_i$'s on $(\Lambda_1')_{\mathbb{R}} < \mathrm{Cl}^+(T_{\mathbb{R}})$. Then by identifying $((\Lambda_1')_{\mathbb{R}}, J_1)$ with $(\Lambda_1)_{\mathbb{R}} = W_1$ where $\Lambda_1 = \mu(\Lambda_1')$ (see Definition 3.2.9), each matrix N_i is taken to a complex matrix $M_i \in M_8(\mathbb{C})$. Finally, based on the multiplication rules satisfied by the M_i's, we choose the $r_i \in R$ that can be mapped to the M_i's under the algebra homomorphism Φ.

Suppose an element $r \in R$ is given by the action of $x \in \mathrm{Cl}^+(T)$ on Λ_i'. Then by Remark 3.3.15, the matrix $\varphi(x)$ acts by multiplication from the left. On the other hand, there is a natural action of $\mathrm{Cl}^+(T)$ on Λ_i' coming from its $\mathrm{Cl}^+(T)$-module structure. The pushforward of this action under φ is also given by matrix multiplication from the left (Remark 3.3.15). Any endomorphism of A_i should commute with the action of $\varphi(\mathrm{Cl}^+(T))$, which implies that $\varphi(x)$ is a diagonal matrix. Conveniently, we know some diagonal matrices in the image $\varphi(\mathrm{Cl}^+(T))$: for example

$$\mathbf{1}_4 \otimes (\varphi(t_1) \cdot \varphi(t_2)) = \mathbf{1}_4 \otimes \varphi(t_1 t_2) \in M_4(\mathbb{Q}) \otimes M_2(\mathbb{H}_{\mathbb{Q}})$$

for $t_1, t_2 \in T'$. These matrices together with the identity matrix span a $\mathbb{Z}$-module of rank 4. Thus R contains the $\mathbb{Z}$-algebra generated by matrices in this form. In fact by studying our construction of φ's, we can obtain a set of $\widetilde{h}_i$'s in $\mathrm{Cl}^+(T)$ which have the same actions as a set of four primitive generators of R, as well as their actions on W_1 as matrices $M_1, \cdots, M_4 \in M_8(\mathbb{C})$. Let us first explain this in detail for $T' = D_4(-1)$, which includes the case when $T = T_3$.

Example 6.2.3. ***(See variable "tih_mat_8" in file [T3.m])*** *Continue from Example 6.2.1. Define $h_{(-2)} = 2h_1 - h_2 - h_3 - h_4 \in \mathrm{Cl}(T)$. The reason for the notation $h_{(-2)}$ is that, under the identification of $D_4(-1)$ with the order $\mathfrak{o}(-2) < \mathbb{H}_\mathbb{Q}$, the element $h_{(-2)} \in D_4(-1)$ is mapped to $-2 \in \mathfrak{o}(-2)$. Let $\mathbf{1}$ represent the identity element in the any Clifford algebras. Then the images of the elements $(h_{(-2)}h_1)$, $(h_{(-2)}h_2)$, $(h_{(-2)}h_3)$ and $(h_{(-2)}h_4)$ in $\mathrm{Cl}^+(T)$ under φ are diagonal matrices. By definition, the identity $\mathbf{1} \in \mathrm{Cl}^+(T)$ must also belong to R. The element $h_{(-2)}h_4$ is $\mathbb{Z}$-linearly dependent on the other four elements, and the elements*

$$\widetilde{h}_1 := \mathbf{1},\ \widetilde{h}_2 := (h_{(-2)}h_1),\ \widetilde{h}_3 := (h_{(-2)}h_2),\ \widetilde{h}_4 := (h_{(-2)}h_3)$$

together span a primitive lattice of rank 4 in $\mathrm{Cl}^+(T)$. So the set is also the set of actions of the primitive generators of R as a (non-free) $\mathbb{Z}$-algebra.

*The built-in function **Solution** in MAGMA allows one to solve a system of equations over $\mathbb{Z}$. In particular, one can obtain matrices N_i's in $M_{16}(\mathbb{Z})$ (with left multiplication on $\mathbb{R}^{16} \simeq (\Lambda'_1)_\mathbb{R}$) that correspond to the right actions of the elements $\widetilde{h}_i$'s on $(\Lambda'_1)_\mathbb{R} < \mathrm{Cl}^+(T_\mathbb{R})$ **(See function "Get_r_action" in file [T3.m])**. One can also check that the N_i's span a primitive lattice in $M_{16}(\mathbb{Z})$ with the lattices machinery in MAGMA.*

*Finally, to obtain the matrices $M_i \in M_8(\mathbb{C})$, we introduce the complex structure J_1 of A_1 as in Definition 3.2.9. Let W_1 be the $+i$-eigenspace of J_1. One may use **Solution** to find $+i$-eigenvectors of J_1 in $(\Lambda'_1)_\mathbb{R}$. Applying **Solution** again, one may transform the $N_1, \cdots, N_4$ to $M_1, \cdots, M_4 \in M_8(\mathbb{C})$, such that they respectively represent the actions of $\widetilde{h}_i$'s on $W_1 \simeq \mathbb{C}^8$ with respect to the eight $+i$-eigenvectors **(See function "Get_CC8_bas" in file [T3.m])**.*

Similarly, we explain how to identify $\widetilde{h}_i$ with M_i in the case $T' = \langle -2 \rangle^{\oplus 4}$.

Example 6.2.4. *Let $T' = \langle -2 \rangle^{\oplus 4}$ be generated by $h_1, \cdots, h_4$ such that the associated symmetric bilinear form is given by the matrix $-2 \cdot \mathbf{1}_4$. Again, denote by $\mathbf{1}$ the identity element of any Clifford algebra. Then the elements (h_1h_1), (h_3h_4), (h_4h_2) and (h_2h_3) in $\mathrm{Cl}^+(T)$ under φ are diagonal matrices. The identity element $\mathbf{1} \in \mathrm{Cl}^+(T)$ is again contained in R and (h_1h_1) is an integral multiple of $\mathbf{1}$. One can check that the elements*

$$\widetilde{h}_1 := \mathbf{1},\ \widetilde{h}_2 := h_3h_4,\ \widetilde{h}_3 := h_4h_2,\ \widetilde{h}_4 := h_2h_3$$

*together span a primitive lattice in $\mathrm{Cl}^+(T)$. Using **Solution**, one can obtain the actions of the $\widetilde{h}_i$ as matrices N_i in $M_{16}(\mathbb{Z})$. These 16-by-16 matrices associated to the $\widetilde{h}_i$ can be transformed into the matrices M_i by considering the $+i$-eigenvectors of the complex structure J_1 in $(\Lambda'_1)_\mathbb{R}$.*

The final step is to choose and match generators r_i of $R < \mathbb{H}_\mathbb{Q}$ to the matrices M_i's such that they satisfy the same set of multiplication rules. Although the map

$$r_i \longmapsto \widetilde{h}_i \longmapsto \varphi(\widetilde{h}_i)$$

does not necessarily preserve multiplication, it does preserve addition and scalar multiplication, which hints at some good choices of the r_i.

Let us start with the case when $T' = D_4(-1)$.

Example 6.2.5. ***(See variable "bas_R" in file [T3.m])*** *Continue from Example 6.2.3. Note that*

$$\varphi(\widetilde{h}_1) = \mathbf{1}_2, \quad \varphi(\widetilde{h}_2) = \begin{pmatrix} -2\bar{h} & 0 \\ 0 & -2h \end{pmatrix}, \quad \varphi(\widetilde{h}_3) = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}, \quad \varphi(\widetilde{h}_4) = \begin{pmatrix} 2j & 0 \\ 0 & -2j \end{pmatrix}$$

A good guess of R as an order would be $\langle 1, \mathfrak{o}(-2)\rangle$. It is easy to identify some sets of primitive generators $\{r_1, \cdots, r_4\}$. For example $r := \{1, -2h, -2i, -2j\}$ and $\bar{r} := \{1, -2\bar{h}, -2\bar{i}, -2\bar{j}\}$ are some obvious ones. By comparing the multiplication rules within each set of primitive generators R and that of the M_i's, one can select one appropriate set of r_i's such that $r_i \mapsto M_i$ defines an algebra isomorphism $\Phi\colon F \to M_8(\mathbb{C})$.

Let us consider the specific case of $T = T_3$. It is easy to check that in the abelian 8-fold $A_k <$ $\mathrm{KS}(X)$, for $k = 1, 4, 6, 7$ (resp. $k = 2, 3, 5, 8$), the set of generators $\bar{r}$ (resp. r) would define an anti-homomorphism of algebras $F \to M_8(\mathbb{C})$. Anti-homomorphisms instead of homomorphisms arise naturally because the map $\widetilde{h}_i \mapsto M_i$ is an anti-homomorphism itself: the action of $\widetilde{h}_i\widetilde{h}_j$ from the right corresponds to the action of M_jM_i from the left. To obtain a homomorphism instead, we precompose the anti-homomorphism by an anti-isomorphism ι of F:

$$\begin{aligned} \iota\colon \mathbb{H}_{\mathbb{Q}} &\longrightarrow \mathbb{H}_{\mathbb{Q}}^{op} \\ 1, i, j &\longmapsto 1, i, j \text{ respectively} \\ k &\longmapsto -k. \end{aligned}$$

In particular, for A_1, the required real representation $\Phi_{\mathbb{R}}$ of $F = \mathrm{End}_{\mathbb{Q}}(A_1)$ is defined by

$$1 \mapsto N_1 = \mathbf{1}_{16}, \quad (-1 + i + j - k) \mapsto N_2, \quad 2i \mapsto N_3, \quad 2j \mapsto N_4,$$

and the complex representation Φ is defined by

$$1 \mapsto M_1 = \mathbf{1}_8, \quad (-1 + i + j - k) \mapsto M_2, \quad 2i \mapsto M_3, \quad 2j \mapsto M_4.$$

The case for $T' = \langle -2\rangle^{\oplus 4}$ is simpler.

Example 6.2.6. *Continue from Example 6.2.4. We have*

$$\varphi(\widetilde{h}_1) = \mathbf{1}_2, \quad \varphi(\widetilde{h}_2) = \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix}, \quad \varphi(\widetilde{h}_3) = \begin{pmatrix} 2j & 0 \\ 0 & 2j \end{pmatrix}, \quad \varphi(\widetilde{h}_4) = \begin{pmatrix} 2k & 0 \\ 0 & 2k \end{pmatrix}$$

A good guess of R as an order would be $\langle 1, 2i, 2j, 2k\rangle$. One can check that in fact the map $r_i \mapsto \widetilde{h}_i \mapsto \varphi(\widetilde{h}_i)$ also preserves multiplication, so the choice of the generators $\{r_1, \cdots, r_4\} = \{1, 2i, 2j, 2k\}$ defines an anti-homomorphism $F \to M_8(\mathbb{C})$. Precomposing by the anti-isomorphism ι in Example 6.2.5 gives the desired representation $\Phi_{\mathbb{R}}$ and Φ.

Before we move on to the next subsection where we compute the attributes associated to the abelian 8-fold A_1, we will show how to transform the representation Φ to the standard one Φ_{std} by a change of basis of $\mathbb{C}^8$ **(See function "Phi2Chi" in file [T3.m])**. Precisely, we hope to find an 8-by-8 change of basis matrix $Q \in M_8(\mathbb{C})$, such that

$$Q \cdot \Phi(r_i) = (\chi(r_i) \otimes \mathbf{1}_4) \cdot Q \text{ for all } i = 1, \cdots, 4.$$

Consider the $\mathbb{C}$-vector space isomorphism $(\cdot)^{\sim}$ described in [BL, p.252] which identifies a d-by-d matrix to a horizontal vector of length d^2

$$\begin{aligned} (\cdot)^{\sim}\colon M_d(\mathbb{C}) &\longrightarrow \mathbb{C}^{d^2} \\ \{a_{ij}\} &\longmapsto a^{\sim} := (a_{11}, a_{12}, \cdots, a_{dd}). \end{aligned}$$

For each r_i, we find matrices A and B such that for all 8-by-8 matrix M, we have

$$\begin{aligned} (M \cdot \Phi(r_i))^{\sim} &= M^{\sim} \cdot A, \\ ((\chi(r_i) \otimes \mathbf{1}_4) \cdot M)^{\sim} &= M^{\sim} \cdot B. \end{aligned}$$

Using the **KernelMatrix** function in MAGMA, one can build (non-unique) 8-by-8 non-singular matrix out of the kernel space of $A - B$ to be the desired matrix Q.

6.2.3 Compute attributes

In this subsection, we will compute the attributes $\{x_1, \cdots, x_4\}$, $\mathcal{M}$ and $\mathcal{T}$ determining the moduli space of abelian 8-folds with totally definite quaternion multiplication $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ as the target space of the map F and any abelian 8-fold A_1 in the image of F. Suppose A_1 is isomorphic to the complex torus W_1/Λ_1. First, we will compute the attribute $\{x_1, \cdots, x_4\} \subset \mathbb{C}^8$ associated to A_1 satisfying Equation 3.3.3(1):

$$(\Lambda_1)_{\mathbb{Q}} = \sum_{i=1}^{4} \Phi_{\mathrm{std}}(F)x_i.$$

Moreover, we will show that the attribute $\{x_1, \cdots, x_4\}$ associated to a particular member $A_1 \simeq ((\Lambda_1')_{\mathbb{R}}/\Lambda_1', J_1)$ is determined by its complex structure J_1.

Lemma 6.2.7. *The real representation $\Phi_{\mathbb{R}}\colon \mathbb{H}_{\mathbb{Q}} \to M_{16}(\mathbb{Z})$ sending each generator r_i to the matrix $N_i \in M_{16}(\mathbb{Z})$ obtained in the same fashion as in Examples 6.2.3 and 6.2.4 has image in the subset of block diagonal matrices*

$$\{\mathrm{diag}(\mathcal{N}_1, \cdots, \mathcal{N}_4) : \mathcal{N}_j \in M_4(\mathbb{Z})\}$$

with respect to a suitable order of the generators of the lattice Λ_1' defining A_1.

Proof. Recall from Section 6.2.1 that the algebra homomorphism $\varphi\colon \mathrm{Cl}(T) \simeq M_4(\mathbb{H}_{\mathbb{Q}}) \oplus M_4(\mathbb{H}_{\mathbb{Q}})$ is obtained from combining the two homomorphisms

$$\mathrm{Cl}(U \oplus U(n)) \to M_4(\mathbb{Q}) \quad \text{and} \quad \mathrm{Cl}(T') \to M_2(\mathbb{H}_{\mathbb{Q}}).$$

It can be observed that the $\Phi_{\mathbb{R}}(\widetilde{h}_i)$'s, hence the image of $\Phi_{\mathbb{R}}$, are pairs of diagonal matrices in $M_4(\mathbb{H}_{\mathbb{Q}}) \oplus M_4(\mathbb{H}_{\mathbb{Q}})$ for both $T' = D_4(-1)$ and $\langle -2 \rangle^{\oplus 4}$.

On the other hand, recall from Example 6.2.1 that each generator of the lattice Λ_1' is a product of $L_s \in \mathrm{Cl}(U \oplus U(n))$ and $K_w \in \mathrm{Cl}(T')$. In particular when fixing $s = s_0$, the rank 4 sub-lattice generated by $\{L_{s_0}K_w : w = 1, \cdots, 4\}$ corresponds to one of the four entries in the first column of $\varphi(\mathrm{Cl}^+(T))$. Since the action of $\varphi(\langle \widetilde{h}_1, \cdots, \widetilde{h}_4 \rangle)$ on the first column of $\varphi(\mathrm{Cl}^+(T))$ is equivalent to that of $\Phi_{\mathbb{R}}(\mathbb{H}_{\mathbb{Q}})$ on $\mathbb{R}^{16} \simeq \Lambda_1' = \langle L_sK_w : s, w = 1, \cdots, 4 \rangle$, it is clear that under suitable order of the generators L_sK_w, the image $\Phi_{\mathbb{R}}(\mathbb{H}_{\mathbb{Q}})$ lies in the claimed subset of block diagonal matrices. □

We first choose $\{(x_{\mathbb{R}})_1, \cdots, (x_{\mathbb{R}})_4\} \subset M_{16}(\mathbb{Z})$ that satisfy

$$(\Lambda_1')_{\mathbb{Q}} = \sum_{i=1}^{4} \Phi_{\mathbb{R}}(F)(x_{\mathbb{R}})_i.$$

We fix the order of the set of generators Λ_1' such that the image $\Phi_{\mathbb{R}}(\mathbb{H}_{\mathbb{Q}})$ are block diagonal matrices of 4-by-4 blocks. Then it is clear that the attributes $\{(x_{\mathbb{R}})_1, \cdots, (x_{\mathbb{R}})_4\}$ can be chosen to be $\{e_1, e_5, e_9, e_{13}\}$, where $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ is the vector with 1 as its j^{th} entry. The complex vectors x_i that distinguish the members in $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ can then be obtained by multiplying the change of basis Q obtained at the end of Section 6.2.2 to their images in the $+i$-eigenspace of the complex structure J_1 **(See function "Get_xs" in file [T3.m])**. It can be checked that they do satisfy the original equation 3.3.3(1)

Next, we will compute the attributes $\mathcal{M}$ such that Equation 3.3.3(2) is satisfied:

$$\Lambda_1 = \left\{ \sum_{i=1}^{4} \Phi_{\mathrm{std}}(a_i)x_i \colon (a_1, \cdots, a_4) \in \mathcal{M} \right\}.$$

As explained in Remark 3.3.17, it is more intuitive to solve $\mathcal{M}$ from a real version of the equation that does not depend on the complex vectors x_i (or equivalently the complex structure of each member A_1). We will instead solve

$$\Lambda_1' = \left\{ \sum_{i=1}^{4} \Phi_{\mathbb{R}}(a_i)(x_{\mathbb{R}})_i \colon (a_1, \cdots, a_4) \in \mathcal{M} \right\}.$$

In other words, we will identify Λ_1' to a $\mathbb{Z}$-submodule $\mathcal{M}$ of F^4, where $F = \mathbb{H}_{\mathbb{Q}}$. Note that from 3.3.3(1) we may decompose Λ_1' into

$$\Lambda_1' \simeq \bigoplus_{i=1}^{4} \mathcal{L}_i,$$

where each $\mathcal{L}_i := \{\Phi_{\mathbb{R}}(a_i)(x_{\mathbb{R}})_i : (a_1, \cdots, a_4) \in \mathcal{M}\}$ is a $\mathbb{Z}$-module of rank 4 that corresponds to the i^{th} diagonal block of the elements in $\Phi_{\mathbb{R}}(F)$. Let us first focus on one of the blocks $\mathcal{L}_i$. We will prove that $\mathcal{L}_i$ is isomorphic to a $\mathbb{Z}$-submodule $\mathcal{M}_i$ of R, where $R = \operatorname{End}(A_1) = \langle r_1, \cdots, r_4 \rangle$ is the order in F we obtained in Section 6.2.2. Consider the $\mathbb{Z}$-submodule $R(x_{\mathbb{R}})_i < \mathbb{Z}^4$ of $\mathcal{L}_i$ generated by the vectors $\Phi_{\mathbb{R}}(r_1)e_1 = e_1, \cdots, \Phi_{\mathbb{R}}(r_4)e_{13}$ after removing unnecessary zeros. Let $(d_1, \cdots, d_4)$ with $d_j | d_{j+1}$ be the elementary divisors of the matrix

$$\Big(e_1 \mid \Phi_{\mathbb{R}}(r_2)e_5 \mid \Phi_{\mathbb{R}}(r_3)e_9 \mid \Phi_{\mathbb{R}}(r_4)e_{13} \Big) \in M_4(\mathbb{R}),$$

and let $d = d_4$. Then $\mathcal{L}_i$ is isomorphic to the $\mathbb{Z}$-module $d\mathcal{L}_i < R(x_{\mathbb{R}})_i$. We can therefore obtain a R-module $\mathcal{M}_i$ by multiplying $d\mathcal{L}_i$ by $(x_{\mathbb{R}})_i^{-1}$ on the right **(See function "Get_calMkk" in file [T3.m])**. Furthermore, $\mathcal{M}_i$ is torsion free and is isomorphic to $\mathcal{L}_i$.

$$\begin{array}{ccccc} \mathcal{L}_i & \xrightarrow{\quad d\cdot \quad} & d\mathcal{L}_i & \xrightarrow{\quad \cdot (x_{\mathbb{R}})_i^{-1} \quad} & \mathcal{M}_i < R \\ & & \wedge & & \\ & & R(x_{\mathbb{R}})_i & & \end{array}$$

This gives us

$$\Lambda_1' \simeq \bigoplus_{i=1}^{4} \mathcal{M}_i < R^4 < F^4.$$

We may even identify some of these $\mathcal{M}_i$'s if they are isomorphic R-modules.

Lemma 6.2.8. *Two R-modules M and N are isomorphic if and only if there exists $h \in \mathbb{H}_{\mathbb{Q}}$ such that $N = Mh$. The isomorphism preserves the number of minimal vectors (i.e. vectors of smallest norm) in the isomorphic modules.*

Proof. The reverse implication for the first statement is clear as R is torsion free. For the forward implication: suppose $f \colon M \to N$ is an R-module isomorphism. Fix any $m \in M$, so we have $Rm < M$. Similar to the above, by considering the elementary divisors of $\chi_{\mathbb{R}}(Rm)$ in M, we can find an integer d such that any $x \in M$ may be written as $x = rm/d$ for some $r \in R$. Now

$$f(x) = \frac{rf(m)}{d} = \frac{rm \cdot m^{-1} \cdot f(m)}{d} = x(m^{-1} \cdot f(m))$$

where $m^{-1} \cdot f(m) \in \mathbb{H}_{\mathbb{Q}}$.

Norm in R is defined as $\operatorname{Nm}(r) = r\bar{r}$ for all $r \in R$. So if $x \in M$ is a minimal vector, then $xh \in N$ is a minimal vector with norm $\operatorname{Nm}(x)\operatorname{Nm}(h)$. □

With the function **ShortestVectors** in MAGMA, one may obtain a list of minimal vectors in each R-module $\mathcal{M}_i$. Then it can be tested whether those $\mathcal{M}_i$'s with equal number of minimal vectors are isomorphic by brute force. To be specific, suppose $\mathcal{M}_i$ and $\mathcal{M}_j$ have minimal vectors $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ respectively. For any $k, l \in 1, \cdots, n$ with $k < l$, let $h_{kl} := u_k^{-1} v_l$. Then $M_i \simeq M_j$ if and only if there exists $h_{kl} \in \mathbb{H}_{\mathbb{Q}}$ such that right multiplication by h_{kl} is a bijection between the set of generators of $\mathcal{M}_i$ and that of $\mathcal{M}_j$. Note that if $n \geq 4$, then the set of minimal vectors in the module $\mathcal{M}_i$ generates $\mathcal{M}_i$ Moreover, if we let U_i be the 4-by-4 matrix representing the right multiplication of h_{kl} on $\mathcal{M}_i$, then $U := \operatorname{diag}(U_1, \cdots, U_4)$ is a matrix taking any element in Λ_1' with respect to the generators e_j's to its image in $\mathcal{M} := \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_4$, where each $\mathcal{M}_i$ is with respect to the basis $\{r_1, \cdots, r_4\}$ of R. By identifying each $\mathcal{M}_i$ with a submodule in $\mathbb{H}$, then there is a 4-by-16 matrix U' over $\mathbb{H}$ such that $U'(e_j) < \mathbb{H}^4$ represents the same element as $U(e_j)$.

We will compute the $\mathcal{M}_i$'s for the example $T = T_3$.

