Irreducible Holomorphic Symplectic Manifolds and Monodromy Operators

submitted by

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Declaration of any previous Submission of the Work

The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

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Declaration of Authorship

I am the author of this thesis, and the work described herein was carried out by myself personally.

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Abstract

One of the most important tools to study the geometry of irreducible holomorphic symplectic manifolds is the monodromy group. The first part of this dissertation concerns the construction and study of monodromy operators on irreducible holomorphic symplectic manifolds which are deformation equivalent to the 10-dimensional example constructed by O'Grady. The second part uses the knowledge of the monodromy group to compute the number of connected components of moduli spaces of both marked and polarised irreducible holomorphic symplectic manifolds which are deformation equivalent to generalised Kummer varieties.

Contents

Introduction

A Riemannian manifold (M, g) is called irreducible if the holonomy representation associated to the Levi-Civita connection of g is irreducible. An important theorem due to De Rham states that if M is complete and simply connected, then it decomposes as a product of irreducible manifolds and the holonomy representation decomposes accordingly. By a theorem of Berger (e.g. [Bea83b] and [GHJ03, Theorem 3.6]), we have a list of possible (reduced) holonomy groups when M is irreducible and not locally symmetric. Recall that a Riemannian manifold (M, g) is locally symmetric if for every point $p \in M$ there exists a local automorphism of (M, g) fixing p and acting as minus the identity on the tangent space T_pM. Holonomy groups of locally symmetric spaces were already known by the work of Cartan (cf. [GHJ03, Section 3.3]).

The main tool to get geometric information out of the holonomy group is the holonomy principle, which asserts that parallel tensors are those tensors invariant under the holonomy group. So, for example, if the holonomy group of M is contained in the unitary group $\mathrm{U}(n)$, then M is Kähler. The most important example, for our purpose, is the case when the holonomy group is contained in the compact symplectic group $\mathrm{Sp}(n)$, i.e. the group of linear automorphisms of the quaternionic space $\mathbb{H}^n$ preserving the natural Hermitian form. By the holonomy principle then, M has three Kähler structures I, J and K satisfying the usual quaternionic relations. Moreover, if $(a, b, c) \in S^2$ is a point on the unit sphere, then $aI + bJ + cK$ is a Kähler structure on M as well. Because of this sphere of Kähler structures, manifolds with holonomy group contained in $\mathrm{Sp}(n)$ are called *hyper-Kähler* manifolds.

Recall that the compact symplectic group can be written as $\mathrm{Sp}(n) = \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n)$, where $\mathrm{Sp}(2n, \mathbb{C})$ is the symplectic group. If we fix a Kähler structure I on M and we denote by $X = (M, I)$ the corresponding Kähler manifold, then the description of $\mathrm{Sp}(n)$ above implies that there exists a *symplectic structure* on X, i.e. there exists a closed holomorphic 2-form σ_X on X which is non-degenerate at every point. We can explicitely describe σ_X: if g is the Riemannian metric on M, I, J and K are the three Kähler structures on (M, g) and $X = (M, I)$, then $\sigma_X = \omega_J + i\omega_K$, where $\omega_J(u, v) = g(Ju, v)$ is the Kähler form associated to the Kähler structure J (same for K). By [Bea83b, Proposition 4], the converse is also true: if $X = (M, I)$ is a compact Kähler manifold with a symplectic structure, then the holonomy group of M is contained in $\mathrm{Sp}(n)$. Notice that the existence of a symplectic structure forces the complex dimension to be even.

The n-th power of the symplectic form gives a trivialisation of the canonical bundle of M. In particular, the first Chern class of X is trivial and by the Calabi-Yau theorem there exists a Ricci-flat Kähler metric on M with the same Kähler structure. Compact Kähler Ricci-flat manifolds have a refined decomposition theorem (cf. [Bea83b, Théorème 2]) due to, independently, Beauville and Bogomolov: the universal cover $\widetilde{X}$ decomposes as a product

$$\widetilde{X} \cong \mathbb{C}^r \times \prod_i V_i \times \prod_j W_j,$$

where $\mathbb{C}^r$ has the standard metric, V_i has holonomy group equal to $\mathrm{SU}(m)$ and W_j has holonomy group equal to $\mathrm{Sp}(m)$.

Manifolds whose holonomy group is contained in $\mathrm{SU}(m)$ are sometimes called *Calabi-Yau* manifolds (the definition of Calabi-Yau manifolds varies a lot in the literature, depending on the area in which they arise).

As a consequence of the holonomy principle and the Beauville-Bogomolov decomposition theorem, hyper-Kähler manifolds whose holonomy group is equal to $Sp(n)$ are simply connected and the symplectic structure is unique up to scalar (cf. [Bea83b, Proposition 4]). Simply connected compact Kähler manifolds with a unique symplectic structure are called *irreducible holomorphic symplectic* manifolds (see Definition 1.1.1).

We have just seen that *irreducible hyper-Kähler* manifolds, which by definition are compact manifold whose holonomy group is equal to $\mathrm{Sp}(n)$, are the same as irreducible holomorphic symplectic manifolds. We usually refer to irreducible hyper-Kähler manifolds when we want to stress the existence of a sphere of Kähler structures, and irreducible holomorphic symplectic manifolds when we want to stress the existence of the symplectic structure.

The existence of a sphere of Kähler structures has a very strong effect on the geometry of irreducible hyper-Kähler manifolds. For example it implies the existence of twistor families which are the main ingredient in the proof of the surjectivity of the period map of such manifolds (cf. [Huy99, Section 8]). In this dissertation we always keep the point of view of holomorphic symplectic geometry, so we will always talk about irreducible holomorphic symplectic manifolds.

Two-dimensional irreducible holomorphic symplectic manifolds are $K3$ surfaces, which were extensively studied in the last century. Higher dimensional irreducible holomorphic symplectic manifolds share a lot of features with $K3$ surfaces, and in fact the latter were often used as main motivation. The most important example is the Torelli Theorem: all the geometry of a $K3$ surface S is encoded in the cohomology group $H^2(S,\mathbb{Z})$, which has the structure of a unimodular lattice Λ of signature $(3,19)$ given by the intersection product. More precisely, the moduli space of marked $K3$ surfaces (see Section 1.1 for definitions) has two connected components and the restriction of the period map to one such component is an isomorphism into an open subset (the period domain) in the projective space $\mathbb{P}(\Lambda \otimes \mathbb{C})$. The number of connected components is a consequence of the fact that the monodromy group $\mathrm{Mon}^2(S) = \mathrm{O}^+(H^2(S,\mathbb{Z})) \subset \mathrm{O}(H^2(S,\mathbb{Z}))$ has index 2 (see Section A.5 for the definition of $\mathrm{O}^+(H^2(S,\mathbb{Z}))$).

For an irreducible holomorphic symplectic manifold X of higher dimension, the cohomology group $H^2(X,\mathbb{Z})$ is also a lattice, of signature $(3, b_2(X)-3)$ and not unimodular in general. The lattice structure is non-trivial and it is induced by the Beauville-Bogomolov-Fujiki form. Weaker forms of the Torelli Theorem are also known (see Section 1.2).

In the $K3$ surface case above we notice two things, which are closely related to each other: we have a complete description of the monodromy group, and the precise number of connected components. The aim of this dissertation is to investigate these two questions for higher dimensional irreducible holomorphic symplectic manifolds.

There are four kinds of known irreducible holomorphic symplectic manifolds. Two of these arise in any dimension ($K3^{[n]}$-type and Kum^n-type, where $2n$ is the dimension), while the other two exist only in dimension 6 (OG6-type) and in dimension 10 (OG10-type). The question whether this is a complete list or not is still open. The monodromy group is known for manifolds of $K3^{[n]}$-type and Kum^n-type thanks to the work of Markman and Mongardi (see Section 6.2.1). In this case it is natural to ask what is the number of connected components of the moduli space of (both marked and polarised) irreducible holomorphic symplectic manifolds of fixed deformation type. This question was answered by Apostolov for manifolds of $K3^{[n]}$-type as part of his PhD thesis (see [Apo14]), and by the author for manifolds of Kum^n-type (see Section 6.2 and [Ono16]).

The monodromy group of manifolds of OG6-type has been recently announced by Mongardi, Rapagnetta and Saccà, but it is still unpublished. They used their recent work ([MRS18]) to study the monodromy of such manifolds by deducing it from the monodromy of a manifold of $K3^{[3]}$-type.

For manifolds of OG10-type the situation is much more obscure. There are very few known examples of monodromy operators and no clue about the shape of the monodromy group. Markman studied in [Mar10b] some monodromy operators induced by symplectic resolutions of singular symplectic varieties. The result is very important by itself, but unfortunately it does not give a big contribution to the problem. In an unpublished work, Markman also tried to study monodromy operators which arise from auto-equivalences of the derived category of $K3$ surfaces, and this led him to conjecture in [Mar11] that the monodromy group would be the whole group of orientation preserving isometries (see Section A.5 for definitions). Notice that the monodromy group is always contained in the group of orientation preserving isometries (Proposition 1.3.5), so Markman conjectured that it is maximal. The conjecture was recently disproved by Mongardi in [Mon16], but his counter-example sheds no geometric light on the problem since it is purely lattice-theoretic.

A main aim of this thesis is to construct new monodromy operators on manifolds of OG10-type and try to gain as much information as possible in order to determine their monodromy group.

First of all, let us notice that the failure of the previous attempts was caused by the lack of examples: until few years ago the only examples of manifolds of OG10-type were symplectic resolutions of singularities of singular moduli spaces of sheaves on $K3$ surfaces. Such manifolds form a codimension 2 sub-family in the moduli space. Recently a new example appeared, namely the symplectic compactification of the intermediate Jacobian fibration associated to a (generic) smooth cubic fourfold due to Laza, Saccà and Voisin. This construction produces a codimension 1 sub-family in the moduli space and hence it is expected to give new and interesting monodromy operators. We studied the monodromy operators induced by the cubic fourfold and the result is Theorem 5.3.2. The family constructed by Laza, Saccà and Voisin is a polarised family of Lagrangian fibrations, so it was natural to expect that the monodromy operators arising from this family preserve both the class of the polarisation and the class of the fibration. What is new and interesting is the action on the discriminant group, which must be the identity. On the other hand, the monodromy operators studied by Markman have no constraint on their action on the discriminant group. This leads one naturally to expect that the monodromy group must have index 4 in the whole group of isometries. We also studied monodromy operators induced from the $K3$ surface in the codimension 2 family constructed by O'Grady. The result in Theorem 5.1.12 was expected but it was never proved before. The monodromy operators arising from this family also act as the identity on the discriminant group. In this picture, the monodromy operators arising from the family of Laza, Saccà and Voisin seem to suggest that we have now enough monodromy operators to hope for a concrete description of the monodromy group. In order to get a formal statement one needs to write down an explicit parallel transport operator between these two families. This problem is a bit too far off now because the theory has still some gaps. In particular, we do not have a clear description of the birational map between the intermediate Jacobian fibration of a Pfaffian cubic fourfold and the symplectic resolution of singularities of a singular moduli space of sheaves on the $K3$ surface dual to the cubic fourfold, constructed by Laza, Saccà and Voisin in [LSV17] and recalled in Section 3.4. The problem is that they give an explicit description of this map in an open subset whose boundary contains a divisor, and so it is not possible to understand, for example, where the class of this divisor goes via the isometry induced on the Picard lattice (see Remark 3.4.10).

A first attempt to give a geometric description of Mongardi's counter-example led us to study a twisted version of the intermediate Jacobian fibration, introduced by Voisin in [Voi16]. Unfortunately this did not give the hoped result, but our contribution to the theory was a description of a twisted Theta divisor which was highly non-trivial at first (see Section 4.4). Moreover, this makes us also able to explicitly describe the Picard lattice of the very general member of this family.

List of main results

- Theorem 4.4.3: definition of a relative twisted Theta divisor on the compactification $\mathcal{J}_V^T$ of the twisted intermediate Jacobian fibration associated to a generic cubic fourfold V.
- Theorem 4.5.4: variation of Hodge structures for the manifolds $\mathcal{J}_V^T$.
- Theorem 5.1.12: study of monodromy operators on manifolds of OG10-type arising from families of $K3$ surfaces.
- Theorem 5.3.2: study of monodromy operators on manifolds of OG10-type arising from families of smooth cubic fourfolds.
- Theorem 6.2.26: computation of the number of connected components of the moduli space of polarised irreducible holomorphic symplectic manifolds of Kum^n-type.

Description of the thesis chapter by chapter

Chapter 1 contains the background and the standard results about irreducible holomorphic symplectic manifolds which we will use in the remaining chapters. In Section 1.1 we recall the very basic properties and we present the most important examples. We also include a subsection about moduli spaces of sheaves: beyond providing more examples of irreducible holomorphic symplectic manifolds, the theory is important by itself and it is worth having a detailed treatment of the subject. Section 1.2 is a summury of the statements which are known as Torelli theorems for irreducible holomorphic symplectic manifolds. In Section 1.3 we give the definition of parallel transport operators and monodromy operators. We analyse some basic properties and we explain why the knowledge of the monodromy group is important. We also talk about polarised parallel transport operators. In Section 1.4 we recall the construction of the moduli space of polarised irreducible holomorphic symplectic manifolds as an analytic space. This will be the main object in Section 6.2.

In Chapter 2 we recall the general theory of cubic fourfolds as developed by Hassett, and their relation to irreducible holomorphic symplectic manifolds. In Section 2.1.1 we recall Hassett's work on cubic fourfolds, stressing their relation with $K3$ surfaces. In Section 2.1.2 we recall Beauville's computation of the monodromy group of a cubic fourfold. In Section 2.2.1 we recall the Beauville-Donagi result about the Fano variety of lines of a cubic fourfold: this is an irreducible holomorphic symplectic manifold of $K3^{[2]}$-type. These results, together with Section 2.1.2, are used in Section 2.2.2 to compute the (polarised) monodromy group of such irreducible holomorphic symplectic manifolds. The result is not new. As we said the monodromy group of manifolds of

$K3^{[n]}$-type was computed by Markman for every n. Nevertheless, the argument we use is new and it will be used again to study monodromy operators on manifolds of OG10-type.

In Chapter 3 we recall the Laza-Saccà-Voisin construction of the symplectic compactification of the intermediate Jacobian fibration of a (generic) cubic fourfold. This is the main ingredient for our study of the monodromy operators on manifolds of OG10-type. In Section 3.1 we recall the background of the problem. Our contribution is the determination of a distinguished divisor on the total space of the fibration which represents the relative Theta divisor. Section 3.2 deals with 1-nodal sections: here we recall how to partially compactify the intermediate Jacobian fibration to singular sections having only one node as singularity. We also study how the distinguished Theta divisor behaves in this partial compactification. In Section 3.3 we recall the main ideas behind the symplectic compactification (LSV compactification) of the intermediate Jacobian fibration. Section 3.4 recalls the construction of the birational map from the LSV compactification to the O'Grady moduli space when the cubic fourfold is Pfaffian. Finally Section 3.5 contains a result connecting the variation of Hodge structures of the LSV compactification to the variation of Hodge structures of the cubic fourfold. This result was supposed to be part our work, but it was communicated to us by Hulek and Laza that they had already got the result as stated in the section.

Chapter 4 is dedicated to the twisted intermediate Jacobian fibration associated to a cubic fourfold. In Section 4.1 we recall the definition of the twisted intermediate Jacobian of a smooth cubic threefold. In Section 4.2 we recall the construction of the twisted intermediate Jacobian fibration and its partial compactification to 1-nodal sections. In Section 4.3 we recall the construction of the symplectic compactification as developed in [Voi16]. Section 4.4 and Section 4.5 contains new results: we construct a twisted Theta divisor and use it to extend the results in Section 3.5 to the twisted case.

In Chapter 5 we construct and study new monodromy operators on manifolds of OG10-type In Section 5.1 we study monodromy operators in families induced by families of $K3$ surfaces. The isometries arising in this way were already expected to be monodromy operators, but no-one ever produced a rigorous proof of this fact. The main ingredient developed in this section is the construction of a family of Donaldson-Uhlenbeck-Yau moduli spaces and a morphism connecting the family of O'Grady's moduli spaces to this family. Section 5.2 is a recollection of Markman's work on monodromy operators arising from symplectic desingularisations of singular symplectic varieties. In Section 5.3 we study the monodromy operators that arise from from families of manifolds of OG10-type induced by families of cubic fourfolds. These monodromy operators are new and were not known before. Finally, in Section 5.4 we discuss the shape of the monodromy group of manifolds of OG10-type. We recall Markman's conjecture and Mongardi's counter-example to it, and we explain how our work helps to shed some new and interesting light about the shape of the monodromy group.

Chapter 6 contains results about the topology of moduli spaces of irreducible holomorphic symplectic manifolds, in particular their connectedness. In Section 6.1 we recall finiteness results, mostly due to Huybrechts and Verbitsky, for irreducible holomorphic symplectic manifolds in general. This serves as motivation and starting point for the next section. Section 6.2 contains the explicit computation of the number of connected components of moduli spaces of both marked and polarised irreducible holomorphic symplectic manifolds of Kum^n-type. The case of manifolds of $K3^{[n]}$-type was studied by Apostolov in [Apo14], and here we extend this results to manifolds of Kum^n-type. We first recall the known results about the monodromy groups of these manifolds and we give a characterisation of parallel transport operators. Then the last two subsections are dedicated to the actual computation for the two moduli spaces.

In Appendix A we recall some facts from the theory of non-degenerate lattices. We only included basic definitions and result that are used in the body of the dissertation, referring to standard references for the general theory.

Since the protagonist of this dissertation is the OG10-type of irreducible holomorphic symplectic manifolds, we decided to recall in Appendix B the original construction of O'Grady. We also collected results from other authors which are used (explicitly or as a motivation) in the body of the text.

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Chapter 1

Irreducible holomorphic symplectic manifolds

1.1 Generalities and examples

Definition 1.1.1. A compact Kähler manifold X is called *irreducible holomorphic symplectic* if it is simply connected and there exists a unique (up to scalar) closed holomorphic 2-form $\sigma_X \in H^0(X, \Omega_X^2)$ which is nondegenerate at all points $x \in X$. Such a form σ_X is called *symplectic*.

Remark 1.1.2. From an algebraic geometry point of view irreducible symplectic varieties can be defined on any field. Since we will make use of analytic technics and results, we are forced to take the perspective of complex analytic geometry.

Notice that the symplectic 2-form σ_X defines a skew-symmetric isomorphism between the holomorphic tangent bundle T_X and the holomorphic cotangent bundle Ω_X. In particular the complex dimension of X is always even. If X has complex dimension $2n$ then σ_X^n defines a trivialisation of the canonical bundle $K_X = \Omega_X^{2n}$, which is thus trivial as a holomorphic line bundle. It follows that the Kodaira dimension of X is 0.

Compact Kähler simply connected manifolds with trivial canonical bundle are described by the following theorem due to, independently, Beauville and Bogomolov (built on a previous decomposition result by De Rham).

Theorem 1.1.3 (Beauville-Bogomolov-De Rham, [Bea83b],[Bog74b]). *Any compact Kähler simply connected manifolds with trivial canonical bundle can be written as a product of irreducible holomorphic symplectic manifolds and Calabi-Yau manifolds.*

Here we use the strong definition of Calabi-Yau manifold, namely that $H^0(\Omega_Y^p) = 0$ for every $0 < p < \dim Y$.

As we said in the Introduction, this theorem is proved using differential methods, especially the Berger classification of holonomy groups. The following is a direct consequence of the holonomy principle.

Proposition 1.1.4 ([Bea83b, Proposition 3]). *Let X be an irreducible holomorphic symplectic manifold of dimension $2n$. Then*

$$H^0(X,\Omega_X^p) = \begin{cases} \mathbb{C}\,\sigma^k & \text{if } p = 2k \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\chi(X) = n + 1$.

Example 1.1.5. When $\dim X = 2$, irreducible holomorphic symplectic manifolds are $K3$ surfaces.

The theory of $K3$ surfaces is well understood and it is used as an inspiration to study irreducible holomorphic symplectic manifolds.

Examples of irreducible holomorphic symplectic manifolds are difficult to find and for a long time $K3$ surfaces were the only known ones. This led Bogomolov to conjecture ([Bog74a]) that in fact there were no higher dimensional examples.

The first one to disprove this conjecture was Fujiki, who proved that the Hilbert square of a $K3$ surface is an irreducible holomorphic symplectic manifold of dimension 4. This was next generalised by Beauville to any Hilbert power of $K3$ surfaces.

Example 1.1.6 ($K3^{[2]}$ manifolds)**.** Let S be a projective $K3$ surface. The Hilbert scheme $S^{[2]}$ of 0-dimensional subschemes of length 2 on S is a smooth projective irreducible manifold of dimension 4. Beauville noticed that the symplectic form of S lifts to a symplectic form on $S^{[2]}$ ([Bea83b, Proposition 5]) and the simply connectedness of S implies the simply connectedness of $S^{[2]}$ ([Bea83b, Lemme 1]).

The variety $S^{[2]}$ comes equipped with the Hilbert-Chow morphism

$$\epsilon\colon S^{[2]} \longrightarrow S^{(2)}$$

where $S^{(2)}$ is the symmetric product of S. The morphism ϵ associates to a closed subscheme of dimension 0 and length 2 on S the pair of points of its support and it is the blow-up of $S^{(2)}$ along the diagonal. The pullback ϵ^* is an injective morphism of Hodge structures on the degree 2 complex cohomology ([Bea83b, Lemme 2]). The group $H^2(S^{(2)},\mathbb{C})$ is Hodge isomorphic to the invariant part (under the symmetric group) of $H^2(S^2,\mathbb{C})$, giving the decomposition $H^2(S^{[2]},\mathbb{C}) = H^2(S,\mathbb{C}) \oplus \mathbb{C}E$. Here E denotes the exceptional divisor of ϵ. In particular the symplectic structure is unique (up to scalar) and we get a family of examples of irreducible holomorphic symplectic fourfolds. Notice that the second Betti number is $b_2(S^{[2]}) = 23$.

One can show that there exists an integral class δ such that $2\delta = E$. Working with integral cohomology, this eventually yields

$$H^2(S^{[2]},\mathbb{Z}) = H^2(S,\mathbb{Z}) \oplus \mathbb{Z}\delta. \tag{1.1.1}$$

In particular, the cohomology group $H^2(S^{[2]},\mathbb{Z})$ is torsion free and moreover one can define a symmetric bilinear pairing on $H^2(S^{[2]},\mathbb{Z})$ such that δ is orthogonal to $H^2(S,\mathbb{Z})$, the restriction to $H^2(S,\mathbb{Z})$ coincides with the intersection product on S and $\delta^2=-2$. This pairing is well defined and makes $H^2(S^{[2]},\mathbb{Z})$ into an even lattice of signature $(3,20)$ isometric to the abstract lattice (cf. Example A.1.7)

$$U^{\oplus 3}\oplus E_8(-1)^{\oplus 2}\oplus\langle -2\rangle.$$

Example 1.1.7 ($K3^{[n]}$ manifolds)**.** Let $n>2$ and consider the Hilbert scheme $S^{[n]}$. The Hilbert-Chow morphism $\epsilon\colon S^{[n]}\to S^{(n)}$ is still a blow-up of $S^{(n)}$ along the (scheme-theoretic) union of the pairwise diagonals ([Hai01, Proposition 3.8.4]), and it is still a resolution of singularities. We denote by E the exceptional divisor. What we said in Example 1.1.6 holds true for $S^{[n]}$ (see [Bea83b]). So $S^{[n]}$ is an irreducible holomorphic symplectic manifold of dimension $2n$, $H^2(S^{[n]},\mathbb{Z})=H^2(S,\mathbb{Z})\oplus\mathbb{Z}\delta$ and one can define a nondegenerate bilinear form on it, which restricts to the usual intersection product on $H^2(S,\mathbb{Z})$. As before, δ is a class such that $E=2\delta$. With this bilinear form it is an even lattice of signature $(3,20)$ abstractly isometric to

$$U^{\oplus 3}\oplus E_8(-1)^{\oplus 2}\oplus\langle 2-2n\rangle.$$

Remark 1.1.8. If S is a non-projective $K3$ surface, working with the Douady space instead of the Hilbert scheme provides non-projective examples of irreducible holomorphic symplectic manifolds.

The main features in this example are not special, but hold more generally for every irreducible holomorphic symplectic manifold X. First of all, notice that by simply connectedness, $H^2(X,\mathbb{Z})$ is always torsion free.

Theorem 1.1.9 ([Bea83b],[Fuj87])**.** *Let X be an irreducible holomorphic symplectic manifold of dimension $2n$. Then there exists an integral nondegenerate quadratic form q_X on $H^2(X,\mathbb{Z})$ of signature $(3,b_2(X)-3)$ and a positive constant c_X such that*

$$\int_X \alpha^n = c_X q_X(\alpha)^{2n} \tag{1.1.2}$$

for every $\alpha\in H^2(X,\mathbb{Z})$.

The quadratic form q_X is called the *Beauville-Bogomolov-Fujiki form* and the constant c_X is called *Fujiki's constant.* The lattice $(H^2(X,\mathbb{Z}),q_X)$ is called *Beauville-Bogomolov-Fujiki* lattice.

The Beauville-Bogomolov-Fujiki form can be written explicitly as

$$q_X(\alpha)=\frac{n}{2}\int_X(\sigma_X\bar\sigma_X)^{n-1}\alpha^2+(1-n)\left(\int_X\sigma_X^n\bar\sigma_X^{n-1}\alpha\right)\left(\int_X\sigma_X^{n-1}\bar\sigma_X^n\alpha\right) \tag{1.1.3}$$

for every $\alpha\in H^2(X,\mathbb{C})$, where $\sigma_X\in H^0(X,\Omega_X^2)$ is normalised so that $\int_X(\sigma_X\bar\sigma_X)^n=1$.

Remark 1.1.10. For all the known examples of irreducible holomorphic symplectic manifolds X, the lattice $(H^2(X,\mathbb{Z}), q_X)$ is even. Nevertheless it is still unknown whether this is a general feature or whether there may exist examples of manifolds X such that $(H^2(X,\mathbb{Z}), q_X)$ is odd.

Remark 1.1.11. Let $p\colon \mathcal{X} \to B$ be a family of (analytic) varieties, i.e. p is proper and smooth, such that the fibre $\mathcal{X}_0$ over a distinguished point $0 \in B$ is an irreducible holomorphic symplectic manifold. Then by [Bea83b, Proposition 9], up to shrinking B if necessary, for every $b \in B$ the fibres $\mathcal{X}_b$ are irreducible holomorphic symplectic manifolds. Therefore we can talk of families of irreducible holomorphic symplectic manifolds.

Remark 1.1.12. Both the Beauville-Bogomolov-Fujiki form and Fujiki's constant remain invariant in families.

Example 1.1.13. For irreducible holomorphic symplectic manifolds deformation equivalent to the Hilbert scheme of points (Example 1.1.7) the Fujiki constant is $\frac{(2n)!}{n!2^n}$ ([Rap08]).

Example 1.1.14 (Generalised Kummer varieties)**.** Let $n \geq 2$. If A is an abelian surface, the Hilbert scheme $A^{[n+1]}$ is again an irreducible smooth projective variety of dimension $2n+2$ and [Bea83b, Proposition 5] applies as well. On the other hand, [Bea83b, Lemme 1] implies that $A^{[n+1]}$ is not simply connected. The Albanese map

$$\mathfrak{a}\colon A^{[n+1]} \longrightarrow A \tag{1.1.4}$$

is simply the composition of the Hilbert-Chow morphism with the sum map, defined using the (additive) group structure on A.

We denote by $\mathrm{K}^{[n]}(A)$ its fibre. It is an irreducible smooth projective variety of dimension $2n$ and the symplectic form on $A^{[n+1]}$ restricts to a symplectic form on $\mathrm{K}^{[n]}(A)$ ([Bea83b, Proposition 7]). Again we have a decomposition ([Bea83b, Proposition 8])

$$H^2(\mathrm{K}^{[n]}(A),\mathbb{Z}) = H^2(A,\mathbb{Z}) \oplus \bar{\delta}$$

where $\bar{\delta}$ is an integral class such that $2\bar{\delta}$ is the restriction to $\mathrm{K}^{[n]}(A)$ of the exceptional divisor E of the Hilbert-Chow morphism. As before, $\mathrm{K}^{[n]}(A)$ gives another family of examples of irreducible holomorphic symplectic manifolds in any (even) dimension greater or equal to 4.

The second Betti number $b_2(\mathrm{K}^{[n]}(A)) = 7$ is independent of the dimension and different from the one computed before for $S^{[n]}$, so these two families of examples are not birational and not even deformation equivalent.

The Beauville-Bogomolov-Fujiki form on $H^2(\mathrm{K}^{[n]}(A),\mathbb{Z})$ restricts again to the intersection product on $H^2(A,\mathbb{Z})$ and hence the lattice structure is isometric (cf. Example A.1.8) to the abstract lattice

$$U^{\oplus 3} \oplus \langle 2+2n \rangle.$$

For irreducible holomorphic symplectic manifolds deformation equivalent to generalised Kummer varieties, the Fujiki constant is $\frac{(2n)!}{n!2^n}(n+1)$ ([Rap08]).

Let $\pi\colon \mathcal{X} \to \mathrm{Def}(X)$ be the Kuranishi family of an irreducible holomorphic symplectic manifold $X \cong \mathcal{X}_0$. Notice that such a family exists by a theorem of Kuranishi, and moreover the universal deformation π is universal for any of its fibres (see [GHJ03, Theorem 22.3]).

Theorem 1.1.15 ([Bog78],[GHJ03])**.** *Up to shrinking* $\mathrm{Def}(X)$ *if necessary,* $\mathrm{Def}(X)$ *is smooth of dimension* $b_2(X) - 2$.

Definition 1.1.16.

- Any irreducible holomorphic symplectic manifold deformation equivalent to the one constructed in Example 1.1.7 is called of $K3^{[n]}$-type.
- Any irreducible holomorphic symplectic manifold deformation equivalent to the one constructed in Example 1.1.14 is called of Kum^n-type.

If Λ is a fixed lattice, a Λ-*marking* is an isometry $\eta\colon H^2(X,\mathbb{Z}) \to \Lambda$. If there is no confusion, we usually simply call it a marking.

Remark 1.1.17. Notice that, by Remark 1.1.12, the choice of a deformation type fixes a lattice Λ. It is not known if there exist more deformation types with the same lattice structure. We will see in Section 6.1 though that there are only finitely many deformation types of irreducible holomorphic symplectic manifolds of fixed dimension.

Choosing a trivialisation of the local system $R^2\pi_*\mathbb{Z}$, we can extend η to the other members of the family, $\eta_t\colon H^2(\mathcal{X}_t,\mathbb{Z}) \to \Lambda$. We have a holomorphic map ([Voi02, Theorem 10.9])

$$\mathcal{P}\colon \mathrm{Def}(X) \longrightarrow \mathbb{P}(\Lambda \otimes \mathbb{C})$$

called *period map* sending t to the line $\eta_t\left(H^{2,0}(\mathcal{X}_t)\right)$.

Theorem 1.1.18 (Local Torelli Theorem, [Bea83b, Théorème 5])**.** *The period map is a local isomorphism onto the period domain*

$$\Omega_\Lambda = \{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (x,x) = 0 \text{ and } (x,\bar{x}) > 0\}, \tag{1.1.5}$$

where $(\cdot,\cdot)$ *is the bilinear form of* Λ.

1.1.1 Other examples: moduli spaces of sheaves

Let S be an abelian or $K3$ surface. We see in Example A.1.9 that the even cohomology ring $H^{\text{even}}(S,\mathbb{Z})$ of S is an even unimodular lattice, called *Mukai lattice* and denoted by $\widetilde{H}(S)$.

The topological Grothendieck group $\mathrm{K}(S)$ of coherent sheaves is a torsion free $\mathbb{Z}$-module naturally isomorphic to $H^{\text{even}}(S,\mathbb{Z})$ via the Chern character, $ch\colon \mathrm{K}(S) \to H^{\text{even}}(S,\mathbb{Z})$ (cf. [AH61, Section 2.4]). We can define a lattice structure on $\mathrm{K}(S)$ by

$$(E,F) := -\chi(E^\vee \otimes F)$$

where χ is the Euler characteristic. The Chern character ch is not an isometry, but this deficiency is fixed by twisting with the Todd class of the surface. More precisely, define $\mathfrak{v}\colon \mathrm{K}(S) \to H^{\text{even}}(S,\mathbb{Z})$ by sending a class E to $\mathfrak{v}(E) := ch(E)\sqrt{td_S}$. The fact that $\mathfrak{v}$ is an isometry is the statement of the Hirzebruch-Riemann-Roch theorem.

We can define a Hodge structure of weight 2 on $\widetilde{H}(S)\otimes\mathbb{C}$ (and hence on $\mathrm{K}(S)\otimes\mathbb{C}$) by putting $\widetilde{H}^{1,1}(S) = H^0(S,\mathbb{C}) \oplus H^{1,1}(S,\mathbb{C}) \oplus H^4(S,\mathbb{C})$ and $\widetilde{H}^{2,0}(S) = H^{2,0}(S,\mathbb{C})$.