Example 6.2.9. ***(See function "calMkk_2_Ikk" in file [T3.m])*** *Continue from Example 6.2.5. Recall that $R = \langle 1, \mathfrak{o}(-2)\rangle$. Up to reordering the index i for the modules $\mathcal{M}_i$, it can be shown that $\mathcal{M}_1$ and $\mathcal{M}_2$ have 6 minimal vectors, while $\mathcal{M}_3$ and $\mathcal{M}_4$ have 12. On the other hand, the R-modules in $\mathbb{H}_{\mathbb{Q}}$*

$$I_6 := \langle h + i, h + j, i - j, k\rangle$$
$$I_{12} := \mathfrak{o} = \langle h, i, j, k\rangle$$

have 6 and 12 minimal vectors respectively. By brute force, one can show that $\mathcal{M}_1 \simeq \mathcal{M}_2 \simeq I_6$, and $\mathcal{M}_3 \simeq \mathcal{M}_4 \simeq I_{12}$. Therefore Λ_1 is isomorphic to the $\mathbb{Z}$-module in F^4

$$\mathcal{M} = I_6 \oplus I_6 \oplus I_{12} \oplus I_{12}.$$

Next we calculate the matrix $\mathcal{T} = \{t_{ij}\}$ that satisfies Equation 3.3.3(3):

$$E\left(\sum_{i=1}^{4} \Phi_{\text{std}}(a_i)x_i, \sum_{j=1}^{4} \Phi_{\text{std}}(b_j)x_j\right) = \operatorname{tr}_{F|\mathbb{Q}}\left(\sum_{i,j=1}^{4} a_i t_{ij} b_j^{\rho}\right)$$

where E is the alternating form associated to the polarisation of the abelian 8-fold A_1, and $a_i, b_j \in \mathbb{H}$. Again, we solve the "real" version of the equation by considering E as a pairing on $(\Lambda_1')_{\mathbb{R}} \simeq \mathbb{R}^{16}$ given in Section 4.3 by

$$(v, w) \longmapsto \operatorname{tr}(\alpha v^* w)$$

for a suitable choice of $\alpha \in \mathrm{Cl}^+(T)$. Let M_E be the corresponding 16-by-16 real matrix with respect to the basis $\{e_1, \cdots, e_{16}\}$ of Λ_1'. Then $\mathcal{T}$ is the unique 4-by-4 matrix such that

$$(U_h')^t \mathcal{T} \overline{U_l'} = (M_E)_{h,l},$$

where U_h' and U_l' are the h-th and the l-th columns of U'. From Lemma 6.2.7, if $h = 4(s-1) + w$ with $0 \leq w < 4$, then the s-th entry on the column U_h' is the only non-zero entry. The vast number of zeros greatly reduces the difficulties of solving for $\mathcal{T}$.

Let us calculate $\mathcal{T}$ for the main example.

Example 6.2.10. ***(See function "Get_calT" in file [T3.m])*** *Let $T = T_3$. Let $\{f_1, \cdots, f_4, h_1, \cdots, h_4\}$ be the set of generators for the lattice T as in Example 6.2.1. Clearly $f_1 + f_2$ and $f_3 + f_4$ are two positive orthogonal vectors. Choosing for example $\alpha = (f_1 + f_2)(f_3 + f_4)$, then the matrix M_E is in the form*

$$M_E = \begin{bmatrix} 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{bmatrix}$$

where each asterisk represents a non-zero 4-by-4 block. This implies that the matrix $\mathcal{T}$ only has four non-zero entries: $t_{1,2}, t_{2,1}, t_{3,4}$ and $t_{4,3}$. Note that we have an over-determined system of linear equations. To solve for any of these non-zero entries, say $t_{1,2}$, it is enough to consider the four equations

$$U'_{1,1} \cdot t_{1,2} \cdot U'_{2,4+k} = (M_E)_{1,4+k} \text{ where } k = 1, \cdots, 4.$$

The calculation gives

$$\mathcal{T} = \begin{pmatrix} 0 & 256 & 0 & 0 \\ -256 & 0 & 0 & 0 \\ 0 & 0 & 0 & -512 \\ 0 & 0 & 512 & 0 \end{pmatrix}$$

In fact, the matrix $\mathcal{T}$ is the same for all Λ'_1 for $i = 1, \cdots, 8$ up to switching the two copies of I_6 (and/or the two copies of I_{12}) in $\mathcal{M} = I_6 \oplus I_6 \oplus I_{12} \oplus I_{12}$.

Remark 6.2.11. *We may also compute the attribute $\mathcal{H} = (h_{ij}) \in M_4(\mathbb{H})$ which is determined by the other three attributes **(See function "Get_calH" in file [T3.m])**. We again consider the "real" version of Equation 3.3.3(4) satisfied by $\mathcal{H}$:*

$$\sqrt{-1}(x_{\mathbb{R}})_i = \sum_{i=1}^{m} \Phi_{\mathbb{R}}(h_{ij})(x_{\mathbb{R}})_j,$$

where $\sqrt{-1}$ represents the action of the complex structure J_1 of A_1. By writing each h_{ij} as a linear combination of the basis $\boldsymbol{r} = \{r_1, \cdots, r_4\}$ we used in Example 6.2.5, this is then equivalent to solving for $a_k^{ij} \in \mathbb{R}$ such that

$$\sqrt{-1}(x_{\mathbb{R}})_i = \sum_{j=1}^{4} \sum_{k=1}^{4} a_k^{ij} \Phi_{\mathbb{R}}(r_k)(x_{\mathbb{R}})_j.$$

*We can easily solve the system consisting of the four equations when $i = 1, \cdots, 4$ using the function **Solution**.*

6.3 Realise map between period domains

We have computed the attributes $\mathcal{M}$ and $\mathcal{T}$ which determine the target $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ in the map $F \colon \mathcal{K}_P \to \mathcal{A}_{\mathcal{M},\mathcal{T}}$, as well as the attribute $\{x_1, \cdots, x_4\} \subset \mathbb{C}^8$, associated to an abelian 8-fold A_1 in the image of F, that is isomorphic to the complex torus $((\Lambda'_1)_{\mathbb{R}}/\Lambda'_1, J_1)$. Recall from Theorem 5.2.1 that the map F lifts to the map $\widetilde{F} \colon \mathcal{D}_T^+ \longrightarrow \mathcal{D}_{\mathcal{M},\mathcal{T}}$, where

$$\mathcal{D}_T^+ = \{\omega \in \mathbb{P}(T_{\mathbb{C}}) : \omega^2 = 0, \omega\overline{\omega} > 0\}$$
$$\mathcal{D}_{\mathcal{M},\mathcal{T}} = \{Z \in M_d(\mathbb{C}) : -Z = Z^t, 1 - Z\overline{Z}^t > 0\}$$

are the HSD overspaces of the modular varieties $\mathcal{K}_P$ and $\mathcal{A}_{\mathcal{M},\mathcal{T}}$. We would like to realise the map $\widetilde{F}$ and compute $\widetilde{F}(\omega) \in \mathcal{D}_{\mathcal{M},\mathcal{T}}$ for any $\omega \in \mathcal{D}_T^+$, which allows us to study the image of an infinitesimal deformation of $\mathcal{D}_T^+$ under the map F.

As discussed in Section 4.3 and 5.1, a K3 surface $X \in \mathcal{K}_P$ lifts to a point $\omega \in \mathcal{D}_T^+$ which gives the complex structure $J \in \mathrm{Cl}^+(T)$ of the corresponding KS variety $\mathrm{KS}(X, \alpha)$. Its restriction to Λ'_1 gives the complex structure J_1 of $A_1 = F(X)$. We have shown that J_1 gives the attribute $\{x_1, \cdots, x_4\} \subset \mathbb{C}^8$. Carefully following the proof of Theorem 3.3.18 and [Sh, Section 2.3–2.5], we can then obtain the standard normalised form of this attribute which is an element in $\mathcal{D}_{\mathcal{M},\mathcal{T}}$.

We shall explain the steps in greater detail for our main example.

Example 6.3.1. ***(See function "Get_z" in file [T3.m])*** *Let $T = T_3$ and fix ω which gives the complex structure $J \in \mathrm{Cl}^+(T)$. We define the matrix X*

$$X = \begin{bmatrix} U & V \\ \bar{V} & -\bar{V} \end{bmatrix}$$

from the attribute x as in the proof of Theorem 3.3.18 ***(See function "Get_Xmat" in file [T3.m]).***

The next step is to find a suitable basis of $\mathbb{H}$ such that $\mathcal{T}^{-1}$ is the matrix $i\mathbf{1}_4$, or equivalently to find a matrix $W \in M_4(\mathbb{H})$ which satisfies

$$W\mathcal{T}^{-1}W^\rho = i\mathbf{1}_4,$$

where the positive anti-involution ρ acts on the matrix W by transpose of conjugation. Given the expression of $\mathcal{T}$ in Example 6.2.10, it is easy to see that if

$$W' := \begin{pmatrix} -256 & -i & 0 & 0 \\ -256 & i & 0 & 0 \\ 0 & 0 & -i & 512 \\ 0 & 0 & i & 512 \end{pmatrix},$$

then $W'\mathcal{T}^{-1}(W')^\rho = \mathrm{diag}(-2i, 2i, -2i, 2i)$. Thus we may take

$$W = \mathrm{diag}(-j/\sqrt{2}, 1/\sqrt{2}, -j/\sqrt{2}, 1/\sqrt{2}) \cdot W'.$$

If we perform change of basis of X by $\overline{\chi(W)}^{-1}$, i.e. *replace X by $X \cdot \overline{\chi(W)}^{-1}$, then $X \in M_8(\mathbb{C})$ is still a block matrix in the form*

$$\begin{bmatrix} U & V \\ \overline{V} & -\overline{V} \end{bmatrix},$$

and the matrix $Z := -V^{-1}U$ is an element in $\mathcal{D}_{\mathcal{M},\mathcal{T}}$.

In particular, we may compute the image under $\widetilde{F}$ for the point $\omega_0 := [\langle (f_1+f_2)/\sqrt{2} - i(f_3+f_4)/2 \rangle_{\mathbb{C}}]$, which clearly belongs to $\mathcal{D}_T^+$ as $(f_1+f_2)/\sqrt{2}$ and $(f_3+f_4)/2$ are orthonormal vectors in $T_{\mathbb{R}}$.

Then with by the above calculations, we have

$$\widetilde{F}(\omega) = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}$$

where

$$a = \frac{8193 - 128\sqrt{2}}{8191}, \; b = \frac{524289 - 1024\sqrt{2}}{524287}.$$

Remark 6.3.2. *As a sanity check, one can check that $Z^t = -Z$ and $1 - Z\overline{Z}^t > 0$. To check the inequality, it is enough to check that all the eigenvalues of Z are positive, as Z is Hermitian. Note that the* ***Eigenvalues*** *function in MAGMA can only find eigenvalues over the same field as the entries of Z are defined over. So to exhaust all the eigenvalues, we must first convert Z to a complex matrix, paying the price of floating point errors. Since the eigenvalues are always real, the check is the same as checking the real parts of the eigenvalues are all positive.*

In practice, it is hard to determine if the initial choice of ω belongs to $\mathcal{D}_T^+$ or the other connected component $\mathcal{D}_T^-$ of the period domain. The above check therefore serves as a flag for this potential mistake: if the resulting Z does not satisfy $1 - Z\overline{Z}^t > 0$, then $\omega \in \mathcal{D}_T^-$. To fix the problem, we replace ω by $\overline{\omega}$, and J by $-J$.

6.3.1 Local deformation

Having understood the map $\widetilde{F}$, we would like to vary the input $\omega \in \mathcal{D}_T^+$ to see how the image of $\widetilde{F}$ changes accordingly. Recall from the proof of Theorem 5.2.2 that $\widetilde{F}$ is locally the Lie algebra isomorphism $\mathfrak{so}^+(2,6) \to \mathfrak{so}^*(8)$. More specifically, we have

$$\begin{array}{ccccc}
\mathfrak{so}^+(2,6) & \longrightarrow & \mathfrak{so}^+(2,6)/\mathfrak{k}_{\mathfrak{so}^+(2,6)} \simeq T_\omega \mathcal{D}_T^+ & \longrightarrow & T_X \mathcal{K}_P \\
\big\downarrow \simeq & & \big\downarrow d\tilde{F}_\omega & & \big\downarrow dF_X \\
\mathfrak{so}^*(8) & \longrightarrow & \mathfrak{so}^*(8)/\mathfrak{k}_{\mathfrak{so}^*(8)} \simeq T_{\tilde{F}(\omega)} \mathcal{D}_{\mathcal{MT}} & \longrightarrow & T_{F(X)} A_{\mathcal{MT}}
\end{array}$$

where $\mathfrak{k}_{\mathfrak{so}^+(2,6)} = \mathrm{Lie}\,(\mathrm{SO}(2) \times \mathrm{SO}(6))$, $\mathfrak{k}_{\mathfrak{so}^*(8)} = \mathrm{Lie}\,(U(4))$, and all horizontal arrows are quotient maps by the suitable objects. This shows that perturbing the point $X \in \mathcal{K}_P$, which is the same as choosing a tangent vector in the 6-dimensional vector space $T_X\mathcal{K}_P$, is equivalent to picking an element in $\mathfrak{so}^+(2,6)$ followed by some identification. In the following we will discuss how this allows us to study the image under $\widetilde{F}$ of the local deformation of a point $\omega \in \mathcal{D}_T^+$.

From [He, Section X.2.1] the Lie algebra $\mathfrak{so}(2,6)$ is a real vector space

$$\mathfrak{so}(2,6) := \left\{ \begin{bmatrix} M_1 & M_2 \\ M_2^t & M_3 \end{bmatrix} : \begin{array}{c} \text{All } M_i \text{ real; } M_2 \text{ arbitrary;} \\ M_1, M_3 \text{ skew-symmetric of order 2 and 6 resp.} \end{array} \right\}.$$

Thus a basis of $\mathfrak{so}(2,6)$ can be given by the folowing 28 elements

$$M_{ij} := \begin{cases} M_{ij}^- := E_{ij} - E_{ji} & \text{for } i = 1,\ j = 2; \text{ or } i \geq 3,\ j > i; \\ M_{ij}^+ := E_{ij} + E_{ji} & \text{for } i = 1, 2,\ j \geq 3. \end{cases}$$

where E_{ij} has 1 at the (i,j)-th entry being the only non-zero entry in the matrix. These generators correspond to tangent vectors of $\mathrm{SO}^+(2,6)$ in 28 directions via the exponential map exp:

$$\begin{aligned} \exp\colon \mathfrak{so}^+(2,6) &\longrightarrow \mathrm{SO}^+(2,6) \\ M &\longmapsto \sum_{k=0}^{\infty} \frac{M^k}{k!}. \end{aligned}$$

On the other hand, recall in the proof of Theorem 5.2.2 that any element $N \in \mathrm{SO}^+(T_{\mathbb{R}}, q) \simeq \mathrm{SO}^+(2,6)$ acts on $\mathcal{D}_T^+$ by left multiplication:

$$\begin{aligned} m_N\colon \mathcal{D}_T^+ &\longrightarrow \mathcal{D}_T^+ \\ [\langle e_1 + ie_2 \rangle_{\mathbb{C}}] &\longmapsto [\langle N \cdot e_1 + iN \cdot e_2 \rangle_{\mathbb{C}}], \end{aligned}$$

where e_1, e_2 are orthonomal vectors in (T, q). Therefore, for any

$$N = \exp\left(t \cdot \sum_{ij} a_{ij} M_{ij} \right) \in \mathrm{SO}^+(2,6)$$

for a small $t \in \mathbb{R}_{>0}$ and some coefficients $a_{ij} \in \mathbb{R}$, the perturbation of point $\omega \in \mathcal{D}_T^+$ by the vector N is given by $m_N(\omega)$ **(See function "Perturb" in file [T3.m])**.

Remark 6.3.3. *It is clear that the cardinality (without multiplicity) of the set* $\left\{ \widetilde{F}\left(m_{M_{ij}}(\omega) \right) \right\}$ *is at most* 6.

6.3.2 Evaluate exponential map

To compute exp, it is enough to obtain the values of $\exp(tM_{ij}^+)$ and $\exp(tM_{ij}^-)$ for some small t. We will first evaluate exp at tM_{ij}^+. Note that $(M_{ij}^+)^2 = 0$, so

$$\exp(tM_{ij}^+) = \mathbf{1}_8 + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \cdot M_{ij}^+ + \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} \cdot \mathbf{1}_8$$

where $\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}$ and $\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}$ are the Taylor series for $\sinh(t)$ and $\cosh(t)$ respectively. So $\exp(tM_{ij}^+)$ is the matrix with each of the (h,k)-th entry being

$$\begin{cases} 1 & \text{if } h = k \notin \{i,j\}; \\ \cosh(t) & \text{if } h = k \in \{i,j\}; \\ \sinh(t) & \text{if } h = i,\ k = j; \text{ or } h = j,\ k = i; \\ 0 & \text{otherwise.} \end{cases}$$

Similarly for tM_{ij}^-, since $(M_{ij}^-)^4 = 0$,

$$\begin{aligned} \exp(tM_{ij}^+) = \mathbf{1}_8 &+ \sum_{k=0}^{\infty} \frac{t^{4k+1}}{(4k+1)!} \cdot M_{ij}^- + \sum_{k=0}^{\infty} \frac{t^{4k+2}}{(4k+2)!} \cdot (M_{ij}^-)^2 \\ &+ \sum_{k=0}^{\infty} \frac{t^{4k+3}}{(4k+3)!} \cdot (M_{ij}^-)^3 + \sum_{k=1}^{\infty} \frac{t^{4k}}{(4k)!} \cdot \mathbf{1}_8 \\ = \mathbf{1}_8 &+ \frac{1}{2}(\sinh(t) + \sin(t)) \cdot M_{ij}^- + \frac{1}{2}(\cosh(t) - \cos(t)) \cdot (-\mathbf{1}_8) \\ &+ \frac{1}{2}(\sinh(t) - \sin(t)) \cdot (-M_{ij}^-) + \frac{1}{2}(\cosh(t) + \cos(t) - 2) \cdot \mathbf{1}_8 \end{aligned}$$

So $\exp(tM_{ij}^-)$ is the matrix with each of the (h,k)-th entry being

$$\begin{cases} 1 & \text{if } h = k \notin \{i,j\}; \\ \cos(t) & \text{if } h = k \in \{i,j\}; \\ \sin(t) & \text{if } h = i, k = j; \\ -\sin(t) & \text{if } h = j, k = i; \\ 0 & \text{otherwise.} \end{cases}$$

In MAGMA, it is not ideal to use real data type for the values of the trigometric or hyperbolic functions as floating point errors are significant when t is small. In order to work over the rationals, we choose t such that the values of the above trigonometric/hyperbolic functions are rational.

In the case of tM_{ij}^+, we would like to obtain rational values of $\cosh(t)$ and $\sinh(t)$ for small $t \in \mathbb{Q}_{>0}$. Setting $x(t) := \cosh(t)$, $y(t) := \sinh(t)$, this is equivalent to finding a rational point close to the point $(1,0)$ on the hyperbola $x^2 - y^2 = 1$. Since $(-1,0)$ is another obvious rational point on the hyperbola, any line with rational slope m through $(-1,0)$ must intersect the hyperbola at another rational point, whose coordinates can be calculated from solving the system of equations

$$\begin{cases} x^2 - y^2 = 1 \\ y = mx + m \end{cases}$$

Therefore a rational point close to $(1,0)$ on the hyperbola has coordinates

$$\left(\frac{1+m^2}{1-m^2}, \frac{2m}{1-m^2} \right)$$

for some small $m \in \mathbb{Q}_{>0}$. Similarly for evaluating $\exp(tM_{ij}^-)$ for t small, we have

$$(\cos(t), \sin(t)) = \left(\frac{1-m^2}{m^2+1}, \frac{2m}{m^2+1}\right)$$

for suitable small $m \in \mathbb{Q}_{>0}$.

Example 6.3.4. *Continue from Example 6.3.1. Let $m = 10^{-3}$ and $t = \sinh^{-1}(2m/(1-m^2)) \approx 0$. Let ω_0' be the perturbation of ω_0 by the vector $tM_{1,3}' \in \mathrm{SO}^+(2,6)$. Then*

$$\widetilde{F}(\omega_0') = \begin{pmatrix} 0 & a' & 0 & 0 \\ -a' & 0 & 0 & 0 \\ 0 & 0 & 0 & b' \\ 0 & 0 & -b' & 0 \end{pmatrix}$$

where

$$a' = \frac{8209390193 - 127999872\sqrt{2}}{8207394191}, \ b' = \frac{525338098289 - 1023998976\sqrt{2}}{525336102287}.$$

6.4 A special locus

In this subsection we will focus on our main example when $T = T_3$. As seen in Example 6.3.1 and 6.3.4, $\widetilde{F}(\omega_0)$ and $\widetilde{F}(\omega_0')$ are in a particularly nice form $Z(a,b) \in M_4(\mathbb{C})$ where

$$\begin{aligned} &Z(a,b)_{1,2} = -Z(a,b)_{2,1} = a; \\ &Z(a,b)_{3,4} = -Z(a,b)_{4,3} = b; \\ &Z(a,b)_{i,j} = 0 \text{ if } (i,j) \notin \{(1,2),(2,1),(3,4),(4,3)\}. \end{aligned}$$

Furthermore, the condition $1 - Z\bar{Z}^t > 0$ tells us that $|a| < 1$ and $|b| < 1$.

Consider the sublattice $T' = \langle f_1, f_2, f_3, f_4 \rangle = U \oplus U(2)$ of (T, q) and let P' be its complement in the K3 lattice Λ_{K3}. By consulting Table 4, the lattice P' is given by $U \oplus E_8(-1) \oplus D_8(-1)$. Then for any ω in the identity component $\mathcal{D}_{T'}^+$ of the period domain of weight two Hodge structures on T', the image $\widetilde{F}(\omega)$ is in this nice form $Z(a,b)$ with $|a|, |b| < 1$. This gives an inclusion of the 2-dimensional subdomain $\widetilde{F}(\mathcal{D}_{T'}^+)$ of $\widetilde{F}(\mathcal{D}_T^+)$ into $\mathcal{S}_1 \times \mathcal{S}_1$, the product of two Siegel upper-half spaces of degree 1:

$$\begin{array}{ccccc} \widetilde{F}(\mathcal{D}_{T'}^+) & \hookrightarrow & D_1 \times D_1 & \xrightarrow{\simeq} & \mathcal{S}_1 \times \mathcal{S}_1 \\ Z(a,b) & \longmapsto & (a,b) & \longmapsto & (f(a), f(b)) \end{array}$$

where f is the conformal map taking a disc D_1 to $\mathcal{S}_1$ by

$$x \longmapsto \frac{i(1+x)}{1-x}.$$

Recall that $\mathcal{S}_1$ is the parametrisation space for arbitrary elliptic curves. So we may consider $\mathcal{D}_{T'}^+$ as a subset of the parametrisation space of a pair of elliptic curves, which aligns with the row of $r = 18$ in Table 3. This observation can be explained by the geometry of the abelian 8-folds parametrised by the special locus $\widetilde{F}(\mathcal{D}_{T'}^+)$. We will denote by X' a K3 surface in $\mathcal{K}_P$ polarised by $P' \supseteq P$, and use the notation $\mathrm{KS}(X') = \mathrm{KS}(T)$ and $\mathrm{KS}(T')$ to differentiate between KS varieties constructed from different lattices.

Theorem 6.4.1. *Suppose $X' \in \mathcal{K}_P$ is polarised by $P' \supseteq P$. Let $A_1 = F(X') < \mathrm{KS}(X')$. If X' is very general, that is if $\mathrm{Pic}(X') = P'$, then A_1 is isogenous to $E_1^4 \times E_2^4$, where E_1 and E_2 are two non-isogenous elliptic curves.*

We will prove this statement using properties of Clifford algebras only. The first step is to prove the following lemma.

Lemma 6.4.2. *Suppose $X' \in \mathcal{K}_P$ is very general. Let $\mathrm{KS}(X') = \mathrm{KS}(T) \sim A_1 \times \cdots \times A_8$ be the decomposition of the KS variety as obtained in Example 6.2.1. If $\mathrm{Pic}(X') = P'$ then for all $i = 1, \cdots, 8$, there exist elliptic curves E_1, E_2 and an integer k satisfying $0 \le k \le 8$ such that*

$$A_i \sim E_1^k \times E_2^{8-k}.$$

Proof. Let X' be a K3 surface whose transcendental lattice is exactly the rank 4 lattice $T' \subset T \subset \Lambda_{K3}$. By Lemma 4.3.5(i), we have $\mathrm{KS}(T) \sim \mathrm{KS}(T')^{2^4}$.

On the other hand, by Theorem 3.5.25, the K3 surface X' has a Shioda-Inose structure associated to an abelian surface A'. From Lemma 4.3.5(i), we have $\mathrm{KS}(T')^{2^{18}} \sim \mathrm{KS}(H^2(X', \mathbb{Z}))$, and from Lemma 4.3.5(ii), we have $\mathrm{KS}(H^2(X', \mathbb{Z})) \sim (A')^{2^{19}}$. By the Poincarè's Complete Reducibility Theorem, we have $\mathrm{KS}(T') \sim (A')^2$.

Finally, from Table 5, we have $\mathrm{Cl}^+(T') \simeq M_2(\mathbb{R})^{\oplus 2}$. By Theorem 5.1.2, this implies $\mathrm{KS}(T') \sim (E_1 \times E_2)^2$, where E_1 and E_2 are non-isogenous elliptic curves. Combining all statements, this gives $A_i \sim E_1^k \times E_2^{8-k}$. Moreover, four subvarieties in the decomposition of $\mathrm{KS}(X')$ described in Theorem 5.1.2 are isogeneous to $E_1^k \times E_2^{8-k}$, and the other four are isogeneous to $E_1^{8-k} \times E_2^k$. □

Remark 6.4.3. *Since A' has transcendental lattice $U \oplus U(2)$, its Picard lattice is given by $U(2)$ consulting Table 4, which suggests that*

$$A' \simeq (E_1 \times E_2)/\{(P, Q)\} \sim E_1 \times E_2,$$

where $P \in E_1[2]$ and $Q \in E_2[2]$ are 2-torsion points in the elliptic curves E_1 and E_2 respectively.

To prove Theorem 6.4.1, it remains to show $k = 4$ in the above statement.

Proof of Theorem 6.4.1. Let $(T')^\perp$ be the sublattice in T such that $T = T' \oplus (T')^\perp$. *i.e.* Let $(T')^\perp = D_4(-1)$. We recall in Example 6.2.1 that pulling back each pseudo-idempotent $32\varepsilon_i$ along the gluing map

$$\mathrm{Cl}^+(T') \otimes \mathrm{Cl}^+((T')^\perp) \longrightarrow \mathrm{Cl}^+(T)$$

is the tensor product $x_j \otimes y_k$. Then by the same reasoning as in the proof of Lemma 4.3.5(i) we have

$$\begin{aligned}\Lambda_1' \simeq \mathrm{Cl}^+(T) \cdot (32\varepsilon_1) &\simeq \Big(\big(\mathrm{Cl}^+(T') \cdot x_j\big) \otimes \Big(\mathrm{Cl}^+((T')^\perp) \cdot y_k\Big) \oplus \big(\mathrm{Cl}^-(T') \cdot x_j\big) \otimes \Big(\mathrm{Cl}^-((T')^\perp) \cdot y_k\Big)\Big) \\ &\simeq 4\left(\big(\mathrm{Cl}^+(T') \cdot x_j\big) \oplus \big(\mathrm{Cl}^-(T') \cdot x_j\big)\right).\end{aligned}$$

The second isomorphism comes from the fact that under the algebra isomorphism $\varphi\colon \mathrm{Cl}((T'_{\mathbb{R}})^\perp) \to M_2(\mathbb{H})$, the images of both $\big(\mathrm{Cl}^+((T')^\perp) \cdot y_k\big)$ and $\big(\mathrm{Cl}^-((T')^\perp) \cdot y_k\big)$ are rank 4 lattices over $\mathbb{Z}$.

On the other hand, $x_1, \cdots, x_4$ are pseudo-idempotents of $\mathrm{Cl}^+(T')$ by definition. Similarly by studying the algebra isomorphism $\varphi\colon \mathrm{Cl}(T'_{\mathbb{R}}) \to M_2(\mathbb{R})^{\oplus 2}$, the lattices $\mathrm{Cl}^+(T') \cdot x_i$ and $\mathrm{Cl}^-(T') \cdot x_i$ are both of rank 1 over $\mathbb{Z}$. Therefore, they respectively correspond to an elliptic curve E_i^+ and E_i^- in the simple decomposition of $\mathrm{KS}(T')$. This implies $k = 4$ or $k = 8$.

Assume for contradiction that $k = 8$, that is, $A_i \sim (E_i^+)^8$ for all i. From the decomposition of a KS variety associated to a very general member $X \in \mathcal{K}_P$ in Example 6.2.1, $A_1, \cdots, A_4$ (resp. $A_5, \cdots, A_8$) are isogenous abelian 8-folds, so $E_1^+, \cdots, E_4^+$ (resp. $E_5^+, \cdots, E_8^+$) are isogenous elliptic curves. Also, $32\varepsilon_1$ and $32\varepsilon_5$ pulls back to $x_1 \otimes y_1$ and $x_1 \otimes y_2$ respectively, so $E_1^+ \sim E_5^+$. This implies $\mathrm{KS}(X')^{2^4} \sim \mathrm{KS}(X) \sim (E_1^+)^{64}$. However, for a very general X' with $\mathrm{Pic}(X') = P'$, we have shown in the proof of Lemma 6.4.2 that $\mathrm{KS}(X') \sim (E_1 \times E_2)^2$, where E_1 and E_2 are non-isogenous. □

Theorem 6.4.1 implies that $\mathcal{D}^+_{T'}$ cuts out a special locus in $\mathcal{D}^+_T$ whose image under $\widetilde{F}$ corresponds to non-simple abelian 8-folds which are products in the form of $E_1^4 \times E_2^4$, where E_1 and E_2 are generically non-isogenous. Also, we have $A_1 \sim \cdots \sim A_8$. This is an example of the exceptional behaviour described at the end of Section 5.1.

We can similarly find a 2-dimensional locus in $\mathcal{D}^+_{T_i}$ for all $i = 1, \cdots, 6$. If X' has transcendental lattice $T_i' = U \oplus U$ or $U \oplus U(2)$, then X' has a Shioda-Inose structure (Theorem 3.5.25). Otherwise if X' has transcendental lattice $T_i' = U(2) \oplus U(2)$, then $X' = \mathrm{Kum}(A)$ is a Kummer surface with $\mathrm{NS}(A) = U$ by [Mo1, Corollary 4.4], and $\mathrm{KS}(X) \sim (A \times A^\vee)^{2^4} \sim A^{2^5}$ by [Mo2, Corollary 4.6] and Lemma 4.3.5(i). Then in both cases, it is easy to check that all the arguments in the proof of Lemma 6.4.2 apply, as they only depend on the rank and the signature of the sublattice T' in T. The proof of Theorem 6.4.1 also works nicely: by choosing pseudo-idempotents $x_1, \cdots, x_4$ in $\mathrm{Cl}(T')$ such that their images under $\varphi : \mathrm{Cl}(T') \to M_4(\mathbb{Q})$ are some integral multiples of $E_{1,1}$ up to $E_{4,4}$ (see Example 6.2.1), then $\mathrm{Cl}^+(T') \cdot x_i$ and $\mathrm{Cl}^-(T') \cdot x_i$ are both of rank 1 over $\mathbb{Z}$ and correspond to two non-isogenous elliptic curves E_1 and E_2. And by choosing pseudo-idempotents $y_1, y_2 \in \mathrm{Cl}((T')^\perp)$ such that their images under $\varphi : \mathrm{Cl}((T')^\perp) \to M_2(\mathfrak{o})$ are some integral multiples of $\mathrm{diag}(1,0)$ and $\mathrm{diag}(0,1)$, we can rule out the possibility that $A_1 \sim E_1^8$. So in both cases, for all A_1 parametrised by $\widetilde{F}(\mathcal{D}^+_{T'})$, we again have $A_1 \sim E_1^4 \times E_2^4$.

7 Future investigations

7.1 Degeneration problem

We may continue to explore the connections between our special families $\mathcal{K}_P$ of K3 surfaces and the resulting moduli spaces $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ of abelian 8-folds by means of degeneration.

In Section 6.4, we have already applied one method of degenerations, which is to study specialisation of families. We are in a collaboration with A. Malmendier and A. Clingher to describe the loci of specialisation $\mathcal{K}_{P_3'}$ in $\mathcal{K}_{P_3}$ (See Section 6.4) where the K3 surfaces also admit polarisation by the rank 18 lattice

$$P_3' = U \oplus E_8(-1) \oplus D_8(-1).$$

More specifically, we also consider the family $\mathcal{K}_{\mathrm{Kum}(S)}$ of Kummer surfaces $\mathrm{Kum}(S)$ associated to the product of two elliptic curves $S = E_1 \times E_2$, as well as the family $\mathcal{K}_{\mathrm{Kum}(A')}$ of Kummer surfaces $\mathrm{Kum}(A')$ associated to the abelian surface A' with Picard lattice $U(2)$ as described in Remark 6.4.3. The family $\mathcal{K}_{\mathrm{Kum}(S)}$ of $\mathrm{Kum}(S)$ is in fact [CM1] a family of K3 surfaces polarised by the 2-elementary rank 18 lattice

$$U \oplus E_8(-1) \oplus 2D_4(-1).$$

Any element $\mathrm{Kum}(S)$ in $\mathcal{K}_{\mathrm{Kum}(S)}$ has 11 types of elliptic fibrations $\mathcal{J}_1, \cdots, \mathcal{J}_{11}$ as classified in [KuS], and the fibration $\mathcal{J}_6$ induces a van Geemen-Sarti duality between the families $\mathcal{K}_{\mathrm{Kum}(S)}$ and $\mathcal{K}_{P_3'}$. On the other hand, the quotient of S by the diagonal action of (P, Q) where $P \in E_1[2]$ and $Q \in E_2[2]$, induces two 2-isogenies

$$\mathrm{Kum}(S) \longrightarrow \mathrm{Kum}(A')$$
$$\mathrm{Kum}(A') \longrightarrow \mathrm{Kum}(S^\vee),$$

where $S^\vee$ is the dual of of S. We believe this is the specialisation of the 2-isogenies between Kummer surfaces of the Jacobian of a special double sextic curve and Kummer surfaces of a $(1,2)$-polarised abelian surface, mentioned in [BCMS]. Therefore for any K3 surface X in the family $\mathcal{K}_{P_3'}$, there exist three other K3 surfaces $\mathrm{Kum}(S), \mathrm{Kum}(S^\vee) \in \mathcal{K}_{\mathrm{Kum}(S)}$ and $\mathrm{Kum}(A') \in \mathcal{K}_{\mathrm{Kum}(A')}$ that fit into the a diagram (Diagram 4) of 2-isogenies of K3 surfaces. Our goal is to understand the dashed arrow in Diagram 4, which is the 2-isogeny from X to $\mathrm{Kum}(A')$ that describes the Shioda-Inose Structure of X, through understanding all other arrows. We also hope to compute the modular forms that cut out the specialisation locus $\mathcal{K}_{P_3'}$ from $\mathcal{K}_{P_3}$.

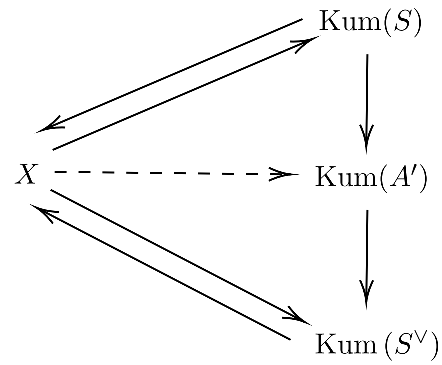


Diagram 4: Some 2-isogenies among K3 surfaces belong to three families of K3 surfaces of Picard rank 18.

Another special locus to study is the ramification locus if the degree of the map from $\mathcal{D}_T^+$ to $\mathcal{D}_{\mathcal{M},\mathcal{T}}$ is different from 1. Specialising to these loci gives information about extra structure carried by the simple abelian 8-folds in $\mathcal{A}_{\mathcal{M},\mathcal{T}}$, which should hint at the correct choice of the arithmetic subgroup in the LSV biquotient of $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ for F to be an isomorphism.

The other method is to degenerate the parametrised varieties into singular ones, which occurs at the cusps of the compactified moduli space. Compactification methods and configuration of cusps for modular varieties of K3 surfaces and abelian varieties are well studied. For certain semi-toroidal compactifications of these modular varieties, it is known which semi-stable K3 surfaces or abelian varieties correspond to each point in the boundary of the modular varieties. One may explore different compactification methods of the moduli spaces $\mathcal{K}_P$ and $\mathcal{A}_{\mathcal{M},\mathcal{T}}$, and study how their boundaries correspond to each other under the map F.

7.2 Special cases of the Hodge conjecture

There have been recent advances in proving special cases of the Hodge conjecture by the study of KS varieties: in [vG2], B. van Geemen has shown that E. Markman's proof of a partial case of the Hodge conjecture for abelian 4-folds of Weil type is related to the KS construction for K3 surfaces of Picard rank 16. We may ask whether the KS construction has similar implications for the other cases of the Hodge conjecture.

Another interesting question is the Kuga-Satake Hodge conjecture [vG1], concerning a certain cycle class of the product of a projective hyperkähler variety with the square of its associated KS variety. The conjecture is proved for a special family of K3 surfaces of Picard rank 16 studied in [Pa], and its specialisation to families of higher Picard ranks [VV]. We may examine the conjecture on the special families of K3 surfaces of Picard rank 14 studied in the thesis.

7.3 Connections with mirror symmetry

Mirror symmetry is an area that lies in the intersection of algebraic geometry and physics, which describes a relation between Calabi-Yau manifolds. In particular Calabi-Yau 2-folds are K3 surfaces, and by the work of I.V. Dolgachev [D], one can create a mirror family of K3 surfaces by associating a rank r lattice polarisation to a rank $20 - r$ polarisation.

A possible direction of research is to study the KS construction for families of K3 surfaces that are Dolgachev mirror duals to the special families of K3 surfaces we have studied. Such a family of mirror duals $\mathcal{K}_P^\vee$ parametrising K3 surfaces of Picard rank 6 is a type IV_{14} LSV, while the original family $\mathcal{K}_P$

of K3 surfaces of Picard rank 14 is a type IV_6 one. With I. Satake's result [Sa], it is expected that one can obtain a map from a type IV_{14} LSV to a type II_{64} LSV, which is a moduli space $\mathcal{A}^{\vee}_{\mathcal{M},\mathcal{T}}$ of polarised abelian 128-fold with totally definite quaternion multiplication. Remarkably, a simple factor of a generic KS variety associated to a K3 surface of Picard rank 6 is also an abelian 128-fold with the expected endomorphism structure. Thus one may construct a map $F^{\vee}\colon \mathcal{K}^{\vee}_P \to \mathcal{A}^{\vee}_{\mathcal{M},\mathcal{T}}$ in the same way as the map $F\colon \mathcal{K}_P \to \mathcal{A}_{\mathcal{M},\mathcal{T}}$ was constructed via the KS construction, and proceed with comparing the geometric properties of $\mathcal{A}^{\vee}_{\mathcal{M},\mathcal{T}}$ and $\mathcal{A}_{\mathcal{M},\mathcal{T}}$.

Moreover, when restricting ourselves to consider $\mathcal{K}_P$ of K3 surfaces polarised by a 2-elementary lattice, the existence of a Dolgachev mirror family $\mathcal{K}_{P^{\vee}}$ is equivalent to the existence of mirror symmetry [Y] between any member X in $\mathcal{K}_P$ and a log del Pezzo surface of index 2. This might provide insights of how to match up members of the Dolgachev mirror families, or a way to relate the maps F and $F^{\vee}$. We hope to develop this idea in another collaboration with A. Malmendier.

Part II
Kuga varieties of polarised abelian surfaces

8 Definitions

In this section, we will give all necessary definitions for our work in Part II.

8.1 Kuga varieties

We introduce our object of interest, a Kuga variety, in greater generality than how it was first introduced in [K].

To begin with, let us first give the definition of a universal family of abelian varieties. Let $\Lambda' \simeq \mathbb{Z}^{2g}$ be a real lattice with an alternating form E. Recall from Section 3.2 that $\mathcal{D}_{(2g,g)}$ is the period domain of $(\Lambda'_{\mathbb{R}}, E)$ of type $(2g, g)$, and that it is biholomorphic to $\mathcal{S}_g$, the Siegel upper half space of degree g. Moreover, each point in the period domain $\mathcal{D}_{(2g,g)}$ is a Hodge filtration $(F^\bullet)$ for $\Lambda'_{\mathbb{C}}$, which is equivalent to a complex structure J on $\Lambda'_{\mathbb{R}}$, or a $\mathbb{R}$-linear map μ from $\Lambda'_{\mathbb{R}}$ to the $+i$-eigenspace of J.

Definition 8.1.1. [Ma, Section 2.1]

*A **universal family of abelian varieties** is a fibred manifold $\mathfrak{X} \to \mathcal{D}_{(2g,g)}$, such that the fibre over the Hodge filtration $(F^\bullet) \in \mathcal{D}_{(2g,g)}$ is the abelian g-fold $A_{(F^\bullet)}$ given by the complex torus*

$$\overline{F^1} / \left(\overline{F^1} \cap \mu\left(\Lambda'\right) \right)$$

*with polarisation E. For any positive integer n, we denote by $\mathfrak{X}^{(n)}$ its n**-fold self fibre product***

$$\mathfrak{X} \times_{\mathcal{D}_{(2g,g)}} \cdots \times_{\mathcal{D}_{(2g,g)}} \mathfrak{X}.$$

In particular, the fibre of $\mathfrak{X}^{(n)} \to \mathcal{D}_{(2g,g)}$ over $(F^\bullet)$ is the n-fold product of the abelian variety $A_{(F^\bullet)}$.

Remark 8.1.2. *We follow Ma's convention to call $\mathfrak{X}$ a universal family instead of a tautological family, but no universality of $\mathfrak{X}$ is to be expected: the base $\mathcal{D}_{2g,g}$ is not a moduli space and the fibres are not distinct.*

Remark 8.1.3. *Consider the following families over $\mathcal{D}_{(2g,g)}$:*

$$\mathbf{W}^{(n)} = \left\{ \left(\overline{F^1}\right)^{\oplus n} \right\}_{(F^\bullet) \in \mathcal{D}_{(2g,g)}} \quad \text{and } \mathbf{\Lambda}^{(n)} = \left\{ \left(\mu(\Lambda')\right)^{\oplus n} \right\}_{(F^\bullet) \in \mathcal{D}_{(2g,g)}}.$$

Then we have $\mathfrak{X}^{(n)} \simeq \mathbf{W}^{(n)} / \mathbf{\Lambda}^{(n)}$.

Remark 8.1.4. *Let $\mathbb{E}$ be the vector subbundle $F^1\mathcal{V}$ of the Hodge bundle $\mathcal{V}$ with respect to its filtration $(F^\bullet)$ [Vo, Section 10.2.1]. It is dual to $\mathbf{W}^{(1)} \otimes \mathcal{O}_{\mathcal{A}}$ under the polarisation form, and both are called a Hodge subbundle. In some literature* e.g. *[Ma] and [vdG], the vector bundle $\mathbb{E}$ instead of $\mathcal{V}$ is called the Hodge bundle.*

Let $f \colon \mathfrak{X} \to \mathcal{A} \simeq \Gamma \backslash \mathcal{S}_g$ be a family of structured abelian g-folds, where the action of $\Gamma := \Gamma(f) < \mathrm{Sp}(2g)$ on $\mathcal{S}_g$ is given in Remark 3.2.24. We will construct an n-fold Kuga variety associated to the family f by defining an extension $\widetilde{\Gamma}^n$ of Γ and a left action of it on $\mathbf{W}^{(n)} \simeq \mathbb{C}^{ng} \times \mathcal{S}_g$ which descends to that of Γ on $\mathcal{S}_g$, with reference to [HKW2, Chapter I.1] and [Na, (2.7)].

First, by identifying $\mathbb{C}^{ng}$ with the set of $n \times g$ complex matrices, we can identify $\mathbb{C}^{ng} \times \mathcal{S}_g$ with a subset of $\mathrm{Gr}(g, \mathbb{C}^{n+2g})$ by sending an element (Z, τ) to a $\mathrm{GL}(g, \mathbb{C})$-equivalence class of block matrices:

$$(Z, \tau) \mapsto \begin{bmatrix} Z \\ \tau \\ \mathbf{1}_g \end{bmatrix}.$$

We define the **integral affine symplectic group** as the semi-direct product $\mathrm{M}_{n\times 2g}(\mathbb{Z}) \rtimes \Gamma$, brought to the form

$$\widetilde{\Gamma}^n = \left\{ (l, \gamma) = \begin{pmatrix} \mathbf{1}_n & l \\ 0 & \gamma \end{pmatrix} \in \mathrm{M}_{n+2g}(\mathbb{Z}) : \gamma \in \Gamma,\ l \in M_{n\times 2g}(\mathbb{Z}) \right\}.$$

The group $\widetilde{\Gamma}^n$ acts on $\mathbb{C}^{ng} \times \mathcal{S}_g$ by left multiplication on the $\mathrm{GL}(g, \mathbb{C})$-equivalence classes of block matrices. Explicitly, if $\widetilde{\gamma} = (l, \gamma) \in \widetilde{\Gamma}^n$ and $\widetilde{\tau} = (Z, \tau) \in \mathbb{C}^{ng} \times \mathcal{S}_g$, then

$$\widetilde{\gamma} \cdot \widetilde{\tau} = \left[\begin{pmatrix} \mathbf{1}_n & l \\ 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} Z \\ \tau \\ \mathbf{1}_g \end{pmatrix} \right] = \begin{bmatrix} Z + l \cdot \tau \\ \gamma \cdot \begin{pmatrix} \tau \\ \mathbf{1}_g \end{pmatrix} \end{bmatrix} = \begin{bmatrix} (Z + l \cdot \tau) \cdot N \\ \gamma \cdot \tau \\ \mathbf{1}_g \end{bmatrix} \tag{7}$$

for some $N \in \mathrm{GL}(g, \mathbb{C})$.

Finally, we define an n-fold Kuga variety.

Definition 8.1.5. [Ma, Section 2.2]

*An n-**fold Kuga variety** associated to a family of structured abelian g-folds f, is the quotient*

$$\mathfrak{X}_\Gamma^{(n)} := \widetilde{\Gamma}^n \backslash \mathfrak{X}^{(n)}.$$

The projection $\mathbb{C}^{ng} \times \mathcal{S}_g \to \mathcal{S}_g$ induces a map $\pi\colon \mathfrak{X}_\Gamma^{(n)} \to \mathcal{A}$. We are interested in the particular kind of n-fold Kuga varieties where the base $\mathcal{A}$ is a modular variety $\mathcal{A}_p$ of abelian surfaces of polarisation type $(1, p)$ for prime $p \geq 3$ with a choice of canonical level structure (see Remark 3.2.8). We denote this n-fold Kuga variety by $\mathfrak{X}_p^n$. The modular group Γ_p associated to $\mathcal{A}_p$ given in Theorem 3.2.26 can be written explicitly [HKW2, Proposition 1.20] as

$$\Gamma_p = \left\{ \gamma \in \mathrm{Sp}(4, \mathbb{Z}) : \gamma - \mathbf{1}_4 \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & p\mathbb{Z} & p^2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \end{pmatrix} \right\}.$$

We denote the associated integral affine symplectic group by $\widetilde{\Gamma}_p^n$.

Remark 8.1.6. *According to Equation (7), the fibre in $\mathfrak{X}_p^n$ over τ is $\mathbb{C}^{ng}$ modulo the lattice given by $l \cdot \tau$. When $p \neq 2$, this fibre is isomorphic to the product of n copies of the torus $\mathbb{C}^g/(\tau, D)\mathbb{Z}^{2g}$ (see [HKW2, Proof of Proposition 2.16]). The (1-fold) Kuga variety $\mathfrak{X}_p^1$ for prime $p \geq 3$ is more commonly known as the universal family of $(1, p)$-polarised abelian surfaces: any other family of $(1, p)$-polarised abelian varieties is a pullback of $\mathfrak{X}_p^1$ up to a base change.*

In the special situation when $p = 2$, so that $-1 \in \Gamma_p$, the fibre over τ is isomorphic to the n^{th} power of the associated Kummer surface.