A vector $v = (r,l,s) \in \widetilde{H}(S)$ is called a *Mukai vector* if ([Yos01, Definition 0.1]) $l \in \mathrm{NS}(S)$ and either:

1. $r > 0$; or
2. $r = 0$, $0 \neq l \in \mathrm{NS}(S)$ is effective and $s \neq 0$; or
3. $r = 0 = l$ and $s < 0$.

Fix now a polarisation H on S and a Mukai vector $v \in \widetilde{H}(S)$. Consider the moduli space $M = M_H(v)$ parametrising Gieseker H-stable sheaves on S and its compactification $\overline{M}_H(v)$ obtained by adding Gieseker H-semistable sheaves ([HL10, Chapter 4]). Recall that $\overline{M}_H(v)$ is projective and $M_H(v)$ is open in it.

Example 1.1.19. The moduli space $M(0,0,-n)$ is empty unless $n=1$, in which case it is isomorphic to S itself. On the other hand $\overline{M}(0,0,-n)$ is isomorphic to the symmetric product $S^{(n)}$.

Example 1.1.20. Let $v = (1,0,\epsilon - n) \in \widetilde{H}(S)$, where $\epsilon = 0,1$ according to S being an abelian or $K3$ surface. Let H be a polarisation on S. Any sheaf $F \in M_H(v)$ is torsion free and hence we have an embedding $F \subset F^{\vee\vee} \cong \mathcal{O}_S$ ([Har80, Corollary 1.4]). It turns out that F is the ideal sheaf of a 0-dimensional closed subscheme of length n in S and so we get an isomorphism $M_H(1,0,\epsilon - n) \cong S^{[n]}$.

If $r > 0$, a celebrated result of Mukai ([Muk84, Corollary 0.2]) states that $M_H(v)$ is either empty or a smooth quasi-projective variety of dimension $(v,v)+2$ which carries a natural symplectic structure. In particular, if H is chosen such that any H-semistable sheaf F with $\mathfrak{v}(F) = v$ is H-stable, then $M_H(v)$ is a smooth projective symplectic manifold.

Remark 1.1.21. The ample cone $\mathrm{Amp}(S)$ has a wall and chamber decomposition with respect to v, such that the set of non-empty v-walls is locally finite.

Suppose v is primitive. The definition of the walls depends on the component r of $v = (r, l, s)$. If $r = 1$ then the decomposition is trivial (cf. Example 1.1.20). The case $r > 1$ is explained in [HL10, Section 4.C] (cf. [Yos96]). The case $r = 0$ and $l \neq 0$ effective is studied in [Yos01, Subsection 1.4]. Finally, the remaining case $r = 0 = l$ is not interesting since we have seen in Example 1.1.19 that the moduli space is well understood in this case.

If v is not primitive, then v-chambers and v-walls are defined in [PR13].

Any polarisation contained in a v-chamber is called *v-generic*.

For the rest of this section we assume that v is primitive. In this case, for any polarisation in a v-chamber, the resulting moduli spaces are compact, i.e. H-semistability coincides with H-stability.

Remark 1.1.22. If $v = mv_0$, where v_0 is primitive and $m > 1$, then even if H belongs to a v-chamber, it is no longer true that $\overline{M}_H(v) = M_H(v)$ (cf. Appendix B). On the other hand, even if v is primitive, one can have the equality $\overline{M}_H(v) = M_H(v)$ for v-special polarisations (i.e. non-generic polarisations). Sometimes in the literature a polarisation H such that $\overline{M}_H(v) = M_H(v)$ is called v-general. Notice that v-generality is stronger than v-genericity.

One of the main tools to study moduli spaces is the existence of a universal family. Sufficient conditions are known for the existence of the universal sheaf: for example it is true whenever there exists a vector $v' \in \widetilde{H}(S)$ such that $(v, v') = 1$ ([Muk87, Theorem A.6]). Nevertheless, Mukai noticed ([Muk87, Theorem A.5]) that even if a universal sheaf does not exist, a twisted version of it always exists (cf. [HL10, Section 4.6]). More precisely, a sheaf $\mathcal{E}$ on $M_H(v) \times S$ is called *quasi-universal of similitude* ρ if $\mathcal{E}$ is flat over $M_H(v)$ and if for every $t \in M_H(v)$ there exists a H-stable sheaf E on S with $\mathfrak{v}(E) = v$ such that $\mathcal{E}_t = E^{\oplus \rho}$. Such a sheaf is universal in the sense that for any sheaf $\mathcal{F}$ on $T \times S$, flat over T and parametrising sheaves in $M_H(v)$, there exists a unique morphism $f : T \to M_H(v)$ such that $\mathcal{F} = f^*\mathcal{E}$. Notice that two quasi-universal sheaves $\mathcal{E}$ and $\mathcal{E}'$ are equivalent if there exist vector bundles V and V' on $M_H(v)$ such that $\mathcal{E} \otimes p^*V \cong \mathcal{E}' \otimes p^*V'$, where $p \colon M_H(v) \times S \to M_H(v)$ is the projection.

Remark 1.1.23. The primitivity of v does not imply the existence of a universal sheaf in general ([Yos98, Example after Theorem 0.1]).

Quasi-universal families are used to link the Hodge theory of the surface to the Hodge theory of the moduli space. Pick a quasi-universal sheaf $\mathcal{E}$ of similitude ρ and define the algebraic cycle $Z_{\mathcal{E}} \in H^*(M_H(v) \times S, \mathbb{Q})$ as

$$Z_{\mathcal{E}} = \frac{1}{\rho}\left(q^*\sqrt{td_S}.ch(\mathcal{E}).p^*\sqrt{td_{M_H(v)}}\right),$$

where p and q are the projections from $M_H(v) \times S$ to $M_H(v)$ and S, respectively.

Define the map

$$\theta_v \colon \widetilde{H}(S) \longrightarrow H^2(M_H(v), \mathbb{Q}) \tag{1.1.6}$$

by sending $x \in \widetilde{H}(S)$ to $\theta_v(x) = p_*\left(Z_{\mathcal{E}}.q^*(x^\vee)\right)$, and notice that this morphism is independent of the choice of the quasi-universal family.

Let us consider now the case in which $v = (r, l, s)$ is isotropic in $\widetilde{H}(S)$, $r > 0$ and $M_H(v)$ is non-empty and compact. By the results of Mukai we recalled before, $M_H(v)$ is a smooth projective symplectic surface, hence it is either an abelian or a $K3$ surface.

Proposition 1.1.24 ([Muk87, Theorem 1.4, Theorem 1.5])**.** *In this situation we have that θ_v is defined over the integers and it induces a Hodge isometry*

$$\theta_v \colon v^\perp/\mathbb{Z}v \longrightarrow H^2(M_H(v), \mathbb{Z}).$$

Moreover,

1. *if S is abelian, then $M_H(v)$ is abelian;*
2. *if S is $K3$, then $M_H(v)$ is $K3$.*

To study the analogous situation in higher dimensions, we divide into two cases.

The $K3$ surface case

Let S be a projective $K3$ surface, $v \in \widetilde{H}(S)$ a primitive Mukai vector and H a v-generic polarisation.

If $(v, v) > 0$, O'Grady noticed that $\theta_v|_{v^\perp}$ is integral, under some assumptions on v, and indeed a Hodge isometry ([O'G97b, Main Theorem]). Moreover, he also proves that in this case $M_H(v)$ is an irreducible holomorphic symplectic manifold deformation equivalent to the Hilbert scheme of points $S^{[(v,v)/2+1]}$.

O'Grady's result has been further generalised by Yoshioka and the final statement can be summarised in the following theorem.

Theorem 1.1.25 ([Yos00, Theorem 0.1], [Yos01, Proposition 5.1, Theorem 8.1])**.** *Let S be a projective $K3$ surface, v a primitive Mukai vector and H a v-generic polarisation.*

1. *The moduli space $M_H(v)$ is non-empty if and only if $(v, v) \geq -2$. In this case, it is irreducible.*
2. *$M_H(v)$ is an irreducible holomorphic symplectic manifold of dimension $(v, v) + 2$ deformation equivalent to $S^{[(v,v)/2+1]}$.*
3. *If $(v, v) > 0$, then the morphism θ_v is defined over the integers and it induces a Hodge isometry*

   $$\theta_v \colon v^\perp \longrightarrow H^2(M_H(v), \mathbb{Z})$$

   where the latter is a lattice with respect to the Beauville-Bogomolov-Fujiki form.

4. *When S varies in families, then the morphisms θ_v form a morphism of local systems.*

We can informally rephrase item 4 by saying that the morphisms θ_v vary continuously in families.

Remark 1.1.26. When v is not primitive, the moduli space of semistable sheaves can be non-empty even for $(v,v) < -2$. For example, the sheaf $\mathcal{O}_S \oplus \mathcal{O}_S$ is (strictly) semistable of Mukai vector $v = (2,0,2)$ but $(v,v) = -8$.

The abelian surface case

Let S be an abelian surface, v a primitive Mukai vector and H a v-generic polarisation. Let $\mathcal{P}$ be the Poincaré bundle over the product $S \times \widehat{S}$, where $\widehat{S} = \mathrm{Pic}^0(S)$ is the dual abelian surface, and consider the Fourier-Mukai equivalence ([Muk81])

$$\mathfrak{F}\colon \mathrm{D}^b(S) \longrightarrow \mathrm{D}^b(\widehat{S}) \tag{1.1.7}$$

$$\mathfrak{F}(E) = R\pi_{\widehat{S}*}\left(\mathcal{P} \overset{\mathrm{L}}{\otimes} \pi^* E\right).$$

Fix a sheaf $E_0 \in M_H(v)$ and define

$$\mathfrak{a}_v\colon M_H(v) \longrightarrow S \times \widehat{S} \tag{1.1.8}$$

$$\mathfrak{a}_v(E) = \left(\det\left(\mathfrak{F}(E) \otimes \mathfrak{F}(E_0)^\vee\right), \det(E) \otimes \det(E_0)^\vee\right).$$

Notice that this definition does not depend on the base point E_0.

Example 1.1.27. Let $v = (1,0,-n)$ be the Mukai vector parametrising ideal sheaves of 0-dimensional closed subschemes of length n, as in Example 1.1.20. Using the exact sequence

$$0 \longrightarrow I_Z \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

and taking determinants in derived categories (cf. [GKZ94, Appendix A]), after few computations one can see that (up to a sign and up to a translation) the component of $\mathfrak{a}_v$ along S coincides with the sum map (1.1.4), while the component along $\widehat{S}$ is constant.

Proposition 1.1.28 ([Yos99, Theorem 3.1, Theorem 3.6], [Yos01, Theorem 0.1])**.** *Assume $(v,v) \geq 2$, then*

1. *the restriction $\theta_v|_{v^\perp}$ is injective;*

2. *$\mathfrak{a}_v$ is the Albanese map;*

3. $\mathrm{NS}(M_H(v)) = \left(v^\perp\right)_{\mathrm{alg}} \oplus \mathfrak{a}_v^* \mathrm{NS}(S \times \widehat{S})$;

4. *$M_H(v)$ is deformation equivalent to $S^{[(v,v)/2]} \times \widehat{S}$.*

The summand $\left(v^{\perp}\right)_{\text{alg}}$ in item 3 of the proposition is the algebraic part of $v^{\perp}$, and it is seen as a subgroup of $\mathrm{NS}(M_H(v))$ via the morphism $\theta_v|_{v^{\perp}}$.

Now let $K_H(v)$ be a fibre of the Albanese map $\mathfrak{a}_v$. Notice that under our assumptions $K_H(v)$ is a smooth projective manifold of dimension $(v,v)-2$.

Remark 1.1.29. If $(v,v)=2$, then $\mathfrak{a}_v$ is an isomorphism ([Yos99, Proposition 4.1, Proposition 4.2]) and so $K_H(v)$ is just a point.

Example 1.1.30. By Example 1.1.27, $K_H(1,0,-n) \cong \mathrm{K}^{[n-1]}(S)$ is an irreducible holomorphic symplectic manifold.

Consider the map $\bar{\theta}_v \colon \widetilde{H}(S) \to H^2(K_H(v),\mathbb{Q})$ obtained by composing θ_v with the restriction map $H^2(M_H(v),\mathbb{Q}) \to H^2(K_H(v),\mathbb{Q})$.

Theorem 1.1.31 ([Yos01, Proposition 5.1, Theorem 0.2])**.** *Assume* $(v,v)\geq 4$. *Then*

1. $K_H(v)$ *is an irreducible holomorphic symplectic manifold manifold of dimension* $(v,v)-2$ *deformation equivalent to the generalised Kummer variety* $\mathrm{K}^{[(v,v)/2-1]}(S)$*;*

2. $\bar{\theta}_v$ *is defined over the integers and*

$$\bar{\theta}_v \colon v^{\perp} \to H^2(K_H(v),\mathbb{Z})$$

   is a Hodge isometry, where the latter is a lattice with the Beauville-Bogomolov-Fujiki bilinear form;

3. *When* A *varies in families, then the morphisms* $\bar{\theta}_v$ *form a morphism of local systems.*

An example coming from a non-primitive vector

Let S be a projective $K3$ surface. In this section we concentrate on the moduli space $\overline{M}_H(2,0,-2)$ parametrising S-equivalence classes of H-semistable rank 2 sheaves E such that $c_1(E)=0$ and $c_2(E)=4$. As in the previous section, H is chosen generic with respect to the Mukai vector $(2,0,-2)$. Notice that this moduli space is a projective variety of dimension 10 and, by Mukai's results, its smooth locus is the open subset $M_H(2,0,-2)$ parametrising stable sheaves. Moreover, $M_H(2,0,-2)$ is symplectic.

Theorem 1.1.32 ([O'G97a], [Rap08])**.** *There exists a symplectic resolution of singularities*

$$\tilde{\pi} \colon \widetilde{M}_S \longrightarrow \overline{M}_H(2,0,-2) \tag{1.1.9}$$

such that:

1. $\widetilde{M}_S$ *is an irreducible holomorphic symplectic manifold of dimension* 10*;*

2. $b_2(\widetilde{M}_S)=24$*;*

3. the Beauville-Bogomolov-Fujiki lattice $H^2(\widetilde{M}_S, \mathbb{Z})$ is abstractly isometric to

$$U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1) \tag{1.1.10}$$

(see Appendix A for a definition of these lattices);

4. the Fujiki constant is 945.

Remark 1.1.33. Since $b_2(\widetilde{M}_S) = 24$, this irreducible holomorphic symplectic manifold is not deformation equivalent to any of the previous examples.

Any manifold deformation equivalent to $\widetilde{M}_S$ is said to be of *OG10-type*.

Remark 1.1.34. Notice that the Fujiki constant is the same as for $K3^{[5]}$-type manifolds, pointing out to the fact that the Fujiki constant does not distinguish different deformation classes.

This example is treated in more detail in Appendix B.

1.2 Moduli spaces and Torelli theorems

Let X be an irreducible holomorphic symplectic manifold of fixed deformation type and such that $H^2(X, \mathbb{Z})$ is isometric to the abstract lattice Λ. Let $\pi\colon \mathcal{X} \to \mathrm{Def}(X)$ be the smooth Kuranishi family considered in Theorem 1.1.15. In particular, since π is a universal deformation for any of its fibres, and thanks to the Local Torelli Theorem 1.1.18, the $\mathrm{Def}(X)$s can be glued together to form a smooth complex manifold $\mathfrak{M}_\Lambda$ of dimension $b_2(X) - 2$. This is called the *moduli space of marked* irreducible holomorphic symplectic manifolds.

Remark 1.2.1. As already noticed in Remark 1.1.17, it is not known if the lattice Λ fixes the deformation type. The notation $\mathfrak{M}_\Lambda$ is then misleading since it should contain a label indicating the deformation type. Nevertheless, this is the standard notation used everywhere in the literature (see [Huy99], [Mar11], [Huy12],[GHS13]) and we decided to follow this convention.

Remark 1.2.2. Note that $\mathfrak{M}_\Lambda$ is not Hausdorff in general.

Remark 1.2.3 ([Bea83a, Proposition 9])**.** Let $n \geq 2$. If A is an abelian surface and A_{n+1} indicates the subgroup of points of order $n+1$, let A_{n+1} act on $A^{[n+1]}$ by translation. This action restricts to $\mathrm{K}^{[n]}(A)$ and acts trivially on $H^2(\mathrm{K}^{[n]}(A), \mathbb{Z})$.

The existence of non-trivial automorphisms acting trivially on the second integral cohomology group obstructs the existence of a universal family on $\mathfrak{M}_\Lambda$, i.e. the universal families $\mathcal{X}$ over each $\mathrm{Def}(X)$ do not glue to form a universal family on $\mathfrak{M}_\Lambda$.

The period map $\mathcal{P}$ extends to a *global* period map

$$\mathcal{P}\colon \mathfrak{M}_\Lambda \longrightarrow \Omega_\Lambda \tag{1.2.1}$$

which is a local isomorphism.

In the 2-dimensional case the Global Torelli Theorem for $K3$ surfaces states that two $K3$ surfaces whose second integral cohomologies are Hodge isometric are isomorphic. Counter-examples to a similar statement in higher dimensions are known (cf. [Deb84]). Moreover, counter-examples to the weaker statement requiring just birationality are known (cf. [Nam02]). Nevertheless, as we now see, these are the only pathologies.

Let $\mathfrak{M}_\Lambda^0$ be the connected component of $\mathfrak{M}_\Lambda$ containing the pair (X, η) and $\mathcal{P}_0$ the restriction of the period map. The following is a summary of the statements which are known as Global Torelli Theorem for irreducible holomorphic symplectic manifolds.

Theorem 1.2.4 (Global Torelli Theorem, [Huy99, Theorem 4.3, Theorem 8.1], [Ver13, Theorem 1.16])**.** *The period map $\mathcal{P}_0$ is surjective. Moreover, for every $x \in \Omega_\Lambda$, the fibre $\mathcal{P}_0^{-1}(x)$ consists of pairwise birational manifolds.*

For a more detailed statement, see [Mar11, Theorem 2.2].

Remark 1.2.5. Hidden in this statement there is a very important result due to Huybrechts ([Huy99, Theorem 4.3]), which is worth mentioning. Given two birational irreducible holomorphic symplectic manifolds, there exists a family of deformations over the disc with two origins (notice that this is a non-Hausdorff space) such that the two origins correspond to the two birational manifolds. In particular, they are deformation equivalent. This result has been recently generalised to some degenerations of irreducible holomorphic symplectic manifolds in [KLSV17].

1.3 Monodromy operators

Let X_1 and X_2 be two deformation equivalent irreducible holomorphic symplectic manifolds and $g\colon H^k(X_1, \mathbb{Z}) \to H^k(X_2, \mathbb{Z})$ an isomorphism of abelian groups.

Definition 1.3.1. We say that g is a *parallel transport operator* of degree k if there exists a family $p\colon \mathcal{X} \to B$, points $b_1, b_2 \in B$ and isomorphisms $\varphi_i\colon X_i \xrightarrow{\sim} \mathcal{X}_{b_i}$ such that the composition $(\varphi_2^*)^{-1} \circ g \circ \varphi_1^*$ is the parallel transport ([GHJ03, Definition I.2.1]) inside the local system $R^k p_* \mathbb{Z}$ along a path γ from b_1 to b_2. Here $R^k p_* \mathbb{Z}$ is endowed with the Gauss-Manin connection ([Voi02, Section 9.2]).

When $X_1 = X_2 = X$ and γ is a loop we talk about *monodromy operators*. If g_1 and g_2 are two monodromy operators of degree k, the composition $g_1 \circ g_2$ is also a monodromy operator. To see this, if $p_j\colon \mathcal{X}_j \to B_j$ and $[\gamma_j] \in \pi_1(B_j)$ are associated to g_j, then we can form the family $p\colon \mathcal{X} \to B$ and the loop $[\gamma] \in \pi_1(B)$ by gluing B_1 and B_2 along the point b_j corresponding to X, $\mathcal{X}_1$ and $\mathcal{X}_2$ along $\mathcal{X}_{b_j}$ and concatenating the loops γ_1 and γ_2. The parallel transport in this family along γ is by construction the composition of g_1 and g_2. It follows that monodromy operators form a group that we denote $\mathrm{Mon}^k(X)$.

Remark 1.3.2. The same argument shows that parallel transport operators form a groupoid.

Remark 1.3.3. Notice that, by definition, the monodromy group is invariant under deformation. Given a deformation type, we can then talk of the monodromy group of the deformation type (strictly speaking it is defined up to conjugacy).

Because of the Torelli theorem, the case $k = 2$ is the most interesting one. We have seen before that $H^2(X, \mathbb{Z})$ has the lattice structure given by the Beauville-Bogomolov-Fujiki bilinear form and so we can talk about isometries of $H^2(X, \mathbb{Z})$. By construction, parallel transport operators of degree 2 are isometries and so $\mathrm{Mon}^2(X) \subset \mathrm{O}\left(H^2(X, \mathbb{Z})\right)$.

Remark 1.3.4. As we will see in Section 6.1, the monodromy group $\mathrm{Mon}(X)$ (i.e. the entire monodromy group acting on $H^*(X, \mathbb{Z})$) coincides with the image of the mapping class group associated to the underlying differential manifold. The fact that the degree-two part of the mapping class group acts via isometries is item 1 of [Ver13, Theorem 3.5]. Moreover, item 3 of the same theorem also states that, under this identification, the natural projection $\mathrm{Mon}(X) \to \mathrm{Mon}^2(X)$ has finite kernel, strengthening the claim that $\mathrm{Mon}^2(X)$ is actually the most interesting part of $\mathrm{Mon}(X)$.

Terminology. *From now on, parallel transport operators (and monodromy operators) are always to be understood to be of degree* 2.

We think of monodromy operators as geometric isometries. By the Torelli Theorem, we know that most of the geometry of an irreducible holomorphic symplectic manifold X is encoded inside the lattice and Hodge structures of the second integral cohomology group. Considering $H^2(X, \mathbb{Z}) = \Lambda$ as an abstract lattice, the group of isometries $\mathrm{O}(\Lambda)$ forgets everything about the geometry of X, while $\mathrm{Mon}^2(X)$ by definition does remember the geometry.

Let us give a concrete example. Intuitively the isometry $-id \in \mathrm{O}(H^2(X, \mathbb{Z}))$ is not geometric: for example its action on the moduli space $\mathfrak{M}_\Lambda$ only changes the sign of the marking and so it gives no information about the geometry of the manifold. This observation is formally stated in the following proposition (cf. Section A.5 for the definition of the group $\mathrm{O}^+(H^2(X, \mathbb{Z}))$ of orientation preserving isometries).

Proposition 1.3.5. $\mathrm{Mon}^2(X) \subset \mathrm{O}^+(H^2(X, \mathbb{Z}))$.

Proof. The geometry of X fixes in a canonical way a choice of orientation, which is constant in families. More precisely, if $\sigma_X \in H^{2,0}(X)$ is the symplectic form and $\omega \in H^{1,1}(X, \mathbb{Z})$ is a Kähler class, then $\{\mathrm{Re}(\sigma_X), \mathrm{Im}(\sigma_X), \omega\}$ is a basis of the positive real 3-space W in Section A.5. This construction clearly works in families. □

Remark 1.3.6. Notice that the same argument as before can be used to make sense of the notion of orientation preserving/reversing for any isometry $g \colon H^2(X, \mathbb{Z}) \to$

$H^2(Y,\mathbb{Z})$. In particular, since the positive 3-space $\{\mathrm{Re}(\sigma_X), \mathrm{Im}(\sigma_X), \omega\}$ varies in families of irreducible holomorphic symplectic manifolds, parallel transport operators are orientation preserving.

The following is a very useful formulation of the Global Torelli Theorem 1.2.4 in Hodge-theoretic terms.

Theorem 1.3.7 (Hodge-theoretic Torelli Theorem, [Mar11, Theorem 1.3])**.** *Let X and Y be two deformation equivalent irreducible holomorphic symplectic manifolds.*

1. *X and Y are birational if and only if there exists a parallel transport operator $g\colon H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ which is an isomorphism of Hodge structures.*

2. *if $g\colon H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ is a parallel transport operator which is an isomorphism of Hodge structures, then g is induced by an isomorphism $X \xrightarrow{\sim} Y$ if and only if g maps a Kähler class on X to a Kähler class on Y.*

If $g\colon H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ is an orientation preserving Hodge isometry between two non-birational irreducible holomorphic symplectic manifolds, then the theorem above says that g cannot be a parallel transport operator. On the other hand, since X and Y are deformation equivalent, we can always find a parallel transport operator $f\colon H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ and so the composition $f \circ g \in \mathrm{O}^+(H^2(X,\mathbb{Z}))$ is not in $\mathrm{Mon}^2(X)$. The philosophy is that the index of $\mathrm{Mon}^2(X)$ inside $\mathrm{O}^+(H^2(X,\mathbb{Z}))$ counts (roughly speaking) the number of pairs (Y,Z) such that Y and Z are deformation equivalent to X, and they are Hodge-isometric to each other but not birational. In other words, we expect that any (non-trivial) class in the quotient $\mathrm{O}^+(H^2(X,\mathbb{Z}))/\mathrm{Mon}^2(X)$, has a representative which comes from two deformation equivalent but non-birational irreducible holomorphic symplectic manifolds having the same periods, as explained above. This is indeed the case for manifolds of $K3^{[n]}$ and Kum^n type. We will see in Section 6.1 that this number is actually finite.

Example 1.3.8. Let A be an abelian surface, $\widehat{A}$ its dual and assume that they are not isomorphic. The Fourier-Mukai equivalence (1.1.7) induces an isomorphism between the cohomology rings, which restricts to an isometry $\tilde{\tau}_A\colon H^2(A,\mathbb{Z}) \to H^2(\widehat{A},\mathbb{Z})$. This isometry is described more geometrically (up to a sign) in terms of Poincaré duality (cf. [Huy06, Lemma 9.23]). One can extend $\tilde{\tau}_A$ to an isometry

$$\tau_A\colon H^2(\mathrm{K}^{[n]}(A),\mathbb{Z}) \longrightarrow H^2(\mathrm{K}^{[n]}(\widehat{A}),\mathbb{Z})$$

just putting $\tau_A(\delta) = \hat{\delta}$. Notice that τ_A is clearly a Hodge isomorphism and by [MM17, Lemma 4.5] it is also orientation preserving. On the other hand, $\mathrm{K}^{[n]}(A)$ and $\mathrm{K}^{[n]}(\widehat{A})$ are not birational ([Nam02]) and so τ_A is not a parallel transport operator. This implies that $\mathrm{Mon}^2(\mathrm{K}^{[n]}(A))$ has always index ≥ 4 inside $\mathrm{O}(H^2(\mathrm{K}^{[n]}(A),\mathbb{Z}))$ and in fact, thanks to the computation of the monodromy group done by Markman and Mongardi (cf. Section 6.2), it has index 4 if $n+1$ is a power of a prime number.

Example 1.3.9. Let X be an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type. By [Mar10a, Lemma 4.11], any class in the quotient $\mathrm{O}^+(H^2(X,\mathbb{Z}))/\operatorname{Mon}^2(X)$ has a representative induced by two non-birational irreducible holomorphic symplectic manifolds of $K3^{[n]}$-type with the same periods.

Polarised monodromy operators

A polarised irreducible holomorphic symplectic manifold is a pair (X, H) where H is an ample line bundle on X. Notice that the first Chern class $c_1\colon \operatorname{Pic}(X) \to H^2(X,\mathbb{Z})$ is injective, so we can replace H by $h = c_1(H) \in H^{1,1}(X,\mathbb{Z})$ without loss of generality. By *degree of the polarisation* we always mean the Beauville-Bogomolov-Fujiki degree $q(h) = d$. In the following, we always write (X, h) for a polarised pair.

Suppose now that (X_1, h_1) and (X_2, h_2) are two *polarised deformation equivalent* irreducible holomorphic symplectic manifolds. This means that there exists a family $p\colon \mathcal{X} \to B$ as in Definition 1.3.1, with distinguished points b_1 and b_2 corresponding to X_1 and X_2, and a section $\tilde{h} \in R^2p_*\mathbb{Z}$ that is flat with respect to the Gauss-Manin connection, such that $\tilde{h}(b_i) = h_i$ and $\tilde{h}(b)$ is of type $(1,1)$ and ample for every $b \in B$.

Definition 1.3.10. A parallel transport operator is called *polarised* if it is obtained by parallel transport inside a polarised deformation family. Let us denote by $\operatorname{Mon}^2(X)_h$ the polarised monodromy group of a polarised pair (X, h).

Notice that, by definition, if $g\colon H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ is a polarised parallel transport operator, then $g(h_1) = h_2$. The converse is also true.

Proposition 1.3.11 ([Mar11, Proposition 7.4])**.** *A parallel transport operator*

$$g\colon H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$$

is polarised if and only if $g(h_1) = h_2$.

As a corollary we get that

$$\operatorname{Mon}^2(X)_h = \operatorname{Mon}^2(X) \cap \mathrm{O}(H^2(X,\mathbb{Z}))_h \tag{1.3.1}$$

where $\mathrm{O}(H^2(X,\mathbb{Z}))_h$ is the subgroup of isometries fixing h.

1.4 Polarised irreducible holomorphic symplectic manifolds

The coarse moduli space $\mathcal{V}_{n,d}$ of polarised irreducible holomorphic symplectic manifolds of dimension $2n$ and polarisation of degree d can be shown to be a quasi-projective algebraic variety using GIT methods. This is a special case of a general result of Viehweg about moduli spaces of projective varieties (cf. [Vie95]). If (X, h) is a polarised pair, we denote by $\mathcal{V}^{\mathrm{irr}}_{n,d}$ the irreducible component of $\mathcal{V}_{n,d}$ containing (X, h). Following [Mar11], we are now going to describe $\mathcal{V}^{\mathrm{irr}}_{n,d}$ as an analytic space.

Fix a primitive class $h \in \Lambda$ with $(h,h) = d > 0$ and consider $\Omega_h = \Omega_\Lambda \cap h^\perp$.

Remark 1.4.1. Note that Ω_h has two connected components. A point $p \in \Omega_h$ defines a canonically oriented positive 3-space $W_p = \operatorname{span}\{\operatorname{Re}(p), \operatorname{Im}(p), h\}$ which varies continuosly with p. Remember that W_p determines an orientation of Λ (cf. Section A.5): therefore the two connected components of Ω_h are in natural correspondence with the set of orientations of Λ.

An irreducible holomorphic symplectic manifold comes with a preferred orientation on $H^2(X,\mathbb{Z})$. Fixing a marking $\eta\colon H^2(X,\mathbb{Z}) \to \Lambda$ then fixes an orientation on Λ and hence a connected component Ω_h^+ of Ω_h. Notice that such a component is constant as (X,η) varies inside a connected component $\mathfrak{M}_\Lambda^0$. Define

$$\mathfrak{M}_h^{0,+} := \mathcal{P}_0^{-1}(\Omega_h^+).$$

A pair (X,η) belongs to $\mathfrak{M}_h^{0,+}$ if it belongs to $\mathfrak{M}_\Lambda^0$, and $\eta^{-1}(h)$ is of type $(1,1)$ and belongs to the positive cone of X. Note that $\mathfrak{M}_h^{0,+}$ is path connected ([Mar11, Proposition 7.1]). Inside $\mathfrak{M}_h^{0,+}$ there is a (non-empty) open subset $\mathfrak{M}_h^{0,a}$ consisting of pairs (X,η) such that $\eta^{-1}(h)$ is ample.

Notice that, for any $(X,\eta) \in \mathfrak{M}_\Lambda^0$, the group $\operatorname{Mon}^2(\mathfrak{M}_\Lambda^0) := \eta \circ \operatorname{Mon}^2(X) \circ \eta^{-1} \subset \operatorname{O}(\Lambda)$ is well defined (up to conjugation) and independent of the choice of (X,η). Define $\operatorname{Mon}(\mathfrak{M}_\Lambda^0)_h$ as the subgroup of $\operatorname{Mon}(\mathfrak{M}_\Lambda^0)$ stabilising h and consider its natural action on $\mathfrak{M}_\Lambda^{0,+}$.

Proposition 1.4.2 ([Mar11, Corollary 7.3])**.** *$\mathfrak{M}_h^{0,a}$ is a path connected $\operatorname{Mon}(\mathfrak{M}_\Lambda^0)_h$-invariant Hausdorff open subset of $\mathfrak{M}_h^{0,+}$. Moreover, the period map $\mathcal{P}_0$ restricts to an injective $\operatorname{Mon}(\mathfrak{M}_\Lambda^0)_h$-equivariant map onto an open dense subset of Ω_h^+.*

The Hausdorff property follows from the Hodge Theoretic Torelli Theorem 1.3.7. The fact that the restriction of the period map is injective follows from the Global Torelli Theorem 1.2.4. The image of $\mathfrak{M}_h^{0,a}$ is dense by an important projectivity criterion due to Huybrechts ([Huy99, Theorem 3.11] and [Huy03a]).