Remark 8.1.7. *We may also construct the modular variety of $(1, p)$-polarised abelian surfaces without any choice of canonical level structure of the abelian surfaces, as the quotient $\Gamma[p] \backslash \mathcal{S}_2$, where [HKW2, Proposition 1.20]*

$$\Gamma[p] = \left\{ \gamma \in \mathrm{Sp}(4, \mathbb{Q}) : \gamma \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} & p^2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \frac{1}{p}\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}$$

is called the ***paramodular group****. Similarly, we can construct the associated integral affine symplectic group* $\widetilde{\Gamma[p]}^n$ *and the n-fold Kuga variety.*

Remark 8.1.8. *Kuga varieties can be defined even more generally. For example in [A], an n-fold Kuga variety is a pullback of* $\mathfrak{X}_\Gamma^{(n)} \to \mathcal{A}$*, an n-fold Kuga variety in the sense of Definition 8.1.5, along an embedding* $\mathcal{M} \to \mathcal{A}$ *of locally symmetric varieties (see Section 2.3) up to base change.*

Let us see an example: recall that the type II_4 *LSV* $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ *constructed in Section 3.3.3 is a modular variety of abelian 8-folds with polarisation type D and totally definite quaternion multiplication. Upon base change by a cover of high enough degree for* $\mathcal{A}_{\mathcal{M},\mathcal{T}}$*, there exists an embedding of* $\mathcal{A}_{\mathcal{M},\mathcal{T}}$ *into the modular variety* $\mathcal{A}_D$ *of abelian 8-folds with polarisation type D. Then the pullback of the universal family of D-polarised abelian 8-folds along this embedding is a Kuga variety over* $\mathcal{A}_{\mathcal{M},\mathcal{T}}$*.*

8.2 Kodaira dimension

In this section, we will give an introduction to Kodaira dimension of a complex variety, and state S. Ma's theorem (Theorem 8.2.10) which bounds the Kodaira dimension of a n-fold Kuga variety.

Let X be a normal compact complex variety (irreducible but not necessarily smooth). Recall that a line bundle is an invertible $\mathcal{O}_X$-module. Similar to the definitions given in Section 3.5.1, we say that an $\mathcal{O}_X$-module L is a **$\mathbb{Q}$-line bundle** if there exists an integer $m > 0$ such that $L^{\otimes m}$ is a line bundle. And again, a basis of $\Gamma(X, L)$ of a $\mathbb{Q}$-line bundle L gives a rational map $\varphi_L \colon X \dashrightarrow \mathbb{P}^r_{\mathbb{C}}$ defined away from the base locus of L.

Definition 8.2.1. [U, Definition 5.1]
Let L be a $\mathbb{Q}$*-line bundle on a normal complex variety X. Define*

$$\mathbb{N}(L) := \left\{ m > 0 : h^0(X, L^{\otimes m}) \geq 1 \right\}.$$

*Then the L-**dimension** of L is defined to be*

$$\kappa(X, L) := \begin{cases} -\infty & \text{if } \mathbb{N}(L) = \varnothing, \\ \max_{m \in \mathbb{N}(L)} \left\{ \dim \varphi_{(L^{\otimes m})}(X) \right\} & \text{otherwise.} \end{cases}$$

Remark 8.2.2. *It is clear that the L-dimension of any* $\mathbb{Q}$*-line bundle L on X is at most* $\dim(X)$*. If L is a very ample line bundle, then* φ_L *is the embedding described in Theorem 3.1.3. So the L-dimension of an ample line bundle on X is* $\dim(X)$ *([U, Example 5.4]). In fact [HaKo, Section 2B], a* $\mathbb{Q}$*-line bundle L on X satisfies* $\kappa(L) = \dim(X)$ *if and only if it is big. On the other hand,* $\kappa(\mathcal{O}(-D)) = -\infty$ *if D is an effective divisor.*

Alternatively, the L-dimension can be considered as the rate of growth of $h^0(X, L^{\otimes m})$ with respect to m.

Lemma 8.2.3. [HaKo, Section 2C]
Let L be a $\mathbb{Q}$*-line bundle on a normal compact complex variety X. If* $\mathbb{N}(L) \neq \varnothing$*, then*

$$\kappa(X, L) = \max \left\{ k : \limsup \frac{h^0(X, L^{\otimes m})}{m^k} > 0 \right\}.$$

Similar to Section 3.5.1, we have a correspondence between $\mathbb{Q}$-line bundles and **$\mathbb{Q}$-Cartier divisors**. Therefore, we can easily define the D-dimension [U, Definition 5.1] of a $\mathbb{Q}$-Cartier divisor D replacing $L^{\otimes m}$ by mD and $h^0(X, L)$ by $h^0(X, \mathcal{O}(mD))$. We will use the notions of L-dimension for $\mathbb{Q}$-line bundles and D-dimension for $\mathbb{Q}$-Cartier divisors interchangeably.

Now let us require X to be smooth. Recall [GrifH, Chapter 1.1 Chern Classes of Line Bundles] the **canonical bundle** ω_X of a smooth complex variety X is the top exterior power of its cotangent bundle T^*_X, or equivalently the bundle of differential n-forms. Let K_X be the **canonical divisor**, defined up to linear equivalence, the associated divisor of ω_X.

Definition 8.2.4. *The **Kodaira dimension** $\kappa(X)$ of a smooth complex variety X is the D-dimension $\kappa(K_X) := \kappa(X, K_X)$ of its canonical divisor K_X.*

For a singular normal complex variety X with smooth locus X_0, one may also define [Reid, Section 1.5] its canonical sheaf to be

$$\omega_X = j_*\left(\omega_{X_0}\right),$$

which is the pushforward of the sheaf of regular $\dim(X)$-forms on X_0 by the inclusion $j\colon X_0 \hookrightarrow X$. Similarly, we define its r^{th}-tensor power to be $\omega_X^{\otimes r} = j_*\left(\omega_{X_0}^{\otimes r}\right)$ for any integer $r > 0$. If ω_X is a $\mathbb{Q}$-line bundle, then we define the canonical divisor K_X to be its corresponding $\mathbb{Q}$-Cartier divisor. With the following theorem which shows that the Kodaira dimension is a birational invariant, we may also define the Kodaira dimension of a singular complex variety to be the Kodaira dimension of any of its smooth birational models.

Theorem 8.2.5. [U, Lemma 6.3]
Let $f\colon X \to Y$ be a birational morphism of smooth complex varieties. Then there is a natural isomorphism of $\mathbb{C}$-vector spaces

$$f^*\colon H^0(Y, \mathcal{O}(mK_Y)) \longrightarrow H^0(X, \mathcal{O}(mK_X)).$$

Remark 8.2.6. *Again, $\kappa(X)$ of a complex variety X is at most* $\dim(X)$. *If $\kappa(X) = \dim(X)$, then we say X is **of general type** [HaKo, Section 2C]. On the other end of the spectrum lie varieties with Kodaira dimension $-\infty$, such as rational varieties. Indeed, the canonical bundle of $\mathbb{P}^n$ is $\mathcal{O}(-n-1)$ and any positive power of it has no global section [Hart, Example 8.20.1].*

Remark 8.2.7. *The number $h^0(X, \omega_X^{\otimes m})$ is called [I, Section 2] the m^{th}-**plurigenus** of X and, by Lemma 8.2.3, the Kodaira dimension is the measure of the rate of growth of the plurigenera with respect to m.*

The aim of this work in Part II is to calculate the Kodaira dimension $\kappa(\mathfrak{X}_p^n)$. Since $\mathfrak{X}_p^n \to \mathcal{A}_p$ has connected fibres, the following theorem, which is a generalisation of S. Iitaka's fundamental theorem of the pluricanonical fibrations, applies.

Theorem 8.2.8. [U, Theorem 6.12]
Let $f\colon X \to Y$ be a surjective morphism of complex varieties with connected fibres. Then there exists a dense open subset W of Y such that for all $w \in W$, we have

$$\kappa(X) \leq \kappa(X_w) + \dim(Y),$$

where X_w denotes the fibre $f^{-1}(w)$.

This implies that the Kodaira dimension of any (n-fold) Kuga variety over a modular variety Y is at most $\dim Y$: the general fibre is the n^{th} power of an abelian variety, and an abelian variety always has Kodaira dimension 0 because it has trivial canonical bundle.

Definition 8.2.9. *We say that an n-fold Kuga variety $\mathfrak{X}_\Gamma^{(n)} \to \mathcal{A}$ is of **relative general type** if $\kappa(\mathfrak{X}_\Gamma^{(n)}) = \dim(\mathcal{A})$.*

In the case of $\mathfrak{X}_p^n$, it is of relative general type when $\kappa(\mathfrak{X}_p^n)$ equals the dimension of $\mathcal{A}_p$, which is 3 (see Section 3.2.4).

Part II of the thesis is based on the work [Ma] of S. Ma where a connection between Siegel modular forms and differential forms on arbitrary Kuga varieties is established. In particular, he gives [Ma, Theorem 1.3] a lower bound of the Kodaira dimension of a Kuga variety, assuming the existence of a specific compactification for the Kuga variety, which is referred to as a Namikawa compactification (see Definition 8.3.9) in [PSMS]. In terms of $\mathfrak{X}_p^n$, this result translates to:

Theorem 8.2.10. *Let X be a Namikawa compactification of $\mathfrak{X}_p^n$. Then*

$$\kappa(\overline{\mathcal{A}_p}, (n+3)\mathcal{L} - \Delta_{\mathcal{A}}) \leq \kappa(K_X) \leq 3$$

where $\overline{\mathcal{A}_p}$ is a toroidal compactification of $\mathcal{A}_p$, $\mathcal{L}$ is the $\mathbb{Q}$-line bundle of weight 1 modular forms of Γ_p and $\Delta_{\mathcal{A}}$ is the boundary divisor of $\overline{\mathcal{A}_p}$.

Note that by [PSMS, Theorem 1.2], X is $\mathbb{Q}$-Gorenstein, so its canonical sheaf is a $\mathbb{Q}$-line bundle, and K_X is indeed a $\mathbb{Q}$-Cartier divisor. We will see that $\kappa(X) = \kappa(K_X)$ if every singularity on X is a canonical singularity (Lemma 8.4.2). Therefore, if we have a Namikawa compactification X of $\mathfrak{X}_p^n$ with canonical singularities, then Theorem 8.2.10 gives us a lower bound for $\kappa(\mathfrak{X}_p^n)$. Such a compactification is constructed in [PSMS]. Now, we would like to find out for which n and p the lower bound $\kappa(\overline{\mathcal{A}_p}, (n+3)\mathcal{L} - \Delta_{\mathcal{A}})$ is equal to 3, or equivalently when $\mathfrak{X}_p^n$ is of relative general type. Our main theorem is the following:

Theorem 8.2.11. *A Kuga variety $\mathfrak{X}_p^n$ is of relative general type if*

- *$p \geq 3$ and $n \geq 4$; or*
- *$p \geq 5$ and $n \geq 3$.*

Before moving on to the explicit computations, we will give the necessary definitions and known results about toroidal compactifications and Namikawa compactifications (Section 8.3); singularities on a Namikawa compactification, and when are they canonical (Section 8.4); and Siegel modular forms and cusp forms (Section 8.5).

8.3 Compactification of Kuga varieties

The Namikawa compactification mentioned in Theorem 8.2.10 can be constructed as a toroidal compactification [Na], which is a common method of compactification for LSVs and universal families over a LSV. For the purpose of Section 9, we will give a brief description of the cusps of an n-fold Kuga variety and the main steps involved in its toroidal compactification. We will specifically describe the Namikawa compactification X of $\mathfrak{X}_p^n$.

8.3.1 Boundary components and cusps

Let us first describe the cusps on an n-fold Kuga variety, which is the locus to be added in the compactification process. Consider the Siegel upper half space $\mathcal{S}_g$ of degree g.

Definition 8.3.1. [Na2, (4.1)], [HKW2, Definition 3.5]
The Siegel upper half space $\mathcal{S}_g$ is isomorphic to the bounded symmetric domain

$$\mathcal{D}_g = \left\{ Z \in \mathrm{Sym}(g, \mathbb{C}) : \mathbf{1}_g - Z\overline{Z} > 0 \right\},$$

whose closure is

$$\overline{\mathcal{D}_g} = \left\{ Z \in \mathrm{Sym}(g, \mathbb{C}) : \mathbf{1}_g - Z\overline{Z} \geq 0 \right\}.$$

Define an equivalence relation $\sim$ on $\overline{\mathcal{S}_g} := \overline{\mathcal{D}_g}$: for two points $p, q \in \overline{\mathcal{S}_g}$, we say $p \sim q$ if and only if they can be connected by finitely many holomorphic curves. The equivalence classes of points in $\overline{\mathcal{S}_g}$ with respect to $\sim$ are called ***boundary components on $\mathcal{S}_g$****. Boundary components in $\overline{\mathcal{S}_g} \backslash \mathcal{S}_g$ are called* ***proper****.*

We can [HKW2, Proposition 3.6] associate to a boundary component of $\mathcal{S}_g$ an isotropic subspace of $\mathbb{R}^{2g}$. A subspace $V \subset \mathbb{R}^{2g}$ is **isotropic** if for all $u, v \in V$, we have $uJ_g^t v = 0$ with J_g being the symplectic form of degree g.

Proposition 8.3.2. [HKW2, Proposition 3.12]

There is a one-to-one correspondence between the set of boundary components of $\mathcal{S}_g$ and the set of isotropic subspaces in $\mathbb{R}^{2g}$ with respect to J. Moreover, this correspondence is $\mathrm{Sp}(2g)$*-equivariant: it respects the action of $\gamma \in \mathrm{Sp}(2g)$ on $\mathcal{S}_g$ given in Remark 3.2.24 and the action of γ^{-1} on $\mathbb{R}^{2g}$ by left multiplication.*

Under this correspondence, we may translate the usual properties of real vector spaces into properties of boundary components.

- A boundary component F of $\mathcal{S}_g$ is said to be of **corank** $g'' \leq g$ if its corresponding isotropic subspace in $\mathbb{R}^{2g}$ has rank g''. The symplectic group $\mathrm{Sp}(2g)$ acts transitively on the set of boundary components of the same corank.
- A boundary component F is said to be **adjacent** to a second boundary component F', or $F' \succ F$, if $F \neq F'$ and $\overline{F'} \supset F$. If V' and V are the corresponding isotropic subspaces of F' and F in $\mathbb{R}^{2g}$, then $F' \succ F$ if and only if $V' \subsetneq V$.

Remark 8.3.3. *In fact, $\mathcal{S}_g$ is a boundary component of corank* 0 *of itself. Every proper boundary component of $\mathcal{S}_g$ is adjacent to $\mathcal{S}_g$.*

Let $\mathcal{A}$ be a LSV of type III, *i.e.* it is an arithmetic quotient of $\mathcal{S}_g$ by the arithmetic subgroup Γ. We are interested in the Γ-orbits of the rational boundary components of $\mathcal{S}_g$.

Definition 8.3.4. *[HKW2, Definition 3.17]*

A ***rational boundary component*** *F is a boundary component whose stabiliser subgroup*

$$\mathcal{P}(F) = \{\gamma \in \mathrm{Sp}(2g) : \gamma(F) = F\}$$

is defined over $\mathbb{Q}$. That is, there exists an algebraic subgroup $\mathcal{P}_{\mathbb{Q}}(F) \subset \mathrm{Sp}(2g, \mathbb{Q})$ such that $\mathcal{P}(F) = (\mathcal{P}_{\mathbb{Q}}(F))(\mathbb{R})$, the $\mathbb{R}$-valued points of the algebraic group $\mathcal{P}_{\mathbb{Q}}(F)$.

The modular group Γ sends a rational boundary component to a rational boundary component. If $V \subset \mathbb{R}^{2g}$ is the isotropic subspace that corresponds to a rational boundary component F, then the integral points $\mathbb{X} := V(\mathbb{Z})$ form a lattice in V, satisfying $V = \mathbb{X} \otimes_{\mathbb{Z}} \mathbb{R}$. We call $\mathbb{X}$ the **isotropic lattice** associated to F. Moreover, the action of the integral group $\mathcal{P}(F) \cap \Gamma$ on $\mathbb{R}^{2g}$ preserves $\mathbb{X}$.

We are now ready to define a cusp in $\mathcal{A}$.

Definition 8.3.5. *Let F be a proper rational boundary component of $\mathcal{S}_g$. Then the* ***cusp*** *in $\mathcal{A} = \Gamma \backslash \mathcal{S}_g$ associated to F is given by the quotient $(\Gamma \cap \mathcal{P}(F)) \backslash F$.*

Before moving on to defining the cusps in Kuga varieties, let us give a geometrical description of the cusps of $\mathcal{A}_p$ with $p \geq 3$ prime [HKW2, Section 3B]. The closure of each Γ_p-orbit of corank 1 boundary components is a modular curve. There are a total of $(1 + (p^2 - 1)/2)$ such modular curves in $\mathcal{A}_p$, and their configuration is given in Diagram 5.

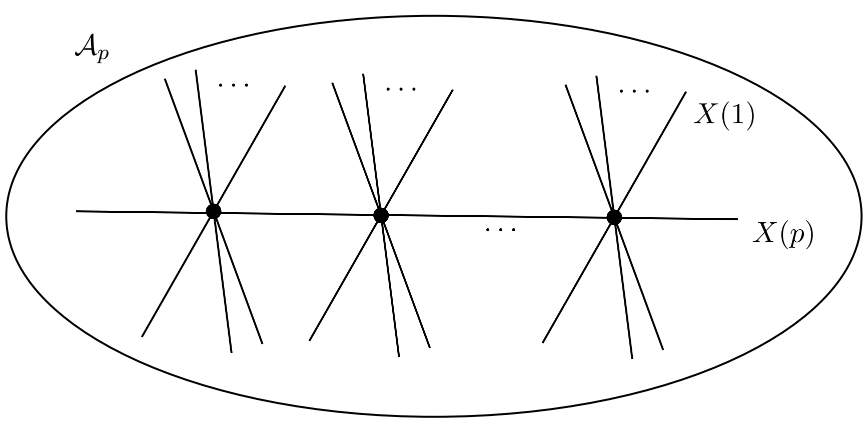


Diagram 5: Configuration of the cusps in $\mathcal{A}_p$.

There are $p+1$ nodes in the diagram: each corresponds to a Γ_p-orbit of corank 2 boundary components. Every node is the intersection of $(1+(p-1)/2)$ modular curves, which indicates adjacency of the corresponding boundary components. The action of the group $\mathrm{Sp}(2,\mathbb{F}_p) \simeq \Gamma[p]/\Gamma_p$ permutes those modular curves with only one node on them: they are all isomorphic to $X(1)$, the modular curve of elliptic curves. The remaining curve $X(p)$ with $p+1$ nodes is the modular curve of elliptic curves of level p. In particular,

$$X(p) = \Gamma(p)\backslash\mathcal{S}_1,$$

where $\mathcal{S}_1$ is the usual upper half space, and

$$\Gamma(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : a, d \equiv 1 \bmod p;\ b, c \equiv 0 \bmod p \right\}$$

is the **principal congruence subgroup**. A cusp isomorphic to $X(1)$ is called a **peripheral** boundary component of $\mathcal{A}_p$. The cusp $X(p)$ is called the **central** boundary component of $\mathcal{A}_p$.

We give similar definitions of cusps for a Kuga variety. Let $\widetilde{\Gamma}^n$ be the integral affine symplectic group extension of Γ. Consider the n-fold Kuga variety

$$\mathfrak{X}_\Gamma^{(n)} = \widetilde{\Gamma}^n \backslash \left(\mathbb{C}^{ng} \times \mathcal{S}_g\right).$$

A **rational boundary component of** $\mathbb{C}^{ng} \times \mathcal{S}_g$ is in the form $\widetilde{F} \simeq \mathbb{C}^{ng} \times F$, where F is a rational boundary component of $\mathcal{S}_g$. The **corank** of $\widetilde{F}$ is defined to be the corank of F. The **cusp** of $\mathfrak{X}_\Gamma^{(n)}$ associated to $\widetilde{F}$ is given by the quotient of $\widetilde{F}$ by its stabiliser subgroup in $\widetilde{\Gamma}$. Since we can separately describe the action of $\widetilde{\Gamma}$ on the two components $\mathcal{S}_g$ and on $\mathbb{C}^{ng}$ (see Equation (7)), a cusp of $\mathfrak{X}_\Gamma^{(n)}$ projects down to a cusp of $\mathcal{A}$.

8.3.2 Toroidal compactification

A toroidal compactification for a type III LSV is a modification of a natural compactification called **Satake compactification**. The intuitive idea behind Satake compactification is to add in the cusps of the LSV. This compactification is unique, but it has some undesirable properties such as being highly singular and having boundary components with codimension strictly greater than 1. A toroidal compactification of a type III LSV is obtained by blowing up the Satake compactification in the boundary. It is normal, has purely 1-codimensional boundaries and it can be chosen to have at worst finite quotient singularities, *i.e.* singularities that arise from fixed points of the action of a finite group on an affine space (to be discussed in Section 8.4). As we will explain below, one can also extend the method of toroidal compactification to Kuga varieties.

We start by describing a partial toroidal compactification for a cusp $\widetilde{F} = \mathbb{C}^{ng} \times F$ on $\mathfrak{X}_\Gamma^{(n)}$ (or $\mathcal{A}$, substituting $n=0$). Suppose the cusp $\widetilde{F}$ has corank $g'' \leq g$. We think of F as given by its corresponding isotropic lattice $\mathbb{X} \simeq \mathbb{Z}^{g''}$ in $\mathbb{R}^{2g}$. Let $\widetilde{\mathcal{P}}(\widetilde{F})$ be the stabiliser subgroup of $\widetilde{F}$ in $\mathbb{R}^{2ng} \rtimes \mathrm{Sp}(2g)$, which can be embedded in $\mathrm{GL}_{n+2g}(\mathbb{R})$. Then we choose a small neighbourhood $N(F)$ of the cusp F in $\mathcal{S}_g$ satisfying

$$\left(P \cdot \widetilde{N}(\widetilde{F})\right) \cap \widetilde{N}(\widetilde{F}) \neq \varnothing$$

for all $P \in \widetilde{\mathcal{P}}(\widetilde{F})$ where $\widetilde{N}(\widetilde{F}) = \mathbb{C}^{ng} \times N(F)$. We also define the following subgroups of $\widetilde{\mathcal{P}}(\widetilde{F})$ (see [Na, (2.5)] and [Na2, (7.1)] for more details):

$$\begin{aligned} \widetilde{\mathcal{P}}'(\widetilde{F}) &:= \text{ the centre of the unipotent radical of } \widetilde{\mathcal{P}} \\ \widetilde{\Upsilon}^n &:= \widetilde{\mathcal{P}}'(\widetilde{F}) \cap \widetilde{\Gamma}^n \\ \widetilde{P''}(\widetilde{F}) &:= \left(\widetilde{\mathcal{P}}(\widetilde{F}) \cap \widetilde{\Gamma}^n\right)/\widetilde{\Upsilon}^n \end{aligned}$$

Note that the groups $\widetilde{\Upsilon}^n$ and $\widetilde{P''}(\widetilde{F})$ both inherit the action of $\widetilde{\Gamma}^n$ on $\mathbb{C}^{ng} \times \mathcal{S}_g$. The reason to introduce these subgroups is that we can separate out a factor of $\widetilde{N}(\widetilde{F})$ where the action of $\widetilde{\mathcal{P}}(\widetilde{F}) \cap \widetilde{\Gamma}^n$ is given by that of $\widetilde{\Upsilon}^n$.