Let Ξ be the set of connected components of $\mathfrak{M}_\Lambda$. The group $\operatorname{O}(\Lambda)$ acts diagonally on the product $\Lambda \times \Xi$ and we pick the $\operatorname{O}(\Lambda)$-orbit $\bar{h}$ spanned by $(h,t) \in \Lambda \times \Xi$ (notice that the action on Ξ is transitive). For every $(h,t) \in \bar{h}$, define $\mathfrak{M}_h^{t,a}$ as before. Then

$$\mathfrak{M}_{\bar{h}}^a := \bigcup_{(h,t)\in\bar{h}} \mathfrak{M}_h^{t,a}$$

is the *moduli space of marked polarised irreducible holomorphic symplectic manifolds* of deformation type $\bar{h}$.

The moduli space $\mathcal{V}_{n,d}^{\mathrm{irr}}$ of polarised manifolds is then recovered by getting rid of the markings.

Theorem 1.4.3 ([Mar11, Lemma 8.1, Lemma 8.3], [GHS10, Theorem 1.5])**.** *There exists an isomorphism*

$$\mathcal{V}^{\mathrm{irr}}_{n,d} \to \mathfrak{M}^{a}_{\bar{h}} / \operatorname{O}(\Lambda)$$

in the category of analytic spaces.

Moreover, for any $(h,t) \in \bar{h}$,

$$\mathfrak{M}^{a}_{\bar{h}} / \operatorname{O}(\Lambda) \cong \mathfrak{M}^{t,a}_{h} / \operatorname{Mon}^2(\mathfrak{M}^{t}_{\Lambda})_{h}.$$

Chapter 2

Cubic fourfolds and irreducible holomorphic symplectic manifolds

2.1 Cubic fourfolds

2.1.1 Special cubic fourfolds

A *cubic fourfold* is a cubic hypersurface $V \subset \mathbb{P}^5$. From now on we always assume that V is smooth. Thanks to the Lefschetz Hyperplane Section Theorem, all the non-trivial cohomological information of V is contained in the middle cohomology group $H^4(V, \mathbb{Z})$, whose Hodge theory is understood via Griffiths' residue theory ([Voi03, Section 6.1.3]). The Hodge diamond of V is

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 21 & & 1 & & 0. \end{array}$$

The intersection product on $H^4(V, \mathbb{Z})$ is a non-degenerate symmetric bilinear form and hence it is a lattice. In the rest of the dissertation we will always denote by $H^4(V, \mathbb{Z})_{\text{prim}}$ the primitive cohomology of V. Notice that, if $h \in H^2(V, \mathbb{Z})$ is the class of a hyperplane in V, then $H^4(V, \mathbb{Z})_{\text{prim}}$ is a lattice isometric to the orthogonal complement of h^2 in $H^4(V, \mathbb{Z})$.

Proposition 2.1.1 ([Has00, Proposition 2.1.2])**.** *Let V be a smooth cubic fourfold.*

1. *$H^4(V, \mathbb{Z})$ is an odd lattice isometric to the abstract lattice*

$$\langle 1 \rangle^{\oplus 21} \oplus \langle -1 \rangle^{\oplus 2}.$$

2. $H^4(V,\mathbb{Z})_{\mathrm{prim}}$ is an even lattice isometric to the abstract lattice

$$L = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2.$$

We refer to Appendix A for the definition of these lattices.

Remark 2.1.2. The proof of this result relies on the close analysis of the Hodge theory of the Fano variety of lines of V done by Beauville and Donagi in [BD85]. We will recall in Section 2.2.1 the outline of their work.

Let $\mathcal{U} \subset \mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3))^*$ be the Zariski open subset parametrising smooth cubic fourfolds. As noted by Hassett in [Has00, Section 2.2], smooth cubic fourfolds are all GIT-stable under the natural SL(6)-action and so the GIT quotient $\mathcal{C} = \mathcal{U} /\!\!/ \mathrm{SL}(6)$ exists as a quasi-projective variety of dimension 20. Moreover, the projection map $\mathcal{U} \to \mathcal{C}$ is a principal SL(6)-bundle and $\mathcal{C}$ parametrises orbits, i.e. it is a geometric quotient.

$\mathcal{C}$ contains irreducible divisors parametrising so-called special cubic fourfolds. Following Hassett, a smooth cubic fourfold is called *special* if it contains a surface which is not homologous to a complete intersection. Alternatively, one can describe special cubic fourfolds V in purely Hodge-theoretic terms by saying that $H^{2,2}(V,\mathbb{Z})_{\mathrm{prim}} := H^4(V,\mathbb{Z})_{\mathrm{prim}} \cap H^{2,2}(V)$ is non-empty.

Proposition 2.1.3 ([Has00, Proposition 3.1.3])**.** *Special cubic fourfolds form a countable union of irreducible divisors in $\mathcal{C}$.*

More precisely, if V is a special cubic fourfold, $Z \subset V$ is a surface not homologous to a complete intersection and $[Z]$ is its class in $H^{2,2}(V,\mathbb{Z})$, then the lattice Λ_Z generated by h^2 and $[Z]$ is a positive definite rank 2 sublattice of $H^4(V,\mathbb{Z})$. The determinant of Λ_Z is called the *discriminant* of the pair (V,Z) (or simply the discriminant of the special cubic fourfold V).

Proposition 2.1.4 ([Has00, Theorem 3.2.3])**.** *If non-empty, the set $\mathcal{C}_d$ of special cubic fourfolds of discriminant d is an irreducible divisor in $\mathcal{C}$.*

Remark 2.1.5. These results follow from the Torelli Theorem for cubic fourfolds ([Voi86]), which makes it possible to state everything in purely lattice-theoretic terms.

Not all the values of $d > 0$ appear as discriminants of special cubic fourfolds. Nevertheless, Hassett proved ([Has00, Theorem 4.3.1]) that if $d > 6$ and $d \equiv 0, 2 \pmod 6$, then there exist special cubic fourfolds of discriminant d.

Example 2.1.6 (Cubic fourfolds containing a plane)**.** Let V be a smooth cubic fourfold and suppose that there exists a plane $P \subset V$. The self-intersection $P \cdot P = c_2(N_{P/V}) = 3$ can be computed using the normal bundle sequence. It follows that

$$\Lambda_P \cong \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

so it has determinant 8. The divisor $\mathcal{C}_8$ parametrises smooth cubic fourfolds containing a plane. Special cubic fourfolds of this kind are the ones used by Voisin to prove the Torelli Theorem ([Voi86]).

Example 2.1.7 (Pfaffian cubic fourfolds)**.** Let V be a smooth cubic fourfold and suppose that there exists a quintic Del Pezzo surface $Z \subset V$. As before, we can compute the self-intersection $Z \cdot Z$ using the normal bundle sequence and we find that

$$\Lambda_Z \cong \begin{pmatrix} 3 & 5 \\ 5 & 13 \end{pmatrix},$$

which has determinant 14. Such cubic fourfolds are related to *Pfaffian cubics* which will be discussed further in Section 2.2.1. We notice though that, by [Has00, Section 4.1.3], Pfaffian cubic fourfolds only form an open and dense subset of $\mathcal{C}_{14}$.

These two examples are particularly important for two reasons.

The first reason is the existence of associated $K3$ surfaces. In Example 2.1.6 every cubic in $\mathcal{C}_8$ contains an octic $K3$ surface (cf. [Has00, Section 4.1.1]), while in Example 2.1.7 the $K3$ surfaces is associated in a more complicated way (cf. Section 2.2.1). From a Hodge-theoretical point of view this is not a surprise because we can see the Hodge diamond of a $K3$ surface inside the Hodge diamond of a smooth cubic fourfold. Nevertheless, this $K3$-like Hodge structure is not reflected in the lattice structure in general. In particular, there are cubic fourfolds without associated $K3$ surfaces. More precisely, if V is a special cubic fourfold and Λ_Z is the lattice associated to some surface Z not homologous to a complete intersection, we say that V has an *associated $K3$ surface* if there exists a projective $K3$ surface S such that $\Lambda_Z^{\perp}$ is Hodge-isometric to the primitive cohomology $H^2(S,\mathbb{Z})_{\text{prim}}$ (i.e. the orthogonal complement of the given polarisation on S). A special cubic fourfold of discriminant d has an associated $K3$ surface if and only if d is not divisible by $4, 9$ or any odd prime p such that $p \equiv -1 \pmod 3$ ([Has00, Theorem 5.2.1]). The proof of this result again boils down to the Torelli Theorem for $K3$ surfaces.

Remark 2.1.8. The existence of associated $K3$ surfaces can be understood in terms of derived categories. In fact Kuznetsov noticed in [Kuz10] that the derived category of a smooth cubic fourfold has an orthogonal decomposition formed of exceptional objects and an indecomposable subcategory $\mathcal{A}$ which is of CY_2-type. If there exists a projective $K3$ surface S such that $\mathcal{A}$ is actually equivalent to $\mathrm{D}^b(S)$, then $\mathcal{A}$ is called *geometric*. It is now known that a special cubic fourfold has an associated $K3$ surface in the sense of Hassett if and only if the corresponding CY_2-category $\mathcal{A}$ is geometric ([AT14], [BLM+18]).

The second reason why Example 2.1.6 and Example 2.1.7 are important is that smooth cubic fourfolds in $\mathcal{C}_8$ and $\mathcal{C}_{14}$ are both rational (cf. [Has99] and [BD85, Proposition 5]). It is conjectured that the very general cubic fourfold (i.e. non-special) is not rational, but there are not even examples of cubic fourfolds known to be non-rational. More generally, Harris-Hassett and Kuznetsov conjecture that a cubic fourfold is rational if and only if there exists an associated $K3$ surface.

2.1.2 Monodromy of a cubic fourfold

Let us start by noticing that the open subset $\mathcal{U} \subset \mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3))^*$ of smooth cubic fourfolds has a universal family $\mathcal{V}$. By the universal property, the monodromy group of a smooth cubic fourfold V is then defined as (cf. Section 1.3)

$$\mathrm{Mon}^4(V) := \mathrm{Im}\left(\rho\colon \pi_1(\mathcal{U}) \longrightarrow \mathrm{O}\left(H^4(V,\mathbb{Z})\right)\right), \tag{2.1.1}$$

where ρ is the monodromy representation.

Let us consider a generic Lefschetz pencil $(V_t)_{t\in\mathbb{P}^1}$ of cubic fourfolds such that $V_0 = V$. Recall that this means that V_t has at worst ordinary double points as singularities, and that the number of singular fibres is finite. If $B \subset \mathbb{P}^1$ is the (finite) set of points such that the corresponding cubic fourfold is singular, then by a theorem of Zariski ([Voi03, Theorem 3.22]) the natural map

$$\pi_1(\mathbb{P}^1 \setminus B) \longrightarrow \pi_1(\mathcal{U})$$

is surjective. Moreover, the monodromy action is irreducible, $H^4(X,\mathbb{Z})$ is generated by vanishing cycles and $\mathrm{Mon}^4(X)$ coincides with the group generated by reflections by vanishing cycles (cf. [Voi03, Section 3.2]).

If L is a lattice, we denote by $\widetilde{\mathrm{O}}(L) \subset \mathrm{O}(L)$ the subgroup of isometries that act as the identity on the discriminant group. In other words, $\widetilde{\mathrm{O}}(L)$ is the kernel of the natural map $\mathrm{O}(L) \to \mathrm{Aut}(A_L)$ (cf. Section A).

Theorem 2.1.9 ([Bea86]). $\mathrm{Mon}^4(V) \cong \widetilde{\mathrm{O}}^+(H^4(V,\mathbb{Z})_{\mathrm{prim}})$.

Remark 2.1.10. Here the orientation is defined with respect to a negative definite subspace (cf. Remark A.5.4). In particular, reflections around +2-classes are orientation preserving. Analogously to Section 1.3, if V is a smooth cubic fourfold and the class $\eta_V \in H^{1,3}(V)$ is a generator, then $\{\mathrm{Re}(\eta_V), \mathrm{Im}(\eta_V)\}$ is a canonical basis for the negative definite subspace in the definition of orientation. This is why we changed from positive to negative subspaces. Fortunately, the Abel-Jacobi map (2.2.1), which is used to compare the Hodge structures of cubic fourfolds and irreducible holomorphic symplectic manifolds, is an anti-isometry, making this change painless.

Remark 2.1.11. It follows that $\widetilde{\mathrm{O}}^+(H^4(V,\mathbb{Z})_{\mathrm{prim}}) \cong \mathrm{O}^+(H^4(V,\mathbb{Z}))_h$, where the latter is the group of isometries g such that $g(h^2) = h^2$. In fact, let us consider the restriction map

$$r\colon\ \mathrm{O}^+(H^4(V,\mathbb{Z}))_h \longrightarrow \mathrm{O}^+(H^4(V,\mathbb{Z})_{\mathrm{prim}}).$$

This map is injective and the image is contained in $\widetilde{\mathrm{O}}^+(H^4(V,\mathbb{Z})_{\mathrm{prim}})$ by Example A.3.3. On the other hand, the monodromy group $\mathrm{Mon}^4(X)$ naturally lives inside $\mathrm{O}^+(H^4(V,\mathbb{Z}))_h$ and its image under r is $\widetilde{\mathrm{O}}^+(H^4(V,\mathbb{Z})_{\mathrm{prim}})$ by Theorem 2.1.9.

2.2 Fano varieties of lines

Let V be a smooth cubic fourfold and let us denote by $F = F(V)$ the Fano variety of lines on V. By [CG72, Theorem 7.8], F is a smooth projective variety of dimension 4. It turns out that F is actually a symplectic variety deformation equivalent to a Hilbert scheme of two points on a projective $K3$ surface. This result follows from Huybrechts' deformation criterion (cf. Remark 1.2.5) once one can prove the result for a particular class of cubic fourfolds. This particular class is contained in the special divisor $\mathcal{C}_{14}$ and consists of *Pfaffian* cubic fourfolds, first studied by Beauville and Donagi in [BD85].

2.2.1 The Beauville-Donagi construction

Let $W = W_6$ be a 6-dimensional vector space. The Pfaffian variety associated to W is the subvariety

$$\mathrm{Pf}(W) = \left\{ \phi \in \mathbb{P}\left(\bigwedge^2 W^*\right) \,\middle|\, \mathrm{Pf}(\phi) = 0 \right\}$$

where we see ϕ as a skew-symmetric 6×6 matrix and $\mathrm{Pf}(\phi)$ is its Pfaffian, i.e. $\mathrm{Pf}(\phi)^2 = \det(\phi)$. In particular, $\mathrm{Pf}(W) \subset \mathbb{P}^{14}$ is a cubic hypersurface. Notice that $\mathrm{Pf}(W)$ can be thought of as the subvariety of $\mathbb{P}\left(\bigwedge^2 W^*\right)$ consisting of tensors of rank ≤ 4. Now if $L_6 \subset \bigwedge^2 W^*$ is a 6-dimensional vector subspace, then $V := \mathrm{Pf}(W) \cap \mathbb{P}(L_6)$ is a Pfaffian cubic fourfold. When L_6 is chosen generically, then V is smooth.

Dually, we can consider the locus $G \subset \mathbb{P}\left(\bigwedge^2 W\right)$ consisting of tensors of rank ≤ 2. It is well known that $G = \mathrm{Gr}(2, W)$ is the Grassmannian of 2-planes in W embedded via the Plücker embedding. The dual variety of V is then

$$S = \{[P] \in G \mid \phi|_P = 0,\ \forall \phi \in L_6\}\,.$$

Notice that $S = G \cap \mathbb{P}(L_6^\perp)$. Again, S is smooth when L_6 is generic and by adjunction S is a projective $K3$ surface of degree 14.

Now, if $x \neq y$ are two points in S, we denote by P_x and P_y the corresponding planes. The sum $P_x + P_y$ is a linear space of dimension 4 and we can consider the set N of linear forms $\phi \in L_6$ such that $\phi|_{P_x+P_y}$ is zero. The last assumption imposes four linear conditions on the coefficients of ϕ and hence $N = \mathbb{P}^1$ is a line inside V. In this way we have constructed a map $S^{[2]} \setminus \Delta \to F$, where Δ is the exceptional divisor of the Hilbert-Chow morphism. On the other hand, points in Δ come with an extra piece of information, namely the tangent direction of the non-reduced subscheme of length 2 on S. In this way, we can carry out the same construction as above for any point in Δ and so we get a map

$$f\colon S^{[2]} \longrightarrow F.$$

Proposition 2.2.1 ([BD85, Proposition 5]). *f is an isomorphism of algebraic varieties.*

Let us consider the incidence variety $Z \subset V \times F$ together with the natural projections p_V and p_F. Notice that $p_F\colon Z \to F$ is a $\mathbb{P}^1$-bundle. The Abel-Jacobi map is

$$\alpha := p_{F*} \circ p_V^*\colon H^4(V,\mathbb{Z}) \longrightarrow H^2(F,\mathbb{Z}). \tag{2.2.1}$$

Let $g = \alpha(h^2)$ and define $H^2(F,\mathbb{Z})_{\mathrm{prim}} := g^\perp$.

Remark 2.2.2. We have seen that $F \cong S^{[2]}$, where S is the $K3$ surface dual to X. In particular we have a canonical isometry (cf. Example 1.1.7)

$$H^2(F,\mathbb{Z}) = H^2(S,\mathbb{Z}) \oplus \mathbb{Z}\delta.$$

If $l \in H^2(S,\mathbb{Z})$ is the class of the degree 14 polarisation on S, then Beauville and Donagi remarked that $g = 2l - 5\delta$ ([BD85]).

Remark 2.2.3. The Abel-Jacobi map (2.2.1) gives a morphism of local systems in any family of smooth cubic fourfolds and associated family of Fano variety of lines.

Proposition 2.2.4 ([BD85, Proposition 6]). *The restriction*

$$\alpha\colon H^4(V,\mathbb{Z})_{\mathrm{prim}} \to H^2(F,\mathbb{Z})_{\mathrm{prim}}$$

is an anti-isometry, where the latter is endowed with the Beauville-Bogomolov-Fujiki form.

Anti-isometry means that $(\alpha(x),\alpha(y)) = -x\cdot y$ for every $x,y \in H^4(V,\mathbb{Z})_{\mathrm{prim}}$.

Remark 2.2.5. In a similar way, we can consider the incidence variety $P \subset \mathbb{P}(W) \times V$ of pairs (w,ϕ) such that $w \in \ker\phi$. The projection $p_V\colon P \to V$ is a $\mathbb{P}^1$-bundle and the projection $p_W\colon P \to \mathbb{P}(W)$ is birational. It follows that V is birational to a linear section of $\mathbb{P}(W)$ and hence rational.

2.2.2 Monodromy of $F(V)$

We want to use the Beauville-Donagi contruction to study monodromy operators of polarised manifolds which are deformation equivalent to the Fano variety of lines $F = F(V)$ of a cubic fourfold V. This computation is a simple exercise, which will be used again in Section 5.3 to study new monodromy operators on irreducible holomorphic symplectic manifolds of OG10-type.

Theorem 2.2.6. *If $F = F(V)$ is the Fano variety of lines of a smooth cubic fourfold V, then*

$$\mathrm{Mon}^2(F)_g = \mathrm{O}^+(H^2(F,\mathbb{Z}))_g,$$

where the last group is the group of orientation preserving isometries fixing the polarisation g.

Proof. The first remark is that the Abel-Jacobi map (2.2.1) induces an isomorphism

$$\widetilde{\mathrm{O}}^+(H^4(V,\mathbb{Z})_{\mathrm{prim}}) \cong \widetilde{\mathrm{O}}^+(H^2(F,\mathbb{Z})_{\mathrm{prim}})$$

and, by Remark 2.1.11, $\widetilde{\mathrm{O}}^+(H^4(V,\mathbb{Z})_{\mathrm{prim}}) \cong \mathrm{O}^+(H^4(V,\mathbb{Z}))_h$.

Consider the injective restriction map (cf. Example A.3.3)

$$r\colon\ \mathrm{O}^+(H^2(F,\mathbb{Z}))_g \longrightarrow \widetilde{\mathrm{O}}^+(H^2(F,\mathbb{Z})_{\mathrm{prim}}).$$

Let now $f \in \mathrm{O}^+(H^2(F,\mathbb{Z}))_g$ and consider its image $\bar{f} \in \mathrm{O}^+(H^4(V,\mathbb{Z}))_h = \mathrm{Mon}^4(V)$ (see Theorem 2.1.9). Then there exists a loop γ in $\mathcal{U}$ inducing $\bar{f}$. Notice that $\mathcal{U}$ is also the base of a deformation family of Fano varieties of lines and hence we can consider the monodromy operator $P_\gamma \in \mathrm{Mon}^2(F)$. By Remark 2.2.3, Proposition 2.2.4 and the fact that these families are naturally polarised, it follows that $P_\gamma = f$ and hence $\mathrm{O}^+(H^2(F,\mathbb{Z}))_g \subset \mathrm{Mon}^2(F)_g$, where the latter is the polarised monodromy group. Since the other inclusion is always true by Proposition 1.3.5 and equality (1.3.1), we get the equality

$$\mathrm{Mon}^2(F)_g = \mathrm{O}^+(H^2(F,\mathbb{Z}))_g.$$

□

Remark 2.2.7. Notice that, as in Remark 2.1.11, as a by-product we get the isomorphism

$$\widetilde{\mathrm{O}}^+(H^2(F,\mathbb{Z})_{\mathrm{prim}}) \cong \mathrm{O}^+(H^2(F,\mathbb{Z}))_g.$$

Geometrically one would expect that, by deforming to a very general irreducible holomorphic symplectic manifold, one gets $\mathrm{Mon}^2(F) = \mathrm{O}^+(H^2(F,\mathbb{Z}))$. This is indeed the case (cf. Theorem 6.2.3), but unfortunately it cannot be derived directly from the computation above. In fact, a priori, the polarisation g can be constrained to move in a special sub-orbit and one needs more (geometric) information to get around this problem.

Chapter 3

Compactified intermediate Jacobian fibrations

3.1 Intermediate Jacobian fibration associated to a cubic fourfold

Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold and let $Y \subset V$ be a smooth linear section, i.e. a smooth cubic threefold. By Griffiths' residue theory, the Hodge structure on the middle cohomology of Y has level one. More precisely

$$h^{p,q}(Y) = \begin{cases} 0 & (p,q) = (3,0),(0,3) \\ 5 & (p,q) = (2,1),(1,2). \end{cases}$$

This implies that the intermediate Jacobian

$$J_Y := \frac{H^{2,1}(Y)^*}{H_3(Y,\mathbb{Z})} \tag{3.1.1}$$

is algebraic. More precisely, it is a principally polarised abelian variety of dimension 5 (see [CG72, Section 3]).

Notice that, by the Lefschetz Theorem on $(1,1)$-classes, the Abel-Jacobi map

$$\phi_Y \colon \mathrm{CH}^2(Y)_{\mathrm{hom}} \longrightarrow J_Y \tag{3.1.2}$$

is surjective. Here $\mathrm{CH}^2(Y)_{\mathrm{hom}}$ stands for the Chow group of codimension 2 cycles homologous to zero and ϕ_Y is defined by sending a cycle Z homologous to 0 to the operator $\int_\Gamma$, where Γ is such that $\partial\Gamma = Z$ (cf. [Voi02, Section 12.1]).

Now let $\mathcal{T} \subset \mathbb{P}\left(H^0(V, \mathcal{O}_V(1))^*\right)$ be the Zariski open subset parametrising smooth linear sections, and consider the universal family

$$p_{\mathcal{T}} \colon \mathcal{Y}_{\mathcal{T}} \longrightarrow \mathcal{T}. \tag{3.1.3}$$

To this we associate the intermediate Jacobian fibration

$$\pi_{\mathcal{T}} \colon \mathcal{J}_{\mathcal{T}} \longrightarrow \mathcal{T}. \tag{3.1.4}$$

The first remark is the following proposition.

Proposition 3.1.1 ([LSV17, Theorem 1.2, Proposition 1.6])**.** *There exists a closed non-degenerate holomorphic 2-form $\sigma_{\mathcal{T}}$ on $\mathcal{J}_{\mathcal{T}}$, such that the fibres of $\pi_{\mathcal{T}}$ are $\sigma_{\mathcal{T}}$-isotropic.*

We want to recall the construction of $\sigma_{\mathcal{T}}$. First of all, let us work fibrewise. Since the Abel-Jacobi map (3.1.2) is surjective, there exist a variety W and a cycle $Z'' \in \mathrm{CH}^2(W \times Y)$ such that the induced map

$$\phi_{Z''} \colon W \longrightarrow J_Y$$

is surjective. Up to replacing W with a linear section, we can suppose that $\phi_{Z''}$ is generically finite of degree N'. Pushing forward Z'' via the map $(\phi_{Z''}, id)$, we then get a cycle $Z' \in \mathrm{CH}^2(J_Y \times Y)$ and a map

$$\phi_{Z'} \colon J_Y \longrightarrow J_Y,$$

which is N' times the identity map. Notice that this amounts to saying that the correspondence

$$[Z']^* \colon H^3(Y, \mathbb{Z}) \longrightarrow H^1(J_Y, \mathbb{Z})$$

is N' times the natural isomorphism $H^3(Y, \mathbb{Z}) \cong H^1(J_Y, \mathbb{Z})$.

We want to spread out the cycles Z' to give rise to a cycle $\mathcal{Z} \in \mathrm{CH}^2(\mathcal{J}_{\mathcal{T}} \times_{\mathcal{T}} \mathcal{Y}_{\mathcal{T}})$. This can be done only up to replacing $\mathcal{T}$ by a (smooth) generically finite cover $\mathcal{S} \to \mathcal{T}$ (see Proposition 4.4.1). Working over $\mathcal{S}$ we eventually get a cycle $\mathcal{Z}' \in \mathrm{CH}^2(\mathcal{J}_{\mathcal{S}} \times_{\mathcal{S}} \mathcal{Y}_{\mathcal{S}})$ such that the correspondence $[\mathcal{Z}']^*$ is N times the natural isomorphism $R^3 p_{\mathcal{S}*}\mathbb{Z} \cong R^1 \pi_{\mathcal{S}*}\mathbb{Z}$. Here $p_{\mathcal{S}}$ and $\pi_{\mathcal{S}}$ are the base changes of (3.1.3) and (3.1.4), respectively.

Finally, let us take a smooth completion $\overline{\mathcal{S}}$ of $\mathcal{S}$ such that the morphism $r \colon \overline{\mathcal{S}} \to \mathcal{T}$ is proper and finite of degree M. The cycle $\mathcal{Z}'$ naturally extends to a cycle $\overline{\mathcal{Z}'}$ and then we put

$$\mathcal{Z} := \frac{1}{MN} \tilde{r}_* \overline{\mathcal{Z}'} \in \mathrm{CH}^2(\mathcal{J}_{\mathcal{T}} \times_{\mathcal{T}} \mathcal{Y}_{\mathcal{T}})_{\mathbb{Q}}. \tag{3.1.5}$$

Here $\tilde{r} \colon \mathcal{J}_{\overline{\mathcal{S}}} \times_{\overline{\mathcal{S}}} \mathcal{Y}_{\overline{\mathcal{S}}} \to \mathcal{J}_{\mathcal{T}} \times_{\mathcal{T}} \mathcal{Y}_{\mathcal{T}}$ is the map induced by r. Notice that $[\mathcal{Z}]^*$ coincides with the natural isomorphism $R^3 p_{\mathcal{T}*}\mathbb{Q} \cong R^1 \pi_{\mathcal{T}*}\mathbb{Q}$ by construction (see [LSV17, Lemma 1.1]).

Now, there is a natural map $q' \colon \mathcal{Y}_{\mathcal{T}} \to V$, which is the inclusion on any fibre $\mathcal{Y}_t \subset X$. Consider then the induced map

$$q \colon \mathcal{J}_{\mathcal{T}} \times_{\mathcal{T}} \mathcal{Y}_{\mathcal{T}} \longrightarrow \mathcal{J}_{\mathcal{T}} \times V$$

and the cycle $q_*\mathcal{Z} \in \mathrm{CH}^3(\mathcal{J}_\mathcal{T} \times V)_\mathbb{Q}$. This induces a Hodge homomorphism

$$[q_*\mathcal{Z}]^*\colon H^4(V,\mathbb{Q}) \longrightarrow H^2(\mathcal{J}_\mathcal{T},\mathbb{Q})$$

and we define the holomorphic form $\sigma_\mathcal{T}$ as

$$\sigma_\mathcal{T} := [q_*\mathcal{Z}]^*\eta \tag{3.1.6}$$

where η is a generator of $H^{3,1}(V)$. If p_i is the projection from $\mathcal{J}_\mathcal{T} \times V$ to the i-th factor, then we can explicitly write

$$\sigma_\mathcal{T} = p_{1*}([q_*\mathcal{Z}]^{3,3} \cup p_2^*\eta)$$

where $[\mathcal{W}]^{3,3} \in H^3(\mathcal{J}_\mathcal{T} \times V, \Omega^3)$ is the degree $(3,3)$ part of the cohomology class associated to $\mathcal{W}$.

Remark 3.1.2.

1. Notice that, by definition, the form $\sigma_\mathcal{T}$ naturally extends to any smooth partial compactification $\overline{\mathcal{J}}$ of $\mathcal{J}_\mathcal{T}$, just by closing the cycle $\mathcal{Z}$ in $\overline{\mathcal{J}}$. This also implies that $\sigma_\mathcal{T}$ is closed. In fact, this is true on any smooth and projective compactification by Hodge theory, and hence must be true on $\mathcal{J}_\mathcal{T}$ by restriction.

2. Formula (3.1.6) can be also written as

$$\sigma_\mathcal{T} = [\mathcal{Z}]^*(q')^*\eta.$$

   This implies that the restriction $\sigma_\mathcal{T}|_{\mathcal{J}_t} = [\mathcal{Z}|_{\mathcal{J}_t \times \mathcal{Y}_t}]^*(\eta|_{\mathcal{Y}_t}) = 0$ (as $H^4(\mathcal{Y}_t) = 0$) for every $t \in \mathcal{T}$. In particular, the fibration $\pi_\mathcal{T}$ is Lagrangian with respect to $\sigma_\mathcal{T}$.

Remark 3.1.3. Let F_Y be the Fano surface of lines in Y. By [CG72, Lemma 11.16, Theorem 11.19] we have an isomorphism of principally polarised abelian varieties

$$J_Y \cong \mathrm{Pic}^0(F_Y).$$

Hence one can study the family $\mathcal{J}_\mathcal{T}$ from the point of view of line bundles on the Fano surface of lines. This approach was the one used by Donagi and Markman in [DM96], who first studied this situation and stated the existence of the holomorphic form $\sigma_\mathcal{T}$.

Remark 3.1.4. The symplectic form $\sigma_\mathcal{T}$ has been constructed in a different way. Let M_Y be the moduli space of rank 2 vector bundles on Y with trivial determinant and second Chern class of degree 2. There is a natural morphism

$$\varphi\colon M_Y \longrightarrow J_Y$$

defined by sending any such sheaf to the Abel-Jacobi invariant of its Chow-theoretic second Chern class. This moduli space, and its relation to the intermediate Jacobian, has been studied by several authors ([IM00a], [MT01], [IM00b]). The final statement, contained in [Dru00], is that M_Y is identified with the blow-up of J_Y along (a translation of) the Fano surface of lines F_Y. We will give more details about this construction in Section 3.4.

Running everything in families, we get a relative moduli space $\mathcal{M}_\mathcal{T}$ of vector bundles with a natural map to $\mathcal{J}_\mathcal{T}$. The symplectic structure on $\mathcal{M}_\mathcal{T}$ (and its relation with the Donagi-Markman symplectic structure on $\mathcal{J}_\mathcal{T}$) has been studied by [MT03] and [KM09, Section 7]. In particular, Kuznetsov and Markushevich conjectured in [KM09] that, if a smooth and symplectic compactification $\overline{\mathcal{J}}$ of $\mathcal{J}_\mathcal{T}$ exists, then $\overline{\mathcal{J}}$ is an irreducible holomorphic symplectic manifold of OG10-type.

Finding smooth compactifications of $\mathcal{J}_\mathcal{T}$ may not be difficult in general: for example one could blow up (several times if necessary) the boundary of $\mathcal{T}$. But in this case the form $\sigma_\mathcal{T}$ will surely become degenerate. So the problem is finding a smooth and projective compactification which remains symplectic.

We explain in the next section how to partially compactify $\pi_\mathcal{T}\colon \mathcal{J}_\mathcal{T} \to \mathcal{T}$ to linear sections admitting one ordinary double point. Before that, we want to make a digression about the relative Theta divisor.