Definition 8.3.6. [T, Section 5]
Let $g' := g - g''$. Identify $\mathbb{C}^{ng} \times \mathcal{S}_g$ with

$$\left\{ \left(\begin{bmatrix} M & N \end{bmatrix}, \begin{bmatrix} \tau' & \omega \\ \omega^t & \tau'' \end{bmatrix} \right) : \begin{matrix} M \in \mathbb{C}^{ng'}, \ N \in \mathbb{C}^{ng''}, \\ \tau' \in \mathcal{S}_{g'}, \ \tau'' \in \mathcal{S}_{g''}, \\ \omega \in M_{g' \times g''}(\mathbb{C}) \end{matrix} \right\}.$$

*Then the **Siegel domain realisation** for $\mathbb{C}^{ng} \times \mathcal{S}_g$ associated to $\widetilde{F}$ is the embedding*

$$\begin{aligned} \mathbb{C}^{ng} \times \mathcal{S}_g &\hookrightarrow (\mathbb{C}^{ng'} \times \mathbb{C}^{ng''}) \times (\mathcal{S}_{g'} \times M_{g' \times g''}(\mathbb{C}) \times M^{\mathrm{sym}}_{g'' \times g''}(\mathbb{C})) \\ \left(\begin{bmatrix} M & N \end{bmatrix}, \begin{bmatrix} \tau' & \omega \\ \omega^t & \tau'' \end{bmatrix} \right) &\mapsto \left((M, N), (\tau', \omega, \tau'') \right) \end{aligned}$$

This has the property that the action of $\widetilde{\mathcal{P}}(\widetilde{F})$ on $\mathbb{C}^{ng} \times \mathcal{S}_g$ preserves the factors of the Siegel domain realisation.

Remark 8.3.7. *When $n = 0$, the embedding of $\mathcal{S}_g$ into $\mathcal{S}_{g'} \times M_{g' \times g''}(\mathbb{C}) \times M^{\mathrm{sym}}_{g'' \times g''}(\mathbb{C})$ described in Definition 8.3.6 is called a **tube domain realisation** associated to a corank g'' cusp. The image in the factor $M^{\mathrm{sym}}_{g'' \times g''}(\mathbb{C})$ is the cone of positive semi-definite forms on $\mathbb{C}^{g''}$. Its product with the vector space $\mathcal{S}_{g'} \times M_{g' \times g''}(\mathbb{C})$ realises the image of the embedding as a tube domain.*

The action of the group $\widetilde{\mathcal{P}}(\widetilde{F}) \cap \widetilde{\Gamma}^n$ on $\mathbb{C}^{ng} \times \mathcal{S}_g$ can also be described independently on each of the factors in the Siegel domain realisation. In particular, we will describe the action of its subgroup $\widetilde{\Upsilon}^n$ for the factor $\mathbb{C}^{ng''} \times M^{\mathrm{sym}}_{g'' \times g''}(\mathbb{C})$. This factor can be identified to $(\mathbb{X}^\vee_{\mathbb{C}})^n \times \mathrm{Sym}^2_{\mathbb{C}}(\mathbb{X}^\vee)$, which is the set of n-tuple of $\mathbb{C}$-linear forms and a bilinear symmetric form over $\mathbb{X}_{\mathbb{C}}$. This is a vector space in the boundary $\mathbb{C}^{ng''} \times \left(\overline{\mathcal{D}_{g''}} \setminus \mathcal{D}_{g''} \right)$ of the domain $\mathbb{C}^{ng''} \times \mathcal{D}_{g''}$.

Proposition 8.3.8. [Na2, Section 3], [PSMS, Section 1]
The group $\widetilde{\Upsilon}^n$ is isomorphic to $(\mathbb{X}^\vee)^n \times \mathrm{Sym}^2(\mathbb{X}^\vee)$, and it acts on $\mathbb{C}^{ng} \times \mathcal{S}_g$ by real translation in the imaginary direction of the factor $\mathbb{C}^{ng''} \times M^{\mathrm{sym}}_{g'' \times g''}(\mathbb{C})$ of the Siegel domain realisation.

We consider the partial quotient of $\widetilde{N}(\widetilde{F})$ by $\widetilde{\Upsilon}^n$. Then by Proposition 8.3.8, we can embed the partial quotient into

$$T(\widetilde{F}) = \mathbb{C}^{ng'} \times (\mathbb{C}^*)^{ng''} \times \mathcal{S}_{g'} \times \mathbb{C}^{g' \times g''} \times (\mathbb{C}^*)^{g'' \times g''}_{\mathrm{sym}}. \tag{8}$$

This is a torus bundle *i.e.* a fibre space over the smooth complex manifold $\mathbb{C}^{ng'} \times \mathcal{S}_{g'} \times \mathbb{C}^{g' \times g''}$ where each fibre is isomorphic to the torus $(\mathbb{C}^*)^{g'g''+g''(g''+1)/2}$. The image of the partial quotient in $T(\widetilde{F})$ is highly singular. By choosing a fan $\Sigma(\widetilde{F})$ (see [HKW2, Section 2A]) for $(\mathbb{X}^\vee_{\mathbb{R}})^n \times \mathrm{Sym}^2_{\mathbb{R}}(\mathbb{X}^\vee)$, we may extend the torus part in $T(\widetilde{F})$ to a smooth torus embedding $\mathrm{Temb}(\Sigma(\widetilde{F}))$. In particular, we obtain torus bundle

$$\widetilde{X}(\widetilde{F}) := \mathcal{S}_{g'} \times \mathbb{C}^{g'g''} \times \mathbb{C}^{ng'} \times \mathrm{Temb}(\Sigma(\widetilde{F})).$$

Moreover if the cone decomposition $\Sigma(\widetilde{F})$ chosen is **admissible** (see [Na2, Definition 7.3(i)]), then the action of $\widetilde{P''}(\widetilde{F})$ on the torus part of $T(\widetilde{F})$ can be extended to an action on $\mathrm{Temb}(\Sigma(\widetilde{F}))$. By taking the interior of the closure of $\widetilde{\Upsilon}^n \backslash \widetilde{N}(\widetilde{F})$ in $\widetilde{X}(\widetilde{F})$ and quotienting it by $\widetilde{P''}$, we arrive at the **partial toroidal compactification** at the cusp $\widetilde{F}$. If in addition $\Sigma(\widetilde{F})$ is **simplicial** (see [HKW2, Definition 3.61(c)]), the partial toroidal compactification has finite quotient singularities which arise from fixed points by the action of $\widetilde{P''}(\widetilde{F})$.

Finally, if the fans chosen for the partial compactification at each cusp are compatible, *i.e.* they form an **admissible family** (see [Na2, Definition 7.3(ii)]), then we can obtain a **toroidal compactification** of the n-fold Kuga variety $\mathfrak{X}_\Gamma^{(n)}$ by gluing the partial toroidal compactification for each cusp $\widetilde{F}$ to $\mathfrak{X}_\Gamma^{(n)}$ along $(\widetilde{\mathcal{P}}(\widetilde{F}) \cap \widetilde{\Gamma}^n) \backslash \widetilde{N}$.

8.3.3 Namikawa compactification

An explicit construction of a Namikawa compactification X (of a special kind) for an n-fold Kuga variety is the main purpose of Section 1 in [PSMS]. Here we briefly introduce the definition and the idea of construction of a Namikawa compactification X of $\mathfrak{X}_p^n$ over $\mathcal{A}_p$.

Definition 8.3.9. *A **Namikawa compactification** of $\mathfrak{X}_p^n$ is an irreducible normal projective variety X containing $\mathfrak{X}_p^n$ as an open subset, together with a projective toroidal compactification $\overline{\mathcal{A}_p}$ of $\mathcal{A}_p$ for which the following conditions hold.*

1. *$\pi\colon \mathfrak{X}_p^n \to \mathcal{A}_p$ extends to a projective morphism $\overline{\pi}\colon X \to \overline{\mathcal{A}_p}$;*
2. *every irreducible component of $\Delta_X := X \smallsetminus \mathfrak{X}_p^n$ dominates an irreducible component of $\Delta_{\mathcal{A}} := \overline{\mathcal{A}_p} \smallsetminus \mathcal{A}_p$.*

Therefore X sits inside the commutative diagram

$$\begin{array}{ccc} \mathfrak{X}_p^n & \hookrightarrow & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} \\ \mathcal{A}_p & \hookrightarrow & \overline{\mathcal{A}_p} \end{array}$$

and $\overline{\pi}$ does not contract any divisor.

Namikawa compactifications are constructed by toroidal methods in [Na]. For the Kuga variety $\mathfrak{X}_p^n$, every rational boundary component $\widetilde{F}$ is of corank $g'' \leq g = 2$, which corresponds to a rank g'' isotropic lattice in $\mathbb{R}^2$. We may construct a partial toroidal compactification at a cusp $\widetilde{F}$ which leads to a Namikawa compactification X, by choosing a suitable cone decomposition $\Sigma(\widetilde{F})$ in $(\mathbb{X}_\mathbb{R}^\vee)^n \times \mathrm{Sym}_\mathbb{R}^2(\mathbb{X}^\vee)$. In [PSMS], the cone decomposition is chosen to be the perfect cone decomposition [HKW2, Remark 3.127]. Briefly, when $n = 0$, *i.e.* $\mathfrak{X}_p^n = \mathcal{A}_p$, the perfect cone decomposition of the cone of positive semi-definite symmetric bilinear forms with rational radical in $\mathrm{Sym}_\mathbb{R}^2(\mathbb{X}^\vee)$ is given by the convex hull of the rank one 1-forms on the faces. When $n > 0$, in [PSMS] we extend the perfect cone decomposition to the Siegel domain realisation. Moreover, $\Sigma(\widetilde{F})$ satisfies the conditions as listed in [PSMS, Proposition 1.4], which ensures that the local uniformising space $\widetilde{X}(\widetilde{F})$ of X has canonical singularities (Definition 8.4.1). We complete the construction of a Namikawa compactification X for $\mathfrak{X}_p^n$ by gluing up the the partial toroidal compactifications coming from the special choice of cone decomposition for each cusp $\widetilde{F}$ as described in Section 8.3.2.

We will prove in the Section 9 that for any p and $n > 2$, this Namikawa compactification X of $\mathfrak{X}_p^n$ has canonical singularities.

8.4 Canonical singularities and the RST criterion

In this section, we will discuss canonical singularities on normal complex varieties, and the Reid–Shepherd-Barron–Tai (RST) criterion for checking if a finite quotient singularity is canonical. The main reference in this section is [Reid].

First, we will give the definition of a normal variety with canonical singularities.

Definition 8.4.1. [Reid, Definitions 1.1]

A normal variety X has ***canonical singularities*** *if it satisfies the following two conditions:*

(i) the canonical divisor K_X is $\mathbb{Q}$-Cartier (or $\mathbb{Q}$-Gorenstein), i.e. *the Weil divisor rK_X is Cartier for some integer $r \geq 1$.*

(ii) if $f\colon Y \to X$ is a resolution of X and $\{E_i\}$ the set of all exceptional prime divisors of f, then

$$rK_Y = f^*(rK_X) + \sum a_i E_i$$

with $a_i \geq 0$.

Part (i) in Definition 8.4.3 says [Reid, Section 1.7] that $\mathcal{O}(rK_X) = \omega_X^{\otimes r}$ extends to a line bundle over X. Part (ii) says [Reid, Section 1.9] that any regular canonical r-form in $\omega_X^{\otimes r}$ is still regular when considered as a rational canonical r-form on the resolution Y of X: it has no poles along the exceptional divisors of f. Therefore, the r^{th}-plurigenera of X and Y agree. In particular, we have the following lemma.

Lemma 8.4.2. *If X is a normal variety with canonical singularities, and $f\colon Y \to X$ is a resolution, then*

$$\kappa(K_X) = \kappa(K_Y).$$

By choosing a simplicial cone decomposition, we may construct a toroidal compactification of a universal family over a type III LSV that locally looks like a quotient of an affine space by the action of a finite group everywhere: away from the cusps, it agrees with the universal family itself; and near the cusp, it is given by the partial toroidal compactification. Singularities on such a quotient arise from fixed points of the action of the finite group: these are called **finite quotient singularities**. The **Reid–Shepherd-Barron–Tai (RST) criterion** is a simple tool for checking if a finite quotient singularity is canonical.

We will need the following set-up to state the RST criterion [Reid, Section 4]. Suppose G is a finite group acting on the complex vector space $\mathbb{C}^m$ linearly. For a non-trivial element $\gamma \in G$ of order k, the eigenvalues of its action on $\mathbb{C}^m$ can be expressed as an m-tuple $(\xi^{\alpha_1}, \dots, \xi^{\alpha_m})$, with ξ being a primitive k-th root of unity and α_j being a non-negative integer less than k for any j. We define, with dependence on the choice of ξ, the **type** of γ to be

$$\frac{1}{k}(\alpha_1, \dots, \alpha_m)$$

and its associated **RST sum** to be

$$\mathrm{RST}(\gamma) := \sum_{i=1}^{m} \frac{\alpha_i}{k}.$$

Furthermore, we say that γ is a **quasi-reflection** if all but one α_j are 0, or equivalently γ preserves a divisor.

The RST criterion is then given by the following:

Theorem 8.4.3. [Reid, 4.11]

Let G be a finite group which acts on $\mathbb{C}^m$ as above. Then $\mathbb{C}^m/G$ has a canonical singularity if G contains no quasi-reflection and every non-trivial element $\gamma \in G$ satisfies the inequality

$$\mathrm{RST}(\gamma) \geq 1.$$

Note that, since we need to check the above inequality involving the RST sum for every element in G, it does not matter which root of unity ξ was chosen to give the type of a generator γ of G.

Remark 8.4.4. *One may visualise the inequality in Theorem 8.4.3. For a non-trivial element $\gamma \in G$ with type $1/k(\alpha_1, \cdots, \alpha_m)$, its RST sum is at least 1 if and only if the point $(\alpha_1/k, \cdots, \alpha_m/k) \in [0,1]^m$ lies on or above the hyperplane $\sum y_i = 1$. For example when $m = 3$, the point in $[0,1]^3$ corresponding to $\gamma \in G$ that satisfies the inquality, has to lie in the shaded area in Diagram 6.*

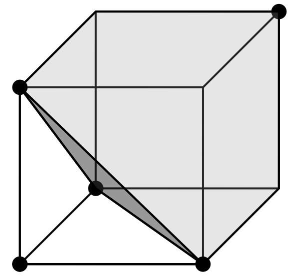


Diagram 6: Pictorial description of the RST inequality when $m = 3$.

Notice that as m increases, the volume below the hyperplane $\sum y_i = 1$ inside the hypercube decreases. Thus finite quotient singularities are more likely to be canonical when the dimension m of the variety increases.

8.5 Modular forms

In this section, we give the minimal definitions and explanations of some terminologies related to modular forms for use in Section 10.

8.5.1 Siegel modular forms

We begin by introducing Siegel modular forms associated to an arithmetic subgroup Γ of $\mathrm{Sp}(2g, \mathbb{Z})$.

We first give an analytic definition of Siegel modular forms. Consider the modular variety $\mathcal{A} := \Gamma \backslash \mathcal{S}_g$ of structured abelian g-folds with modular group Γ. Let k be a positive integer. A weight k Siegel modular form of Γ is a holomorphic function on $\mathcal{S}_g$ that satisfies certain rules with respect to the Γ action.

Definition 8.5.1. [F, Definition I.3.1]

A holomorphic function $f\colon \mathcal{S}_g \to \mathbb{C}$ for $g > 1$ is a ***Siegel modular form*** *of a finite index subgroup $\Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ of weight k if it satisfies the* ***automorphy condition****:*

$$f(\gamma(\tau)) = \det(C\tau + D)^k f(\tau) \text{ for all } \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma.$$

In the case of $g = 1$, f is a Siegel modular form if it satisfies both the automorphy condition and the ***growth condition****: for all projective rational matrices $\gamma \in \mathrm{Sp}(2, \mathbb{R})$,* i.e. *$r\gamma \in \mathrm{Sp}(2, \mathbb{Q})$ for some $r \in \mathbb{R}^*$, we have*

$$(c\tau + d)^{-k} f(\gamma(\tau)) \text{ is bounded as } \mathrm{Im}(\tau) \to \infty \text{ if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2, \mathbb{R}).$$

We denote by $M_k(\Gamma)$ the vector space of weight k Siegel modular forms of Γ.

The automorphy condition is simply a transformation rule for f with respect to the action of Γ on $\mathcal{S}_g$. The group of projective rational matrices in $\mathrm{Sp}(2, \mathbb{R})$ acts transitively on the set of rational boundary components in $\mathcal{S}_g$, so the growth condition is checking whether f exhibits controlled behaviour near the cusps in $\mathcal{A}$, that is, whether f is holomorphic at the cusps.

Remark 8.5.2. *The automorphy condition implies that f is holomorphic near the cusps when $g > 1$ by the Koecher principle [vdG, Theorem 2]. Also due to the automorphy condition, it is in fact enough to check the growth condition for one γ in each coset of Γ in the group of projective rational matrices in* $\mathrm{Sp}(2, \mathbb{R})$ *for the case $g = 1$.*

Remark 8.5.3. *When $g = 2$, we have* $\mathrm{Sp}(2, \mathbb{Z}) \simeq \mathrm{SL}(2, \mathbb{Z})$*, and a modular form in* $M_k(\mathrm{SL}(2, \mathbb{Z}))$ *is called a weight k elliptic modular form.*

Remark 8.5.4. *The vector spaces $M_k(\Gamma)$ for $k \geq 0$ form a graded ring $\bigoplus_k M_k(\Gamma)$ under multiplication: if $f_1 \in M_{k_1}(\Gamma)$ and $f_2 \in M_{k_2}(\Gamma)$ for some $k_1, k_2 \geq 0$, then $f_1 f_2 \in M_{k_1+k_2}(\Gamma)$.*

From now on, we will just write modular forms for Siegel modular forms. Modular forms for Γ are closely related to differential forms of the Siegel modular variety $\mathcal{A}$. Let $\mathcal{A}^0$ be the unramified part of $\mathcal{A}$. Recall that a canonical form on $\mathcal{A}^0$ is a section on $H^0(K_{\mathcal{A}^0})$.

Proposition 8.5.5. [vdG, Section 11]

Let $\{\tau_{ij}\}$ for $1 \leq i, j, \leq g$ be a set of coordinates for $\mathcal{S}_g \subset M_g(\mathbb{C})$. Consider the volume form $\omega = \bigwedge_{1 \leq i \leq j \leq g} d\tau_{ij}$ on $\mathcal{S}_g$. If f is a weight $k(g+1)$ modular form of Γ, then $f\omega^{\otimes k} \in H^0(kK_{\mathcal{S}_g})$ is a Γ-invariant canonical k-form on $\mathcal{S}_g$, which descends to a canonical k-form on $\mathcal{A}^0$. This describes an isomorphism of graded rings

$$\bigoplus_k M_{k(g+1)}(\Gamma) \longrightarrow \bigoplus_k H^0(kK_{\mathcal{A}^0}),$$

where multiplication in $\bigoplus_k H^0(kK_{\mathcal{A}^0})$ is the tensor product.

The above isomorphism allows us to give an alternative definition of a modular form from the geometrical perspective. Recall that the Hodge subbundle $\mathbb{E}$ defined in Remark 8.1.4 is isomorphic to $H^0(\mathcal{A}, \Omega_{\mathcal{A}})$.

Definition 8.5.6. *The $\mathbb{Q}$-line bundle $\mathcal{L} := \omega_{\mathcal{A}}/(g+1) = \det(\mathbb{E})$ is called the **line bundle of weight** 1 **modular forms of** Γ.*

In particular, a weight k modular form of Γ is a section of $k\mathcal{L}$ regular on $\mathcal{A}^0$.

Remark 8.5.7. [vdG, Theorem 7]

To end this subsection, let us discuss more about the geometry of $\mathcal{A}$ in relation to this line bundle $\mathcal{L}$. Recall the Baily-Borel Theorem says that $\mathcal{A}$ is a quasi-projective variety. In fact, the embedding of $\mathcal{A}$ into the projective space comes from the ample $\mathbb{Q}$-line bundle $\mathcal{L}$. Moreover, the Satake compactification of $\mathcal{A}$ is the closure of the image of $\mathcal{A}$ under this embedding, and is given by $\mathrm{Proj}\left(\bigoplus_k M_k(\Gamma)\right)$.

8.5.2 Siegel cusp forms

Let us move on to define a Siegel cusp form. One considers the Siegel operator Φ: roughly, $\Phi(f)(\tau)$ for $f \in M_k(\Gamma)$ and $\tau \in \mathcal{S}_g$ is the limit of $f(\tau)$ as τ approaches some standard corank 1 boundary component. Then a weight k **Siegel cusp form** of Γ is defined to be a modular form $f \in M_k(\Gamma)$ with $\Phi(f)(N(\cdot)) = 0$ on $\mathcal{S}_g$ for every projective rational matrix $N \in \mathrm{Sp}(2g, \mathbb{R})$. For the precise definition of a Siegel operator or a Siegel cusp form, see [F, Section I.3, Definition II.6.9]. We denote by $S_k(\Gamma)$ the vector space of weight k Siegel cusp forms of Γ. From now on, we write cusp forms instead of Siegel cusp forms.

Remark 8.5.8. *As in remark 8.5.2, a cusp form is a modular form that vanishes at every cusp that comes from a corank 1 boundary component, and it is enough to check that $\Phi(f)(N(\cdot))$ vanishes for one N in each coset of Γ in the group of projective rational matrices in* $\mathrm{Sp}(2g, \mathbb{R})$ *because of the automorphy condition. Since for each boundary component of corank greater than 1 there exists a boundary component of corank 1 that is adjacent to it, a cusp form in fact vanishes at every cusp of $\mathcal{A}$.*

By considering a cusp form as a modular form that vanishes at every (corank 1) cusp, we have an alternative definition of a cusp form involving the line bundle $\mathcal{L}$. Consider the **Mumford compactification** [AMRT] where we take the union of $\mathcal{A}$ with the corank 1 boundary components in the boundary of the Satake compactification, and then blow up the boundary. The Mumford compactification is canonical: to construct the partial compactification of a corank 1 boundary component F of $\mathcal{S}_g$, the only way to compactify the partial quotient $T(F)$ of the associated tube domain (see Equation 8), is to add the point 0 to the $\mathbb{C}^*$ factor.

Remark 8.5.9. *The Mumford compactification is the common open part in all toroidal compactifications. Despite its name, the Mumford compactification is not compact.*

If $\mathcal{A}^{\#}$ is the Mumford compactification of $\mathcal{A}$, then $\mathcal{A}^{\#} \setminus \mathcal{A}$ is a divisor $\Delta_{\mathcal{A}}$ which we call the **the boundary divisor** of $\mathcal{A}$. A cusp form of Γ of weight k can then be considered as a global section of the divisor $k\mathcal{L} - \Delta_{\mathcal{A}}$.

In fact, if $f \in M_{k(g+1)}(\Gamma)$ is a cusp form, then the canonical k-form $f\omega^{\otimes k}$ on $\mathcal{A}^0$ mentioned in Proposition 8.5.5 extends to a canonical k-form on the Mumford compactification $\mathcal{A}^{\#}$.

Theorem 8.5.10. [AMRT, Section IV.1.2] *Let $\mathcal{A} \simeq \Gamma\backslash\mathcal{S}_g$ be a moduli space of structured abelian g-folds, and let $\mathcal{A}^{\#}$ be the Mumford compactification of $\mathcal{A}$. Then there is an isomorphism of vector spaces*

$$S_{(g+1)}(\Gamma) \simeq H^0(K_{\mathcal{A}^{\#}}).$$

Remark 8.5.11. *The lower bound for $\kappa(K_X) = \kappa(K_{X^0})$ in Ma's theorem (Theorem 8.2.10) is determined by proving extended versions of Theorem 8.5.5 and Theorem 8.5.10 for a Namikawa compactification of Kuga families [Ma, Theorem 1.2]: for any Namikawa compactification X of $\mathfrak{X}_\Gamma^{(n)}$ which projects to a toroidal compactification $\overline{\mathcal{A}}$ of $\mathcal{A}$, we have an isomorphism of graded rings*

$$\bigoplus_{k\geq 0} H^0(X, kK_X + k\Delta_X)) \simeq \bigoplus_{k\geq 0} M_{k(n+g+1)}(\Gamma),$$

where Δ_X is the boundary divisor $X \setminus \mathfrak{X}_\Gamma^{(n)}$. Moreover for every positive integer k, there an injection

$$H^0(k(n+g+1)\mathcal{L} - \Delta_X) \simeq S_{k(n+g+1)} \hookrightarrow H^0(kK_X + (k-1)\Delta_X),$$

which results in the injection

$$H^0(k(n+g+1)\mathcal{L} - k\Delta_X) \hookrightarrow H^0(kK_X).$$

By Lemma 8.2.3, we have $\kappa((n+g+1)\mathcal{L} - \Delta_X) \leq \kappa(K_X)$.

8.5.3 Jacobi cusp forms

In this section, we will define Jacobi forms and Jacobi cusp forms. The standard reference for the topic is [EZ].

The notion of Jacobi forms arises from the modular forms of a special parabolic subgroup Γ_∞ of $\mathrm{Sp}(4,\mathbb{Z})$. For this we consider $\mathcal{S}_2$ as its image under the tube domain realisation for a corank 1 boundary component F

$$\tau \mapsto (\tau', \omega, \tau'') \in \mathcal{S}_1 \times \mathbb{C} \times \mathbb{C}.$$

Definition 8.5.12. [Grit]

The group Γ_∞ is the set of matrices in $\mathrm{Sp}(4,\mathbb{Z})$ in the form

$$\begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

The group Γ_∞ preserves the $\mathbb{C}$ factor in the tube domain realisation of $\mathcal{S}_2$ associated to F, *i.e.* Γ_∞ acts on $\mathbb{C}$ by translation. In particular, the action of Γ_∞ on the tube domain can be interpreted as the extended action of the stabiliser subgroup $\mathcal{P}(F)$ on the torus bundle $\widetilde{X}(F) \simeq \mathcal{S}_1 \times \mathbb{C} \times \mathrm{Temb}(\Sigma(F))$, where $\mathrm{Temb}(\Sigma(F)) \simeq \mathbb{C}$ is the unique smooth torus embedding of $\mathbb{C}^*$ mentioned in Section 8.5.2. For more details of Γ_∞, see [Grit] and [vdG, Section 11].

Definition 8.5.13. [Grit]

Let k and m be positive integers. A holomorphic function ϕ on $\mathcal{S}_1 \times \mathbb{C}$ is a ***Jacobi form*** *of weight k and index m if the function on $\mathcal{S}_2$ given by*

$$\tau \mapsto \phi(\tau', \omega) \exp(2\pi i m \tau'')$$

is a weight k modular form for Γ_∞.

We denote by $J_{k,m}$ the vector space of Jacobi forms of weight k and index m.

A Jacobi form $\phi \in J_{k,m}$ can also be characterised by its Fourier expansion

$$\phi(\tau', \omega) = \sum_{n=0}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ r^2 \leq 4mn}} c(n,r) \exp\big(2\pi(n\tau' + r\omega)\big)$$

together with a list of transformation rules with respect to the (restricted) action of Γ_∞ on $\mathcal{S}_1 \times \mathbb{C}$ (see [vdG, Section 8]). The Jacobi form ϕ is therefore [vdG, Section 11] a local section of a certain $\mathbb{Q}$-line bundle $\mathcal{J}$ on the Kuga variety

$$\mathfrak{X}^{(1)}_{\Gamma_\infty} \simeq \Gamma_\infty \backslash (\mathbb{C} \times \mathcal{S}_1).$$

This Kuga variety is the boundary divisor in the Mumford compactification $\mathcal{A}^{\#}$, and $\mathcal{J} = 3k\mathcal{L} + m\mathcal{N}$, where $\mathcal{L}$ is the line bundle of weight 1 modular forms of Γ_∞, and $\mathcal{N}$ is the normal bundle of $\mathfrak{X}^{(1)}_{\Gamma_\infty}$ in $\mathcal{A}^{\#}$.

To end this section, we give the definition of a Jacobi cusp form.

Definition 8.5.14. [Grit, Section 1]

A Jacobi form ϕ of weight k and index m is a ***Jacobi cusp form*** *of the same weight and index if in its Fourier expansion, a summand is non-zero only if $r < 4mn$. We denote the vector space of Jacobi cusp form of weight k and index m by $J^{\mathrm{cusp}}(k,m)$.*

9 Canonical singularities

In this section, we will show that the L-dimension $\kappa(\overline{\mathcal{A}_p}, (n+3)\mathcal{L} - \Delta_{\mathcal{A}})$ is a lower bound for $\kappa(\mathfrak{X}^n_p)$ for any $n \geq 3$ and for any p, by showing that any Namikawa compactification X of $\mathfrak{X}^n_p$ as constructed in [PSMS] has canonical singularities.

9.1 Strategy

We will separately examine the singularities in the interior and the boundary of X, and check if they are canonical by applying the RST criterion.

9.1.1 Strategy in the interior

In the interior $\mathfrak{X}^n_p$ of X, a singularity corresponds to a point $\widetilde{\tau} = (Z, \tau)$ in $\mathbb{C}^{2n} \times \mathcal{S}_2$ fixed by $\widetilde{\Gamma}^n_p$. We can apply the RST criterion to check if $\widetilde{\tau}$ corresponds to a canonical singularity: suppose $\widetilde{\gamma}$ is an element in the **isotropy group** $\mathrm{iso}(\widetilde{\tau}) < \widetilde{\Gamma}^n_p$ of $\widetilde{\tau}$. *i.e.* $\widetilde{\gamma}$ fixes $\widetilde{\tau}$. Equation (7) in Section 8 shows the action of $\widetilde{\gamma}$ separately as the action of γ on the $\mathcal{S}_2$ factor and that of $\widetilde{\gamma}$ on the $\mathbb{C}^{2n}$ factor. Also, $\widetilde{\gamma}$ fixes $\widetilde{\tau}$

only if γ fixes τ. The isotropy group $\mathrm{iso}(\widetilde{\tau})$ of $\widetilde{\tau}$ in $\widetilde{\Gamma}_p^n$ is finite, so any nontrivial element $\widetilde{\gamma} = (l, \gamma)$ in $\mathrm{iso}(\widetilde{\tau})$ is a torsion element and $l = 0$. As a result of [T, Theorem 4.1], the induced action of any element $\gamma \in \mathrm{iso}(\tau) \leq \Gamma_p$ of order k on the tangent space $T_\tau(\mathcal{S}_2)$ can be diagonalised under suitable local coordinates. It will be shown that $\widetilde{\gamma}$ also acts diagonally on $T_Z(\mathbb{C}^{2n})$. This gives us the finite dimensional representation of $\mathrm{iso}(\widetilde{\tau})$ required for the application of the RST criterion.

Note that it suffices to apply the RST criterion at a limited number of singularities in $\mathfrak{X}_p^n$:

Lemma 9.1.1. *Let $\widetilde{\tau} = (Z, \tau)$ be a point in $\mathbb{C}^{2n} \times \mathcal{S}_2$ that corresponds to a canonical singularity in $\mathfrak{X}_p^n$. Then either τ corresponds to a canonical singularity in $\mathcal{A}_p$, or* $\mathrm{iso}(\widetilde{\tau}) = \langle \widetilde{\sigma} := (0, -\mathbf{1}_4) \rangle < \widetilde{\Gamma}_p^n$. *In the latter case, τ corresponds to a smooth point.*

Proof. The isotropy group of $\widetilde{\tau}$, $\mathrm{iso}(\widetilde{\tau})$, cannot contain a quasi-reflection: according to [Ma, Lemma 7.1], a non-trivial element $\widetilde{\gamma} \in \mathrm{iso}(\widetilde{\tau})$ does not fix any divisor in $\mathfrak{X}_p^n$.

Consider any nontrivial $\widetilde{\gamma} := (0, \gamma) \in \mathrm{iso}(\widetilde{\tau})$. If γ acts trivially on $\mathcal{S}_2$, then $\gamma = -\mathbf{1}_4$.

Moreover, by the definition of RST sums, we have

$$\mathrm{RST}(\widetilde{\gamma}) \geq \mathrm{RST}(\gamma)$$

So $\widetilde{\tau}$ corresponds to a canonical singularity in $\mathfrak{X}_p^n$ if τ corresponds to a canonical singularity in $\mathcal{A}_p$. □

9.1.2 Strategy in the boundary

A singularity in the boundary of X correspond to a point $\widetilde{\tau}$ in $\widetilde{X}(\widetilde{F})$ fixed by $\widetilde{P''}(\widetilde{F})$ near a proper boundary component $\widetilde{F}$ of corank g'. Again, the RST criterion can be applied to check if $\widetilde{\tau}$ corresponds to a canonical singularity: Let $\widetilde{\tau} := (Z, \tau)$, where $Z \in \mathbb{C}^{2n}$ and $\tau \in \mathcal{S}_{g'} \times \mathrm{Temb}(\Sigma(\widetilde{F}))$. As mentioned in Definition 8.3.6, $\widetilde{P''}(\widetilde{F})$ preserves the decomposition of $\widetilde{X}(\widetilde{F})$, so $\widetilde{\gamma}$ acts on each factors of $\widetilde{X}(\widetilde{F})$ separately. A calculation similar to Section 8 Equation (7) shows that locally at $\widetilde{\tau}$, $\widetilde{\gamma} = (l, \gamma) \in \widetilde{P''}(\widetilde{F})$ fixes $\widetilde{\tau}$ only if γ fixes τ. However, different from what we had in Section 9.1.1, $\widetilde{\gamma}$ may not be a torsion element, i.e. l could be non-zero. Nevertheless, the action of $\widetilde{\gamma}$ on the tangent space of a resolution of $\widetilde{X}(\widetilde{F})$ at $\widetilde{\tau}$ at $\widetilde{\tau}$ is of finite order, so the RST criterion can be applied there.

The following observations are useful for checking whether these singularities are canonical:

1. [PSMS, Lemma 1.3]: Let $(\widetilde{X}(\widetilde{F}))^*$ be a smooth $\widetilde{P''}(\widetilde{F})$-equivariant resolution of $\widetilde{X}(\widetilde{F})$. If $\widetilde{P''}(\widetilde{F})$ has no quasireflection, then the partial compactification $\widetilde{P''}(\widetilde{F}) \backslash \widetilde{X}(\widetilde{F})$ has canonical singularities if $\widetilde{P''}(\widetilde{F}) \backslash (\widetilde{X}(\widetilde{F}))^*$ has canonical singularities. In particular, this implies that we can apply the RST criterion at the singularities in $\widetilde{P''}(\widetilde{F}) \backslash (\widetilde{X}(\widetilde{F}))^*$ instead.

2. Let $\widetilde{\tau} = (Z, \tau)$ correspond to a canonical singularity near $\widetilde{F}$. Then either τ corresponds to a canonical singularity in the boundary of $\overline{\mathcal{A}_p}$, or $\mathrm{iso}(\widetilde{\tau}) = \langle \widetilde{\sigma} := (l, -\mathbf{1}_4) \rangle < \widetilde{P''}(\widetilde{F})$ for some $l \in L$. In the latter case, τ corresponds to a smooth point. The proof is similar to that in Lemma 9.1.1. Again, this implies that we only need to apply the RST criterion at a limited number of singularities.

9.2 Singularities in the interior of compactification

In this section, we will identify the singularities in $\mathfrak{X}_p^n$ and show that for $n > 2$, they are all canonical.

First we identify singularities that project to non-canonical singularities in $\mathcal{A}_p$. It is given in the proof of [HKW1, Theorem 1.8] that for any odd prime p, the singular points in $\mathcal{A}_p$ are exactly the points that lie on two disjoint curves which we call C_1 and C_2. Any point on one of these curves corresponds to a point τ in $\mathcal{S}_2$, whose isotropy group in Γ_p is generated by a single generator. Its induced action on the tangent space of $\mathcal{S}_2$ at τ is also given there: one can write any point in the tangent space $T_\tau(\mathcal{S}_2)$ in the form

$$\begin{pmatrix} \tau_1 + x & \tau_2 + y \\ \tau_2 + y & \tau_3 + z \end{pmatrix}.$$

So the tuple (x, y, z) can be considered as the local coordinates for $T_\tau \mathcal{A}_p$, and the respective action of a generator of $\mathrm{iso}(\tau)$ on $T_\tau(\mathcal{S}_2)$ with these coordinates is given by

$$(x, y, z) \mapsto (-x, -iy, z) \text{ along } C_1;$$
$$(x, y, z) \mapsto (\rho^2 x, -\rho y, z) \text{ along } C_2, \text{ where } \rho = e^{2\pi i/3}.$$

Therefore, the chosen generators are of types $\frac{1}{4}(2, 3, 0)$ and $\frac{1}{6}(4, 5, 0)$ when the root of unity ξ is chosen to be i and $e^{2\pi i/6}$ respectively on each curve C_1 and C_2.

Note that both cyclic groups generated by i and by $e^{2\pi i/6}$ contain a reflection. However as mentioned in the proof of [HKW1, Proposition 1.8], quotients by reflections are smooth. On C_1, dividing the isotropy group by the reflection group gives an order 2 cyclic group with a generator of type $\frac{1}{2}(1, 1, 0)$; whereas on C_2, the quotient gives us an order 3 cyclic group with a generator of type $\frac{1}{3}(1, 1, 0)$. By applying the RST criterion on these cyclic groups of order 2 and 3, it is clear that the singularities on C_1 are canonical, whereas those on C_2 are not.

Let $\widetilde{\tau} := (Z, \tau) \in \mathbb{C}^{2n} \times \mathcal{S}_2$ such that τ corresponds to a point in C_2. Let $\widetilde{\sigma} := (0, -\mathbf{1}_4)$ and $\widetilde{\gamma} := (0, \gamma)$, where γ is the generator of $\mathrm{iso}(\tau)$ with the action on $T_\tau(\mathcal{S}_2)$ described above. Then either $\mathrm{iso}(\widetilde{\tau}) = \langle \widetilde{\gamma}, \widetilde{\sigma} \rangle$ or $\mathrm{iso}(\widetilde{\tau}) = \langle \widetilde{\gamma} \rangle$.

We shall first compute the type of $\widetilde{\gamma}$. We only need to understand the action of $\widetilde{\gamma}$ at a point $\widetilde{Y} = (Z + Y, \tau)$ on the tangent space $T_{\widetilde{\tau}}(\mathbb{C}^{2n} \times \{\tau\}) \simeq T_Z(\mathbb{C}^{2n})$ to complete the type of $\widetilde{\gamma}$. To do this, we need the explicit expressions of the set C_2 and its isotropy group $\mathrm{iso}(\tau)$ from [HKW1, Definition 1.5]:

$$C_2 = \left\{ \begin{pmatrix} \rho & 0 \\ 0 & \tau_3 \end{pmatrix} : \rho = e^{2\pi i/3}, \tau_3 \in \mathcal{S}_1 \right\},$$

$$\mathrm{iso}(\tau) = \left\langle \gamma = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Following Equation (7) in Section 8, the action of $\widetilde{\gamma}$ at $\widetilde{Y}$ in $T_{\widetilde{\tau}}(\mathbb{C}^{2n} \times \{\tau\})$ is given by

$$\widetilde{\gamma} \cdot \widetilde{Y} = \begin{bmatrix} (Z + Y) \cdot N \\ \tau \\ \mathbf{1}_2 \end{bmatrix}, \text{ where } N = \begin{pmatrix} (\rho + 1)^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\widetilde{\gamma}$ fixes (Z, τ), $Z \cdot N = Z$ and $\widetilde{\gamma}$ acts on $T_Z(\mathbb{C}^{2n})$ diagonally by sending the set of local coordinates Y to $Y \cdot N$.

Note that $(\rho + 1)^{-1} = e^{2\pi i \cdot (5/6)}$. So by choosing the primitive root of unity to be $e^{2\pi i/6}$, which is the same as that for the $\mathcal{S}_2$ factor, we have an extra n copies of $5/6$'s and n copies of 0's in the RST sum of $\widetilde{\gamma}$. In other words, the type of $\widetilde{\gamma}$ is $\frac{1}{6}(4, 5, 0, 5, \ldots, 5, 0, \ldots, 0)$.

As for the type of $\widetilde{\sigma}$, since $\widetilde{\sigma}$ acts trivially on $T_\tau(\{Z\} \times \mathcal{S}_2)$, the first entries in the type of $\widetilde{\sigma}$ which correspond to the $\mathcal{S}_2$ factor are all 0's. On the other hand, the calculation in Equation (7) shows that $\widetilde{\sigma}$ acts on the set of local coordinates in $T_Z(\mathbb{C}^{2n})$ diagonally by $X \mapsto -X$. So the type of $\widetilde{\sigma}$ is $\frac{1}{2}(0, 0, 0, 1, \ldots, 1)$ when the primitive root of unity ξ is chosen to be -1.

Since $\widetilde{\sigma}$ commutes with $\widetilde{\gamma}$, we can draw the following table which shows the type of a non-trivial element $\widetilde{\gamma}^{k_1} \widetilde{\sigma}^{k_2} \in \mathrm{iso}(\widetilde{\tau})$, where $0 \leq k_1 \leq 5$ and $0 \leq k_2 \leq 1$.

k_1 \ k_2	0	1
0	N/A	$\frac{1}{2}(0,0,0,1,\dots,1,1,\dots,1)$
1	$\frac{1}{6}(4,5,0,5,\dots,5,0,\dots,0)$	$\frac{1}{6}(4,5,0,2,\dots,2,3,\dots,3)$
2	$\frac{1}{6}(2,4,0,4,\dots,4,0,\dots,0)$	$\frac{1}{6}(2,4,0,1,\dots,1,3,\dots,3)$
3	$\frac{1}{6}(0,3,0,3,\dots,3,0,\dots,0)$	$\frac{1}{6}(0,3,0,0,\dots,0,3,\dots,3)$
4	$\frac{1}{6}(4,2,0,2,\dots,2,0,\dots,0)$	$\frac{1}{6}(4,2,0,5,\dots,5,3,\dots,3)$
5	$\frac{1}{6}(2,1,0,1,\dots,1,0,\dots,0)$	$\frac{1}{6}(2,1,0,4,\dots,4,3,\dots,3)$

The types of all non-trivial elements in $\langle\widetilde{\gamma}\rangle$ are given by the first column of the table, while that in $\langle\widetilde{\gamma},\widetilde{\sigma}\rangle$ are given by the entire table. Notice the RST criterion only fails when $n \leq 2$:

$$\mathrm{RST}(\widetilde{\gamma}^5) < 1.$$

We conclude that for $n > 2$, both $\langle\widetilde{\gamma}\rangle$ and $\langle\widetilde{\gamma},\widetilde{\sigma}\rangle$ satisfy the RST criterion, and therefore $\widetilde{\tau}$ is a canonical singularity in $\mathfrak{X}_p^n$, no matter $\mathrm{iso}(\widetilde{\tau}) = \langle\widetilde{\gamma},\widetilde{\sigma}\rangle$ or $\mathrm{iso}(\widetilde{\tau}) = \langle\widetilde{\gamma}\rangle$.

Finally, for any singularity that corresponds to a point in $\mathbb{C}^{2n} \times \mathcal{S}_2$ whose isotropy group is $\langle\widetilde{\sigma}\rangle$, we only need to study the first row of the table: there is no quasi-reflection and the RST inequality is satisfied for any n. Therefore such singularity is always canonical.

9.3 Singularities in the boundary of compactification

In this section we will check that every singularity in the boundary of X is canonical.

First, we identify all the non-canonical singularities in $\overline{\mathcal{A}_p}$. Consider the compact curves C_1^* and C_2^* containing C_1 and C_2 in $\overline{\mathcal{A}_p}$. Then from [HKW1, Propositions 2.15 and 3.4], for any odd prime p, the complement $\overline{\mathcal{A}_p} \smallsetminus (C_1^* \cup C_2^*)$ contains only isolated singularities. The types of a generator in the respective isotropy groups are given as $\frac{1}{2}(1,1,1)$ or $\frac{1}{3}(1,2,1)$. So both isotropy groups satisfy the RST criterion, and these singularities in X are canonical. Therefore, any non-canonical singularity in X has to project down to $C_1^* \smallsetminus C_1$ or $C_2^* \smallsetminus C_2$.

From the same source above, each set $C_1^* \smallsetminus C_1$ and $C_2^* \smallsetminus C_2$ consists of $(p^2-1)/2$ points, one in each of the corank 1 peripheral boundary components. [HKW1, Proposition 2.8] further says that near one of these boundary component $\widetilde{F}$, the singularities in C_1^* and C_2^* are represented by $Q_1 = (i,0,0)$ and $Q_2 = (\rho,0,0)$ as points in $\mathcal{S}_1 \times \mathbb{C} \times \mathbb{C}$, the tube domain realisation of $\mathcal{S}_2$, with $\rho = e^{2\pi i/3}$.

First consider the singularity in X associated to Q_2: let $\widetilde{\tau} := (Z,\tau) \in \widetilde{X}(\widetilde{F})$ such that $\tau = Q_2$. From [HKW1, Propositions 2.5 and 2.8], the stabiliser subgroup of τ in $P''(F)$ is generated by the order 6 element

$$\gamma = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\widetilde{\gamma} := (l,\gamma)$ be the corresponding generator in $\mathrm{iso}(\widetilde{\tau})$, and let $\widetilde{\sigma} := (l, -\mathbf{1}_4)$ for some $l \in L$. Then again $\mathrm{iso}(\widetilde{\tau}) = \langle\widetilde{\gamma}\rangle$ or $\mathrm{iso}(\widetilde{\tau}) = \langle\widetilde{\gamma},\widetilde{\sigma}\rangle$. To find the types of elements in $\mathrm{iso}(\widetilde{\tau})$, we consider their actions on each factor of $\widetilde{X}(\widetilde{F})^*$, a $\widetilde{P''}(\widetilde{F})$-equivariant resolution of $\widetilde{X}(\widetilde{F})$. [T, Lemmas 5.1 and 5.2] describes such a resolution of singularities for the moduli space of principally polarised abelian g-folds whose modular group is $\mathrm{Sp}(2g,\mathbb{Z})$, as well as a formula for the RST sum of a generator γ in the isotropy group. This result applies to our work as $\Gamma < \mathrm{Sp}(2g,\mathbb{Z})$. Explicitly when $g = 2$, there are three factors in the resolution of $\mathcal{S}_2$: $\mathcal{S}_{g'}, \mathbb{C}^{g'g''}$ and a torus at infinity. The following submatrices are extracted from the entries γ_{ij} of γ:

$$\gamma' = \begin{pmatrix} \gamma_{11} & \gamma_{13} \\ \gamma_{31} & \gamma_{33} \end{pmatrix}, \qquad U = \begin{pmatrix} \gamma_{22} \end{pmatrix}.$$

Suppose γ' has eigenvalues $\lambda^{\pm 1}$ and U has eigenvalue μ. Then the eigenvalues of the action of γ on the tangent space of the $\mathcal{S}_{g'}$ factor, the $\mathbb{C}^{g'g''}$ factor and the torus at infinity in the resolution of $\mathcal{S}_2$ are λ, $\lambda\mu$ and 0 respectively.

In our case, γ' has eigenvalues $e^{\pm 2\pi i/6}$ and U has eigenvalue 1. Therefore, when $e^{2\pi i/6}$ is chosen to be the primitive root of unity, the $\mathcal{S}_{g'}$, $\mathbb{C}^{g'g''}$ and the torus at infinity factors contribute $\frac{2}{6}$, $\frac{1}{6}$ and 0 to the RST sum respectively.

For the RST sum over the remaining $\mathbb{C}^n \times (\mathbb{C}^*)^n$ factor of $\widetilde{\tau}$, follow Section 8 Equation (7) and consider the action of $\widetilde{\gamma}$ on $\widetilde{Y} := (Z+Y, \tau)$ in the tangent space at Z of the resolved $\mathbb{C}^{2n}$ factor:

$$\widetilde{\gamma} \cdot \widetilde{Y} = \begin{bmatrix} (Z'+Y) \cdot N \\ \tau \\ \mathbf{1}_2 \end{bmatrix} \text{ where } Z' = Z + l \cdot \begin{pmatrix} \tau \\ \mathbf{1}_2 \end{pmatrix} \text{ and } N = \begin{pmatrix} \frac{1}{\rho+1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Again $\widetilde{\gamma}$ fixes $\widetilde{\tau}$, so $Z' \cdot N = Z$ and $\widetilde{\gamma}$ acts on the tangent space diagonally by sending the local coordinates Y to $Y \cdot N$. The eigenvalues of the action are the eigenvalues of N, which are $e^{2\pi i \cdot (5/6)}$ and 1. When we choose $e^{2\pi i/6}$ to be the primitive root of unity for $\widetilde{\gamma}$, which is the same choice as the other factors, they contribute n copies of $\frac{5}{6}$ and n copies of 0 to the RST sum.