3.1.1 A distinguished Theta divisor on $\mathcal{J}_\mathcal{T}$

For every $t \in \mathcal{T}$, $J_{\mathcal{Y}_t} = \mathcal{J}_t$ is a principally polarised abelian fivefold and the principal polarisation is given by the Theta divisor θ_t. This gives a relatively ample line bundle $\Theta_\mathcal{T} \in \mathrm{Pic}(\mathcal{J}_\mathcal{T}/\mathcal{T})$, which is defined up to tensoring with a line bundle on $\mathcal{T}$. This indeterminacy is annoying, for example because the degree of $\Theta_\mathcal{T}$ is not determined. We want then a distinguished divisor on $\mathcal{J}_\mathcal{T}$ whose restriction to any fibre $\mathcal{J}_t$ is exactly θ_t, in order to have a preferred representative of $\Theta_\mathcal{T}$. Describing such a divisor is the goal of this sub-section.

First of all, let us recall the following result in [CG72, Section 13]. We work over a fibre $\mathcal{Y}_t = Y$. Let F_Y be the Fano surface of lines on Y and recall that the Abel-Jacobi map $\alpha\colon F_Y \to \mathrm{Alb}(F_Y)$ is an immersion.

Remark 3.1.5. The Abel-Jacobi map depends on a base point $l \in F_Y$, so one should really write α_l. We prefer to drop l from the notation.

Composing with the isomorphism $\mathrm{Alb}(F_Y) \cong J_Y$ ([CG72, Lemma 11.16]), we get an immersion $F_Y \subset J_Y$. The Poincaré dual of the class $[F_Y]$ is represented by $\theta^3/3!$ ([CG72, Proposition 13.1]).

Define now a morphism

$$\varphi\colon F_Y \times F_Y \longrightarrow J_Y \tag{3.1.7}$$

by sending any two lines l_1 and l_2 to the difference $\alpha(l_1) - \alpha(l_2)$.

Remark 3.1.6. Notice that φ is now independent of any base point $l \in F_Y$.

If we restrict to the subset of lines l_1 and l_2 such that $l_1 \cap l_2 = \emptyset$, then $d\varphi$ is injective ([CG72, Theorem 12.37]). Hence φ is an immersion and the image of φ (with reduced structure) is a divisor inside J_Y.

Proposition 3.1.7 ([CG72, Theorem 13.4]). *The Poincaré dual of the image of φ (with reduced structure) is θ.*

We want to recall the proof of this result because it will be useful later. First of all, Clemens and Griffiths explicitly construct six pairs of (disjoint) lines which are mapped into the same point via φ, so the morphism has degree at least 6. Now, the direct image $\varphi_*[F_Y \times F_Y] \in \mathrm{CH}^1(J_Y)$ can be identified with the Pontryagin product $[F_Y] \star [F_Y]$ and, by the Pontryagin-Poincaré formula [BL04, Corollary 16.5.8], we see that

$$\varphi_*[F_Y \times F_Y] = 6\theta.$$

Since θ is indivisible, this implies that the degree of φ is exactly 6 and the claim follows.

We want to run this argument relatively in the family $p_{\mathcal{T}} \colon \mathcal{Y}_{\mathcal{T}} \to \mathcal{T}$. First of all, let

$$\mathcal{F}_{\mathcal{T}} \longrightarrow \mathcal{T}$$

be the relative Fano surface and

$$\alpha_{\mathcal{T}} \colon \mathcal{F}_{\mathcal{T}} \longrightarrow \mathcal{J}_{\mathcal{T}}$$

the relative Abel-Jacobi map. Then we can define as before the morphism

$$\varphi_{\mathcal{T}} \colon \mathcal{F}_{\mathcal{T}} \times_{\mathcal{T}} \mathcal{F}_{\mathcal{T}} \longrightarrow \mathcal{J}_{\mathcal{T}}. \tag{3.1.8}$$

The reduced image $\Theta_{\mathcal{F}_{\mathcal{T}}}$ of $\varphi_{\mathcal{T}}$ is then an effective divisor on $\mathcal{J}_{\mathcal{T}}$ and $\Theta_{\mathcal{F}_{\mathcal{T}}}|_{\mathcal{J}_t} = \theta_t$ for every $t \in \mathcal{T}$. Hence $\Theta_{\mathcal{F}_{\mathcal{T}}}$ is a distinguished (geometric) representative of the relative Theta divisor $\Theta_{\mathcal{T}}$.

3.2 Extension to 1-nodal cubic threefolds

Let Y be a 1-nodal cubic threefold in $\mathbb{P}^4$ and pick a hyperplane H which intersects Y in a smooth cubic surface. Without loss of generality, we may choose homogeneous coordinates $[x_0, \cdots, x_4]$ so that $H = V(x_0)$ and the node is $q = [1 : 0 : 0 : 0 : 0]$. Then the equation of Y is

$$x_0 f_2(x_1, \cdots, x_4) + f_3(x_1, \cdots, x_4) = 0$$

where f_i is a homogeneous polynomial of degree i and the quadric $V(f_2)$ is smooth. Let $D = V(f_2, f_3)$ be the residual curve and notice that the embedding $D \subset H$ is canonical, in particular D has genus 4.

Remark 3.2.1. The converse is also true. Let D be a non-hyperelliptic curve of genus 4, canonically embedded in $\mathbb{P}^3$, and let $P \cong \mathbb{P}^4$ be the linear system of cubics in $\mathbb{P}^3$ containing D. Then the image of the induced rational map $\mathbb{P}^3 \dashrightarrow P$ is a cubic threefold Y. If Q is the unique quadric containing D (see [Har77, Example 5.2.2]), then Y is 1-nodal if and only if Q is smooth (see [vdGK10, Proposition 2.1]).

Remark 3.2.2. These cubic threefolds are rational. In fact, projecting from q gives a birational morphism

$$\lambda \colon \widetilde{Y} \longrightarrow H$$

where $\widetilde{Y}$ is the blow-up of Y at q.

By [CG72, Lemma 6.5], the set of lines in Y passing through q is a cone over D and so λ can be thought of as the blow-up of H at D. Together with [CG72, Lemma 3.11], this gives an isomorphism

$$J_D \cong J_{\widetilde{Y}}. \tag{3.2.1}$$

To any 1-nodal cubic threefold Y with a node at q, one can associate a semi-abelian variety J^o_Y. As before, if $\widetilde{Y}$ is the blow-up of Y at q, then ([Gri69, Corollary 16.4])

$$h^{3,0}(\widetilde{Y}) = 0 \qquad \text{and} \qquad h^{2,1}(\widetilde{Y}) = 4.$$

Moreover, $H_\bullet(\widetilde{Y}, \mathbb{Z})$ is torsion free,

$$h_1(\widetilde{Y}, \mathbb{Z}) = h_5(\widetilde{Y}, \mathbb{Z}) = 0, \qquad h_3(\widetilde{Y}, \mathbb{Z}) = 8 \qquad \text{and} \qquad h_2(\widetilde{Y}, \mathbb{Z}) = h_4(\widetilde{Y}, \mathbb{Z}) = 2.$$

In fact, $H_4(\widetilde{Y}, \mathbb{Z})$ is generated by the classes of the exceptional divisor and the hyperplane section (cf. [Gri69, Claim 15.8, Claim 15.10, Claim 15.12]). Notice that the intermediate Jacobian $J_{\widetilde{Y}}$ is algebraic of dimension 4. All these statements are proved using Picard-Lefschetz theory.

We have a surjective map

$$H_3(Y \setminus \{q\}, \mathbb{Z}) \longrightarrow H_3(\widetilde{Y}, \mathbb{Z})$$

with kernel isomorphic to $\mathbb{Z}$ and generated by the vanishing cycle of the Lefschetz pencil. Using his residue theory, Griffiths proves that there is an injective homomorphism

$$H_3(Y \setminus \{q\}, \mathbb{Z}) \longrightarrow H^1(\widetilde{Y}, \Omega^2_{\widetilde{Y}}(\log E_q))^*$$

where E_q is the exceptional divisor (cf. [Gri69, Equation 16.15 and discussion at pages 526–527]). Moreover, there is a short exact sequence ([Gri69, Sequence 16.13])

$$0 \longrightarrow \mathbb{C} \longrightarrow H^1(\widetilde{Y}, \Omega^2_{\widetilde{Y}}(\log E_q))^* \longrightarrow H^1(\widetilde{Y}, \Omega^2_{\widetilde{Y}})^* \longrightarrow 0.$$

If we define
$$J_Y^o := H^1(\widetilde{Y}, \Omega^2_{\widetilde{Y}}(\log E_q))^* / H_3(Y \setminus \{q\}, \mathbb{Z})$$
then there is a short exact sequence ([Gri69, Theorem 16.16])
$$1 \longrightarrow \mathbb{G}_m = \mathbb{C}/\mathbb{Z} \longrightarrow J_Y^o \longrightarrow J_{\widetilde{Y}} \longrightarrow 0. \tag{3.2.2}$$
Recall that $J_{\widetilde{Y}} \cong J_D$. Let Q be the unique smooth quadric containing D (cf. Remark 3.2.1). Let $|W_1|$ and $|W_2|$ be the two g^1_3s on D corresponding to the two rulings of Q (in particular $W_1 + W_2 = K_D$). As explained in [vdGK10, Corollary 6.3], the extension (3.2.2) is determined by the line bundle $L = \mathcal{O}(W_1 - W_2) \in \mathrm{Pic}^0(J_D)$.

Remark 3.2.3. This is also implicit in [Gri69]. In fact in [Gri69, Claim 5.10] the classes A and B correspond to the two divisors W_1 and W_2.

There is a canonical way to compactify J_Y^o to a projective variety J_Y. Consider the projective bundle $p\colon \mathbb{P} = \mathbb{P}(L \oplus \mathcal{O}) \to J_{\widetilde{Y}}$. Let us denote by Σ_1 and Σ_2 the two divisors (isomorphic to $J_{\widetilde{Y}}$) inside $\mathbb{P}$ associated to the projections $L \oplus \mathcal{O} \to L$ and $L \oplus \mathcal{O} \to \mathcal{O}$, respectively. Notice that each $\mathcal{O}(\Sigma_i) \otimes \mathcal{O}(-1)$ is a line bundle on $\mathbb{P}$, trivial on the fibres, and by the See-Saw Theorem, it comes from a line bundle L_i on $J_{\widetilde{Y}}$. Pulling back the relation $\mathcal{O}(\Sigma_i) \otimes \mathcal{O}(-1) = p^* L_i$ on Σ_j, it turns out that
$$\mathcal{O}(\Sigma_1) = \mathcal{O}(1) \qquad \text{and} \qquad \mathcal{O}(\Sigma_2) = \mathcal{O}(1) \otimes p^* L^\vee. \tag{3.2.3}$$
We define J_Y then as the (non-normal) projective variety obtained from $\mathbb{P}$ by gluing Σ_1 and Σ_2 via the translation L.

Remark 3.2.4. In the definition of J_Y, the line bundle $L \in \mathrm{Pic}^0(J_D)$ is seen as a translation thanks to the principal polarisation (Theta divisor) on J_D. This natural identification is the condition that ensures that J_Y has a polarisation which is the flat limit of the Theta divisors on the smooth fibres of a Lefschetz pencil. This polarisation is still called Theta divisor and denoted by θ_Y.

Remark 3.2.5. Deleting Σ_1 and Σ_2 from $\mathbb{P}$, one gets J_Y^o back. Hence the smooth locus of J_Y is naturally identified with J_Y^o.

Remark 3.2.6. Topologically, J_Y looks like the product of an abelian fourfold and a nodal cubic. This follows from the exact sequence (3.2.2): the abelian fourfold is $J_{\widetilde{Y}}$ and the nodal cubic is obtained by compactifying the $\mathbb{G}_m = \mathbb{C}^*$ to $\mathbb{P}^1$ by adding the point at 0 and infinity, and then gluing them.

Everything we said so far can be run in families. So, let $\mathcal{T}_1 \subset \mathbb{P}(H^0(V, \mathcal{O}(1))^*)$ be the open subset parametrising linear sections of V with at worst an ordinary double point. Then there is a fibration (cf. [Gri69, Theorem 17.1])
$$\pi_{\mathcal{T}_1} \colon \mathcal{J}_{\mathcal{T}_1} \longrightarrow \mathcal{T}_1$$

partially compactifying the fibration (3.1.4).

Remark 3.2.7. By Remark 3.1.2, the symplectic form $\sigma_{\mathcal{T}}$ has a natural extension $\sigma_{\mathcal{T}_1}$ to $\mathcal{J}_{\mathcal{T}_1}$. This extension is still nondegenerate by [LSV17, Proposition 1.23].

3.2.1 Extension of the distinguished Theta divisor

We want to study here the extension of the Theta divisor $\Theta_{\mathcal{F}_{\mathcal{T}}}$ defined in Section 3.1.1 to the boundary $\mathcal{J}_{\mathcal{T}_1} \setminus \mathcal{J}_{\mathcal{T}}$. The result we want to obtain is the following.

Proposition 3.2.8. *There exists a natural map*

$$\varphi_{\mathcal{T}_1} \colon \mathcal{F}_{\mathcal{T}_1} \times_{\mathcal{T}_1} \mathcal{F}_{\mathcal{T}_1} \longrightarrow \mathcal{J}_{\mathcal{T}_1} \tag{3.2.4}$$

extending the morphism (3.1.8) and such that its image (with reduced structure) coincides with the closure of $\Theta_{\mathcal{F}_{\mathcal{T}}}$ *inside* $\mathcal{J}_{\mathcal{T}_1}$.

We denote by $\Theta_{\mathcal{F}_{\mathcal{T}_1}}$ the closure of $\Theta_{\mathcal{F}_{\mathcal{T}}}$ inside $\mathcal{J}_{\mathcal{T}_1}$.

Let us first see what happens on a fibre, so let Y be a cubic threefold with an ordinary double point q. Let F_Y be the Fano surface of lines and $D \subset Y$ be the non-hyperelliptic curve of genus 4 determined by Y as in Section 3.2. By [CG72, Theorem 7.8], F_Y is singular along a curve isomorphic to D. More precisely, the singular locus of F_Y is the curve of lines passing through the node q and the latter is a cone over D.

Remark 3.2.9. As explained in [CG72, Section 8], Y contains no planes, so we have a well-defined morphism

$$\nu \colon \operatorname{Sym}^2 D \longrightarrow F_Y$$

by sending the point $(p_1, p_2) \in \operatorname{Sym}^2 D$ to the residual line $\nu(p_1, p_2)$ of the intersection of Y with the plane determined by the lines l_{p_1} and l_{p_2} (for $p \in F_Y$, we denote by l_p the corresponding line in Y).

This map is the normalisation of F_Y. To understand it better, consider the two embeddings of D inside $\operatorname{Sym}^2 D$ defined by sending $p \in D$ to the support of $W_i - p$ (here, as before, W_i are the two rulings of the unique smooth quadric containing D). Denote by D_i the image of D via these two morphisms. Then F_Y is the (non-normal) variety obtained from $\operatorname{Sym}^2 D$ by gluing D_1 and D_2 together. More precisely, if we say that $p_1, p_2 \in D_1$ and $p_3, p_4 \in D_2$ are complementary if $l_{\nu(p_1,p_2)} = l_{\nu(p_3,p_4)}$, then ν identifies complementary points (cf. [vdGK10, Section 2]).

Identifying $J_{\widetilde{Y}} \cong J_D$ and fixing two base points $p, p' \in J_D$, there is an Abel-Jacobi map

$$\phi \colon \operatorname{Sym}^2 D \longrightarrow J_{\widetilde{Y}}$$

sending the point $(p_1, p_2) \in \operatorname{Sym}^2 D$ to the Abel-Jacobi image of $p_1 + p_2 - p - p'$. This map lifts to a map

$$\tilde{\phi} \colon \operatorname{Sym}^2 D \longrightarrow \mathbb{P}$$

once we choose a line bundle $M \in \mathrm{Pic}(\mathrm{Sym}^2 D)$ and a surjective morphism

$$\tau\colon \phi^*(L \oplus \mathcal{O}) \to M$$

(cf. [Har77, Proposition II.7.12]). If we choose $M = \mathcal{O}(D_1)$, then $\tilde{\phi}^*(L\oplus\mathcal{O})^\vee\otimes\mathcal{O}(D_1) = \mathcal{O}(D_2)\oplus\mathcal{O}(D_1)$ and the latter has a distinguished section giving the desired surjective morphism.

Remark 3.2.10. Notice that $\mathcal{O}(D_1) = \mathcal{O}(D_2) \otimes \phi^* L$.

This particular choice of M also implies that $\tilde{\phi}^{-1}(\Sigma_i) = D_i$ and that $\tilde{\phi}^*\mathcal{O}(1) = M$ (cf. equalities (3.2.3)). Moreover, if $p_1, p_2 \in D_1$ and $p_3, p_4 \in D_2$ are complementary points, then $p(\tilde{\phi}(p_1, p_2)) - p(\tilde{\phi}(p_3, p_4)) = L$ (recall that $p\colon \mathbb{P} \to J_{\widetilde{Y}} \cong \mathrm{Pic}^0(J_D)$). Therefore $\tilde{\phi}$ descends to an Abel-Jacobi map (cf. [vdGK10, Section 9] for a deeper analysis of this map)

$$\varphi\colon F_Y \longrightarrow J_Y. \tag{3.2.5}$$

Proposition 3.2.11 ([vdGK10, Proposition 10.1 and its proof])**.** *The image (with reduced structure) $\varphi(F_Y)$ coincides with the flat limit of Fano surfaces of lines of smooth cubic threefolds. Moreover, it is Poincaré dual to the class $\theta_Y^3/3!$, where θ_Y is the Theta divisor on J_Y defined in Remark 3.2.4.*

Now define the map

$$\varphi_1\colon F_Y \times F_Y \longrightarrow J_Y \tag{3.2.6}$$

as we did in the smooth case.

Proposition 3.2.12. *The image (with reduced structure) $\varphi_1(F_Y \times F_Y)$ is Poincaré dual to θ_Y.*

Proof. This follows from Proposition 3.1.7, Remark 3.2.4 and Proposition 3.2.11, passing to the flat limit of a simple (Lefschetz) degeneration. □

Proof of Proposition 3.2.8. The map (3.2.4) is the relative version of the map (3.2.6). The second claim is a direct consequence of Proposition 3.2.12.

3.3 LSV compactification

We have said that $\mathcal{J}_{\mathcal{T}}$ has a partial compactification $\mathcal{J}_{\mathcal{T}_1}$ such that the symplectic form on $\mathcal{J}_{\mathcal{T}}$ extends to a symplectic form on $\mathcal{J}_{\mathcal{T}_1}$ (see Remark 3.2.7). Since $\mathcal{J}_{\mathcal{T}_1}$ is flat over $\mathcal{T}_1$ and $\mathcal{T}_1 \subset \mathrm{P} := \mathbb{P}\left(H^0(V, \mathcal{O}_V(1))^*\right)$ has boundary of codimension 2, the idea is to compactify $\mathcal{J}_{\mathcal{T}_1}$ instead, since then the symplectic form will automatically extend to a symplectic form on the compactification.

Theorem 3.3.1 ([LSV17, Main Theorem]). *Suppose V is a generic smooth cubic fourfold. Then there exists a smooth, projective and symplectic compactification $\mathcal{J}_V$ of $\mathcal{J}_{\mathcal{T}_1}$ such that the projection*

$$\pi_V \colon \mathcal{J}_V \longrightarrow \mathrm{P}$$

is a Lagrangian fibration. Moreover, $\mathcal{J}_V$ is an irreducible holomorphic symplectic manifold of OG10-type.

Remark 3.3.2. Here generic means in an open and dense Zariski subset. In particular this means that there are special cubic fourfolds (in the sense of Hassett) such that this construction specialises well. A concrete example is represented by Pfaffian cubic fourfolds (see [LSV17, Section 3.2]).

It is not our intention to explain the construction of this compactification, but we want anyway to recall the main idea.

Remark 3.3.3. Let Y be a smooth cubic threefold and $l \subset Y$ a generic line. Let Y_l be the blow-up of Y at l and consider the conic bundle

$$\pi_l \colon Y_l \longrightarrow \mathbb{P}^2$$

obtained by projecting from l onto a plane. The discriminant locus, i.e. the locus where the conics degenerate, is a quintic curve $C \subset \mathbb{P}^2$. The preimage $\widetilde{C}$ of C is the curve of lines in Y intersecting l and the restriction

$$\pi_l \colon \widetilde{C} \longrightarrow C$$

is an étale double cover.

To this étale double cover, Mumford associates a principally polarised abelian variety $\mathrm{Prym}(\widetilde{C}/C)$, defined as the connected component of the identity of the kernel of the norm map

$$\mathrm{Nm} \colon J_{\widetilde{C}} \longrightarrow J_C.$$

Notice that $\mathrm{Prym}(\widetilde{C}/C)$ has dimension 5 (cf. [CG72, Appendix C]). It turns out that

$$\mathrm{Prym}(\widetilde{C}/C) \cong J_Y$$

as principally polarised abelian varieties.

The idea in [LSV17] is to compactify the analogous fibration by Prym varieties instead of the fibration (3.1.4). Notice that, to run Mumford's argument, we need to choose a line. This means that the fibration by Prym varieties is actually a fibration over the relative Fano surface of lines, i.e.

$$\mathrm{Prym}(\widetilde{\mathcal{C}}_{\mathcal{T}}, \mathcal{C}_{\mathcal{T}}) \longrightarrow \mathcal{F}_{\mathcal{Y}_{\mathcal{T}}}.$$

The first step is to compactify this family to a projective variety fibred over $\mathcal{F}_{\mathrm{P}}$, where $\mathrm{P} = \mathbb{P}(H^0(V, \mathcal{O}(1))^*$ as before,

$$\mathrm{Prym}(\widetilde{\mathcal{C}}_{\mathrm{P}}, \mathcal{C}_{\mathrm{P}}) \longrightarrow \mathcal{F}_{\mathrm{P}} \tag{3.3.1}$$

(cf. [LSV17, Section 4]).

Remark 3.3.4. To be more precise, not any line is good to run Mumford's argument, especially if we want to deal with singular cubic threefolds. Hence one should work with a subset $\mathcal{F}^o_{\mathrm{P}}$ of $\mathcal{F}_{\mathrm{P}}$ consisting of *very good lines* (in the sense of [LSV17, Section 2]). Notice that this is not an issue because $\mathrm{Prym}(\widetilde{\mathcal{C}}_{\mathrm{P}}, \mathcal{C}_{\mathrm{P}}) \longrightarrow \mathcal{F}^o_{\mathcal{Y}_{\mathrm{P}}}$ is still surjective for generic V.

The second step is to descend this fibration to a projective variety $\mathcal{J}_V$ over P.

The smoothness of $\mathcal{J}_V$ follows from a general smoothness criterion for relative Prym varieties ([LSV17, Theorem 4.20]).

The claim that $\mathcal{J}_V$ is irreducible holomorphic symplectic is a direct check. We already discussed the existence of a symplectic form. Now the claim follows from the fact that any symplectic form on any étale cover $\widetilde{\mathcal{J}}_V$ of $\mathcal{J}_V$ is a multiple of the symplectic form induced by σ_V ([LSV17, Lemma 5.8]). From the Beauville-Bogomolov-De Rham Decomposition Theorem, it will follow that $\mathcal{J}_V$ is simply connected and has a unique symplectic form. This statement is proved by comparison with an irreducible holomorphic symplectic eight-fold Z_V of $K3^{[4]}$-type associated to V, discovered by C. Lehn, M. Lehn, Sorger and van Straten ([LLSvS17]). More precisely, Laza, Saccà and Voisin prove that the relative Theta divisor $\Theta \subset \mathcal{J}_V$ is birational to a $\mathbb{P}^1$-bundle $\mathbb{P}$ over Z_V, giving so a rational map $\mathbb{P} \dashrightarrow \mathcal{J}_V$. Since $\mathbb{P}$ is simply connected, this lifts to a rational map $\mathbb{P} \dashrightarrow \widetilde{\mathcal{J}}_V$. Now, since Z_V is irreducible symplectic and $\mathbb{P}$ is a $\mathbb{P}^1$-bundle over it, the claim follows from the fact that, if α is a holomorphic 2-form on $\widetilde{\mathcal{J}}_V$ which vanishes on the image of Θ in $\mathbb{P}$, then $\alpha = 0$.

Once we know that $\mathcal{J}_V$ is an irreducible holomorphic symplectic manifold, the claim that it is of OG10-type follows from an explicit birational map to the O'Grady moduli space. We will recall this map in Section 3.4.

Definition 3.3.5. Any projective variety isomorphic to $\mathcal{J}_V$, for some smooth cubic fourfold V, will be called a LSV manifold.

Remark 3.3.6. Let $\Theta_{\mathcal{T}_1}$ be a relative Theta divisor on $\mathcal{J}_{\mathcal{T}_1}$ and Θ_V its closure inside $\mathcal{J}_V$. A priori Θ_V is only relatively ample. On the other hand, by [LSV17, Theorem 5.11], the compactification $\mathcal{J}_V$ has the explicit form

$$\mathcal{J}_V = \mathrm{Proj}\left(\bigoplus_{k\geq 0} R^0 j_{1*}\left(R^0\pi_{\mathcal{T}_1 *}\mathcal{O}(kd\Theta_{\mathcal{T}_1})\right)\right)$$

where $j_1 \colon \mathcal{T}_1 \to \mathrm{P}$ is the inclusion and $d \gg 0$ is fixed. In particular Θ_V is ample on $\mathcal{J}_V$.

Remark 3.3.7. We have seen in Section 3.1.1 how to construct a distinguished divisor $\Theta_{\mathcal{F}_{\mathcal{T}}}$ on $\mathcal{J}_{\mathcal{T}}$ and we studied in Section 3.2.1 how it extends to the partial compactification $\mathcal{J}_{\mathcal{T}_1}$. We close it now in the LSV compactification $\mathcal{J}_V$ and denote by $\Theta_{\mathcal{F}_V}$ this closure. This is a distinguished representative of the relative Theta divisor $\Theta_V \in \mathrm{Pic}(\mathcal{J}_V/\mathrm{P})$ and it gives a distinguished polarisation of such manifolds.

3.4 Relation with O'Grady moduli spaces

In this section we want to recall the following result.

Theorem 3.4.1 ([LSV17, Theorem 6.2])**.** *If V is a Pfaffian cubic fourfold, then $\mathcal{J}_V$ is birational to $\widetilde{M}_S$, where $S = V^\vee$ is the $K3$ surface dual to V and $\widetilde{M}_S$ is the O'Grady symplectic resolution of singularities of the moduli space of semistable sheaves of rank 2 on S, with trivial first Chern class and second Chern class of degree 4.*

As a consequence we get that, for generic V, the varieties $\mathcal{J}_V$ are all of OG10-type.

Moduli space of instantons on a smooth cubic threefold

Let Y be a smooth cubic threefold. We consider the moduli space M_Y of semistable rank 2 sheaves E such that $c_1(E) = 0$, $c_2(E) = 2$ and $c_3(E) = 0$. This moduli space has been studied by several authors ([MT01], [IM00a], [Kuz04], [Dru00]). The first remark (cf. [Dru00]) is that M_Y has a stratification

$$M_Y = M_Y^{\mathrm{lf}} \sqcup A_0 \sqcup B$$

where

- M_Y^{lf} is the open subset parametrising locally free sheaves. These sheaves are stable and they all come from smooth elliptic quintic curves in Y via Serre's construction (cf. [MT01]).
- A_0 is the locally closed subset of sheaves E of the form

  $$0 \longrightarrow E \longrightarrow H^0(C, L) \otimes \mathcal{O}_Y \longrightarrow L \longrightarrow 0$$

  where $C \subset Y$ is a smooth conic and $L \in \mathrm{Pic}(C)$ is such that $L^{\otimes 2} = \mathcal{O}_Y(1)|_C$ (i.e. L is a theta characteristic). These sheaves are stable and A_0 has codimension 1 in M_Y.
- B is the closed subset of sheaves E which are S-equivalent to the sum $\mathcal{I}_{l_1} \oplus \mathcal{I}_{l_2}$, where l_1 and l_2 are two lines in Y. These sheaves are strictly semistable and B is a divisor in M_Y.

Remark 3.4.2. The closure A of A_0 intersects B in the locus corresponding to incident lines.

Proposition 3.4.3 ([Dru00, Théorème 4.6])**.** *M_Y is smooth of dimension* 5.

If $E \in M_Y$, then the Chow-valued second Chern class $\mathfrak{c}_2(E) \in \mathrm{CH}^2(Y)$ has (cohomological) degree 2. If α is a class of degree 2, then we can send any sheaf $E \in M_Y$ to the point $\phi_Y(\mathfrak{c}_2(E) - \alpha) \in J_Y$, where ϕ_Y is the Abel-Jacobi map (3.1.2). With a slight abuse of notation, we write

$$\mathfrak{c}_2 \colon M_Y \longrightarrow J_Y \tag{3.4.1}$$

for the morphism described above.

Remark 3.4.4. If we want to make this construction canonical, we should really work with a torsor over J_Y and replace the Abel-Jacobi map with the twisted Abel-Jacobi map (see Section 4.1). Nevertheless, if Y is Pfaffian, we have a distinguished degree 2 class, which provides a trivialisation of the torsor (cf. Example 4.3.6) and therefore a distinguished morphism to J_Y.

As already notice by [MT01] and [IM00a], $\mathfrak{c}_2$ is 1-to-1 on M_Y^{lf}. Moreover, $\mathfrak{c}_2(B)$ is mapped onto (a translation of) the image of the sum map (3.1.7); recall that this is a divisor in the same equivalence class as the Theta divisor on J_Y. On the other hand $\mathfrak{c}_2$ contracts the divisor A to a surface (the Fano surface of conics). The final statement is the following.

Theorem 3.4.5 ([Dru00, Théorème 1.4])**.** *$\mathfrak{c}_2 \colon M_Y \to J_Y$ is the blow-up of J_Y along a translation of the Fano surface of lines of Y.*

As a corollary, we get that J_Y is birational to a moduli space of sheaves on Y.

If now V is a smooth cubic fourfold, one can consider the moduli space $\mathcal{M}_V$ parametrising semistable torsion sheaves with invariants

$$(\mathrm{rk}, c_1, c_2, c_3, c_4) = (0, 2h, 3h^2, 8l, 1)$$

where $h = c_1(\mathcal{O}_V(1))$ and l is the class of a line in V. The generic point of $\mathcal{M}_V$ is of the form i_*E where $i \colon Y \to V$ is a smooth linear section and $E \in M_Y$.

Remark 3.4.6. $\mathcal{M}_V$ can be thought of as a relative moduli space of sheaves on a family of cubic threefolds.

Let $\mathcal{M}_V^0 \subset \mathcal{M}_V$ be the open subset parametrising torsion sheaves supported on smooth linear sections of V and locally free on their support. If $\mathcal{T} \subset \mathrm{P} \cong \mathbb{P}^5$ is the subset of smooth linear sections of V, let us denote by

$$q \colon \mathcal{M}_V^0 \to \mathcal{T}$$

the morphism associating to any vector bundle its Fitting support.

Theorem 3.4.7 ([MT03, Corollary 1.6, Proposition 2.1, Theorem 2.4],[KM09, Section 7]). *$\mathcal{M}_V^0$ is smooth of dimension 10. Moreover, there exists a symplectic form on $\mathcal{M}_V^0$ such that the morphism $q\colon \mathcal{M}_V^0 \to \mathcal{T}$ is a Lagrangian fibration.*

Theorem 3.4.5 and Theorem 3.4.7 implies that $\mathcal{J}_{\mathcal{T}}$ is birational to $\mathcal{M}_V^0$. The next step is to show that, when V is Pfaffian, $\mathcal{M}_V^0$ is birational to $\widetilde{M}_S$.

Outline of the proof of Theorem 3.4.1

In the following we use the same notation as in Section 2.2.1. Let $W = W_6$ be a 6-dimensional vector space and let $L_6 \subset \bigwedge^2 W^*$ be a 6-dimensional vector subspace. Then $V = \mathrm{Pf}(W) \cap \mathbb{P}(L_6)$ and

$$S = \left\{ [P] \in \mathrm{Gr}(2, W) \subset \mathbb{P}\left(\bigwedge^2 W\right) \middle| \phi|_P \equiv 0 \quad \forall \phi \in L_6 \right\}.$$

A smooth linear section of V is given by a general 5-dimensional vector subspace $L_5 \subset L_6$ and the cubic threefold is $Y = V \cap \mathbb{P}(L_5)$. The dual variety

$$R = \left\{ [P] \in \mathrm{Gr}(2, W) \subset \mathbb{P}\left(\bigwedge^2 W\right) \middle| \phi|_P \equiv 0 \quad \forall \phi \in L_5 \right\}.$$

is a smooth Fano threefold of degree 14.

Remark 3.4.8. Notice that all these Fano threefolds R contain the $K3$ surface S by construction.

Proposition 3.4.9 ([IM00a], [Kuz04]). *R is birational to Y. More precisely, they are related by a flop.*

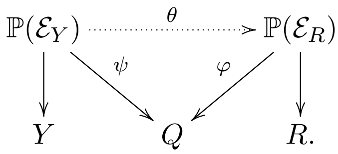
Let us briefly recall the construction of this flop. First of all, on R we have the tautological bundle $\mathcal{E}_R$, which is the restriction of the tautological bundle on $\mathrm{Gr}(2, W)$. On the other hand, on Y we have the Pfaffian bundle $\mathcal{E}_Y$, such that $(\mathcal{E}_Y)_\phi = (\ker \phi)^*$ on each point $\phi \in Y$. By [Kuz04, Theorem 2.2], $H^0(R, \mathcal{E}_R) \cong W^* \cong H^0(Y, \mathcal{E}_Y)$. Moreover, $\mathcal{E}_Y$ is generated by global sections and induces an embedding $Y \subset \mathrm{Gr}(2, W)$. There are two natural projections

$$\psi\colon \mathbb{P}(\mathcal{E}_Y) \to \mathbb{P}(W) \qquad \text{and} \qquad \varphi\colon \mathbb{P}(\mathcal{E}_R) \to \mathbb{P}(W)$$

such that $\psi(\mathbb{P}(\mathcal{E}_Y)) = \varphi(\mathbb{P}(\mathcal{E}_R)) = Q \subset \mathbb{P}(W)$ is a quartic hypersurface singular along a curve C of degree 25 and arithmetic genus 26 ([Kuz04, Proposition 2.11, Proposition 2.15]). The composition

$$\theta = \varphi^{-1} \circ \psi\colon \mathbb{P}(\mathcal{E}_Y) \dashrightarrow \mathbb{P}(\mathcal{E}_R)$$

is a flop in the surface $\Sigma_Y = \psi^{-1}(C)$ (see [Kuz04, Theorem 2.17]). This situation is described by the following diagram (cf. [Kuz04, Theorem 2.18])



This birational morphism is the bridge which makes it possible to pass from sheaves on Y to sheaves on R. Finally, any sheaf on R can be restricted to a sheaf on S and the fact that these sheaves have the right invariants is a direct check.

A generic point in $\mathcal{M}_V^0$ is of the form i_*E where $i\colon Y \to V$ is a generic (smooth) linear section and $E \in M_Y^{\mathrm{lf}}$ is also generic. By [LSV17, Section 6.2], E is uniquely determined by a genus 5 curve $C \subset \mathbb{P}(\mathcal{E}_Y)$ such that the image C' of C under the projection $\mathbb{P}(\mathcal{E}_Y) \to Y$ is a curve of degree 9, the restriction $C \to C'$ is an isomorphism and the image of C in Q is a linearly normal canonical curve. The correspondence is summarised by the short exact sequence

$$0 \longrightarrow E \xrightarrow{\sigma} \mathcal{E}_Y \longrightarrow \mathcal{O}_S(C) \longrightarrow 0$$

where $\sigma \in H^0(Y, E^* \otimes \mathcal{E}_Y) \neq 0$ is generic, $\Sigma \in |\mathcal{O}_Y(2)|$ is a smooth $K3$ surface and $C \subset \Sigma$ (see [LSV17, Lemma 6.7]). More precisely, one can see that $\psi(C) \subset Q$ is a complete intersection of three quadrics in a hyperplane section of Q (cf. beginning of [LSV17, Section 6.4]). If the equation of Q is $q = 0$, the equations of $\psi(C)$ are $q_1 = q_2 = q_3 = 0$ and $H \subset \mathbb{P}(W)$ is a hyperplane, then $q|_H = s_1q_1 + s_2q_2 + s_3q_3$, where s_1, s_2 and s_3 are quadratic polynomials on H. Define the curve C_1 by $q_1 = q_2 = s_3 = 0$. This curve does not meet the indeterminacy locus of φ^{-1} and the inverse image $C' = \varphi^{-1}(C_1) \subset \mathbb{P}(\mathcal{E}_R)$ is a genus 5 curve such that its image under the projection $\mathbb{P}(\mathcal{E}_R) \to R$ has degree 13 in R ([LSV17, Lemma 6.12]). When C' is generic enough, by [LSV17, Lemma 6.9] there exists a unique smooth $K3$ surface $\Sigma' \subset R$ such that $C' \subset \Sigma'$. Eventually one concludes that there exists a rank 2 vector bundle F on R fitting in the short exact sequence

$$0 \longrightarrow F \longrightarrow \mathcal{E}_R \longrightarrow \mathcal{O}_{\Sigma'}(C') \longrightarrow 0.$$

By direct computations one sees that $c_1(F) = 0$ and that $c_2(F) = 4$ ([LSV17, Section 6.3]).

Finally, restricting F to the $K3$ surface $S = V^\vee$, one gets a rank 2 vector bundle with trivial determinant and second Chern class of degree 4. If S is generic, then $F|_S$ is also stable, i.e. $F|_S \in M_S^{\mathrm{lf}} \subset \widetilde{M}_S$.

The fact that this correspondence produces a rational map (birational actually) between $\mathcal{M}_V^0$ and M_S^{lf} is highly non-trivial and we refer to [LSV17, Section 6.4] for the details.

Remark 3.4.10. The birational isomorphism described above is not completely satisfying. For many useful applications it would be necessary to have a clear understanding of what this map does to the generic point of the boundary of $\mathcal{J}_{\mathcal{T}}$ to the partial compactification $\mathcal{J}_{\mathcal{T}_1}$ (for example, because the boundary of $\mathcal{J}_{\mathcal{T}}$ in $\mathcal{J}_V$ is a divisor). The first problem in this direction is that we do not know whether the extension of $\mathcal{M}^0$ to 1-nodal cubic threefolds is smooth or not (and we do not even know if $\mathcal{M}_{\mathcal{T}} = \mathcal{M}_V \times_{\mathrm{P}} \mathcal{T}$ is smooth or not). These questions are the objects of a joint work with Giulia Saccà. Our conjecture is that $\mathcal{M}_V$ is smooth (at least for generic V).

Remark 3.4.11. The fact that the varieties $\mathcal{J}_V$ are of OG10-type can be seen more easily using a recent deformation criterion by Kollár, Laza, Saccà and Voisin ([KLSV17]). In this case one first degenerates the cubic fourfold to a chordal cubic fourfold, which is very singular. We recall that chordal cubic fourfolds have associated $K3$ surfaces of genus 2. The relative intermediate Jacobian degeneration has central fibre which has an irreducible component birational to the O'Grady moduli space (cf. [KLSV17, Section 5.3]). The claim then follows from [KLSV17, Theorem 0.1].

3.5 VHS induced by the cubic fourfold

Let V be a generic smooth cubic fourfold and $\pi_V \colon \mathcal{J}_V \to \mathbb{P}^5$ the compactified intermediate Jacobian fibration. Let Θ_V be the relative Theta divisor and $b_V = \pi_V^*(\mathcal{O}(1))$ the class of the fibration.

Proposition 3.5.1 ([HLS17])**.** *The lattice $\langle \Theta_V, b_V \rangle$ generated by Θ_V and b_V is saturated, primitive and isometric to the hyperbolic plane.*

We reproduce the proof here because we will refer to it later.

Proof. First of all, notice that $b_V^{10} = 0$ and so from Fujiki's relation (1.1.2) it follows that $q(b_V) = 0$.

Consider now the class $b_V + t\Theta_V$. Comparing the coefficients of t^5 in Fujiki's relation, it follows that $5!q(\Theta_V, b_V) = \Theta_V^5 b_V^5 = 5!$. The last equality follows from the fact that Θ_V is a Theta divisor on each fibre. The Gram matrix of the lattice generated by Θ_V and b_V is then

$$\begin{pmatrix} * & 1 \\ 1 & 0 \end{pmatrix}.$$

Independently of the value of $q(\Theta_V)$, this is a saturated and primitive lattice isometric to a hyperbolic plane. □

Remark 3.5.2. Notice that we do not need to know the degree of the relative Theta divisor to state this result.

Now let $p_V \colon \mathcal{Y}_V \to \mathrm{P}$ be the universal family of linear sections of V. There is a natural morphism $q \colon \mathcal{Y}_V \to V$ which is the inclusion of each fibre $\mathcal{Y}_t$ into V. For any

$x \in V$, the fibre $q^{-1}(x)$ is the $\mathbb{P}^4$ of hyperplane sections passing through x (in fact, $\mathcal{Y}_V$ can be thought of as the $\mathbb{P}^4$-bundle $T_{\mathbb{P}^5}(-1)|_V$). Now let $\mathcal{Z} \in \mathrm{CH}^2(\mathcal{Y}_V \times_V \mathcal{J}_V)_{\mathbb{Q}}$ be the cycle constructed in (3.1.5). Up to replacing $\mathcal{Z}$ with a multiple, we can suppose that it is integral.

Define the map

$$\alpha \colon H^4(V, \mathbb{Z}) \longrightarrow H^2(\mathcal{J}_V, \mathbb{Z})$$

as the composition of the pullback q^* and the correspondence $[\mathcal{Z}]^*$. Notice that α is a morphism of Hodge structures.

Proposition 3.5.3 ([HLS17])**.** *Using the notation as above, the following holds:*

1. *if $h^2 \in H^4(V, \mathbb{Z})$ is the square of the hyperplane class on V, then $\alpha(h^2) \in \langle \Theta_V, b_V \rangle$;*
2. *the restriction*

$$\alpha \colon H^4(V, \mathbb{Z})_{\text{prim}} \longrightarrow \langle \Theta_V, b_V \rangle^{\perp} \tag{3.5.1}$$

is an anti-similitude.

Anti-similitude means that there exists $N > 0$ such that $(\alpha(x), \alpha(y)) = -N x.y$ for every $x, y \in H^4(V, \mathbb{Z})_{\text{prim}}$.

Remark 3.5.4. There is no reason to expect $N = 1$ and actually one can show that N must be a multiple of 3. In fact, $H^4(V, \mathbb{Z})$ is an overlattice of $h^2 \oplus (h^2)^{\perp}$ and if N were equal to 1 then the image of the element $(h^2 + t)/3 \in H^4(V, \mathbb{Z})$ could never be integral.

Chapter 4

Compactified intermediate Jacobian fibrations: twisted case

4.1 Twisted intermediate Jacobians of cubic threefolds

Let Y be a smooth cubic threefold. Recall that the Deligne complex is

$$\mathbb{Z}(p)\colon \qquad 0 \longrightarrow \mathbb{Z} \stackrel{(2\pi i)^p}{\longrightarrow} \mathcal{O}_Y \longrightarrow \Omega_Y^1 \longrightarrow \cdots \longrightarrow \Omega_Y^{p-1} \longrightarrow 0$$

and then the Deligne cohomology is defined as the hyper-cohomology of $\mathbb{Z}(p)$, more precisely

$$H^k_D(Y, \mathbb{Z}(p)) := \mathbb{H}^k(Y, \mathbb{Z}(p)).$$

Example 4.1.1. $H^2_D(Y, \mathbb{Z}(1)) = H^1(Y, \mathcal{O}_Y^*) = \mathrm{Pic}(Y)$.

The main result we need is the existence of a short exact sequence ([Voi02, Corollary 12.27])

$$0 \longrightarrow J_Y \longrightarrow H^4_D(Y, \mathbb{Z}(2)) \stackrel{c}{\longrightarrow} H^4(Y, \mathbb{Z}) \longrightarrow 0. \qquad (4.1.1)$$

The Deligne cohomology is useful to study cycles on a variety. In fact, to any codimension p algebraic cycle Z on Y, one can associate a Deligne class $[Z]_D \in H^{2p}_D(Y, \mathbb{Z}(p))$ ([Voi02, Section 12.3.3]) producing a cycle class map

$$cl_D\colon \mathrm{CH}^2(Y) \longrightarrow H^4_D(Y, \mathbb{Z}(2))$$

which lifts the standard cohomological cycle class map. If Z is homologous to zero, then $[Z]_D \in J_Y$ and cl_D is the Abel-Jacobi map. In particular $J_Y = c^{-1}(0)$.

Definition 4.1.2 (Cf. [Voi13])**.** For any $k \in H^4(Y, \mathbb{Z}) = \mathbb{Z}$, we define the k-twisted intermediate Jacobian as

$$J^k_Y := c^{-1}(k).$$

If $\mathrm{CH}^2(Y)_k$ is the subgroup of cycles Z such that $c([Z]_D) = k$, then the restriction

$$cl_D\colon \mathrm{CH}^2(Y)_k \longrightarrow J_Y^k \tag{4.1.2}$$

is the twisted Abel-Jacobi map.

Notice that $J_Y^k \cong J_Y$ for every k, but the isomorphism is not canonical, since we must fix a class. On the other hand, there is always the canonical isomorphism $J_Y^k \cong J_Y^{-k}$. Moreover, since Y is a cubic threefold, we have the degree 3 distinguished class $[\mathcal{O}_Y(1)]^2 \in H^4(Y, \mathbb{Z})$, which we can use to translate everything back to the origin, i.e. we have a canonical isomorphism $J_Y \cong J_Y^3$. These two remarks imply that, up to canonical isomorphism, there exists only one twisted intermediate Jacobian,

$$J_Y^T := J_Y^1 \cong J_Y^2. \tag{4.1.3}$$

Remark 4.1.3. J_Y^T can be algebraically thought of as a torsor over J_Y, under translation.

4.2 Twisted intermediate Jacobian fibration

As in Chapter 3, we fix a smooth cubic fourfold V and we look at the open subset $\mathcal{T} \subset \mathbb{P}(H^0(V, \mathcal{O}_V(1))^*)$ parametrising smooth linear sections. The exact sequence (4.1.1) can be relativised over $\mathcal{T}$ to get an exact sequence

$$0 \longrightarrow \mathcal{J}_{\mathcal{T}} \longrightarrow \mathcal{H}_D^4 \overset{c}{\longrightarrow} R^4 p_{\mathcal{T}*}\mathbb{Z} \longrightarrow 0$$

where $p_{\mathcal{T}}\colon \mathcal{Y}_{\mathcal{T}} \to \mathcal{T}$ is the universal family of smooth linear sections. The *twisted intermediate Jacobian fibration* is then

$$\mathcal{J}_{\mathcal{T}}^T := c^{-1}(1) \longrightarrow \mathcal{T}, \tag{4.2.1}$$

where we are using the fact that $R^4 p_{\mathcal{T}*}\mathbb{Z}$ is canonically isomorphic to the trivial sheaf $\mathbb{Z}$.

The aim of the next section is to explain how to compactify $\mathcal{J}_{\mathcal{T}}^T$ in such a way that the compactification is an irreducible holomorphic symplectic manifold. As before, it is more convenient to work on a partial compactification to linear sections with at worst one simple node as singularity. At this stage it is not even clear how to get this partial compactification though. The aim of this section is to explain this first step.

As before, $\mathcal{T}_1 \supset \mathcal{T}$ is the set of linear sections which are at worst 1-nodal.

Proposition 4.2.1 ([Voi16, Proposition 3.1])**.** *There exists a quasi-projective variety $\mathcal{J}_{\mathcal{T}_1}^T$ and a projective morphism $\pi_{\mathcal{T}_1}^T : \mathcal{J}_{\mathcal{T}_1}^T \to \mathcal{T}_1$ such that:*

1. *$\mathcal{J}_{\mathcal{T}_1}^T$ is étale locally isomorphic to $\mathcal{J}_{\mathcal{T}_1}$ over $\mathcal{T}_1$.*

2. *If $\mathcal{T}' \to \mathcal{T}$ is a base change with $\mathcal{T}'$ smooth and if there exists $\mathcal{Z} \in \mathrm{CH}^2(\mathcal{Y}_{\mathcal{T}'})$ such that $\mathcal{Z}$ has degree 1 on the fibres of the base change $\mathcal{J}^T_{\mathcal{T}'} \to \mathcal{T}'$, then there exists a canonical section $\mathcal{T}' \to \mathcal{J}^T_{\mathcal{T}'}$.*

By the discussion in Section 4.1, we want that the torsor $\mathcal{J}^T_{\mathcal{T}}$ parametrises 2-cycles of degree 1 on the fibres of $p_{\mathcal{T}}$, and the second condition in the proposition is saying that the object we constructed is actually what we were looking for.

We refer to [Voi16] for the complete proof of Proposition 4.2.1. In the rest of this section we want to recall how to construct $\mathcal{J}^T_{\mathcal{T}_1}$ as a torsor over $\mathcal{J}_{\mathcal{T}_1}$ and state some important properties.

Recall from Section 3.2 that $\mathcal{J}_{\mathcal{T}_1}$ has an open subset $\mathcal{J}^0_{\mathcal{T}_1}$ which is a group scheme over $\mathcal{T}_1$. From the point of view of Section 3.2, and using Picard-Lefschetz theory, we can write

$$\mathcal{J}^0_{\mathcal{T}_1} = \mathcal{H}^{1,2}_{\mathcal{T}_1} / R^3 p_{\mathcal{T}_1 *}\mathbb{Z},$$

where $\mathcal{H}^{1,2}_{\mathcal{T}_1}$ is Deligne's extension of Hodge bundles (cf. [Del71, Section 3.1]). The main remark is that $\mathcal{J}^0_{\mathcal{T}_1}$ acts on $\mathcal{J}_{\mathcal{T}_1}$ ([Voi16, Lemma 3.2]) and the torsor $\mathcal{J}^T_{\mathcal{T}_1}$ will be defined as a class inside $H^1_{\mathrm{t}}(\mathcal{T}_1, R^3 p_{\mathcal{T}_1 *}(\mathbb{Z}/3\mathbb{Z})) = H^1(\mathcal{T}_1, R^3 p_{\mathcal{T}_1 *}(\mathbb{Z}/3\mathbb{Z}))$. Notice that

$$R^3 p_{\mathcal{T}_1 *}(\mathbb{Z}/3\mathbb{Z}) \subset \mathcal{J}^0_{\mathcal{T}_1} \subset \mathcal{A}ut_{\mathcal{T}_1}(\mathcal{J}_{\mathcal{T}_1}/\mathcal{T}_1).$$

Remark 4.2.2. The action of $\mathcal{J}^0_{\mathcal{T}_1}$ on $\mathcal{J}_{\mathcal{T}_1}$ is easier to understood in terms of relative Picard schemes. In fact, $\mathcal{J}_{\mathcal{T}}$ is identified with $\mathrm{Pic}^0(\mathcal{J}_{\mathcal{T}}/\mathcal{T})$, and so $\mathcal{J}^0_{\mathcal{T}_1}$ is identified with $\mathrm{Pic}^0(\mathcal{J}_{\mathcal{T}_1}/\mathcal{T}_1)$. In particular, $\mathcal{J}^0_{\mathcal{T}_1}$ naturally acts on the compactification $\overline{\mathrm{Pic}^0(\mathcal{J}_{\mathcal{T}_1}/\mathcal{T}_1)}$ by torsion free rank 1 sheaves on the fibres of $\pi_{\mathcal{T}_1} \colon \mathcal{J}_{\mathcal{T}_1} \to \mathcal{T}_1$, which coincides with $\mathcal{J}_{\mathcal{T}_1}$ by uniqueness. Under this identification the action is reduced to the tensor product.

Proposition 4.2.3 ([Voi16, Lemma 3.3])**.** *There exists an element*

$$\xi \in H^1_{\mathrm{t}}(\mathcal{T}_1, R^3 p_{\mathcal{T}_1 *}(\mathbb{Z}/3\mathbb{Z}))$$

induced by V in a canonical way.

The existence of ξ is easy to prove. By Picard-Lefschetz theory, since every fibre of $p_{\mathcal{T}_1} \colon \mathcal{Y}_{\mathcal{T}_1} \to \mathcal{T}_1$ is at worst 1-nodal, for every $t \in \mathcal{T}_1$ one has that $H^4(\mathcal{Y}_t, \mathbb{Z}) = \mathbb{Z}$ and is generated by the class of a line (not passing through the node). In particular, the local system $R^4 p_{\mathcal{T}_1 *}\mathbb{Z} = \mathbb{Z}$ is trivial and we denote by ζ a generator of $H^0(\mathcal{T}_1, R^4 p_{\mathcal{T}_1 *}\mathbb{Z})$.

Consider the Leray spectral sequence

$$E_2^{ij} = H^i(\mathcal{T}_1, R^j p_{\mathcal{T}_1 *}\mathbb{Z}) \Longrightarrow H^{i+j}(\mathcal{Y}_{\mathcal{T}_1}, \mathbb{Z})$$

and in particular the class $d_2(\zeta) \in H^2(\mathcal{T}_1, R^3 p_{\mathcal{T}_1 *}\mathbb{Z})$. The first remark is that this class is 3-torsion. In fact if $h_{\mathcal{Y}_{\mathcal{T}_1}}$ is the pullback of the polarisation $\mathcal{O}_V(1)$ of V via the natural map $\mathcal{Y}_{\mathcal{T}_1} \to X$ and $h^2_{\mathcal{Y}_{\mathcal{T}_1}} \in H^4(\mathcal{Y}_{\mathcal{T}_1}, \mathbb{Z})$, then since V is a cubic hypersurface, the image

$\tilde{h}$ of $h^2_{\mathcal{Y}_{\mathcal{T}_1}}$ in $H^0(\mathcal{T}_1, R^4 p_{\mathcal{T}_1 *}\mathbb{Z})$ is 3ζ and so $3d_2(\zeta) = d_2(\tilde{h}) = 0$.

Now, from the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$, using the fact that both $R^3 p_{\mathcal{T}_1 *}\mathbb{Z}$ and $R^4 p_{\mathcal{T}_1 *}\mathbb{Z}$ are torsion free, we get the short exact sequence

$$0 \longrightarrow R^3 p_{\mathcal{T}_1 *}\mathbb{Z} \xrightarrow{\cdot 3} R^3 p_{\mathcal{T}_1 *}\mathbb{Z} \longrightarrow R^3 p_{\mathcal{T}_1 *}(\mathbb{Z}/3\mathbb{Z}) \longrightarrow 0.$$

Consider the long exact sequence in cohomology associated to this short exact sequence. Since $d_2(\zeta)$ is 3-torsion, its image under the induced map $H^2(\mathcal{T}_1, R^3 p_{\mathcal{T}_1 *}\mathbb{Z}) \xrightarrow{\cdot 3} H^2(\mathcal{T}_1, R^3 p_{\mathcal{T}_1 *}\mathbb{Z})$ is zero and hence there exists a class

$$\xi \in H^1(\mathcal{T}_1, R^3 p_{\mathcal{T}_1 *}(\mathbb{Z}/3\mathbb{Z}))$$

which is mapped to it.

The fact that ξ is canonical is more difficult to prove. In fact the cohomology group $H^1(\mathcal{T}_1, R^3 p_{\mathcal{T}_1 *}\mathbb{Z})$ is not zero and so there may be another class ξ' mapping to $d_2(\zeta)$. This is due to the fact that the fibres of $p_{\mathcal{T}_1}$ are "special" cubic threefolds, i.e. they are all contained in a fixed cubic fourfold. Voisin's trick is to consider a universal version of this problem. Namely, let us consider the universal family $p_{W_1} \colon \mathcal{Y}^{\text{univ}}_{W_1} \to W_1$ parametrising all the cubic threefolds in $\mathbb{P}^5$ supported on a hyperplane $H \cong \mathbb{P}^4 \subset \mathbb{P}^5$ and with at worst one simple node. Since now the cubic threefolds in the fibres are not special, one can eventually prove that $H^1(W_1, R^3 p_{W_1 *}\mathbb{Z}) = 0$ (see [Voi16, Sublemma 3.4]). The class ξ in Proposition 4.2.3 is then obtained by pulling back the analogous (canonical) class in the universal case, via the natural morphism $\mathcal{T}_1 \to W_1$.

This proves the first part of Proposition 4.2.1. The fact that this torsor is really the object we want follows from the fact that it becomes trivial once base changed to the relative Fano surface of lines. Let $\mathcal{F}_1^o \subset \mathcal{F}_1$ be the open subset of the relative Fano surface consisting of very good lines (see Remark 3.3.4). We denote by $v_1 \colon \mathcal{F}_1^o \to \mathcal{T}_1$ the natural projection.

Proposition 4.2.4 ([Voi16, Lemma 3.5])**.** *There exists a trivialisation*

$$\mathcal{J}_{\mathcal{T}_1} \times_{\mathcal{T}_1} \mathcal{F}_1^o \cong \mathcal{J}^T_{\mathcal{T}_1} \times_{\mathcal{T}_1} \mathcal{F}_1^o.$$

Remark 4.2.5. The proposition above is proved by showing that the pullback $v_1^*(\xi)$ is zero in $H^1(\mathcal{F}_1^o, v_1^* R^3 p_{\mathcal{T}_1 *}(\mathbb{Z}/3\mathbb{Z}))$. Again, the trick is to show this vanishing in the universal case and then pull it back to the special case.

Remark 4.2.6. The isomorphism in Proposition 4.2.4 is an isomorphism of quasi-projective varieties. Notice that it is not canonical, since $\mathcal{J}_{\mathcal{T}_1} \times_{\mathcal{T}_1} \mathcal{F}_1^o$ has a section while $\mathcal{J}^T_{\mathcal{T}_1} \times_{\mathcal{T}_1} \mathcal{F}_1^o$ has not.

Let us very briefly explain why Proposition 4.2.4 induces item 2 of Proposition 4.2.1. If $\mathcal{T}' \to \mathcal{T}$ is a base change as in Proposition 4.2.1, and we further suppose that there exists a family of lines $L_{\mathcal{T}'} \subset \mathcal{Y}_{\mathcal{T}'}$, then by Proposition 4.2.4 we get a section $\mathcal{T}' \to \mathcal{J}^T_{\mathcal{T}}$. The fact that we can get the same conclusion with the weaker hypothesis of having a family of 1-cycles $\mathcal{Z} \in \mathrm{CH}^2(\mathcal{Y}_{\mathcal{T}'})$ of degree 1 on the fibres, follows from the Universal Generation Theorem of Shen (see [She16, Theorem 1.7, Section 3]).

We conclude this section with the following remark. The universal property of $\mathcal{J}^T_{\mathcal{T}_1}$ (item 2 of Proposition 4.2.1) makes us able to run the same argument we used in Section 3.1 to construct the symplectic form on $\mathcal{J}_{\mathcal{T}}$ (and hence on $\mathcal{J}_{\mathcal{T}_1}$). More precisely, if $Y \subset V$ is a smooth linear section, then the twisted Abel-Jacobi map to J^T_Y is surjective and the proof of [LSV17, Lemma 1.1] works without changes. The outcome is a cycle $\mathcal{Z} \in \mathrm{CH}^2(\mathcal{J}^T_{\mathcal{T}} \times V)_{\mathbb{Q}}$ which can be used to define a holomorphic 2-form $\sigma^T_{\mathcal{T}}$ as the image of a generator of $H^{3,1}(V)$ under the correspondence associated to $\mathcal{Z}$. Moreover, the 2-form $\sigma^T_{\mathcal{T}}$ vanishes when restricted to a fibre of $\pi^T_{\mathcal{T}} \colon \mathcal{J}_{\mathcal{T}} \to \mathcal{T}$ and naturally extends to any (partial) compactification (cf. Remark 3.1.2). Finally, it is non-degenerate because $\mathcal{J}^T_{\mathcal{T}_1}$ is locally étale isomorphic to $\mathcal{J}_{\mathcal{T}_1}$.

4.3 The symplectic compactification

Recall the notation $\mathrm{P} = \mathbb{P}\left(H^0(V, \mathcal{O}_V(1))^*\right)$.

Theorem 4.3.1 ([Voi16])**.** *Let V be a generic smooth cubic fourfold. There exists a smooth and projective compactification $\mathcal{J}^T_V$ of $\mathcal{J}^T_{\mathcal{T}_1}$ such that $\mathcal{J}^T_V$ is an irreducible holomorphic symplectic manifold of $OG10$-type. Moreover, the induced morphism*

$$\pi^T_V \colon \mathcal{J}^T_V \longrightarrow \mathrm{P}$$

is a Lagrangian fibration.

For the genericity assumption, see Remark 3.3.4. The idea of the compactification is the same as in Section 3.3. In particular, using Proposition 4.2.4, the base change $\mathcal{J}^T_{\mathcal{T}_1} \times_{\mathcal{T}_1} \mathcal{F}^o_{\mathcal{T}_1}$ is compactified via Prym varieties by [LSV17, Section 4]. Using the same notation as in (3.3.1), we write

$$\mathrm{Prym}(\widetilde{\mathcal{C}}_{\mathrm{P}}, \mathcal{C}_{\mathrm{P}}) \longrightarrow \mathcal{F}_{\mathrm{P}}$$

for the compactified Prym fibration. Notice that by the smoothness criterion [LSV17, Theorem 4.20], this compactification is smooth and projective. So the only problem is to descend $\mathrm{Prym}(\widetilde{\mathcal{C}}_{\mathrm{P}}, \mathcal{C}_{\mathrm{P}})$ to a smooth and projective variety over P.

The descent argument is tackled and solved in the same way as the non-twisted case. The main ingredient is the following result.

Proposition 4.3.2 ([Voi16, Lemma 4.2])**.** *There exists a line bundle $\mathcal{L}$ on $\mathcal{J}^T_{\mathcal{T}_1}$ such that the pullback $v_1^*\mathcal{L}$ extends (in a unique way) to a relatively ample line bundle on* $\mathrm{Prym}(\widetilde{\mathcal{C}}_{\mathrm{P}}, \mathcal{C}_{\mathrm{P}})$.

Once we have such a line bundle, we can define

$$\mathcal{J}_V^T = \operatorname{Proj}\left(\bigoplus_{k\geq 0} R^0 j_{1*}\left(R^0\pi^T_{\mathcal{T}_1 *}\mathcal{L}^{\otimes kd}\right)\right) \tag{4.3.1}$$

where $j\colon \mathcal{T}_1 \to \mathrm{P}$ is the natural inclusion and $d \gg 0$ is fixed (cf. Remark 3.3.6).

Remark 4.3.3. Since the fibres of $\pi^T_{\mathcal{T}}\colon \mathcal{J}^T_{\mathcal{T}} \to \mathcal{T}$ are not abelian varieties, it makes no sense to talk about Theta divisor. Nevertheless by construction $\mathcal{J}^T_{\mathcal{T}_1}$ is a quasi-projective variety and so there exists a relatively ample line bundle $\mathcal{L}$ that we can use to implement the descent argument. Notice that, since $\mathcal{J}^T_{\mathcal{T}_1} \times_{\mathcal{T}_1} \mathcal{F}^o_{\mathcal{T}_1} \subset \mathrm{Prym}(\widetilde{\mathcal{C}}_{\mathrm{P}}, \mathcal{C}_{\mathrm{P}})$ has boundary of codimension 2, the pullback $v_1^*\mathcal{L}$ extends to a line bundle on $\mathrm{Prym}(\widetilde{\mathcal{C}}_{\mathrm{P}}, \mathcal{C}_{\mathrm{P}})$. The actual result in Proposition 4.3.2 is the claim that such an extension is relatively ample. This relatively ampleness is what ensures that the Proj construction (4.3.1) works.

Remark 4.3.4. The closure $\overline{\mathcal{L}}$ in $\mathcal{J}^T_V$ is a relatively ample line bundle which is actually ample by construction. As is Chapter 3, we would like to have a distinguished representative of this relatively ample line bundle, and this will be the goal of Section 4.4.

To conclude this section, we want to explain why $\mathcal{J}^T_V$ is an irreducible holomorphic symplectic manifold of OG10-type. We already said that $\mathcal{J}^T_{\mathcal{T}_1}$ has a symplectic form, and since $\mathcal{J}^T_{\mathcal{T}_1}$ has boundary of codimension 2 in $\mathcal{J}^T_V$, such a symplectic form extends to a symplectic form on $\mathcal{J}^T_V$. Now the claim that $\mathcal{J}^T_V$ is simply connected and the symplectic form is unique up to scalar follows from the fact that it is birational to $\mathcal{J}_V$ for particular choices of V. In fact, if there exists a Hodge class $\alpha \in H^4(V,\mathbb{Z})$ such that its restriction to any smooth linear section has degree 1, then the torsor $\mathcal{J}^T_{\mathcal{T}}$ becomes trivial and so isomorphic to $\mathcal{J}_{\mathcal{T}}$. Notice that this also implies that $\mathcal{J}^T_V$ is of OG10-type.

Remark 4.3.5. As explained in [Voi16], cubic fourfolds with a special class as above form a Zariski dense subset in the moduli space; therefore there exist generic cubic fourfolds which satisfy this condition.

Example 4.3.6. Let V be a smooth Pfaffian cubic fourfold (see Section 2.2.1). If $\mathcal{E}_V$ is the Pfaffian vector bundle, i.e. the rank two vector bundle such that $\mathcal{E}_\phi = (\ker\phi)^*$ for every $\phi \in V$ (we are using the same notation as in Section 2.2.1), then the second Chern class $c_2(\mathcal{E}_V(-2)) \in H^4(V,\mathbb{Z})$ is such that its restriction to any (smooth) linear section has degree -1. It follows that $\mathcal{J}^T_V$ is birational to $\mathcal{J}_V$. Moreover one can see that in this case $\mathcal{J}^T_V \cong \mathcal{J}_V$ is actually a regular isomorphism.