Do the same for $\widetilde{\sigma}$ to find $\mathrm{RST}(\widetilde{\sigma})$: write $\sigma = -\mathbf{1}_4$ and consider the submatrices σ' and U extracted from σ in the same way as above. Their eigenvalues are $\{-1,-1\}$ and -1 respectively, which contribute 0 to the RST sum for all 3 factors of the solution of $\mathcal{S}_2$ after resolving. Following Equation (7), the action of $\widetilde{\sigma}$ on the tangent space of the resolved $\mathbb{C}^n \times (\mathbb{C}^*)^n$ factor at Z is again multiplication by -1 to the local coordinates Y.

Therefore, we can draw a similar table as in the previous subsection for each element $\widetilde{\gamma}^{k_1}\widetilde{\sigma}^{k_2} \in \mathrm{iso}(\widetilde{\tau})$, where $0 \leq k_1 \leq 5$ and $0 \leq k_2 \leq 1$:

k_1 \ k_2	0	1
0	N/A	$\frac{1}{2}(0,0,0,1,\ldots,1,1,\ldots,1)$
1	$\frac{1}{6}(2,1,0,5,\ldots,5,0,\ldots,0)$	$\frac{1}{6}(2,1,0,2,\ldots,2,3,\ldots,3)$
2	$\frac{1}{6}(4,2,0,4,\ldots,4,0,\ldots,0)$	$\frac{1}{6}(4,2,0,1,\ldots,1,3,\ldots,3)$
3	$\frac{1}{6}(0,3,0,3,\ldots,3,0,\ldots,0)$	$\frac{1}{6}(0,3,0,0,\ldots,0,3,\ldots,3)$
4	$\frac{1}{6}(2,4,0,2,\ldots,2,0,\ldots,0)$	$\frac{1}{6}(2,4,0,5,\ldots,5,3,\ldots,3)$
5	$\frac{1}{6}(4,5,0,1,\ldots,1,0,\ldots,0)$	$\frac{1}{6}(4,5,0,4,\ldots,4,3,\ldots,3)$

One can check that there is no quasi-reflection, and the RST sum is at least 1 everywhere on the table. So the RST criterion is satisfied for both $\langle\widetilde{\gamma}\rangle$ and $\langle\widetilde{\gamma},\widetilde{\sigma}\rangle$. Thus for all $n \geq 1$, the singularity in X that corresponds to (Z, Q_2) is canonical.

Now we replace Q_2 by Q_1 everywhere in the above to check whether the other singularity in the boundary component $\widetilde{F}$ is canonical or not. Again, let $\widetilde{\tau} = (Z, \tau)$ such that $\tau = Q_1$. The stabiliser subgroup of $\tau = Q_1$ in $P''(F)$ is generated by the order 4 element

$$\gamma = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

First we caluclate RST(γ). Extract the submatrices γ' and U as before. The eigenvalues of γ' are $\pm i$ and the eigenalue of U is 1. When i is the chosen primitive root of unity, the $\mathcal{S}_{g'}$ factor, the $\mathbb{C}^{g'g''}$ factor and the torus at infinity in the resolution of $\mathcal{S}_2$ contribute a $\frac{2}{4}$, a $\frac{1}{4}$ and a 0 to the RST sum respectively. Consider the action of $\widetilde{\gamma}$ at $\widetilde{Y} = (Z+Y,\tau)$. Then Equation (7) gives:

$$\widetilde{\gamma}\cdot\widetilde{Y} = \begin{bmatrix}(Z'+Y)\cdot N\\ \tau \\ \mathbf{1}_2\end{bmatrix} \text{ where } Z' = Z + l\cdot\begin{pmatrix}\tau\\ \mathbf{1}_2\end{pmatrix} \text{ and } N = \begin{pmatrix}-i & 0\\ 0 & 1\end{pmatrix}.$$

Once more $Z'\cdot N = Z$ and $\widetilde{\gamma}$ acts on the tangent space diagonally by sending the local coordinates Y to $Y\cdot N$. This action has eigenvalues $e^{2\pi i\cdot(3/4)}$ and 1, which contribute n copies of $\frac{3}{4}$ and n copies of 0 to the RST sum over the resolved $\mathbb{C}^n\times(\mathbb{C}^*)^n$ factor when the primitive root of unity chosen is i.

The RST sums of $\widetilde{\sigma}$ restricted to each factor is the same as the case of Q_2.

Therefore we can draw the table for the type of $\widetilde{\gamma}^{k_1}\widetilde{\sigma}^{k_2}\in \mathrm{iso}(\widetilde{\tau})$, where $0\le k_1\le 3$ and $0\le k_2\le 1$:

k_1 \ k_2	0	1
0	N/A	$\frac{1}{2}(0,0,0,1,\dots,1,1,\dots,1)$
1	$\frac{1}{4}(2,1,0,3,\dots,3,0,\dots,0)$	$\frac{1}{4}(2,1,0,1,\dots,1,2,\dots,2)$
2	$\frac{1}{4}(0,2,0,2,\dots,2,0,\dots,0)$	$\frac{1}{4}(0,2,0,0,\dots,0,2,\dots,2)$
3	$\frac{1}{4}(2,3,0,1,\dots,1,0,\dots,0)$	$\frac{1}{4}(2,3,0,3,\dots,3,2,\dots,2)$

The RST criterion is satisfied for both $\langle\widetilde{\gamma}\rangle$ and $\langle\widetilde{\gamma},\widetilde{\sigma}\rangle$, so for all $n\ge 1$, the singularity in X that corresponds to (Z,Q_1) is canonical.

We summarise our findings in the following theorem:

Theorem 9.3.1. *Singularities in the Namikawa compactification X of $\mathfrak{X}_p^n$ are canonical for $n\ge 3$. For $n=1,2$, the set of non-canonical singularities in X is exactly the preimage under $\overline{\pi}$ of the curve C_2 in $\mathfrak{X}_p^1$ and $\mathfrak{X}_p^2$ respectively.*

10 Low weight cusp form trick

In this section, we will prove the following theorem:

Theorem 10.0.1. *The equality $\kappa(\overline{\mathcal{A}_p},(n+3)\mathcal{L}-\Delta_{\mathcal{A}}) = 3$ holds for the following values of n and p:*

- *$p\ge 3$ and $n\ge 4$;*
- *$p\ge 5$ and $n\ge 3$.*

To find a lower bound for $\kappa((n+3)\mathcal{L}-\Delta_{\mathcal{A}})$, which is the rate of growth with respect to m of the dimension of the space of weight $m(n+3)$-cusp forms of Γ_p, we use the "low weight cusp form trick", which has been used in this context in [GritH] and [GritS], and more widely thereafter.

Suppose $n > N$ and there exists a non-zero weight $3+N$ cusp form F of Γ_p, that is, there is a non-zero $F\in H^0((3+N)\mathcal{L}-\Delta_A)$. For any non-zero $F'\in H^0(m(n-N)\mathcal{L})$,

$$F^mF'\in H^0(m((n+3)\mathcal{L}-\Delta_A)).$$

Fixing F, the space of cusp forms in the form of F^mF' then grows at the same rate as $H^0(m(n-N)\mathcal{L})$ with respect to m, which is known to be $\mathcal{O}(m^3)$ (See [GritHS, proof of Theorem 1.1]). So we have $\kappa((n+3)\mathcal{L}-\Delta_A)\ge 3$.

Therefore, $\mathfrak{X}_p^n$ is of relative general type if

$$h := \dim H^0((3+N)\mathcal{L} - \Delta_A) > 0.$$

To find a lower bound for h, we apply Gritsenko's lifting of Jacobi cusp forms mentioned in [Grit, Theorem 3], which states the existence of an injective lifting

$$J^{\text{cusp}}(k,p) \hookrightarrow S_k(\Gamma[p])$$

where $J^{\text{cusp}}(k,p)$ is the space of Jacobi cusp forms of weight k and index $p \geq 1$, and $S_k(\Gamma[p])$ is the space of weight k cusps forms of $\Gamma[p]$, with the paramodular group $\Gamma[p]$ defining the moduli space of $(1,p)$-polarised abelian surfaces without level structure as $\Gamma[p]\backslash\mathbb{H}_2$. But since $\Gamma_p \leq \Gamma[p]$, the image of the lifting is also contained in $S_k(\Gamma_p)$.

From [EZ, Equation (8) in Introduction], dim $J^{\text{cusp}}(k,p) \geq j(k,p)$ (equality holds when $k \geq p$), where

$$j(k,p) := \begin{cases} \sum\limits_{j=0}^{p} \left(\dim M_{k+2j} - \left(\left\lfloor \frac{j^2}{4p} \right\rfloor + 1\right)\right), & \text{if } k \text{ is even} \\ \sum\limits_{j=1}^{p-1} \left(\dim M_{k+2j-1} - \left(\left\lfloor \frac{j^2}{4p} \right\rfloor + 1\right)\right), & \text{if } k \text{ is odd} \end{cases}$$

with M_r being the space of modular forms of weight r for $\mathrm{SL}(2,\mathbb{Z})$.

It is a general fact that

$$\dim M_r = \begin{cases} \left\lfloor \frac{r}{12} \right\rfloor, & \text{if } r \equiv 2 \mod 12 \\ \left\lfloor \frac{r}{12} \right\rfloor + 1, & \text{otherwise.} \end{cases}$$

By a simple computation, it can be found that the first prime p such that $j(k,p) > 0$ for $k = 5$ and 6 are $p = 5$ and 3 respectively. Note:

1. $\dim(S_k(\Gamma_p)) \geq j(k,p)$ for any k, p;
2. $j(k,p)$ increases with p;
3. From [U2], $\kappa(\mathfrak{X}_p^n)$ is non-decreasing with respect to n.

By letting $k = 3 + N = 2 + n$, this shows that for the values of n and p stated in Theorem 10.0.1, $\dim(S_k(\Gamma_p)) \geq j(k,p) \geq 1$. This concludes our proof for Theorem 10.0.1.

Combining the results of [GrifH] and [HS], which say $\mathfrak{X}_p^0 = \mathcal{A}_p$ is of general type for $p \geq 37$, we can mark on the (p,n)-plane a region for which the Kuga varieties are of relative general type as in Diagram 7.

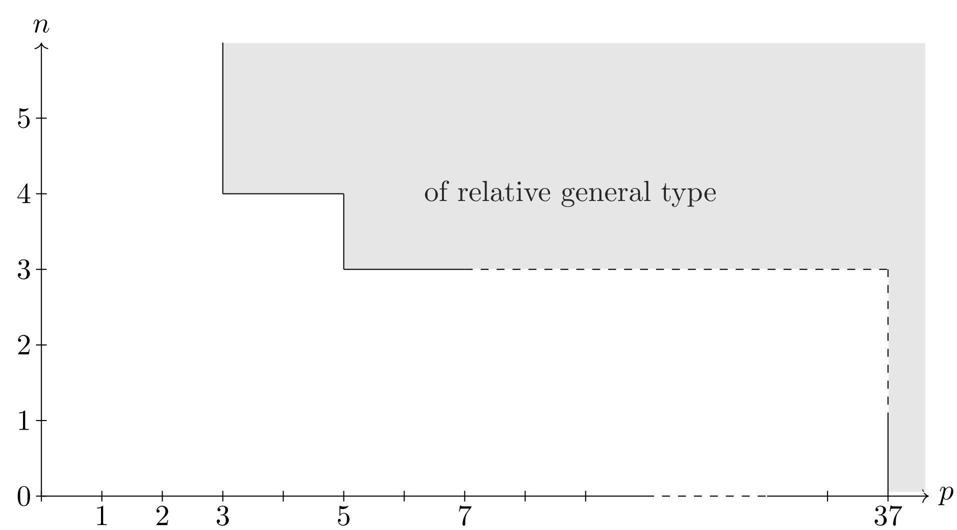


Diagram 7: The pairs (p, n) for which the Kuga variety $\mathfrak{X}_p^n$ is of relative general type.

Lastly, we discuss some possible improvements for Theorem 8.2.11. By following [HS] and applying the Riemann-Roch theorem on the exceptional divisor E of a blow-up at a non-canonical singularity in $\mathfrak{X}_p^1$, we may be able to improve our boundary at $n = 1$ by finding two consecutive primes p' and p'' with $p' < p''$ such that $\kappa(\mathfrak{X}_{p'}^1) < \kappa(\mathfrak{X}_{p''}^1)$. However, that would involve understanding the intersection behaviour of divisors on the 4-fold E, which is expected to be complicated. The low density of prime numbers near 37 makes the quest less promising: the estimate for p' we find by this method may not be smaller than 31.

There are a few more questions that can be asked: for example, whether the boundary we have drawn can be improved for $p = 5$ and $p = 3$. The image of Gritsenko's lift is not the entire $S_k(\Gamma_p)$ or even $S_k(\Gamma[p])$, so we might be able to find a weight 4 cusp form with respect to Γ_p or $\Gamma[p]$ through other means which improves the bound at $p = 5$, and likewise for $p = 3$. Another question is to calculate $\kappa(\mathfrak{X}_p^n)$ for other $\mathfrak{X}_p^n$ not of relative general type by considering the slope of Siegel cusp forms of Γ_p, which is the ratio between weight and vanishing order at ∞, and to draw a boundary on the (p, n)-plane separating the regions with $\kappa(\mathfrak{X}_p^n) = -\infty$ and $\kappa(\mathfrak{X}_p^n) \geq 0$. We can also extend the problem by considering $p = 2$, non-prime p, or abelian surfaces without level structure.

pre-T3.m

```
//Set up T = T_3
U := Matrix(K, 2, 2, [0, 1, 1, 0]);
D4 := Matrix(K, 4, 4, [2, 1, 1, 1, 1, 2, 0, 0, 1, 0, 2, 0, 1 ,0 ,0, 2]);
T:= DiagonalJoin(DiagonalJoin(U, U*2), D4*(-1));

//CU: \Cl(U+U(2))
CU<f1,f2,f3,f4>, VU, fU := CliffordAlgebra(DiagonalJoin(U, U*2));
//CUp: \Cl^+(U+U(2))
CUp, gU := EvenSubalgebra(CU);
iU := MainInvolution(CU);
//CD: Cl(D_4(-1))
CD<h1,h2,h3,h4>, VD, fD := CliffordAlgebra(D4*(-1));
//CDp: \Cl^+(D_4(-1))
CDp, gD := EvenSubalgebra(CD);
iD := MainInvolution(CD);

//Idpt: pseudo-idempotents 32\epsilon_i
x1 := f3*f1*f2*f4;
x2 := 4*f1*f2 - x1;
x3 := 2*f3*f4 - x1;
x4 := 8*One(CU) - x1 - x2 - x3;
X:=[x1, x2, x3, x4];


temp := h1*h2*h3*h4 + h2*h3 + h3*h4 + h4*h2;
y1:= 2*One(CD)-temp;
y2:= 2*One(CD)+temp;
Y:=[y1,y2];

//Get_lambda_U: Calculate lattice \Lambda_i associated to abelian 8-fold A_i ``restricted to'' CU
//input:
//pseudo-idempotent x
//output:
//even degree generators and odd degree generators of \Lambda_i in CU
function Get_lambda_U(x)
I := 8*One(CU) - x;
M := Matrix(K, 16, 16, [Eltseq(Basis(CU)[ii]*I):ii in [1 .. 16]]);
KM := KernelMatrix(Matrix(Integers(),M));
L_CUp := [];
L_CUi := [];
for ii in [1..NumberOfRows(KM)] do
if iU(CU!KM[ii]) eq CU!KM[ii] then L_CUp := L_CUp cat [CU!KM[ii]];
else L_CUi := L_CUi cat [CU!KM[ii]];
end if;
end for;
return L_CUp, L_CUi;
end function;

function Get_lambda_D(y)
I := 4*One(CD) - y;
M := Matrix(K, 16, 16, [Eltseq(Basis(CD)[ii]*I):ii in [1 .. 16]]);
KM := KernelMatrix(Matrix(Integers(),M));
L_CDp := [];
L_CDi := [];
for ii in [1..NumberOfRows(KM)] do
if iD(CD!KM[ii]) eq CD!KM[ii] then L_CDp := L_CDp cat [CD!KM[ii]];
else L_CDi := L_CDi cat [CD!KM[ii]];
end if;
end for;
return L_CDp, L_CDi;
end function;
```

```
Idpt := [[1,1],[2,2],[3,2],[4,1],[1,2],[2,1],[3,1],[4,2]];

for n in [1..8] do

L_CUp, L_CUi := Get_lambda_U(X[Idpt[n][1]]);
L_CDp, L_CDi := Get_lambda_D(Y[Idpt[n][2]]);

/*
printf "[";
for ii in [1..#L_CUp] do
AsPolynomial(L_CUp[ii]);
if ii ne #L_CUp then printf ",";
end if;
end for;
if n ne 8 then printf "],";
else printf "]";
end if;
*/

/*
printf "[";
for ii in [1..#L_CUi] do
AsPolynomial(L_CUi[ii]);
if ii ne #L_CUi then printf ",";
end if;
end for;
if n ne 8 then printf "],";
else printf "]";
end if;
*/

printf "[";
for ii in [1..#L_CDp] do
AsPolynomial(L_CDp[ii]);
if ii ne #L_CDp then printf ",";
end if;
end for;
if n ne 8 then printf "],";
else printf "]";
end if;

/*
printf "[";
for ii in [1..#L_CDi] do
AsPolynomial(L_CDi[ii]);
if ii ne #L_CDi then printf ",";
end if;
end for;
if n ne 8 then printf "],";
else printf "]";
end if;
*/
end for;
```