4.4 Distinguished twisted Theta divisor

As already said in Remark 4.3.3, a priori it is not clear what a twisted (relative) Theta divisor is. The aim of this section is to construct an ample divisor on $\mathcal{J}^T_V$ and explain why it actually deserves the name of twisted Theta divisor.

We start by recalling the situation on a single smooth fibre.

Let Y be a smooth cubic threefold and $\mathcal{J}_Y^T$ the twisted intermediate Jacobian (see Section 4.1). Following Section 3.1.1, we define a sum map

$$\psi\colon F_Y \times F_Y \longrightarrow J_Y^T \tag{4.4.1}$$

by sending two lines l_1 and l_2 to the image under the twisted Abel-Jacobi map (4.1.2) of their sum.

By [Dru00, Théorème 1.4] (cf. also [Bea02, Remark 6.5]), ψ is generically 2-to-1 onto its image. Moreover, since the Néron-Severi group is invariant under translations, we have that $\mathrm{NS}(J_Y^T) = \mathrm{NS}(J_Y)$ and under this identification the class $[\mathrm{Im}(\psi)]$ is equal to $3\theta_Y$, where θ_Y is the Theta divisor of J_Y. This computation is done exactly as in the non-twisted case: we identify $\psi_*(F_Y \times F_Y)$ with the Pontryagin product $[F_Y] \star [F_Y]$, and then we use the Pontryagin-Poincaré formula to conclude. If we define

$$L_Y := [\mathrm{Im}(\psi)]/3,$$

then L_Y deserves the name of twisted Theta divisor.

Let us go back to the relative situation. Define the morphism

$$\psi_{\mathcal{T}}\colon \mathcal{F}_{\mathcal{T}} \times_{\mathcal{T}} \mathcal{F}_{\mathcal{T}} \longrightarrow \mathcal{J}_{\mathcal{T}}^T \tag{4.4.2}$$

as the relative version of (4.4.1). It is generically 2-to-1 onto its image and by what we said above, $\mathrm{Im}(\psi_{\mathcal{T}})_t = 3L_{\mathcal{Y}_t}$ for every $t \in \mathcal{T}$. Recall that here $\mathrm{Im}(\psi_{\mathcal{T}})$ denotes the image with the reduced scheme structure. We want to spread the cycle $L_{\mathcal{Y}_t}$ in order to get a cycle $L_{\mathcal{T}}$ on $\mathcal{J}_{\mathcal{T}}^T$: this cycle will have the property that, at least locally, $\mathrm{Im}(\psi_{\mathcal{T}}) = 3L_{\mathcal{T}}$. Before stating the result, we want to recall the spreading principle.

Spreading cycles

Let $p\colon X \to S$ be a smooth projective morphism and suppose that S is a smooth variety over $\mathbb{C}$. Notice that in particular this implies that X is also smooth. We denote by $\mathcal{Z}^k(X)$ the group of codimension k cycles. For every $s \in S$, we write X_s for the fibre $p^{-1}(s)$ and, for any cycle $Z \in \mathcal{Z}^k(X)$, we write Z_s for the restriction of Z to X_s.

Suppose that $s \in S$ is a very general point and $Z_s \in \mathcal{Z}^k(X_s)$. By very general we mean that s is contained in the complement of a countable union of Zariski closed subsets of S.

Proposition 4.4.1 (cf. [Voi14, Section 1.1.1])**.** *There exists a Zariski open subset $U \subset S$, a finite cover $W \to U$ and a cycle $Z'_W \in \mathcal{Z}^k(X_W)$ such that $Z'_{W,s} = Z_{W,s}$.*

Here X_W is the base change of X to W.

Proof. There exist countably many (relative) Hilbert schemes $H_i \to S$ parametrising all the subvarieties in the fibres of $p\colon X \to S$. In particular, there are countably many varieties $p_j\colon Y_j \to S$ parametrising all the cycles in the fibres of $p\colon X \to S$. Notice that the projections $Y_j \to S$ are proper.

Now, let E be the set of indices j such that $p_j\colon Y_j \to S$ is not surjective. Since we are working over $\mathbb{C}$ (which is an uncountable field), the subset $\bigcup_{j\in E} \operatorname{Im}(p_j\colon Y_j \to S)$ is strictly contained in S and, if S' is its complement, then $s \in S'$ by hypothesis.

By construction, there exist universal cycles $\mathcal{Z}_{Y_j} \subset X \times_S Y_j$ and, for any $s \in S'$ and any $Z_s \in \mathcal{Z}^k(X_s)$, there exists an index j and a point $y \in Y_j$ such that $p_j(y) = s$ and $Z_s = \mathcal{Z}_{Y_j,y}$.

To conclude the proof, notice that we can replace Y_j with a linear section and suppose then that $p_j\colon Y_j \to S$ is generically finite. □

Suppose that Z_s is rationally equivalent to zero for every $s \in S$. In general, it is not true that the spreaded cycle Z is rationally equivalent to zero. Nevertheless, if we allow ourselves to work with rational coefficients, then it is true (cf. [Voi14, Theorem 1.2]). On the other hand, if the cycle has codimension 1, then the same claim with integral coefficients can be achieved, up to working locally. The following result is modelled out of the proof of [Voi03, Lemma 10.22].

Lemma 4.4.2. *Suppose that $Z \in \mathcal{Z}^1(X)$ is such that Z_s is rational equivalent to zero for every $s \in S$. Then, there exists a Zariski open subset U of S such that Z_U is rationally equivalent to zero.*

Proof. Since X is smooth and Z is codimension 1, there exists a line bundle $L \in \operatorname{Pic}(X)$ such that $L = \mathcal{O}(Z)$. By hypothesis, L_s is trivial on each fibre X_s, hence, by the Base Change Theorem, the pushforward p_*L is a line bundle on S. Moreover, there exists a canonical isomorphism

$$L \cong p^*p_*L.$$

In particular we can pick any Zariski open subset U where p_*L is trivial to get the claim. □

Theorem 4.4.3. *There exists a cycle $L_{\mathcal{T}} \in \mathrm{CH}^1(\mathcal{J}^T_{\mathcal{T}})$ with the property that there exists a Zariski open subset $U \subset \mathcal{T}$ such that $\operatorname{Im}(\psi_U) \sim_{\mathrm{rat}} 3L_U$. Moreover, the associated line bundle $\mathcal{L}_{\mathcal{T}} = \mathcal{O}(L_{\mathcal{T}})$ is relatively ample.*

Proof. Let $t \in \mathcal{T}$ be a very general point and consider the cycle $L = \operatorname{Im}(\psi)_t/3$ in $\mathrm{CH}^1(\mathcal{J}^T_t)$ as defined above. By Proposition 4.4.1, there exists a Zariski open $U \subset \mathcal{T}$ and a finite morphism $W \to U$ such that there exists a cycle $T_W \in \mathrm{CH}^1(\mathcal{J}^T_W)$ satisfying $(T_W)_t = L$. By Lemma 4.4.2, up to shrinking U if necessary (and hence up to shrinking W) we can suppose that $\operatorname{Im}(\psi_W)_t \sim_{\mathrm{rat}} 3(T_W)_t$, for every $t \in W$. (Here ψ_W is the base change of ψ to W.)

Let $T \in \mathrm{CH}^1(\mathcal{J}^T_{\mathcal{T}})$ be the closure in $\mathcal{T}$ of the reduced image of T_W under the finite cover $W \to U$. Then $L_{\mathcal{T}} = T$ has the properties of Theorem 4.4.3.

Let $\mathcal{L}_{\mathcal{T}} = \mathcal{O}(L_{\mathcal{T}})$. If $t \in \mathcal{T}$ is generic, then by construction $(\mathcal{O}(\mathrm{Im}(\phi)) - 3\mathcal{L}_{\mathcal{T}})_t$ is topologically trivial and by [Voi16, Lemma 3.4] we get that $(\mathcal{O}(\mathrm{Im}(\phi)) - 3\mathcal{L}_{\mathcal{T}})_t$ is numerically trivial for every $t \in \mathcal{T}$. This implies that $\mathcal{L}_{\mathcal{T}}$ is relatively ample over $\mathcal{T}$. □

Remark 4.4.4. We can close $L_{\mathcal{T}}$ inside the partial compactification $\mathcal{J}^T_{\mathcal{T}_1}$. Again by [Voi16, Lemma 3.4] and by Section 3.2.1, the line bundle $\mathcal{O}(L_{\mathcal{T}_1})$ is relatively ample and hence can be used to define the symplectic compactification (4.3.1).

4.5 Some theorems

Let $b^T = (\pi^T)^*\mathcal{O}_{\mathrm{P}}(1)$ be the class of the fibration.

Theorem 4.5.1. *The lattice $U_{\mathcal{L}_V, b^T}$ generated by $\mathcal{L}_V$ and b^T is saturated and isometric to the hyperbolic plane.*

Proof. If $\mathcal{U} \subset \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}(3))^*)$ is the parameter space of smooth cubic fourfolds and $\mathcal{U}'$ is a Zariski open subset of it, we can consider the induced family of (twisted) intermediate Jacobian fibrations

$$\upsilon \colon \mathcal{J}^T_{\mathcal{U}'} \longrightarrow \mathcal{U}'.$$

Notice that this is a polarised family of Lagrangian fibrations, so the sections $\{\mathcal{L}_V\}_{V \in \mathcal{U}'}$ and $\{b^T_V\}_{V \in \mathcal{U}'}$ are flat. In particular, the lattices they generate also form a sub-local system of $R^2\upsilon_*\mathbb{Z}$ and the claim follows if we can prove it for a special member of the family.

If V is a Pfaffian cubic fourfold, then the twisted intermediate Jacobian $\mathcal{J}^T_V$ exists, and $\mathcal{J}^T_V \cong \mathcal{J}_V$, $b^T = b$ and $\mathcal{L}_V = \Theta_{\mathcal{F}_V}$ (cf. Example 4.3.6). In particular, $\langle \mathcal{L}_V, b^T \rangle \cong \langle \Theta_{\mathcal{F}_V}, b \rangle$ is saturated and isometric to the hyperbolic plane by Theorem 3.5.3. □

Remark 4.5.2. The argument by specialising to a Pfaffian cubic fourfold also implies that the degrees of $\Theta_{\mathcal{F}_V}$ and $\mathcal{L}_V$ are the same.

Remark 4.5.3. This answer a question of Claire Voisin.

The cycle $\mathcal{Z} \in \mathrm{CH}^2(\mathcal{J}^T_{\mathcal{T}} \times_{\mathcal{T}} V)_{\mathbb{Q}}$ we constructed at the end of Section 4.2 can be closed in the compactification $\mathcal{J}^T_V \times_{\mathrm{P}} V$ and gives a morphism of Hodge structures between the cohomologies of V and $\mathcal{J}^T_V$. Up to replace $\mathcal{Z}$ with a multiple (as in Section 3.5), we can suppose that it is integral and hence it gives a morphism

$$\alpha^T \colon H^4(V, \mathbb{Z}) \longrightarrow H^2(\mathcal{J}^T_V, \mathbb{Z}).$$

The following is a twisted version of the results in Section 3.5.

Theorem 4.5.4. *The induced map*

$$\alpha^T\colon H^4(V,\mathbb{Z})_{\text{prim}} \longrightarrow \langle \mathcal{L}_V, b^T\rangle^{\perp} \subset H^2(\mathcal{J}_V^T,\mathbb{Z})$$

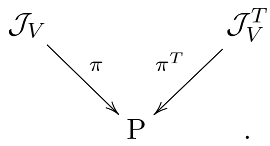
is a (anti-)similitude of lattices and an isomorphism of Hodge structures.

Proof. First of all, let us prove that the class $h^2 \in H^4(V,\mathbb{Z})$ is sent into the lattice $\langle \mathcal{L}_V, b^T\rangle$. If V is very general, i.e. not special, then its algebraic middle cohomology is generated by h^2. Moreover, the algebraic part of $H^2(\mathcal{J}_V^T,\mathbb{Z})$ is generated by Θ_V and b_V. Since α is a morphism of Hodge structures, the claim follows.

Now let V be generic: its primitive cohomology has an irreducible Hodge structure. Any morphism of Hodge structures is either zero or injective on irreducible Hodge structures, and since α is not zero, it follows that it is injective. The claim then follows by Schur Lemma and the fact that the two lattices are abstractly isometric (up to the sign). □

We want to finish this section with a remark about a conjecture of Voisin.

Let V be a generic cubic fourfold. Then we have two Lagrangian fibrations



As noticed by Voisin in [Voi16], if V is generic enough (see below), then $\mathcal{J}_V$ and $\mathcal{J}_V^T$ are not birational as Lagrangian fibrations, since one has a (rational) section while the other does not.

Conjecture (Voisin)**.** If V is generic enough, then $\mathcal{J}_V$ and $\mathcal{J}_V^T$ are not birational as algebraic varieties.

Remark 4.5.5. Here generic enough means that V is generic and has no Hodge classes in $H^4(V,\mathbb{Z})$ of degree 1 on any smooth hyperplane section (cf. [Voi16], discussion before Remark 3.5).

If we further assume that V is very general (i.e. not special in the sense of Hassett), then Theorem 3.5.3 and Theorem 4.5.4 implies that

$$\mathrm{Pic}(\mathcal{J}_V) = \langle \Theta_{\mathcal{F}_V}, b\rangle \qquad \text{and} \qquad \mathrm{Pic}(\mathcal{J}_V^T) = \langle \mathcal{L}_V, b^T\rangle$$

are two hyperbolic planes. There are four primitive isotropic classes in a hyperbolic plane and any Lagrangian fibration structure on either $\mathcal{J}_V$ or $\mathcal{J}_V^T$ would give an effective such class. This reduces the Conjecture to the statement: there are no primitive effective isotropic classes other than b and b^T in $\mathrm{Pic}(\mathcal{J}_V)$ and $\mathrm{Pic}(\mathcal{J}_V^T)$. This last question is still open.

Chapter 5

Monodromy operators on manifolds of OG10-type

5.1 Monodromy operators coming from the $K3$ surface

In this section we want to show that monodromy operators on $K3$ surfaces lift to monodromy operators on OG10 manifolds. The relation between $K3$ surfaces and OG10 manifolds is made explicit via O'Grady's desingularisation of singular moduli spaces of sheaves. We refer to Appendix B for notations and results.

Let S be a projective $K3$ surface, H a generic polarisation (in the sense of Section B.1), M_S the moduli space of rank 2 semistable sheaves on S with trivial determinant and second Chern class equal to 4 and $\widetilde{\pi}\colon \widetilde{M}_S \to M_S$ the O'Grady symplectic desingularisation. Recall that (Theorem B.4.2) $\widetilde{M}_S$ is isomorphic to the blow-up of M_S along the (reduced) singular locus $\Sigma_S = M_S \setminus M_S^s$, where $M_S^s \subset M_S$ is the open subset parametrising stable sheaves.

If $B_S \subset M_S$ is the closed (Weil) divisor parametrising non-locally free sheaves and $\widetilde{B}_S$ the strict transform, then the lattice generated by $\widetilde{\Sigma}_S$ and $\widetilde{B}_S$ is saturated and isometric to the $G_2(-1)$ lattice, i.e.

$$\langle \widetilde{\Sigma}_S, \widetilde{B}_S \rangle = \begin{pmatrix} -6 & 3 \\ 3 & -2 \end{pmatrix} \cong G_2(-1).$$

Here $\widetilde{\Sigma}_S$ is the exceptional divisor of the desingularisation. In this setting we have a decomposition

$$H^2(\widetilde{M}_S, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus G_2(-1).$$

By Corollary A.3.2, the restriction map

$$r\colon \ \mathrm{O}^+(H^2(\widetilde{M}_S, \mathbb{Z}), G_2(-1)) \longrightarrow \mathrm{O}^+(H^2(S, \mathbb{Z})) \tag{5.1.1}$$

is surjective; and notice that $\mathrm{O}^+(H^2(S,\mathbb{Z})) = \mathrm{Mon}^2(S)$ ([Huy16, Proposition 7.5.5.5]).

We want to show that, given a monodromy operator $g \in \mathrm{Mon}^2(S)$, there exists a canonical isometric extension $\tilde{g} \in \mathrm{O}^+(H^2(\widetilde{M}_S, \mathbb{Z}), G_2(-1))$ such that $\tilde{g} \in \mathrm{Mon}^2(\widetilde{M}_S)$. As we will see, this extension is given by the identity on $G_2(-1)$.

Let T be a curve and (S, H) a polarised $K3$ surface. Let $\mathcal{S}_T \to T$ be a deformation family such that $\mathcal{S}_0 = S$ for a base point $0 \in T$ and let $\mathcal{H}_T$ be a line bundle on $\mathcal{S}_T$, flat over T, such that $\mathcal{H}_0 = H$. It is known that the set of points $t \in T$ such that $\mathcal{H}_t$ is not ample is finite. Moreover, Perego and Rapagnetta notice in [PR13, Proposition 2.20] that the set of points $t \in T$ such that $\mathcal{H}_t$ is not generic is also finite. We summarise this remark in the following statement for future reference.

Lemma 5.1.1. *Up to removing a finite number of points from T, we can suppose that $\mathcal{H}_t$ is a ample and generic for every $t \in T$*

In the following we assume that $\mathcal{H}_T$ satisfies the conditions above, i.e. $\mathcal{H}_t$ is ample and generic for every $t \in T$.

Consider the relative moduli space $\mathcal{M}_T \to T$ (resp. $\mathcal{M}_T^s$) parametrising rank 2 semistable (resp. stable) sheaves on the fibres of $\mathcal{S}_T \to T$ with trivial determinant and second Chern class equal to 4 (cf. [Mar78] and [HL10, Theorem 4.3.7]). Notice that $\mathcal{M}_T^s \subset \mathcal{M}_T$ is open.

Remark 5.1.2. $\mathcal{M}_T$ can be thought of as a moduli space of torsion sheaves on $\mathcal{S}_T$.

Since $\mathcal{M}_t$ is reduced and irreducible for every $t \in T$, $\mathcal{M}_T$ is flat over T ([EH00, Proposition II.2.19] and cf. [PR13, Lemma 2.21]) and we can think of it as a deformation of (singular) moduli spaces. Now, define $\Sigma_T := \mathcal{M}_T \setminus \mathcal{M}_T^s$. As explained in the proof of [PR13, Proposition 2.20], since $\mathcal{H}_t$ is generic for every $t \in T$, Σ_t is an irreducible closed subvariety which coincides with the singular locus of $\mathcal{M}_t$. Moreover, $\Sigma_t = \Sigma_{\mathcal{S}_t}$.

Remark 5.1.3. Notice that Σ_T has a modular description as the relative second symmetric product $\mathcal{S}ym^2\mathcal{S}_T^{[2]}$. The singular locus of Σ_T is then identified with $\mathcal{S}_T^{[2]}$. This implies that $(\Sigma_{\mathrm{red}})_t = (\Sigma_t)_{\mathrm{red}}$ for every $t \in T$.

By [EH00, Proposition II.2.19] we have that Σ_T and $(\Sigma_T)_{\mathrm{red}}$ are flat over T. Blowing up $\mathcal{M}_T$ at $(\Sigma_T)_{\mathrm{red}}$ yields a projective and flat projection

$$p \colon \widetilde{\mathcal{M}}_T \longrightarrow T \tag{5.1.2}$$

such that $\widetilde{\mathcal{M}}_t = \widetilde{M}_{\mathcal{S}_t}$.

Remark 5.1.4. Notice that a priori it is not true that the blow-up of the family is the family of the blow-ups. The key result here is [PR13, Proposition 2.22], which states that the projection $\mathcal{M}_T \to T$ is a product locally at any point in any fibre (as germs of complex spaces).

The family (5.1.2) is the deformation family of O'Grady manifolds associated to a deformation of polarised $K3$ surfaces.

The first remark is the following.

Lemma 5.1.5. *Let $\widetilde{M}_S$ be the O'Grady desingularisation of M_S and $\widetilde{\Sigma}_S$ the exceptional divisor. Any monodromy operator g arising from a deformation family (5.1.2), as constructed before, must satisfy the equality $g(\widetilde{\Sigma}_S) = \widetilde{\Sigma}_S$.*

Proof. This is clear from the discussion above. In fact, on $\widetilde{\mathcal{M}}_T$ there is the relative exceptional divisor $\widetilde{\Sigma}_T$ which is flat over T. The associated class in cohomology is then flat in the local system $R^2p_*\mathbb{Z}$ and hence preserved by any parallel transport in the same local system. □

Next, we want to understand what is the orbit of the divisor $\widetilde{B}_S$ under monodromy operators arising from this kind of family. This is more subtle, because the locus $\mathcal{B}_T := \mathcal{M}_T \setminus \mathcal{M}_T^{\mathrm{lf}}$ does not have a modular description as in Remark 5.1.3. Here and in the following $\mathcal{M}_T^{\mathrm{lf}} \subset \mathcal{M}_T$ is the open subset parametrising locally free sheaves on the fibres of $\mathcal{S}_T \to T$. We need to work with the Donaldson-Uhlenbeck-Yau moduli space $\overline{\mathcal{N}}_\infty(g)$ of anti-self-dual connections on S. As explained in Section B.2, $\overline{\mathcal{N}}_\infty(g)$ exists as a (reduced) projective scheme and there is a regular morphism of schemes

$$\phi\colon M_S \longrightarrow \overline{\mathcal{N}}_\infty(g).$$

Moreover, $\overline{\mathcal{N}}_\infty(g) = \mathcal{N}_\infty(g) \coprod S^{(4)}$ where $S^{(4)}$ is the fourth symmetric product of S. The morphism ϕ restricts to an isomorphism $M_S^{\mathrm{lf}} \cong \mathcal{N}_\infty(g)$, where $M_S^{\mathrm{lf}} \subset M_S$ is the open subset parametrising locally free sheaves.

We want to relativise this construction to the family $p\colon \mathcal{M}_T \to T$. For this, we need to run the same arguments as in [Li93, Section 1, Section 2] (cf. Section B.2) in families.

Let $\mathrm{Quot}_{\mathcal{S}/T}$ be the Quot scheme of sheaves on the fibres of $\mathcal{S}_T \to T$ and $Q_T \subset \mathrm{Quot}_{\mathcal{S}/T}$ the open subset of semistable ones. Then the moduli space $\mathcal{M}_T$ is obtained by GIT quotient by the algebraic group $G = \mathrm{PGL}(N)$ (for a suitable integer N). On $\mathcal{S}_T \times Q_T$ there is a universal quotient sheaf F_T, flat over T (cf. [HL10, Theorem 2.2.4]).

Now let $k \geq 1$ and $D_T \in |k\mathcal{H}_T|$ be a divisor which is smooth over T. Notice that such a divisor D_T always exists, up to shrinking the base T. Since the fibres of D_T over T are smooth algebraic curves, we can consider the relative Jacobian $J^{g(D_T)-1}(D_T)$. Here $g(D_T)$ means the genus of the general fibre of D_T over T. Let $\theta_{D_T} \in J^{g(D_T)-1}(D_T)$ be flat over T. Then we define the line bundle

$$\widetilde{\mathcal{L}}_k(D_T, \theta_{D_T}) := \det\left(R^\bullet q_{1*}(F_T|_{D_T} \otimes q_2^*\theta_{D_T})\right)^{-1} \tag{5.1.3}$$

where q_i is the projection from $Q_T \times D_T$ to the i-th factor and $F_T|_{D_T}$ is the restriction of F_T to $Q_T \times D_T$. Notice that by construction $\widetilde{\mathcal{L}}_k(D_T, \theta_{D_T})$ is flat over T.

Lemma 5.1.6. *$\widetilde{\mathcal{L}}_k(D_T, \theta_{D_T})$ descends to a line bundle $\mathcal{L}_k(D_T, \theta_{D_T})$ on $\mathcal{M}_T$.*

Proof. Since $\mathcal{M}_T$ is constructed as a G-quotient from Q_T, [Li93, Lemma 1.6] says that a G-bundle E on Q_T descends to $\mathcal{M}_T$ if and only if for every closed point $x \in Q_T$ with closed orbit, the stabiliser G_x acts trivially on E_x. Notice that $\widetilde{\mathcal{L}}_k(D_T, \theta_{D_T})$ is a G-bundle on Q_T because we have chosen D_T such that the Euler characteristic of the fibres over T is zero (cf. proof of [Li93, Proposition 1.7]).

Now, as noted in Remark 5.1.2, closed points in Q_T are all of the form $i_{t*}F_t$, where $i_t\colon \mathcal{S}_t \to \mathcal{S}_T$ is the inclusion and $F_t \in \mathcal{M}_t$. Moreover, since $\mathcal{H}_t$ is assumed to be generic for every $t \in T$, Q_t satisfies the hypotheses of [Li93, Proposition 1.7] and therefore the proof is reduced to the proof of [Li93, Proposition 1.7]. □

With an abuse of notation, we denote by $\mathcal{L}_k$ the line bundle $\mathcal{L}_k(D_T, \theta_{D_T})$.

Proposition 5.1.7. *Let $(\mathcal{S}_T, \mathcal{H}_T)$ be a polarised family of $K3$ surfaces over a curve T. Let $\mathcal{L}_k = \mathcal{L}_k(D_T, \theta_{D_T})$ be the line bundle on $\mathcal{M}_T$ constructed above and suppose $k > 5$. Then there exists a positive integer $\bar{m}$ such that $(\mathcal{L}_k^m)_t$ is generated by global sections for every $t \in T$ and for every $m \geq \bar{m}$.*

Proof. For any $t \in T$, there exists a positive integer m_t such that $(\mathcal{L}_k^{m_t})_t$ is generated by global sections for every $m \geq m_t$ and $k > 5$ ([Li93, Theorem 3]). Now, define

$$\bar{m} := \sup_{t \in T}\{m_t\}$$

and since T is quasi-compact, $\bar{m} < \infty$. □

The pushforward $p_*\mathcal{L}_k^{\bar{m}}$ is not locally free in general, but its double dual $p_*(\mathcal{L}_k^{\bar{m}})^{\vee\vee}$ is always locally free by [Har80, Corollary 1.4]. The proposition above says then that the induced map

$$\varphi_T\colon \mathcal{M}_T \longrightarrow \mathbb{P}\left(p_*(\mathcal{L}_k^{\bar{m}})^{\vee\vee}\right) \tag{5.1.4}$$

is a regular morphism of schemes. Notice that $\mathbb{P}\left(p_*(\mathcal{L}_k^{\bar{m}})^{\vee\vee}\right)$ is flat over T (it is a projective bundle) and that φ_T is defined fibrewise.

Let us define $\overline{\mathcal{N}}_T$ as the image of $\mathcal{M}_T$ via φ_T. By construction (or by [EH00, Proposition II.2.19]) $\overline{\mathcal{N}}_T$ is flat over T and, for every $t \in T$, $\overline{\mathcal{N}}_t$ is the Donaldson-Uhlenbeck-Yau moduli space associated to the $K3$ surface $\mathcal{S}_t$. The natural projection

$$\overline{\mathcal{N}}_T \longrightarrow T \tag{5.1.5}$$

is then a family of Donaldson-Uhlenbeck-Yau moduli spaces. If we put $\mathcal{N}_T = \varphi_T(\mathcal{M}_T^{\mathrm{lf}})$, then we get a relative Uhlenbeck decomposition

$$\overline{\mathcal{N}}_T = \mathcal{N}_T \coprod \mathcal{S}_T^{(4)}$$

where $\mathcal{S}_T^{(4)}$ is the relative symmetric product, i.e. $(\mathcal{S}_T^{(4)})_t = \mathcal{S}_t^{(4)}$.

Remark 5.1.8. Notice that $\mathcal{S}_T^{(4)}$ is flat over T.

Remark 5.1.9. The construction above is not canonical: it depends on the choice of both D_T and θ_{D_T}, so one should really write $\varphi_{T,D_T,\theta_{D_T}}$. This is not an issue, because we want a morphism to a (relative) Donaldson-Uhlenbeck-Yau moduli space, but for our purposes we do not need that such a morphism is unique. We suppress this dependence from the notation.

Anyway, when $T = \mathrm{Spec}(\mathbb{C})$ is a point, Li noticed that $\mathcal{L}_k(D_T, \theta_{D_T})$ does not depend on D_T and θ_{D_T}. In particular, for a general base T, the claim is true fibrewise and so, if D'_T is another smooth divisor on $\mathcal{M}_T$ and $\theta_{D'_T} \in J^{g(D'_T)-1}(D'_T)$, then

$$\mathcal{L}_k(D_T, \theta_{D_T}) \cong \mathcal{L}_k(D'_T, \theta_{D'_T}) \otimes p^*A$$

where A is a line bundle on T.

Proposition 5.1.10. *$\mathcal{B}_T = \mathcal{M}_T \setminus \mathcal{M}_T^{\mathrm{lf}}$ is flat over T.*

Proof. Consider the surjective morphism

$$\varphi_T \colon \mathcal{B}_T \longrightarrow \mathcal{S}_T^{(4)}$$

obtained by restricting the morphism (5.1.4). By [EH00, Proposition II.2.19], it is enough to show that there are no embedded components of $\mathcal{B}_T$ supported on a fibre $\mathcal{B}_t$, for every $t \in T$. Suppose such a component $\overline{\mathcal{B}}_T \subset \mathcal{B}_T$ exists and is supported on the fibre $\mathcal{B}_{t_0}$. Since φ_T is defined fibrewise, $\varphi_T(\overline{\mathcal{B}}_T) = \varphi_{t_0}(\overline{\mathcal{B}}_T) \subset \mathcal{S}_{t_0}^{(4)}$. Since $\mathcal{S}_T^{(4)}$ is flat over T (Remark 5.1.8), $\varphi_{t_0}(\overline{\mathcal{B}}_T)$ cannot be an embedded component of $\mathcal{S}_T^{(4)}$.

On the other hand, for every $t \in T$, φ_t is the morphism constructed by Li, and we know its fibres have the same behaviour independently of $t \in T$.

We conclude that such an embedded component $\overline{\mathcal{B}}_T$ cannot exist and that $\mathcal{B}_T$ is flat over T.

□

Lemma 5.1.11. *Let $\widetilde{M}_S$ be the O'Grady desingularisation of M_S, $B_S = M_S \setminus M_S^{\mathrm{lf}}$ and $\widetilde{B}_S$ its proper transform. Any monodromy operator g arising from a deformation family (5.1.2) must satisfy the equality $g(\widetilde{B}_S) = \widetilde{B}_S$.*

Proof. It follows directly from Proposition 5.1.10 as in the proof of Lemma 5.1.5. □

The main result of this section is the following proposition.

Theorem 5.1.12. *Let $g \in \mathrm{O}^+(H^2(\widetilde{M}_S, \mathbb{Z}))$ be such that $g(\widetilde{\Sigma}_S) = \widetilde{\Sigma}_S$ and $g(\widetilde{B}_S) = \widetilde{B}_S$. Then g is a monodromy operator.*

Proof. Let g be as in the statement. In particular $g \in \mathrm{O}^+(H^2(\widetilde{M}_S, \mathbb{Z}), G_2(-1))$ and so its image $r(g)$ under the restriction map (5.1.1) is a monodromy operator on S. This means that there exists a family of deformations $\mathcal{S}_T \to T$ such that $r(g)$ is obtained by parallel transport along a loop γ such that $[\gamma] \in \pi_1(T, 0)$. By the proof of [Huy16, Proposition 7.5.5.5], it follows that T can be taken to be a curve (in fact, T can be taken to be the disc Δ). Let $\mathcal{H}$ be a line bundle on $\mathcal{S}$ such that $\mathcal{H}_0$ is the polarisation H on S. The number of points $t \in T$ where $\mathcal{H}_t$ is either not ample or not generic is finite (cf. Lemma 5.1.1). Let us denote by T' the complement in T of these points. By [God71, Théorème 2.3], the induced map

$$\pi_1(T', 0) \longrightarrow \pi_1(T, 0)$$

is surjective and so we can assume that $[\gamma] \in \pi_1(T', 0)$.

By construction the parallel transport along γ in the family $\widetilde{\mathcal{M}}_{T'} \to T'$ is an isometry g' such that $r(g') = r(g)$ and moreover, by Lemma 5.1.5 and Lemma 5.1.11, $g'(\widetilde{\Sigma}_S) = \widetilde{\Sigma}_S$ and $g'(\widetilde{B}_S) = \widetilde{B}_S$. Therefore $g = g'$. □

Remark 5.1.13. The statement of Theorem 5.1.12 was expected, but a rigorous proof has always been missing. We remark that, to the best of the author's knowledge, the proof presented here is new and original, and the technique used here is also new. In particular, it seems that this is the first time that the construction of a family of Donaldson-Uhlenbeck-Yau moduli spaces (5.1.5) appear.

5.2 Markman's result about symplectic resolutions of singularities

There are other monodromy operators coming from the family of OG10 manifolds constructed by O'Grady. These operators have been studied by Markman in [Mar10b] via an interesting construction which we will recall here. We use the same notation as in the previous section. So S is a projective $K3$ surface, M_S is the singular moduli space considered by O'Grady and $\widetilde{M}_S$ is its symplectic resolution of singularities.

The first remark is that M_S is a *symplectic variety*, i.e. a normal projective variety with rational Gorenstein singularities, whose smooth locus has a symplectic 2-form. In the following we work with a symplectic variety Y and with its symplectic resolution $\pi\colon X \to Y$. From now on we assume that X is an irreducible holomorphic symplectic manifold.

Remark 5.2.1. The definition of symplectic variety used here is the one introduced by Beauville in [Bea00]. Recently, because of the minimal model program in higher dimensions, interest in symplectic varieties has increased and people worried that the hypotheses on the singularities that Beauville imposed are too restrictive. In fact, any symplectic variety (according to this definition) admits a symplectic resolution, but there are examples of varieties which deserve the name of singular irreducible holomorphic symplectic manifolds and which do not admit any resolution of singularities (e.g. Theorem B.4.2). For a recent account of the theory of symplectic varieties and examples, we refer to [PR18].

Let $\mathcal{X} \to \mathrm{Def}(X)$ and $\mathcal{Y} \to \mathrm{Def}(Y)$ be the semi-universal deformations of X and Y, respectively. Let us denote by $0_X \in \mathrm{Def}(X)$ and $0_Y \in \mathrm{Def}(Y)$ the base points corresponding to X and Y.

Theorem 5.2.2 ([Nam01, Theorem 2.2], [Mar10b, Lemma 1.2])**.** *Let X and Y be as above.*

1. *Both $\mathrm{Def}(X)$ and $\mathrm{Def}(Y)$ are smooth and of the same dimension . If $\bar{\pi}\colon \mathrm{Def}(X) \to \mathrm{Def}(Y)$ is the map induced by the resolution π, then up to shrinking both $\mathrm{Def}(X)$ and $\mathrm{Def}(Y)$ in a neighbourhood of 0_X and 0_Y, $\bar{\pi}$ is a branched Galois cover. Moreover, for a generic point $\eta \in \mathrm{Def}(X)$, we have $\mathcal{X}_\eta \cong \mathcal{Y}_{\bar{\pi}(\eta)}$.*

2. *If G is the Galois group of $\bar{\pi}$, then G acts on $H^\bullet(X, \mathbb{Z})$ via monodromy operators which respect the Hodge structure. Moreover, the action is faithful on $H^{1,1}(X, \mathbb{Z})$ and trivial on $H^{2,0}(X, \mathbb{Z})$.*

The next step is to understand the Galois group. Let $\Sigma \subset Y$ be the singular locus and $\Sigma_0 \subset \Sigma$ the *dissident locus*, i.e. the locus where the singularities are not of ADE type.

Example 5.2.3. Let $Y = M_S$ and $X = \widetilde{M}_S$. The singular locus Σ is the locus parametrising strictly semistable sheaves. Recall that $\Sigma \cong \mathrm{Sym}^2 S^{[2]}$, and so the dissident locus is $\Sigma_0 = \Omega \cong S^{[2]}$. Notice that M_S has singularities of type A_1 along $\Sigma \setminus \Sigma_0$ (cf. Appendix B).

Let $\widetilde{U} = X \setminus \pi^{-1}(\Sigma_0)$. By a result of Namikawa ([Nam01, Proposition 1.6]), the codimension of Σ_0 inside Y is at least 4 and hence the codimension of $\pi^{-1}(\Sigma_0)$ inside X is at least 2 (here we are using a result of Kaledin, see [Kal06], which states that π is semi-small). This implies that $H^2(\widetilde{U}, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ and so, from now on, we can safely ignore the dissident locus.

The complement $\Sigma \setminus \Sigma_0$, if not empty, is a disjoint union of smooth and symplectic subvarieties of codimension 2 inside $Y \setminus \Sigma_0$ ([Nam01, Proposition 1.6]). Let $\mathcal{D}_\Sigma$ be the set of connected components of $\Sigma \setminus \Sigma_0$, $B \in \mathcal{D}_\Sigma$ and $b \in B$. The fibre $\pi^{-1}(b)$ is a tree of smooth rational curves, with dual graph a Dynkin diagram of type ADE by definition. The fundamental group $\pi_1(B, b)$ acts on the set of irreducible components of $\pi^{-1}(b)$ in such a way that the dual action on the dual graph is via automorphisms. The quotient of this Dynkin diagram by this action is called the folded Dynkin diagram. Denote by W_B the Weyl group of the folded Dynkin diagram associated to B.

Theorem 5.2.4 ([Mar10b, Theorem 1.4])**.** *The Galois group of $\bar{\pi}\colon \mathrm{Def}(X) \to \mathrm{Def}(Y)$ is isomorphic to $\prod_{B \in \mathcal{D}_\Sigma} W_B$.*

Let us explain a bit better how the Weyl group W_B acts on $H^2(X, \mathbb{Z})$. Notice that here the remark that $H^2(\widetilde{U}, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ is important.

First of all, notice that if $B \in \mathcal{D}_\Sigma$, then $E_B = \pi^{-1}(B)$ is connected and of pure codimension 1 inside X. If E_B^i is an irreducible component of E_B, then we denote by e_B^i the Poincaré dual of its closure. Let $\Lambda_B = \mathrm{span}\{e_B^i\}$ be the sublattice of $H^2(X,\mathbb{Z})$ generated by the irreducible components of E_B.

Proposition 5.2.5 ([Mar10b, Lemma 4.10, Lemma 4.14, Section 4.7])**.** *The sublattice Λ_B is saturated and the reflection $r_{e_B^i}$ is integral for every i, i.e. $r_{e_B^i} \in \mathrm{O}(H^2(X,\mathbb{Z}))$. The Weyl group W_B is isomorphic to the subgroup of $\mathrm{O}(H^2(X,\mathbb{Z}))$ generated by the reflections $r_{e_B^i}$.*

Example 5.2.6. Let us come back to Example 5.2.3. In this case $B = \Sigma \setminus \Sigma_0$ has only one connected component and, since $\widetilde{M}_S$ is obtained by blowing up M_S along B, the dual graph of a fibre is of type A_1. The folded Dynkin diagram coincides with the Dynkin diagram and the Weyl group is generated by a single reflection. By Proposition 5.2.5, this reflection is the reflection around the exceptional divisor $\widetilde{\Sigma}$ of the resolution. Finally, Theorem 5.2.2 states that this reflection is a monodromy operator.

Remark 5.2.7. Notice that $\Lambda_B = \mathbb{Z}\widetilde{\Sigma}$ and W_B is isomorphic to $\mathrm{O}(\Lambda_B)$ and it is realised as a subgroup of $\mathrm{O}(H^2(\widetilde{M}_S,\mathbb{Z}))$ by extending the isometries by the identity.

The example above is not satisfactory, since we only constructed one monodromy operator. One way to obtain more monodromy operators is by considering the natural morphism

$$\phi\colon M_S \longrightarrow \overline{\mathcal{N}}_\infty(g)$$

where $\overline{\mathcal{N}}_\infty(g)$ is the Donaldson-Uhlenbeck-Yau moduli space (cf. Section B.2).

We want to apply Markman's results to the symplectic resolution of singularities

$$\pi := \phi \circ \tilde{\pi}\colon \widetilde{M}_S \longrightarrow \overline{\mathcal{N}}_\infty(g). \tag{5.2.1}$$

The singular locus of $\overline{\mathcal{N}}_\infty(g)$ is $S^{(4)}$ and the dissident locus Σ_0 is identified with the big diagonal. The complement $B = S^{(4)} \setminus \Sigma_0$ is connected. Recall (Proposition B.2.11) that the restriction of ϕ to $\phi^{-1}(B)$ is a $\mathbb{P}^1$-bundle over B. Moreover, if $b \in B$, then the intersection $\Sigma \cap \phi^{-1}(b)$ consists of three points. Since Σ is the singular locus of M_S and has singularities of type A_1 outside the diagonal, it follows that the Dynkin diagram dual to the tree of rational curves $\pi^{-1}(b)$ (see Figure 5.1) is of type D_4. By [Mar10b, Example 3.4], the folded Dynkin diagram associated to D_4 is a Dynkin diagram of type G_2.

Theorem 5.2.8 ([Mar10b, Section 5.2])**.** *Let $g \in \mathrm{O}^+(H^2(\widetilde{M}_S,\mathbb{Z}))$ be an isometry such that the restriction of g to the sublattice $H^2(S,\mathbb{Z})$ is the identity. Then g is a monodromy operator.*

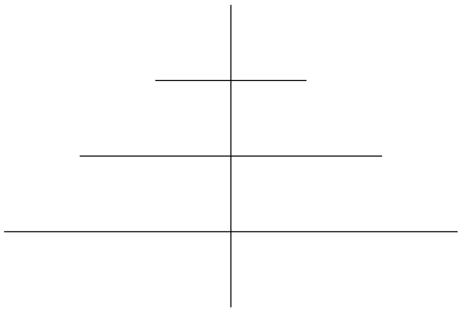


FIGURE 5.1: Tree of rational curves in $\widetilde{M}_S$.

Proof. This is a direct consequence of Proposition 5.2.5. In fact, $\Lambda_B = \mathrm{span}\{\widetilde{\Sigma}, \widetilde{B}\}$ and the group described in the statement is $\mathrm{O}(\Lambda_B)$, which coincides with the Weyl group of the G_2 Dynkin diagram by Remark B.3.3. □

5.3 Monodromy operators coming from cubic fourfolds

In this section V is a generic cubic fourfold, $\pi_V\colon \mathcal{J}_V \to \mathrm{P}$ is the compactified intermediate Jacobian fibration, Θ_V is a relative Theta divisor and b_V is the class of the fibration. We denote by U_{Θ_V,b_V} the hyperbolic plane generated by Θ_V and b_V. Recall from Section 3.5 that

$$H^2(\mathcal{J}_V,\mathbb{Z}) \cong U_{\Theta_V,b_V} \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$$

The restriction map

$$r\colon\ \mathrm{O}^+\left(H^2(\mathcal{J}_V,\mathbb{Z}), U_{\Theta_V,b_V}\right) \longrightarrow \mathrm{O}^+\left(U^{\perp}_{\Theta_V,b_V}\right)$$

is surjective (cf. Section A.3) and the kernel is a finite group of order two.

By Theorem 3.5.3 and Theorem 2.1.9,

$$\widetilde{\mathrm{O}}^+\left(U^{\perp}_{\Theta_V,b_V}\right) \cong \widetilde{\mathrm{O}}^+(H^4(V,\mathbb{Z})_{\mathrm{prim}}) \cong \mathrm{Mon}^4(V).$$

The preimage $r^{-1}\left(\widetilde{\mathrm{O}}^+\left(U^{\perp}_{\Theta_V,b_V}\right)\right)$ is the subgroup

$$\widetilde{\mathrm{O}}^+\left(H^2(\mathcal{J}_V,\mathbb{Z}), U_{\Theta,b}\right) = \left\{g \in \mathrm{O}^+\left(H^2(\mathcal{J}_V,\mathbb{Z}), U_{\Theta,b}\right) \mid \bar{g} = id_{A_{\mathcal{J}_V}}\right\}.$$

In the following we adopt the notation of Section 2.1.1, so $\mathcal{U} \subset \mathbb{P}(H^0(\mathbb{P}^5,\mathcal{O}(3))^*)$ is the universal family of smooth cubic fourfolds. We denote by $\mathcal{U}' \subset \mathcal{U}$ the open subset of non-special cubic fourfolds, so in particular $\mathcal{U}'$ is the complement in $\mathcal{U}$ of the union of countably many divisors. We introduce $\mathcal{U}'$ because it gives a family of LSV manifolds

$$\upsilon\colon \mathcal{J}_{\mathcal{U}'} \longrightarrow \mathcal{U}'.$$

Remark 5.3.1. Notice that υ is a family of polarised Lagrangian fibrations, and therefore any monodromy operator aring from this family must preserve both Θ_V and b_V.

Let $\widetilde{\mathrm{O}}_b^+\left(H^2(\mathcal{J}_V,\mathbb{Z})\right)_\Theta$ be the subgroup of $\widetilde{\mathrm{O}}^+\left(H^2(\mathcal{J}_V,\mathbb{Z}),U_{\Theta,b}\right)$ that restricts to the identity on $U_{\Theta,b}$.

Theorem 5.3.2. $\widetilde{\mathrm{O}}_b^+\left(H^2(\mathcal{J}_V,\mathbb{Z})\right)_\Theta$ *acts via monodromy operators.*

Proof. Let $g\in\widetilde{\mathrm{O}}_b^+\left(H^2(\mathcal{J}_V\mathbb{Z})\right)_\Theta$. In particular, its restriction $r(g)$ induces the isometry $\tilde{g}\in\widetilde{\mathrm{O}}^+\left(H^4(V,\mathbb{Z})_{\mathrm{prim}}\right)=\mathrm{Mon}^4(V)$. Then there exists a loop $[\gamma]\in\pi_1(\mathcal{U})$ such that $\tilde{g}$ is the image of γ via the monodromy representation (2.1.1). $\mathcal{U}$ is a Zariski open subset of $\mathbb{P}^5$, hence it is open in the standard topology. The restriction to U of the Fubini-Study metric on $\mathbb{P}^5$ can be non-complete on U: we can make such a metric complete by multiplying it with a smooth (scalar) function which diverges to infinity at least quadratically when approaching the boundary of U. Lemma 5.3.3 below ensures then that we can move γ in such a way that $[\gamma]\in\pi_1(U')$. The parallel transport along γ inside the local system $R^2\upsilon_*\mathbb{Z}$ coincides with g by construction (cf. Remark 5.3.1). ☐

Lemma 5.3.3. *Let M be a connected and complete Riemannian manifold. Let $\{D_k\}_{k\in I}$ be a countable set of closed submanifolds in M of (real) codimension strictly greater than 1. Let $M'=M\setminus\bigcup_{k\in I}D_k$ and $i:M'\to M$ be the inclusion. Then the induced map*

$$i_*:\pi_1(M',p)\longrightarrow\pi_1(M,p)$$

is surjective for every $p\in M'$.

Proof. Let $\gamma\in\pi_1(M,p)$. Let $\mathcal{L}_\gamma$ denote the set of loops δ in M based at $p\in M'$ such that $[\delta]=\gamma\in\pi_1(M,p)$. When endowed with the Hausdorff distance (induced by the complete metric on M), $\mathcal{L}_\gamma$ becomes a complete metric space. For a closed submanifold $D\subset M$, let $\mathcal{L}_\gamma(D)$ be the open subset of $\mathcal{L}_\gamma$ consisting of loops disjoint from D. If the codimension of D is strictly greater than 1, Sard's theorem, applied to the inclusion map $\mathcal{L}_\gamma\setminus\mathcal{L}_\gamma(D)\to\mathcal{L}_\gamma$, implies that $\mathcal{L}_\gamma(D)$ is dense in $\mathcal{L}_\gamma$. By Baire's Category theorem it follows then that $\cap_{k\in I}\mathcal{L}(D_k)$ is dense in $\mathcal{L}$ and hence there exists a loop $\bar{\delta}$ in M which is disjoint from all the $D_k s$, i.e. $[\bar{\delta}]\in\pi_1(M',p)$. By construction $i_*([\bar{\delta}])=\gamma$. ☐

Remark 5.3.4. Consider the chain of inclusions

$$\widetilde{\mathrm{O}}_b^+\left(H^2(\mathcal{J}_V,\mathbb{Z})\right)_\Theta\subset\mathrm{Mon}^2(\mathcal{J}_V)_\Theta\subset\mathrm{O}^+\left(H^2(\mathcal{J}_V,\mathbb{Z})\right)_\Theta. \tag{5.3.1}$$

Suppose that $q(\Theta_V)=a$. If $g\in\mathrm{O}^+\left(H^2(\mathcal{J}_V,\mathbb{Z})\right)_\Theta\setminus\mathrm{O}^+\left(H^2(\mathcal{J}_V,\mathbb{Z}),U_{\Theta,b}\right)_\Theta$, then

$$g(b)=a\Theta+(1-2a)b+v$$

where v satisfies:

$$(v,U_{\Theta,b})=0\qquad\text{and}\qquad v^2=2(a-1). \tag{5.3.2}$$

In particular the index of $\mathrm{O}^+\left(H^2(\mathcal{J}_V,\mathbb{Z}),U_{\Theta,b}\right)_\Theta$ in $\mathrm{O}^+\left(H^2(\mathcal{J}_V,\mathbb{Z})\right)_\Theta$ is infinite and so is the index of $\widetilde{\mathrm{O}}_b^+\left(H^2(\mathcal{J}_V,\mathbb{Z})\right)_\Theta$ in $\mathrm{O}^+\left(H^2(\mathcal{J}_V,\mathbb{Z}),U_{\Theta,b}\right)_\Theta$.

5.4 Mongardi's counter-example to Markman's conjecture and new improvements

So far we constructed monodromy operators on manifolds of OG10-types using two different families. In this section we want to report some known facts about the shape of the monodromy group of such manifolds and explain how the operators in the previous sections of this chapter help to shed some light on this problem.

Markman conjectured in [Mar11, Conjecture 10.7] that if X be an irreducible holomorphic symplectic manifold of OG10-type, then $\mathrm{Mon}^2(X)=\mathrm{O}^+(H^2(X,\mathbb{Z}))$.

The conjecture was motivated by the results of Section 5.2 and an unpublished work of Markman, whose idea was to study monodromy operators on $\widetilde{M}_S$ induced by auto-equivalences of the derived category of the $K3$ surface S (this was the technique used by Markman to study monodromy operators on manifolds of $K3^{[n]}$- and Kum^n-type, cf. [Mar08] and [Mar18]).

The conjecture was recently disproved by Mongardi using a completely different approach, which we are going to describe now.

Let $\mathcal{BK}(X)$ be the birational Kähler cone of X, i.e. the union $\bigcup f^{-1}\mathcal{K}(X')$, where f runs through all the birational maps between X and any other irreducible holomorphic symplectic manifold X'. Here $\mathcal{K}(X)$ is the Kähler cone of X.

A divisor D on X is called a *wall divisor* if $q_X(D)<0$ and $g(D^\perp)\cap\mathcal{BK}(X)=\emptyset$ for every $g\in\mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ (cf. [Mon14, Definition 1.2]). Here $\mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ is the subgroup of monodromy operators which are isomorphism of Hodge structures (in the following we simply call such isometries Hodge monodromy operators). Wall divisors decompose the positive cone into chambers, and one of these chamber is the Kähler cone (cf. [Mon14, Lemma 1.4] together with the remark that the Kähler cone is dual to the Mori cone). From our point of view, the most important feature of wall divisors is that they are preserved by Hodge monodromy operators.

Proposition 5.4.1 ([Mon14, Theorem 1.3]). *Let (X,η) and (X',η') be two deformation equivalent marked irreducible holomorphic symplectic manifolds. Assume that the composition $(\eta')^{-1}\circ\eta$ is a parallel transport operator which is an isomorphism of Hodge structures. If D is a wall divisor on X, then $(\eta')^{-1}\circ\eta(D)$ is a wall divisor on X'.*

This result gives a useful criterion to determine if a given Hodge isometry is not monodromy, provided that one has a good understanding of wall divisors.

The last problem is not easy to solve in general. But when X is a moduli space of sheaves (or the Albanese fibre of a moduli space of sheaves or the O'Grady symplectic desingularisation), using results from Bridgeland stability condition theory as developed by Bayer and Macrì (cf. [BM14], [Yos16], [MZ16]), one can give a useful numerical characterisation of such wall divisors in purely lattice-theoretic terms. We will only state the result for the generalised O'Grady moduli spaces (as defined in Section B.4) and refer to [Mon16, Theorem 1.3, Theorem 1.4] for other deformation types.

Let S be a projective $K3$ surface and $w \in \widetilde{H}(S)$ a Mukai vector such that $(w, w) = 2$. Let $M_S(2w)$ be the singular moduli space of semistable sheaves (with respect to a generic polarisation) on S with invariants fixed by w and let $\widetilde{M}_S(2w)$ be its symplectic desingularisation (see Theorem B.4.2). Let D' be a divisor on $M_S(2w)$ such that $q_{M_S(2w)}(D') < 0$ and let D be its pullback to $\widetilde{M}_S(2w)$. In the following T is the saturated (hyperbolic) sub-lattice of $\widetilde{H}(S)$ generated by w and D'.

Proposition 5.4.2 ([Mon16, Theorem 1.5])**.** *With the above notations, D is a wall divisor if and only if one of the following properties holds:*

- *there exists $s \in T$ such that $s^2 = -2$ and $(s, w) = 0$;*
- *there exists $s \in T$ such that $s^2 = -2$ and $(s, w) = 1$.*

Moreover, if one of the above properties holds, and if E is a primitive generator of $\langle s, w\rangle \cap w^{\perp}$, then E is a wall divisor.

Suppose that S is a projective $K3$ surface and H a polarisation on it, and suppose that there is an automorphism $\varphi \in \mathrm{Aut}(S)$ of order 3 such that $\varphi^*(\sigma_S) = \sigma_S$ and $\varphi^*(H) = H$ (here σ_S is the symplectic form on S). Let us further assume that $\varphi^*(2w) = 2w$. By [MW15, Proposition 4.3], φ induces an automorphism $\tilde{\varphi} \in \mathrm{Aut}(\widetilde{M}_S(2w))$ whose action on cohomology preserves the symplectic form (i.e. $\tilde{\varphi}$ is a symplectic automorphism). The co-invariant lattice $\Omega_{\tilde{\varphi}}(\widetilde{M}_S(2w))$ is defined as the orthogonal complement of the invariant lattice and by assumption it is contained in $H^{1,1}(\widetilde{M}_S(2w))$. Again by [MW15, Proposition 4.3], it follows that $\Omega_{\tilde{\varphi}}(\widetilde{M}_S(2w)) \cong \Omega_{\varphi}(S)$. The possible co-invariant lattices for symplectic automorphisms of prime order on $K3$ surfaces have been completed classified in [GS07, Theorem 4.1]. It follows that there exists a $(1,1)$-class $F \in \Omega_{\tilde{\varphi}}(\widetilde{M}_S(2w))$ such that $q(F) = -10$ and $\mathrm{div}(F) = 1$. Moreover, by construction F is orthogonal to the polarisation H and hence it is not a wall divisor.

Let us fix some numerical parameters to make us able to do computations. So let us assume that $H^2 = 2$ and that $w = (1, 0, -1)$. Let us also assume that $\mathrm{Pic}(S) = \mathbb{Z}H$. By direct computation, one checks that the class $s = (2, H, 1) \in \widetilde{H}(S)$ is such that $(s, s) = -2$ and $(s, w) = 1$. Hence the pullback D on $\widetilde{M}_S(2w)$ of the generator D' of $\langle s, w\rangle \cap w^{\perp}$ is a wall divisor (cf. item c in Theorem B.4.2). By direct check again, $q(D) = -10$ and $\mathrm{div}(D) = 1$ (but notice that $\mathrm{div}(D') = 2$: here we are using the results of [Per10]). Since D and F have the same degree and the same divisibility, the Eichler criterion (Proposition A.4.1) says that there exists an orientation preserving isometry sending D to F. By Proposition 5.4.1, this isometry cannot be a monodromy operator and so the following result is proved.

Proposition 5.4.3 ([Mon16, Theorem 3.3])**.** *If X is an irreducible holomorphic symplectic manifold of OG10-type. Then the inclusion* $\mathrm{Mon}^2(X) \subset \mathrm{O}^+(H^2(X,\mathbb{Z}))$ *is strict.*

Mongardi's counter-example to Markman's conjecture is purely lattice-theoretic and it sheds no light on the geometry of manifolds of OG10-type, nor does it give a clue about the shape of their monodromy group.

Remark 5.4.4. Following the expectation that the index of the monodromy group gives the number of non-birational models which are Hodge-isometric (see discussion after Theorem 1.3.7 in Section 1.3), one would expect that Mongardi's counter-example geometrically corresponds to two non-birational manifolds of OG10-type which are Hodge-isometric. As we have seen in Section 4.5, when V is a very general cubic fourfold, one expects that $\mathcal{J}_V$ and $\mathcal{J}_V^T$ are not birational. It is not clear though if they are Hodge isometric or not, but the most natural choice of Hodge isomorphism between them is not an isometry.

The subgroup of the monodromy group determined in Theorem 5.3.2 arises from a special family of polarised Lagrangian fibration, hence the isometries contained in it preserve both the class of the polarisation and the class of the fibration. It is expected that in this situation, by deforming to a very general manifold, we can relax these conditions and still get monodromy operators. So Theorem 5.3.2 is conjecturally saying that the index of the monodromy group in the group of orientation preserving isometries is at most 3. On the other hand, by Theorem 5.2.8, we know that there exist monodromy operators which have no restrictions on their action on the discriminant group. So we expect that the index is strictly smaller than 3. Finally, by Proposition 5.4.3 we know that the index must be strictly bigger than 1 and it is natural to expect that the index of the monodromy group is 2 inside the group of orientation preserving isometries.

Chapter 6

Application to the topology of moduli spaces

6.1 Finiteness results

Aim of this section is to recall the following result, which is the starting point for the computations of the next section.

Theorem 6.1.1 (Huybrechts, Markman, Verbitsky)**.** *Let Ξ be the set of connected components of $\mathfrak{M}_\Lambda$, the moduli space of marked irreducible holomorphic symplectic manifolds (cf. Section 1.2). Then Ξ is finite.*

This is a folklore result whose proof comes from results of Huybrechts, Markman and Verbitsky. We could not find any satisfying treatment (see for example [Mar11, Lemma 7.5]), so we decided to recall its proof here.

Remark 6.1.2. Only in this section we change our point of view and talk about hyper-Kähler manifolds. These are compact Riemannian manifolds (M, g) of (real) dimension $4n$, whose holonomy group is the symplectic group $\mathrm{Sp}(n)$. By the holonomy principle and [Bea83b, Proposition 4], there exists a Kähler structure I on M such that $X = (M, g, I)$ is an irreducible holomorphic symplectic manifold; the vice versa is also true (see the Introduction).

In the following, Teich_M is the Teichmüller space parametrising irreducible holomorphic symplectic structures on a compact Kähler manifold M. Equivalently, Teich_M parametrises $\mathrm{Sp}(n)$ metrics on M. Let $\Gamma = \mathrm{Diff}^+(M)/\mathrm{Diff}_0(M)$ be the mapping class group, with its natural action on $\mathrm{Aut}(H^*(M,\mathbb{Z}))$. The first remark is the following result.

Proposition 6.1.3 ([Huy03b, Theorem 2.6])**.** *The Teichmüller space* Teich_M *has finitely many connected components.*

This result is a direct consequence of the famous Finiteness Theorem of Huybrechts, which states that there are only finitely many deformation types of irreducible holomorphic symplectic manifolds ([Huy03b, Theorem 2.1]).

The following result of Verbitsky is a particular case of a more general result by Sullivan ([Sul77]).

Proposition 6.1.4 ([Ver13, Theorem 3.5.(iv)])**.** *The action of the mapping class group* Γ *on* $H^2(M,\mathbb{Z})$ *has finite index.*

Now let $I \in \mathrm{Teich}_M$ be a complex structure on M and let $X = (M, I)$ be the corresponding irreducible holomorphic symplectic manifold.

Proposition 6.1.5 ([Ver13, Theorem 7.2])**.** *If* $\Gamma^I \subset \Gamma$ *is the subgroup stabilising the connected component of* Teich_M *containing* I*, then* $\mathrm{Mon}^2(X) \cong \Gamma^I$.

Proof of Theorem 6.1.1. Start by noticing that $\mathrm{O}(\Lambda)$ acts transitively on Ξ: by definition the stabiliser of a connected component $t \in \Xi$ is the monodromy group $\mathrm{Mon}^2(\mathfrak{M}^t_\Lambda)$ defined in Section 1.4. It follows that the cardinality of Ξ is equal to the index of $\mathrm{Mon}^2(X)$ inside $\mathrm{O}(H^2(X,\mathbb{Z}))$ for any $(X,\eta) \in \mathfrak{M}_\Lambda$. Now, by Proposition 6.1.3 and Proposition 6.1.5, the index of $\mathrm{Mon}^2(X) \cong \Gamma^I$ in Γ is finite. By Proposition 6.1.4 the index of Γ in $\mathrm{O}(H^2(X,\mathbb{Z}))$ is also finite and the claim follows.

Remark 6.1.6. Notice that, by Section 1.4, the same holds true for the moduli space of polarised irreducible holomorphic symplectic manifolds.

6.2 Connected components of moduli spaces

In this section we give a concrete example of how the monodromy group can be used to do explicit computations. In particular, we are interested in counting the number of connected components of moduli spaces of both marked and polarised irreducible holomorphic symplectic manifolds. By the results of the previous section, this is equivalent to determining the index of the (polarised) monodromy group inside the group of isometries.

We will focus only on one deformation type, namely the Kum^n type. The knowledge of the monodromy group is fundamental for this kind of computations, so it is natural to restrict to the cases where the monodromy group is known. We point out that the case of manifolds of $K3^{[n]}$-type has been already studied by Apostolov as part of his PhD thesis. The results of this section generalise the result in his paper [Apo14].

Remark 6.2.1. The monodromy group of OG10 manifolds is still unknown and its study has been one of the main contributions of this thesis. The monodromy group of OG6 manifolds has been recently announced by Mongardi and Rapagnetta to be equal to the whole group of orientation preserving isometries, but the result is still unpublished.

6.2.1 Monodromy groups and a characterisation of parallel transport operators

In the following X is an irreducible holomorphic symplectic manifold deformation equivalent to a generalised Kummer variety. If $\dim X = 2n$, then

$$H^2(X,\mathbb{Z}) \cong U^3 \oplus \langle -2-2n\rangle.$$

Notice that U^3 is a unimodular lattice.

Denote by $A_X = A_{H^2(X,\mathbb{Z})}$ the discriminant group: it is a cyclic group of order $2n+2$. There is a natural morphism $t\colon \mathrm{O}(H^2(X,\mathbb{Z})) \to \mathrm{Aut}(A_X)$. Define

$$W(X) = \{g \in \mathrm{O}(H^2(X,\mathbb{Z})) \mid t(g) = \pm id\} \qquad (6.2.1)$$

and consider the associated character $\chi\colon W(X) \longrightarrow \{\pm 1\}$. Let $f\colon W(X) \longrightarrow \{\pm 1\}$ be defined by $f(g) = \chi(g)\det(g)$ and define

$$N(X) = \ker f. \qquad (6.2.2)$$

Remark 6.2.2. Notice that $W(X)$ is the group generated by products of reflections ρ_u (see Equation A.5.3), where $(u,u) = \pm 2$ ([Mar10a, Lemma 4.2]). It follows that $N(X)$ is the group generated by products $\rho_{u_1}\cdots\rho_{u_k}$, where $(u_j,u_j) = -2$ for an even number of indeces, and $(u_j,u_j) = 2$ for the remaining ones.

Theorem 6.2.3 ([Mar18, Section 9.1],[Mon16, Theorem 2.3])**.** *Let X be an irreducible holomorphic symplectic manifold of* Kum^n*-type. Then*

$$\mathrm{Mon}^2(X) = N(X).$$

Remark 6.2.4. If X has $K3^{[n]}$-type, then $\mathrm{Mon}^2(X) = W(X)$ ([Mar08]).

When X is the Albanese fibre of a moduli space of sheaves on an abelian surface S (see Section 1.1.1), with invariants fixed by the Mukai vector $v \in \widetilde{H}(S)$, we have a natural isometry $H^2(X,\mathbb{Z}) \cong v^\perp$ and so a natural primitive embedding $H^2(X,\mathbb{Z}) \to \widetilde{H}(S)$.

Remark 6.2.5. Notice that if $g \in W(X)$ then it extends to the lattice $\widetilde{H}(S)$, i.e. there exists an isometry $\tilde{g} \in \mathrm{O}(\widetilde{H}(S))$ such that $\tilde{g}|_{H^2(X,\mathbb{Z})} = g$ (cf. Section A.3).

In the following we denote by Λ_n the abstract lattice isometric to $H^2(X,\mathbb{Z})$ and by $\widetilde{\Lambda}$ the abstract lattice isometric to $\widetilde{H}(S)$. Remember that (Examples A.1.7 and A.1.8)

$$\widetilde{\Lambda} = U^4.$$

Let $\mathrm{O}(\Lambda_n, \widetilde{\Lambda})$ be the set of primitive embeddings of Λ_n inside $\widetilde{\Lambda}$. Both $\mathrm{O}(\Lambda_n)$ and $\mathrm{O}(\widetilde{\Lambda})$ act on $\mathrm{O}(\Lambda_n, \widetilde{\Lambda})$ by, respectively, pre- and post-composition. As discussed in Section A.2, two primitive embeddings are considered isometric if there exists an isometry of the overlattice exchanging the two embeddings. So we are naturally led to consider the quotient set $\mathrm{O}(\widetilde{\Lambda})\backslash\mathrm{O}(\Lambda_n,\widetilde{\Lambda})$.