T3.m

```
//T_3 = U + U(2) + D_4(-1)

//field K = \QQ[\sqrt(-1), \sqrt(2)] \subset \CC
P<X> := PolynomialRing(Rationals());
K<x,s> := NumberField([X^2+1, X^2-2]:Abs := true);

//Conj: complex conjugate in K
Conj := hom<K -> K|-x,s>;

//Conj_mat: complex conjugate for matrices/K
//input:
//M = matrix over K
//output:
//complex conjugate of M
function Conj_mat(M)
return Matrix(K,Nrows(M),Ncols(M),[Conj(M[i,j]):j in [1..Ncols(M)],i in [1..Nrows(M)]]);
end function;

//Set up T = T_3
U := Matrix(K, 2, 2, [0, 1, 1, 0]);
D4 := Matrix(K, 4, 4, [2, 1, 1, 1, 1, 2, 0, 0, 1, 0, 2, 0, 1 ,0 ,0, 2]);
T:= DiagonalJoin(DiagonalJoin(U, U*2), D4*(-1));

//C: Clifford algebra \Cl(T)
C<f1,f2,f3,f4,h1,h2,h3,h4>, V, f := CliffordAlgebra(T);
basCV := [f1,f2,f3,f4,h1,h2,h3,h4];

//Cp: \Cl^+(T)
Cp, g := EvenSubalgebra(C);

//H: \HH_\QQ
H<i,j,k>:=QuaternionAlgebra< K | -1, -1 >;
h := (1 + i + j + k)/2;

//Gives conjugation of matrices with entries in \HH
function Conj_H_mat(M)
return Matrix(H,Nrows(M),Ncols(M),[Conjugate(M[i,j]):j in [1..Ncols(M)],i in [1..Nrows(M)]]);
end function;

//Quadratic form in H
function quadform(u)
return u*Conjugate(u);
end function;

//Bilinear form in H
function bilform(u,v)
if u eq v then return quadform(u);
else return 1/2*(quadform(u+v)-quadform(u)-quadform(v));
end if;
end function;

//\alpha: product of two orthogonal positive definite vectors in T
alpha:= Cp!((f1+f2)*(f3+f4));

//A: change of basis matrix such that A^t*T*A is Diag([1,1,-1,...,-1])
A1 := Transpose(Matrix(K,4,4,[1/s,1/s,0,0,0,0,1/2,1/2,1/s,-1/s,0,0,0,0,1/2,-1/2]));
A2 := Transpose(Matrix(K,4,4,[s,-1/s,-1/s,-1/s,0,1/s,0,0,0,0,1/s,0,0,0,0,1/s]));
A := DiagonalJoin([A1,A2]);

//J = (f1+f2)/s * (f3+f4)/2: complex structure of KS variety, product of two +ve def vectors
e1 := ColumnSubmatrix(A,1,1);
e2 := ColumnSubmatrix(A,2,1);
```

```

//Perturb \omega = <e_1 + ie_2> in the direction of M_{a,b}
//input: m~0
function Perturb(m,a,b)
ct := (1-m^2)/(m^2+1);
st := 2*m/(m^2+1);
cht := (2+2*m^2)/(2-2*m^2);
sht := 2*m/(1-m^2);
SOt := IdentityMatrix(K,8);
if a lt 3 and b gt 2 then
SOt[a,b] := sht;
SOt[b,a] := sht;
SOt[a,a] := cht;
SOt[b,b] := cht;
else
SOt[a,b] := st;
SOt[b,a] := -st;
SOt[a,a] := ct;
SOt[b,b] := ct;
end if;
//SOV in SO(V)
SOV :=A*SOt*A^(-1);
return SOV*e1, SOV*e2;
end function;

//e1, e2 := Perturb(10^(-3), 1,3);

J := -&+[Eltseq(e1)[ii]*basCV[ii]: ii in [1..8]]*&+[Eltseq(e2)[ii]*basCV[ii]: ii in [1..8]];

L_CUp :=
[
    [f1*f2*f3*f4,f2*f4],
    [-f1*f2*f3*f4 + 4*f1*f2, f2*f3],
    [-f1*f2*f3*f4 + 2*f3*f4, f1*f4],
    [f1*f2*f3*f4 - 4*f1*f2 - 2*f3*f4 + 8, f1*f3],
    [f1*f2*f3*f4, f2*f4],
    [-f1*f2*f3*f4 + 4*f1*f2, f2*f3],
    [-f1*f2*f3*f4 + 2*f3*f4, f1*f4],
    [f1*f2*f3*f4 - 4*f1*f2 - 2*f3*f4 + 8, f1*f3]
];

L_CUi :=
[
    [f2*f3*f4, f1*f2*f4],
    [f1*f2*f3, f2*f3*f4 - 4*f2],
    [f1*f3*f4, f1*f2*f4 - 2*f4],
    [f1*f2*f3 - 2*f3, f1*f3*f4 - 4*f1],
    [f2*f3*f4, f1*f2*f4],
    [f1*f2*f3, f2*f3*f4 - 4*f2],
    [f1*f3*f4, f1*f2*f4 - 2*f4],
    [f1*f2*f3 - 2*f3,f1*f3*f4 - 4*f1]
];

L_CDp :=[
    [h1*h2 + h1*h3 + h3*h4 + 2,h1*h2*h3*h4 - 2*h1*h3 + 2*h2*h3 - 4,h1*h2 - h1*h3 + h2*h3 + h2*h4,
h1*h2 + h1*h4 + h2*h3 + 2],
    [h1*h2*h3*h4 + 2*h1*h2 + 2*h2*h3 + 4,-h1*h2 + h1*h3 - h2*h3 + h3*h4,h1*h2 + h1*h3 + h2*h4 + 2,
h1*h3 + h1*h4 - h2*h3 + 2],
    [h1*h2*h3*h4 + 2*h1*h2 + 2*h2*h3 + 4,-h1*h2 + h1*h3 - h2*h3 + h3*h4,h1*h2 + h1*h3 + h2*h4 + 2,
h1*h3 + h1*h4 - h2*h3 + 2],
    [h1*h2 + h1*h3 + h3*h4 + 2,h1*h2*h3*h4 - 2*h1*h3 + 2*h2*h3 - 4,h1*h2 - h1*h3 + h2*h3 + h2*h4,
h1*h2 + h1*h4 + h2*h3 + 2],
    [h1*h2*h3*h4 + 2*h1*h2 + 2*h2*h3 + 4,-h1*h2 + h1*h3 - h2*h3 + h3*h4,h1*h2 + h1*h3 + h2*h4 + 2,
h1*h3 + h1*h4 - h2*h3 + 2],
    [h1*h2 + h1*h3 + h3*h4 + 2,h1*h2*h3*h4 - 2*h1*h3 + 2*h2*h3 - 4,h1*h2 - h1*h3 + h2*h3 + h2*h4,
h1*h2 + h1*h4 + h2*h3 + 2],
```

```
    [h1*h2 + h1*h3 + h3*h4 + 2,h1*h2*h3*h4 - 2*h1*h3 + 2*h2*h3 - 4,h1*h2 - h1*h3 + h2*h3 + h2*h4, h1*h2 + h1*h4 + h2*h3 + 2],
    [h1*h2*h3*h4 + 2*h1*h2 + 2*h2*h3 + 4,-h1*h2 + h1*h3 - h2*h3 + h3*h4,h1*h2 + h1*h3 + h2*h4 + 2, h1*h3 + h1*h4 - h2*h3 + 2]
];

L_CDi := [
    [-h1*h3*h4 - 2*h1 + 2*h2 + 2*h3,h1*h2*h3 + h1*h3*h4 - h2*h3*h4 - 2*h2,h1*h3*h4 - 2*h1 - h2*h3*h4 + 2*h4,h1*h2*h4 + 2*h1 + h2*h3*h4 - 2*h2],
    [-h1*h3*h4 - 2*h1 + h2*h3*h4 + 2*h3,h1*h2*h3 + 2*h1 - h2*h3*h4 - 2*h2,h1*h3*h4 - 2*h1 + 2*h2 + 2*h4,h1*h2*h4 - h1*h3*h4 + h2*h3*h4 - 2*h2],
    [-h1*h3*h4 - 2*h1 + h2*h3*h4 + 2*h3,h1*h2*h3 + 2*h1 - h2*h3*h4 - 2*h2,h1*h3*h4 - 2*h1 + 2*h2 + 2*h4,h1*h2*h4 - h1*h3*h4 + h2*h3*h4 - 2*h2],
    [-h1*h3*h4 - 2*h1 + 2*h2 + 2*h3,h1*h2*h3 + h1*h3*h4 - h2*h3*h4 - 2*h2,h1*h3*h4 - 2*h1 - h2*h3*h4 + 2*h4,h1*h2*h4 + 2*h1 + h2*h3*h4 - 2*h2],
    [-h1*h3*h4 - 2*h1 + h2*h3*h4 + 2*h3,h1*h2*h3 + 2*h1 - h2*h3*h4 - 2*h2,h1*h3*h4 - 2*h1 + 2*h2 + 2*h4,h1*h2*h4 - h1*h3*h4 + h2*h3*h4 - 2*h2],
    [-h1*h3*h4 - 2*h1 + 2*h2 + 2*h3,h1*h2*h3 + h1*h3*h4 - h2*h3*h4 - 2*h2,h1*h3*h4 - 2*h1 - h2*h3*h4 + 2*h4,h1*h2*h4 + 2*h1 + h2*h3*h4 - 2*h2],
    [-h1*h3*h4 - 2*h1 + 2*h2 + 2*h3,h1*h2*h3 + h1*h3*h4 - h2*h3*h4 - 2*h2,h1*h3*h4 - 2*h1 - h2*h3*h4 + 2*h4,h1*h2*h4 + 2*h1 + h2*h3*h4 - 2*h2],
    [-h1*h3*h4 - 2*h1 + h2*h3*h4 + 2*h3,h1*h2*h3 + 2*h1 - h2*h3*h4 - 2*h2,h1*h3*h4 - 2*h1 + 2*h2 + 2*h4,h1*h2*h4 - h1*h3*h4 + h2*h3*h4 - 2*h2]
];

I_6_H := [h-i, h+j, i-j, k];
I_12_H := [h, i, j, k];


function Get_lambda(L_CUp, L_CUi, L_CDp, L_CDi)

Mpp := Matrix(Integers(), 8, 128, [Eltseq(Cp!(L_CUp[ii]*L_CDp[jj])):jj in [1..#L_CDp], ii in [1..#L_CUp]]);
Mii := Matrix(Integers(), 8, 128, [Eltseq(Cp!(L_CUi[ii]*L_CDi[jj])):jj in [1..#L_CDi], ii in [1..#L_CUi]]);
M := VerticalJoin(Mpp, Mii);
L_matK := Matrix(K,M);
L_Cp := [Cp!M[ii]:ii in [1..16]];
return L_matK, L_Cp;
end function;

//Get_r_action: Get matrix of right action of Cp on L_Cp < Cp
//input:
//xx in Cp, L_Cp, L_matK
//output:
//16x16 matrix/K
function Get_r_action(xx, L_Cp, L_matK)
Lxx := Matrix(K,16,128,[Eltseq(L_Cp[ii]*xx) : ii in [1..16]]);
return Transpose(Solution(L_matK,Lxx));
end function;

//Get_l_action: Get matrix of left action of Cp on L_Cp
function Get_l_action(xx, L_Cp, L_matK)
xxL := Matrix(K,16,128,[Eltseq(xx*L_Cp[ii]) : ii in [1..16]]);
return Transpose(Solution(L_matK,xxL));
end function;

//Get_CC8_bas: Obtain a basis for the \pm sqrt(-1)-eigenspaces for J in \RR^16
//input:
//J: complex structure in \Cl(T), L_Cp, L_matK
//output:
//two 16x8 matrix/K with respect to Lambda_i<Cp
function Get_CC8_bas(J, L_Cp, L_matK)
J_mat := Get_l_action(Cp!J, L_Cp, L_matK);
//C8p: matrix whose columns span +\sqrt(-1)-eigenspace
CC8p := KernelMatrix(Transpose(J_mat) - DiagonalMatrix(K,[x:ii in [1..16]]));//+ve eigenspace
CC8p := Transpose(CC8p);
```

```
//C8n: matrix whose columns span -\sqrt(-1)-eigenspace
CC8n := KernelMatrix(Transpose(J_mat) + DiagonalMatrix(K,[x:ii in [1..16]]));
CC8n := Transpose(CC8n);
return CC8p,CC8n;
end function;

//Get_coeff: get coefficients when xx is written as linear comb. of basis bas / \HH
//input:
//xx, bas
//output:
//seq /K
function Get_coeff(bas,xx)
xx_seq:=Eltseq(xx);
bas_seq:=Transpose(Matrix(K,4,4,[Eltseq(xx):xx in bas]))^(-1)*Matrix(K,4,1,xx_seq);
return Eltseq(bas_seq);
end function;

//Dot_product: product between sequences v, w
function Dot_prod(v,w)
n := #v;
return &+[v[ii]*w[ii]: ii in [1..n]];
end function;

//Chi: M -> \rchi(M): Representation M_d(\HH_\QQ) -> M_2d(\CC)
function Chi(M)
n := NumberOfRows(M);
A := [];
B := [];
M_seq := Eltseq(M);
for ii in [1..#M_seq] do
z := M_seq[ii];
z_seq := Eltseq(z);
A := A cat [z_seq[1]+x*z_seq[2]];
B := B cat [z_seq[3]+x*z_seq[4]];
end for;
AM := Matrix(n,n,A);
BM := Matrix(n,n,B);
return BlockMatrix(2,2,[AM,BM,-Conj_mat(BM),Conj_mat(AM)]);
end function;

//Chi_R: M -> \rchi_\RR(M): Representation M_d(\CC) -> M_2d(\RR)
function Chi_R(M)
nn := NumberOfRows(M);
A := [];
B := [];
M_seq := Eltseq(M);
for ii in [1..#M_seq] do
z := M_seq[ii];
z_seq := Eltseq(z);
A := A cat [z_seq[1]];
B := B cat [z_seq[2]];
end for;
AM := Matrix(K, nn,nn,A);
BM := Matrix(K, nn,nn,B);
return BlockMatrix(2,2,[AM,BM,-BM,AM]);
end function;

//Phi_std: r -> \rchi(r) \otimes Id_4
//input:
//r in \HH_\QQ
//output:
//8x8 matrix/K
function Phi_std(r)
temp := Eltseq(Chi(Matrix(1,1,[r])));
```

```
return BlockMatrix(2,2,[temp[jj]*IdentityMatrix(K,4):jj in [1..4]]);
end function;

//Phi2Chi: find 8x8 change of basis matrix Q, such that Q*Phi(x)*Q^(-1) = Chi(x) \otimes id_4
function Phi2Chi(tih_mat_8,bas_R)

AminB := ZeroMatrix(K,0,64);
for ii in [1..4] do
//Define A such that Q*Phi(x) given by Eltseq(Q)*A
A := DiagonalJoin([Transpose(tih_mat_8[ii]):jj in [1..8]]);
//Similarly define B such that (Chi(x)\otimes id_4)*Q is Eltseq(Q)*B
temp := Eltseq(Phi_std(bas_R[ii]));
B := BlockMatrix(8,8,[temp[jj]*IdentityMatrix(K,8):jj in [1..64]]);
AminB := VerticalJoin(AminB, A-B);
end for;
//KM: the space Eltseq(Q) such that Q*Phi(x) = (Chi(x)\otimes id_4)*Q
KM := KernelMatrix(Transpose(AminB));

Qs := [];
for ii in [1..NumberOfRows(KM)] do
//transform the length 64 vector back to a square matrix
Q_temp := Matrix(K,8,8,Eltseq(KM[ii]));
//sanity check
//&and[Q_temp*tih_mat_8[kk] eq BlockMatrix(2,2,[Eltseq(Chi(Matrix(1,1,[bas_R[kk]])))[ll]*IdentityMatrix(K,4):ll in [1..4]])*Q_temp: kk in [1..4]];
Qs := Qs cat [Q_temp];
end for;

//Create a non-singular matrix Q from the kernel space spanned by KM
Q := ZeroMatrix(K,8,8);
ii := 1;
while Determinant(Q) eq 0 and ii le #Qs do
Q := Q + ii*Qs[ii];
ii := ii + 1;
end while;
//sanity check
//&and[Q*tih_mat_8[kk]*Q^(-1) eq BlockMatrix(2,2,[Eltseq(Chi(Matrix(1,1,[bas_R[kk]])))[ll]*IdentityMatrix(K,4):ll in [1..4]]): kk in [1..4]];
return Q;
end function;

//Get_xs: find image of e1, e5, e9, e13 in CC^8 wrt \Phi_{std}
//input:
//Q: change of basis matrix to use \Phi_std
//output:
//8x4 matrix whose columns are the x_i's
function Get_xs(CC8p, CC8n, L_matK, Q)
S := Solution(Transpose(HorizontalJoin(CC8p,CC8n)), VerticalJoin([IdentityMatrix(K,16)[jj]: jj in [1,5,9,13]]));
return Matrix(H,Q*Transpose(ColumnSubmatrix(S,8)));
end function;

//Get_calM : obtain generators of \calM
//input:
//m: index of x_m
//output:
//4x4 matrix whose columns are generators of calM with respect to bas_R
function Get_calMkk(kk, tih_mat_16)
ii := 4*(kk-1)+1;
Rx_kk :=[[M[ii+jj,ii]:jj in [0..3]]:M in tih_mat_16];
Rx_kk := Transpose(Matrix(Integers(),Rx_kk));
S, P, T := SmithForm(Rx_kk);
n := LeastCommonMultiple(Diagonal(S));
return Transpose(Matrix(Rationals(),Rx_kk)^(-1)*n);
end function;
```

```
//Mult_H: compute xx^d1*yy^d2 in \HH with respect to bas_R
//input:
//xx, yy coefficients wrt bas_R; d1, d2 in \ZZ
//output:
//sequence of length 4/K: coefficients of product wrt bas_R
function Mult_H(bas_R, xx, d1, yy, d2)
xx_R := &+[xx[jj]*bas_R[jj]: jj in [1..4]];
yy_R := &+[yy[jj]*bas_R[jj]: jj in [1..4]];
prod := xx_R^d1*yy_R^d2;
return Get_coeff(bas_R,prod);
end function;

//calMkk_2_Ikk: change of basis matrix from calM_kk to I_kk
//Input:
//Shortest_calMkk, Shortest_Ikk: shortest vectors in respective lattices
//bilform_mat: quadratic form associated to the lattice generated by bas_R
//calMkk: generators of calM_kk in terms of bas_R
//output:
//4-by-4 matrix
function calMkk_2_Ikk(bas_R, Shortest_calMkk, Shortest_Ikk, bilform_mat, calMkk)
calMkkhh := 0;
L_Id := Lattice(IdentityMatrix(Rationals(),4), bilform_mat);
xx := Shortest_calMkk[1];
//Norm(xx) eq Norm(Shortest_I_m[1]);
for yy in Shortest_Ikk do
hh := Mult_H(bas_R, xx, -1, yy, 1);
if {L_Id!Mult_H(bas_R, zz, 1, hh, 1): zz in Shortest_calMkk} eq Seqset(Shortest_Ikk)
then  calMkkhh := Transpose(Matrix([Mult_H(bas_R, calMkk[ii], 1, hh,1): ii in [1..4]]));
break;
end if;
end for;
return calMkkhh;
end function;

//E: to calculate each entry in the matrix of polarisation
function E(xx, yy, L_Cp, L_matK)
zz := alpha * MainAntiautomorphism(C)(xx) * yy;
zzL_Cp := Matrix(K,16,128,[Eltseq(Cp!zz*L_Cp[i]) : i in [1..16]]);
T := Transpose(Solution(L_matK, zzL_Cp));
return Trace(T);
end function;

//Get_calT: calculate matrix calT as imaginary part of polarisation
function Get_calT(calM2I, bas_R, L_Cp, L_matK)
mm := [1, 5, 9, 13];

//mE: matrix of imaginary part of polarisation with respect to L_Cp
mE := Matrix(K,16,16,[E(xx,yy, L_Cp, L_matK):xx,yy in L_Cp]);
bas_4R := Matrix(H,1,16,[bas_R cat bas_R cat bas_R cat bas_R]);
//LmatK_2_I: matrix taking each column in L_matK to a vector in I in terms of 1, i, j, k
LmatK_2_I := bas_4R * Matrix(H,calM2I);
calT := ZeroMatrix(H,4,4);
l_seq := [2, 1, 4, 3];
//the (hh, ll)-th block in Emat is non-zero, corresponds to a non-zero entry calT_hl in calT
for hh in [1..4] do
ll := l_seq[hh];
mE_hl := Submatrix(mE, mm[hh], mm[ll], 4, 4);

//find calT_hl
vectra:=[mE_hl[1,ii]:ii in [1..4]];
genI_h := [LmatK_2_I[1,mm[hh]+kk]:kk in [0..3]];
genI_l := [LmatK_2_I[1,mm[ll]+kk]:kk in [0..3]];
traces:= Matrix(K,4,4,[Trace(xx*yy):xx in bas_R, yy in [Conjugate(zz)*genI_h[1]:zz in genI_l]]);
```

```
seqT_hl:= Eltseq(traces^(-1)*Matrix(K,4,1,vectra));
calT_hl:= &+[seqT_hl[i]*bas_R[i]:i in [1..4]];

//checking
//matEthl:=Matrix(K,4,4,[Trace(genI_h[ii]*calT_hl*Conjugate(genI_l[jj])):ii,jj in [1..4]]);
//matEthl eq mE_hl;  //TRUE !!!!

//update T
calT[hh,ll] := calT_hl;
end for; //hh
return calT;
end function;

//Get_calH: to obtain the matrix \calH
function Get_calH(J, L_Cp, L_matK, tih_mat_16, bas_R, calM2I)
mm := [1, 5, 9, 13];
J_mat := Get_l_action(Cp!J, L_Cp, L_matK);
calM2I_inv := Matrix(K,calM2I)^(-1);

calH := ZeroMatrix(H,4,4);
//solve sqrt(-1)*x_i = sum_{j=1}^4( sum_{k=1}^4( a_jk * Phi(bas_R[k]) ) * x_j )
for ii in [1..4] do
    Phi_kj := ZeroMatrix(K,16,0);
    for jj in [1..4] do
        for kk in [1..4] do
        //the i-th column of IIcalMinv is bas_R[i] in terms of the e_j's
        Phi_kj := HorizontalJoin(Phi_kj, tih_mat_16[kk]*ColumnSubmatrix(calM2I_inv,mm[jj],1));
        end for;//kk
    end for; //jj
    h_ii := Solution(Transpose(Phi_kj),Transpose(J_mat*ColumnSubmatrix(calM2I_inv,mm[ii],1)));
    h_ii := Matrix(H,16,1,Eltseq(h_ii));
    for jj in [1..4] do
        calH[ii,jj] := &+[h_ii[mm[jj]+kk-1][1]*bas_R[kk]: kk in [1..4]];
    end for; //jj
end for;//ii
return calH;
end function;

//Transform a matrix over K = \QQ<\sqrt(-1), \sqrt(2)> to one over \CC
function K2CC_mat(M)
n := NumberOfRows(M);
M_seq := Eltseq(M);
MCC_seq := [];
for ii in [1..#Eltseq(M)] do
z := M_seq[ii];
z_seq := Eltseq(z);
MCC_seq := MCC_seq cat [z_seq[1] + z_seq[2]*Sqrt(-1) + z_seq[3]*Sqrt(2) + z_seq[4]*Sqrt(-1)*Sqrt(2)];
end for;
return Matrix(n,n,MCC_seq);
end function;

//Get_Xmat: To obtain the square matrix associated to the attribute x_i's
function Get_Xmat(xs)
U := RowSubmatrix(xs,1,4);
V := RowSubmatrix(xs,5,4);
return BlockMatrix(2,2,[U,V,Conj_mat(V),-Conj_mat(U)]);
end function;

//obtain the image z in period domain \calD_\calA given J
function Get_z(calT, calH, xs)
X_mat := Get_Xmat(xs);

//oT, oH: \rchi(T) and \rchi(H)
oT := Chi(calT);
```

```
oH := Chi(calH);

//find oW : \rchi(W)
A:= Matrix(H,4,4,[0,1,0,0,1,0,0,0,0,0,1,0,0,0,0,1]);
temp:=A*calT^(-1)*Transpose(Conj_H_mat(A));
W1 := Matrix(H,2,2,[-i, Conjugate(temp[1,2])^(-1), i, Conjugate(temp[1,2])^(-1)]);
W2 := Matrix(H,2,2,[-i, Conjugate(temp[3,4])^(-1), i, Conjugate(temp[3,4])^(-1)]);
W := DiagonalMatrix([-j/s,1/s,-j/s,1/s])*DiagonalJoin(W1,W2)*A;
//W*calT^(-1)*Transpose(Conj_H_mat(W)) eq DiagonalMatrix([i,i,i,i]); //true
oW := Chi(W);
//x*oW*oT^(-1)*Transpose(Conj_mat(oW)) eq DiagonalJoin(-IdentityMatrix(K,4), IdentityMatrix(K,4)); //true

temp := X_mat*Conj_mat(oW)^(-1);
U := Submatrix(temp,1,1,4,4);
V := Submatrix(temp,1,5,4,4);
z := -V^(-1)*U;
//check z is indeed an element in the period domain \DMT
/*
Transpose(z) eq -z; //true
//IE: contains the set of eigenvalues (over CC, without multiplicity)
z_CC := K2CC_mat(Matrix(K,z));
conj_z_CC := Matrix(4,4,[Conjugate(zz): zz in Eltseq(z_CC)]);
IE := Eigenvalues(1-z_CC*Transpose(conj_z_CC));
&and[Real(IE[jj][1]) gt 0 : jj in [1..#IE]]; //true: 1-zz^* is positive definite
//if false, take -J as the complex structure
*/
return z;
end function;

//================================================================================
//Consider A_n, n = 1.
n := 1;

//L_matK, L_Cp := Get_lambda(n);
L_matK, L_Cp := Get_lambda(L_CUp[n], L_CUi[n], L_CDp[n], L_CDi[n]);
//tih: \ti{h}_i \in \Cl^+(T) for i in [1..4]
hm2 := 2*h1-h2-h3-h4;
tih := [1, hm2*h1, hm2*h2, hm2*h3];
//Check tih spans a primitive lattice in \Cl^+(T)
/*
tih_Lat := Lattice(Matrix(Integers(),4,128,[Eltseq(Cp!tih[ii]): ii in [1..4]]));
Id_Lat := Lattice(IdentityMatrix(Integers(), 128));
Id_Lat/tih_Lat; //\ZZ^124
*/
//tih_mat_16: action of tih as 16x16 matrices
tih_mat_16 := [Get_r_action(Cp!tih[ii], L_Cp, L_matK): ii in [1..4]];

//tih_mat_8: action of tih as 8x8 matrices over CC8p
CC8p, CC8n:= Get_CC8_bas(J, L_Cp, L_matK);
tih_mat_8 := [Transpose(Solution(Transpose(CC8p), Transpose(tih_mat_16[ii]*CC8p))): ii in [1..4]];

//bas_R := basis of R
if n in [1,4,6,7] then
bas_R := [1, -1+i+j-k, 2*i, 2*j];
else
bas_R := [1, -1-i-j+k, -2*i, -2*j];
end if;

//Check Phi: bas_R[ii] -> tih_mat_8[i] is a homomorphism (preserves multiplication)
//[[Dot_prod(Get_coeff(bas_R, bas_R[ii]*bas_R[jj]),tih_mat_8) eq tih_mat_8[ii]*tih_mat_8[jj]: jj in [1..4]]: ii in [1..4]];

Q := Phi2Chi(tih_mat_8,bas_R);
//xs: x_i's satisfying equation 3.3.3(1)
```

```
xs := Get_xs(CC8p, CC8n, L_matK, Q);

//bilform_mat: inner product matrix for R wrt bas_R
bilform_mat := Matrix(Rationals(), 4,4, Eltseq([[bilform(bas_R[i],bas_R[j]): i in [1..4]]: j in [1..4]]));
//I_6  = I_6_H wrt bas_R
I_6 := 2*VerticalJoin([Matrix(Rationals(),1,4,Get_coeff(bas_R, I_6_H[jj])): jj in [1..4]]);
I_12 := 2*VerticalJoin([Matrix(Rationals(),1,4,Get_coeff(bas_R, I_12_H[jj])): jj in [1..4]]);
L_I_6 := Lattice(I_6, bilform_mat);
L_I_12 := Lattice(I_12, bilform_mat);
//Shortest_I_6: Shortest vectors in lattice
Shortest_I_6 := ShortestVectors(L_I_6);
Shortest_I_6 := Shortest_I_6 cat [-1*xx : xx in Shortest_I_6];
Shortest_I_12 := ShortestVectors(L_I_12);
Shortest_I_12 := Shortest_I_12 cat [-1*xx : xx in Shortest_I_12];

I := [I_6, I_6, I_12, I_12];
Shortest_I := [Shortest_I_6, Shortest_I_6, Shortest_I_12, Shortest_I_12];;

//calM2I: change of basis matrix from calM to I
calM2I := ZeroMatrix(K,0,0);
for kk in [1..4] do
calMkk := Get_calMkk(kk, tih_mat_16);
L_calMkk := Lattice(calMkk, bilform_mat);
Shortest_calMkk := ShortestVectors(L_calMkk);
Shortest_calMkk := Shortest_calMkk cat [-1*xx : xx in Shortest_calMkk];
calM2I := DiagonalJoin(calM2I, calMkk_2_Ikk(bas_R, Shortest_calMkk, Shortest_I[kk], bilform_mat, calMkk));
end for;
//\calT satisfying equation 3.3.3(3)
calT := Get_calT(calM2I, bas_R, L_Cp, L_matK);

//\calH satisfying equation 3.3.3(4)
calH := Get_calH(J, L_Cp, L_matK, tih_mat_16, bas_R, calM2I);

//Z = \ti{F}(J) \in \DMT
Get_z(calT, calH, xs);
```

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