Proposition 6.2.6 ([Wie16, Theorem 4.9], [Ono16, Proposition 1.4]). *There exists a distinguished* $\mathrm{Mon}^2(X)$*-invariant* $\mathrm{O}(\widetilde{\Lambda})$*-orbit*

$$[i_X] \in \mathrm{O}(\widetilde{\Lambda})\backslash\mathrm{O}(H^2(X,\mathbb{Z}),\widetilde{\Lambda}).$$

Proof. Let $v \in \widetilde{H}(S)$ be a Mukai vector and let $K(v)$ be the Albanese fibre of a moduli space of sheaves on the abelian surface. We can deform X to $K(v)$ and pick a parallel transport operator $P\colon H^2(X,\mathbb{Z}) \to H^2(K(v),\mathbb{Z})$. There exists a distinguished primitive embedding $i_v\colon H^2(K(v),\mathbb{Z}) \to \widetilde{H}(S)$ and hence a distinguished $\mathrm{O}(\widetilde{\Lambda})$-orbit $[i_v] \in \mathrm{O}(\widetilde{\Lambda})\backslash\mathrm{O}(H^2(K(v),\mathbb{Z}),\widetilde{\Lambda})$. We put

$$[i_X] := [i_v \circ P] \in \mathrm{O}(\widetilde{\Lambda})\backslash\mathrm{O}(H^2(X,\mathbb{Z}),\widetilde{\Lambda}).$$

Now, by Theorem 6.2.3 and Remark 6.2.5, $[i_X]$ is $\mathrm{Mon}^2(X)$-invariant. Moreover, its definition is independent of the choice of the parallel transport operator P chosen.

Finally, $[i_X]$ is independent of the choice of the moduli space $K(v)$ (and so of the primitive embedding i_v). In fact, by [Yos01, Proposition 5.1] (cf. Theorem 1.1.31), we can deform such moduli spaces one into the other (via one-dimensional deformation families) and the associated primitive embeddings glue together to give a morphism of local systems with target the local system $\widetilde{\Lambda}$. This concludes the proof. □

Remark 6.2.7. The analog result for manifolds of $K3^{[n]}$-type is [Mar11, Corollary 9.5].

Remark 6.2.8. Notice that $W(X)$ is identified with the stabiliser with respect to the $\mathrm{O}^+(H^2(X,\mathbb{Z}))$-action of $[i_X]$ in $\mathrm{O}(\widetilde{\Lambda})\backslash\mathrm{O}(H^2(X,\mathbb{Z}),\widetilde{\Lambda})$.

This enables us to give a useful and concrete characterisation of parallel transport operators.

Proposition 6.2.9 ([Ono16, Proposition 1.5]). *Let X_1 and X_2 be two irreducible holomorphic symplectic manifolds of* Kum^n*-type, and let $g\colon H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ be an orientation preserving isometry. Then,*

1. *if g is a parallel transport operator, then $[i_{X_1}] = [i_{X_2}] \circ g$;*
2. *if $[i_{X_1}] = [i_{X_2}] \circ g$, then either g is a parallel transport operator or $\tau_{X_2} \circ g$ is, where τ_{X_2} is any element in $W(X_2) \setminus N(X_2)$.*

It makes sense to talk about orientation preserving isometries between X_1 and X_2 in view of Remark 1.3.6. Notice that since $N(X)$ has index 2 in $W(X)$, the choice of τ_{X_2} is essentially unique.

Proof. As before, $K(v)$ is the Albanese fibre of a moduli space of sheaves on an abelian surface.

Assume first that g is a parallel transport operator. Let us deform both X_1 and X_2 to the same moduli space $K(v)$ and pick two parallel transport operators $P_i\colon H^2(X_i,\mathbb{Z}) \to H^2(K(v),\mathbb{Z})$. By assumption on g, we can choose $P_1 = P_2 \circ g$ and then

$$[i_{X_1}] = [i_v \circ P_1] = [i_v \circ P_2 \circ g] = [i_v \circ P_2] \circ g = [i_{X_2}] \circ g$$

by the proof of Proposition 6.2.6.

Vice versa, let us suppose that $[i_{X_1}] = [i_{X_2}] \circ g$. Since X_1 and X_2 are deformation equivalent, we can pick a parallel transport operator $f\colon H^2(X_2,\mathbb{Z}) \to H^2(X_1,\mathbb{Z})$ and by the previous part of the proof we have $[i_{X_2}] = [i_{X_1}] \circ f$. Putting together these two equalities, we get the relation $[i_{X_1}] = [i_{X_1}] \circ (f \circ g)$, that is $f \circ g \in W(X_1)$.

If $f \circ g \in N(X_1)$, then we conclude as before. If $f \circ g \notin N(X_1)$, then there exists $h \in W(X_1) \setminus N(X_1)$ such that $h \circ f \circ g \in N(X_1)$ is a monodromy operator. As before, the composition $(f^{-1} \circ h \circ f) \circ g$ is a parallel transport operator and $f^{-1} \circ h \circ f = \tau_X \in W(X_1) \setminus N(X_1)$. □

Remark 6.2.10. The analog result for manifolds of $K3^{[n]}$-type is [Mar11, Theorem 9.8].

Remark 6.2.11. When $X = \mathrm{K}^{[n]}(A)$ for a (non-principally polarised) abelian surface, one can pick the isometry $\tau_A\colon H^2(\mathrm{K}^{[n]}(A),\mathbb{Z}) \to H^2(\mathrm{K}^{[n]}(\hat{A}),\mathbb{Z})$ defined in Example 1.3.8 and compose it with a parallel transport operator from $\mathrm{K}^{[n]}(\hat{A})$ to $\mathrm{K}^{[n]}(A)$. With an abuse of notation we also denote by τ_A the orientation preserving isometry obtained in this way. Notice that $\tau_A \in W(\mathrm{K}^{[n]}(A)$ but $\tau_A \notin N(\mathrm{K}^{[n]}(A))$.

Remark 6.2.12. By Remark 6.2.2, if $(u,u) = -2$, then $\tau_X = \rho_u$ is a good choice which will be useful for the next result.

Now let (X_1,h_1) and (X_2,h_2) be two polarised deformation equivalent irreducible holomorphic symplectic manifolds. Using Proposition 1.3.11, we get the following corollary.

Corollary 6.2.13. *Suppose X_1 and X_2 are irreducible holomorphic symplectic manifolds of* Kum^n*-type, and let $g\colon H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ be an orientation preserving isometry. Then:*

1. *if g is a polarised parallel transport operator, then $[i_{X_1}] = [i_{X_2}] \circ g$ and $g(h_1) = h_2$;*
2. *if $[i_{X_1}] = [i_{X_2}] \circ g$ and $g(h_1) = h_2$, then either g is a polarised parallel transport operator or there exists an element $u \in H^2(X_2,\mathbb{Z})$, with $(u,u) = -2$ and $(u,h_2) = 0$, such that $\rho_u \circ g$ is a parallel transport operator.*

Proof. The only thing to prove is the existence of the element u such that $(u,u) = -2$ and $(u,h_2) = 0$. Since -2-elements exist in hyperbolic planes, this follows from the Eichler criterion (Proposition A.4.1). □

6.2.2 Moduli spaces of marked irreducible holomorphic symplectic manifolds

Let Λ_n be the abstract lattice of an irreducible holomorphic symplectic manifold of dimension $2n$ of Kum^n-type, and $\mathfrak{M}_{\Lambda_n}$ the moduli space of marked irreducible holomorphic symplectic manifolds. We denote by Ξ_n the set of connected components of $\mathfrak{M}_{\Lambda_n}$.

As explained in Section 1.4, the orientation character (A.5.1) gives a well defined map

$$\mathrm{orient}\colon \Xi_n \longrightarrow \{\pm 1\}$$

by sending each connected component $\mathfrak{M}^t_{\Lambda_n}$ to $\mathrm{orient}(\eta(H^2(X,\mathbb{Z})))$, where $(X,\eta) \in \mathfrak{M}^t_{\Lambda_n}$.

On the other hand we can define the map

$$\mathrm{orb}\colon \Xi_n \longrightarrow \mathrm{O}(\widetilde{\Lambda})\backslash \mathrm{O}(\Lambda_n, \widetilde{\Lambda})$$

by sending $\mathfrak{M}^t_{\Lambda_n}$ to $[i_X] \circ \eta^{-1}$, where $(X,\eta) \in \mathfrak{M}^t_{\Lambda_n}$. This is well defined, because if $(X',\eta') \in \mathfrak{M}^t_{\Lambda_n}$ is another marked pair, then the composition $\eta^{-1} \circ \eta'$ is a parallel transport operator and, by Proposition 6.2.9, $[i_{X'}] \circ \eta'^{-1} = [i_X] \circ \eta^{-1}$.

Proposition 6.2.14 ([Ono16, Proposition 1.6])**.** *The product map*

$$\mathrm{orb} \times \mathrm{orient}\colon \Xi_n \longrightarrow \mathrm{O}(\widetilde{\Lambda})\backslash \mathrm{O}(\Lambda_n, \widetilde{\Lambda}) \times \{\pm 1\}$$

is 2:1 and surjective.

Proof. It directly follows from Proposition 6.2.9. □

Remark 6.2.15. The analog result for manifolds of $K3^{[n]}$-type is [Mar11, Corollary 9.10].

Corollary 6.2.16. *The number of connected components of the moduli space $\mathfrak{M}_{\Lambda_n}$ of irreducible holomorphic symplectic manifolds of* Kum^n*-type is*

$$|\Xi_n| = 2^{\rho(n+1)+1},$$

where $\rho(k)$ is the number of distinct primes in the factorisation of k.

Proof. By [Mar10a, Lemma 4.3.(1)], the cardinality of $\mathrm{O}(\widetilde{\Lambda})\backslash \mathrm{O}(\Lambda_n, \widetilde{\Lambda})$ is equal to $2^{\rho(n+1)-1}$. Hence the claim follows directly from Proposition 6.2.14. □

6.2.3 Moduli spaces of polarised irreducible holomorphic symplectic manifolds

Let Λ_n be the abstract lattice of an irreducible holomorphic symplectic manifold of dimension $2n$ of Kum^n-type and let $h \in \Lambda_n$ be a primitive class such that $(h,h) = 2d > 0$

(notice that Λ_n is even). Remember from Section 1.4 that the moduli space $\mathcal{V}_{n,d}$ of polarised irreducible holomorphic symplectic manifolds of dimension $2n$ and (primitive) polarisation of degree $2d$ has an irreducible component $\mathcal{V}_{n,d}^{\mathrm{irr}}$ isomorphic to the quotient of the moduli space $\mathfrak{M}_{\bar{h}}$ by the natural action of $\mathrm{O}(\Lambda_n)$. As always, $\bar{h}$ is the orbit of h under $\mathrm{O}(\Lambda_n)$. In the following, $\Upsilon_{n,d}$ denotes the set of connected components of $\mathcal{V}_{n,d}$.

Let $(X,h) \in \mathfrak{M}_{\bar{h}}/\mathrm{O}(\Lambda_n)$ be a polarised pair and pick a representative $i \in [i_X]$. The orthogonal complement $i(H^2(X,\mathbb{Z}))^\perp \subset \widetilde{\Lambda}$ is a positive rank 1 sublattice. Therefore the lattice $T_{(X,h)}$, primitively generated by $i(H^2(X,\mathbb{Z}))^\perp$ and $i(h)$, is a positive rank 2 sublattice of $\widetilde{\Lambda}$. We want to use these combinatorial data to study the cardinality of $\Upsilon_{n,d}$.

Notice that if i' is another representative of $[i_X]$, then there exists an isometry $\tilde{g} \in \mathrm{O}(\widetilde{\Lambda})$ which restricts to an isometry $g \in \mathrm{O}(T_{(X,h)})$. Moreover, by construction $g(i(h)) = i'(h)$.

This suggests the definition of the set

$$\Sigma_n = \Big\{(T,h) \mid T \text{ positive rank 2 lattice and } h \in T \text{ primitive s.t. } h^\perp = \langle 2n+2\rangle \Big\} / \sim,$$

where $(T,h) \sim (T',h')$ if there exists an isometry $g\colon T \to T'$ such that $g(h) = h'$. We denote by $[T,h]$ the equivalence classes.

Remark 6.2.17. By Proposition A.2.3, T can be primitively embedded in $\widetilde{\Lambda}$ in a unique way (up to an isometry of $\widetilde{\Lambda}$).

Now let $I(X)$ be the set of positive and primitive classes in $H^2(X,\mathbb{Z})$. There is a well-defined map

$$f_X\colon I(X) \longrightarrow \Sigma_n$$

defined by sending $h \in I(X)$ to $[T_{(X,h)}, i(h)]$ for any $i \in [i_X]$. In the following, we drop the dependence on $[i_X]$ from the notation and we simply write $[T(X,h),h]$.

Proposition 6.2.18 ([Ono16, Proposition 2.3])**.** *Given two polarised pairs (X_1,h_1) and (X_2,h_2) of manifolds of* Kum^n*-type, a polarised parallel transport operator between them exists if and only if $f_{X_1}(h_1) = f_{X_2}(h_2)$.*

Proof. Suppose that $P\colon H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ is a polarised parallel transport operator. By Proposition 1.3.11 and Corollary 6.2.13, we immediately get an isometry $\tilde{g} \in \mathrm{O}(\widetilde{\Lambda})$ which restricts to an isometry $g\colon T_{(X_1,h_1)} \to T_{(X_2,h_2)}$ such that $g(h_1) = h_2$.

Vice versa, suppose that such an isometry g exists. In particular $T_{(X_1,h_1)}$ has two primitive embeddings inside $\widetilde{\Lambda}$, the second one given by composing the natural embedding $T_{(X_2,h_2)} \subset \widetilde{\Lambda}$ with g. By Proposition A.2.3, there exists a unique (up to isometry) such primitive embedding and hence there exists an isometry $\tilde{g} \in \mathrm{O}(\widetilde{\Lambda})$ such that the diagram

$$\begin{array}{ccc} \widetilde{\Lambda} & \xrightarrow{\tilde{g}} & \widetilde{\Lambda} \\ \uparrow & & \uparrow \\ T_{(X_1,h_1)} & \xrightarrow{g} & T_{(X_2,h_2)} \end{array}$$

commutes.

Since $\tilde{g}(i_1(H^2(X_1,\mathbb{Z}))) = i_2(H^2(X_2,\mathbb{Z}))$, it follows that $\tilde{g}$ restricts to an isometry P from $H^2(X_1,h_1)$ to $H^2(X_2,h_2)$. Here $i_1 \in [i_{X_1}]$ and $i_2 \in [i_{X_2}]$, and we have a commutative diagram

$$\begin{array}{ccc} \widetilde{\Lambda} & \xrightarrow{\tilde{g}} & \widetilde{\Lambda} \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_2} \\ H^2(X_1,\mathbb{Z}) & \xrightarrow{P} & H^2(X_2,\mathbb{Z}). \end{array}$$

In particular $[i_{X_2}] \circ P = [i_{X_1}]$ and $P(h_1) = h_2$. By Corollary 6.2.13, we get a polarised parallel transport operator from P as long as P is orientation preserving.

Let us then suppose that P is not orientation preserving. Let us pick an element $u \in H^2(X_2,\mathbb{Z})$ such that $(u,u) = 2$ and $(u,h_2) = 0$, and let us consider the reflection ρ_u. Since ρ_u is orientation preserving by definition and $\rho_u(h_2) = -h_2$, then $P' = -\rho_u \circ P$ is an orientation preserving isometry such that $[i_{X_2}] \circ P' = [i_{X_2}]$ and $P'(h_2) = h_2$, and we can apply Corollary 6.2.13 to produce a polarised parallel transport operator. □

Remark 6.2.19. The analog result for manifolds of $K3^{[n]}$-type is [Apo14, Proposition 1.6].

Remark 6.2.20. f_X is a *faithful monodromy invariant*, as defined in [Mar13, Section 5.3].

The existence of a polarised parallel transport operator between (X_1,h_1) and (X_2,h_2) is equivalent to saying that (X_1,h_1) and (X_2,h_2) belong to the same connected component.

Proposition 6.2.21 ([Ono16, Proposition 2.4, Proposition 2.7])**.** *There exists a well-defined bijective map*

$$f \colon \Upsilon_n \longrightarrow \Sigma_n \tag{6.2.3}$$

defined by sending a connected component $\mathcal{V}^t_{n,d}$ *to* $f_X(h)$*, for any* $(X,h) \in \mathcal{V}^t_{n,d}$*.*

Here $\Upsilon_n = \bigcup_{d>0} \Upsilon_{n,d}$ is the set of connected components of the moduli space $\mathcal{V}_n = \coprod_{d>0} \mathcal{V}_{n,d}$ of polarised irreducible holomorphic symplectic manifolds of Kum^n-type.

Proof. Let us first show that f is well defined, i.e. $f_{X_1}(h_1) = f_{X_2}(h_2)$ if (X_1,h_1) and (X_2,h_2) are in $\mathcal{V}^t_{n,d}$. Pick a marking η_1 of X_1 such that $(X_1,\eta_1,h_1) \in \mathfrak{M}^{a,t}_{\bar{h}}$, where $h_1 \in \bar{h}$. Since $\mathrm{O}(\Lambda_n)$ acts transitively on the set of connected components of the moduli space of marked irreducible holomorphic symplectic manifolds, we can pick a marking

η_2 of X_2 such that $(X_2, \eta_2, h_2) \in \mathfrak{M}_{\bar{h}}^{a,s}$. Recall (Proposition 1.4.2) that $\mathfrak{M}_{\bar{h}}^{a,t}$ is path-connected, so we can choose a path

$$\gamma\colon [0,1] \longrightarrow \mathfrak{M}_{\bar{h}}^{a,t}$$

joining (X_1, η_1, h_1) to (X_2, η_2, h_2). Now, if $\mathfrak{M}_{\bar{h}}^{a,t}$ were a fine moduli space, we could pick any polarised parallel transport operator along γ and use Proposition 6.2.18 to conclude. This is not the case, but we can fix it by following an idea from the proof of [Mar11, Corollary 7.4]. For every $s \in [0,1]$, we can consider a simply connected open neighbourhood U_s of $\gamma(s)$. Up to shrinking U_s we can suppose that there exists a semi-universal family $\chi_s\colon \mathcal{X}_s \to U_s$. Notice that $\{U_s\}_{s\in[0,1]}$ is a covering of $\gamma([0,1])$ and we can pick a finite subcover $\{V_j\}_{j=1}^m$ with the property that

$$\gamma\left(\left[\frac{j-1}{m}, \frac{j}{m}\right]\right) \subset V_j.$$

Each V_j comes with a universal family $\chi_j\colon \mathcal{X}_j \to V_j$. Let B be the analytic space obtained by gluing the V_j's along the edge points $\gamma(\frac{j}{m})$ (with transversal Zariski tangent spaces). Now let

$$\chi\colon \mathcal{X} \longrightarrow B$$

be the family obtained by gluing the families χ_j (notice that this family is not universal). The intervals $[\frac{j-1}{m}, \frac{j}{m}]$ also glue together to create a path

$$\gamma'\colon [0,1] \longrightarrow B$$

and the parallel transport along it in the family χ is a polarised parallel transport operator between (X_1, h_1) and (X_2, h_2). The claim then follows from Proposition 6.2.18.

To prove the injectivity of f, let $(X_k, h_k) \in \mathcal{V}_{n,d}^{t_k}$, for $k = 1, 2$, such that $f_{X_1}(h_1) = f_{X_2}(h_2)$. We want to show that (X_1, h_1) and (X_2, h_2) belong to the same connected component. By Proposition 6.2.18, there exists a polarised parallel transport operator. Unwinding the definition, this means that there exists a family

$$\chi\colon \mathcal{X} \longrightarrow B$$

with distinguished points $b_1, b_2 \in B$ corresponding to X_1 and X_2, a path γ from b_1 to b_2 and a flat section $\tilde{h} \in H^0(B, R^0\chi_*\mathbb{Z})$ such that $\tilde{h}_{b_j} = h_j$ and $\tilde{h}_b$ is ample and of type $(1,1)$ for every $b \in B$. The strategy is to show that (X_2, h_2) is isomorphic (as an algebraic variety) to a pair $(X_3, h_3) \in \mathcal{V}_{n,d}^{t_1}$: this will conclude the proof.
The first remark is that, locally on B, we can lift the section $\tilde{h}$ to the relative Picard group $R^1\chi_*\mathcal{O}_{\mathcal{X}}^*$. We need this lift because a polarised pair was defined to be a pair (X, H) where H is an ample line bundle, and we decided to work with $h = c_1(H)$ for

convenience. This local lift follows from the long exact sequence

$$\cdots \longrightarrow R^1\chi_*\mathcal{O}_{\mathcal{X}} \longrightarrow R^1\chi_*\mathcal{O}^*_{\mathcal{X}} \longrightarrow R^2\chi_*\mathbb{Z} \longrightarrow \cdots$$

since $\tilde{h}_b$ is of type $(1,1)$, and it is unique because $R^1\chi_*\mathcal{O}_{\mathcal{X}} = 0$. In particular we can cover $\gamma([0,1])$ with finitely many Zariski open subsets $\{V_j\}_{j=1}^m$ and, as before, choose trivialisations $\tilde{\eta}_j\colon R^2\chi_{j,*}\mathbb{Z} \cong \Lambda_n$ such that the family $(\mathcal{X}_j = \mathcal{X}|_{V_j}, \tilde{\eta}_j, \tilde{h}_j)$, of marked polarised irreducible holomorphic symplectic manifolds, is mapped to the same connected component $\mathfrak{M}^{a,s}_{\tilde{h}}$, independently of j, by the classifying morphisms $\alpha_j\colon V_j \to \mathfrak{M}_{\Lambda_n}$. Here $\tilde{h}_j \in R^1\chi_{j,*}\mathcal{O}^*_{\mathcal{X}_j}$ is the lifted line bundle on $\mathcal{X}_j$. Let $q\colon \mathfrak{M}^a_{\tilde{h}} \to \mathfrak{M}^a_{\tilde{h}}/\operatorname{O}(\Lambda_n)$ be the quotient map and denote by ϕ_k the analytic isomorphism between $\mathfrak{M}^a_{\tilde{h}}/\operatorname{O}(\Lambda_n)$ and the irreducible component of $\mathcal{V}^{t_k}_{n,d}$ containing (X_k, h_k). Let us define

$$(X_3, h_3) = \phi_1(q(\alpha_m(b_2)))$$

and notice that $(X_3, h_3) \in \mathcal{V}^{t_1}_{n,d}$. By construction, (X_3, h_3) and (X_2, h_2) are isomorphic in the category of analytic spaces, and hence in the category of algebraic varieties by the GAGA principle.

Finally, let us prove the surjectivity of f. Let $[T, h] \in \Sigma_n$. By Proposition A.2.3 there exists a unique primitive embedding $j\colon T \to \widetilde{\Lambda}$. If $v \in T$ is a generator of $h^\perp$, then $j(v) \in \widetilde{\Lambda}$ is an element of square $2n+2$. By construction, the orthogonal complement $j(v)^\perp$ is abstractly isometric to Λ_n. Let us fix such an isometry $i\colon \Lambda_n \to j(v)^\perp$. It induces an embedding $\Lambda_n \to \widetilde{\Lambda}$ which, abusing notation, we still call i. The $\operatorname{O}(\widetilde{\Lambda})$-orbit $[i]$ determines, up to a finite ambiguity, a connected component $\mathfrak{M}^t_{\Lambda_n}$ of the moduli space $\mathfrak{M}_{\Lambda_n}$ of marked pairs (Proposition 6.2.14). Define $h_1 := i^{-1}(j(h)) \in \Lambda_n$. Since $\mathfrak{M}^{a,t}_{h_1}$ is non-empty, there exists an irreducible holomorphic symplectic manifold X, a marking η and an ample divisor H on X such that $\eta^{-1}(c_1(H)) = h_1$. Consider then the connected component $\mathcal{V}^0 \subset \mathcal{V}_n$ containing the pair (X, H). By construction $f(\mathcal{V}^0) = [T, h]$ and the proof is concluded. □

Remark 6.2.22. The analog result for manifolds of $K3^{[n]}$-type is [Apo14, Theorem 1.7, Proposition 2.3].

In the rest of this section we want to compute the cardinality of Σ_n. The first remark is that a pair $[T, h] \in \Sigma_n$ is completely determined by the primitive embedding $j\colon \langle h\rangle \to T$ such that $j(h)^\perp = \langle 2n+2\rangle$. Therefore we want to count the number of such primitive embeddings.

Remark 6.2.23. By Proposition A.2.3, without loss of generality we can think of both $\langle h\rangle$ and T as sublattices of $\widetilde{\Lambda}$.

Recall that the divisibility δ of an element h in a lattice L is the positive generator of the ideal $(h, L) \subset \mathbb{Z}$. Both the degree and the divisibility are isometric invariants.

Remark 6.2.24. Fixing the degree and the divisibility of h does not determine its orbit. However, if $\Sigma_{n,d,\delta} \subset \Sigma_n$ is the subset consisting of pairs $[T,h]$ such that $(h,h) = 2d$ and $\mathrm{div}(h) = \delta$, then

$$\Sigma_n = \coprod \Sigma_{n,d,\delta},$$

and we can restrict our attention to computing $|\Sigma_{n,d,\delta}|$.

Remark 6.2.25. If we think of both Λ_n and T as primitively embedded in $\widetilde{\Lambda}$, then the divisibility of h in T is the same as the divisibility of h in Λ_n.

If we denote by $\mathcal{V}_{n,d,\delta}$ the moduli space of polarised pairs (X,h) for which $q_X(h) = 2d$ and $\mathrm{div}(h) = \delta$, then $\mathcal{V}_n = \coprod \mathcal{V}_{n,d,\delta}$ and $f(\Upsilon_{n,d,\delta}) = \Sigma_{n,d,\delta}$. As usual, $\Upsilon_{n,d,\delta}$ is the set of connected components of $\mathcal{V}_{n,d,\delta}$.

The main result of this section is the following. We use the following notation: for an integer l we write $\phi(l)$ for the Euler function and $\rho(l)$ for the number of distinct primes in the factorisation of l; for w and δ_1 as defined above, we write $w = w_+(\delta_1)w_-(\delta_1)$, where $w_+(\delta_1)$ is the product of all powers of primes in the factorisation of w dividing $\gcd(w,\delta_1)$ (that is, $w_-(\delta_1)$ is the part coprime to δ_1).

Theorem 6.2.26 ([Ono16, Theorem 2.8])**.** *With the notations as above, we have:*

- $|\Upsilon_{n,d,\delta}| = w_+(\delta_1)\phi(w_-(\delta_1))2^{\rho(\delta_1)-1}$ *if* $\delta > 2$ *and one of the following holds:*
  - g_1 *is even,* $\gcd(d_1,\delta_1) = 1 = \gcd(n_1,\delta_1)$ *and* $-d_1/n_1$ *is a quadratic residue mod* δ_1*;*
  - g_1, δ_1 *and* d_1 *are odd,* $\gcd(d_1,\delta_1) = 1 = \gcd(n_1, 2\delta_1)$ *and* $-d_1/n_1$ *is a quadratic residue mod* $2\delta_1$*;*
  - g_1, δ_1 *and* w *are odd,* d_1 *is even,* $\gcd(d_1,\delta_1) = 1 = \gcd(n_1, 2\delta_1)$ *and* $-d_1/4n_1$ *is a quadratic residue mod* δ_1*.*
- $|\Upsilon_{n,d,\delta}| = w_+(\delta_1)\phi(w_-(\delta_1))2^{\rho(\delta_1/2)-1}$ *if* $\delta > 2$*,* g_1 *is odd,* δ_1 *is even,* $\gcd(d_1,\delta_1) = 1 = \gcd(n_1, 2\delta_1)$ *and* $-d_1/n_1$ *is a quadratic residue mod* $2\delta_1$*.*
- $|\Upsilon_{n,d,\delta}| = 1$ *if* $\delta \le 2$ *and one of the following holds:*
  - g_1 *is even,* $\gcd(d_1,\delta_1) = 1 = \gcd(n_1,\delta_1)$ *and* $-d_1/n_1$ *is a quadratic residue mod* δ_1*;*
  - g_1, δ_1 *and* d_1 *are odd,* $\gcd(d_1,\delta_1) = 1 = \gcd(n_1, 2\delta_1)$ *and* $-d_1/n_1$ *is a quadratic residue mod* $2\delta_1$*;*
  - g_1, δ_1 *and* w *are odd,* d_1 *is even,* $\gcd(d_1,\delta_1) = 1 = \gcd(n_1, 2\delta_1)$ *and* $-d_1/4n_1$ *is a quadratic residue mod* δ_1*;*
  - g_1 *is odd,* δ_1 *is even,* $\gcd(d_1,\delta_1) = 1 = \gcd(n_1, 2\delta_1)$ *and* $-d_1/n_1$ *is a quadratic residue mod* $2\delta_1$*.*
- $|\Upsilon_{n,d,\delta}| = 0$ *otherwise.*

Remark 6.2.27. The analog result for manifolds of $K3^{[n]}$-type is [Apo14, Proposition 2.1].

Proof. Using the bijection (6.2.3) and the discussion above, $|\Upsilon_{n,d,\delta}| = |\Sigma_{n,d,\delta}|$ and the latter is the number of primitive embeddings $j\colon \langle 2d\rangle \to T$ such that $j(\langle 2d\rangle)^\perp = \langle 2n+2\rangle$.

By Proposition A.2.2, an embedding $j\colon \langle 2d\rangle \to T$ is determined by the pair (H,γ), where $H \subset A_{2d}$ is a subgroup, $\gamma\colon H \to A_{2n+2}$ is an injective homomorphism and the pushout $\Gamma_\gamma = H \subset A_{2d} \oplus A_{2n+2}$ is isotropic. Since we have also fixed $\mathrm{div}(h) = \delta$, it follows that H must be of order δ (see [Apo14, Proposition 2.2]).

Remark 6.2.28. Recall that two pairs (H,γ) and (H',γ') determine the same primitive embedding j if $H = H'$ and there exist an isometry $\varphi \in \mathrm{O}(\langle 2d\rangle) \cong \mathbb{Z}/2\mathbb{Z}$ and an isometry $\psi \in \mathrm{O}(\langle 2n+2\rangle) \cong \mathbb{Z}/2\mathbb{Z}$ such that $\gamma \circ \overline{\varphi} = \overline{\psi} \circ \gamma'$ (see Section A.2).

Identifying A_{2d} with $\mathbb{Z}/2d\mathbb{Z}$ and picking generators h of $\langle 2d\rangle$ and v of $\langle 2n+2\rangle$, we can write $H = \langle h/\delta\rangle$. Then γ is uniquely determined by the image $\gamma(h/\delta) = cv/\delta$, where c is coprime with δ. The isotropy condition is

$$\frac{2d}{\delta^2} + \frac{c^2(2n+2)}{\delta^2} \equiv 0 \pmod 2. \tag{6.2.4}$$

Put

$$d_1 = \tfrac{2d}{\gcd(2d,2n+2)}, \quad n_1 = \tfrac{2n+2}{\gcd(2d,2n+2)}, \quad g = \tfrac{\gcd(2d,2n+2)}{\delta}$$

$$w = \gcd(g,\delta), \qquad g_1 = \tfrac{g}{w}, \qquad \delta_1 = \tfrac{\delta}{w}$$

and note that equation (6.2.4) is equivalent to

$$\delta_1\left(\frac{2d}{\delta^2} + \frac{c^2(2n+2)}{\delta^2}\right) = g_1(d_1 + c^2 n_1) \equiv 0 \pmod{2\delta_1}. \tag{6.2.5}$$

The problem is now reduced to determine all the solutions c of equation (6.2.5) such that $\gcd(c,\delta) = 1$. This problem has already been solved by Gritsenko, Hulek and Sankaran in [GHS10, Proposition 3.6]. Since we are interested in isometric embeddings, we have to understand which of these solutions are invariant under the isometries in Remark 6.2.28. Both $\mathrm{O}(\langle 2d\rangle)$ and $\mathrm{O}(\langle 2n+2\rangle)$ act on H by changing the sign of the first, respectively the second, coordinate. Moreover, notice that H has a central symmetry, i.e. $(x,y) \in H$ if and only if $(-x,-y) \in H$. We can then distinguish two cases:

- $\delta \le 2$: then any subgroup H is fixed by this action and the number of solutions c corresponds to the number of primitive embeddings;
- $\delta > 2$: then there are no fixed subgroups H and we must divide the number of solutions c by 2.

This concludes the proof. □

Remark 6.2.29. When $w = 1$, the values of d and δ determine the orbit of h and hence $\mathcal{V}_{n,d,\delta} = \mathfrak{M}_{\bar{h}} / \mathrm{O}(\Lambda_n)$ is connected (cf. [GHS10, Corollary 3.7]).

We conclude this section by giving a few examples.

Example 6.2.30. If $\delta = 1$, then the orbit of h is determined and the corresponding moduli space is connected.

Example 6.2.31. Let p and q be two (different) odd primes and put $\delta = d = pq$ and $n+1 = mpq$, where $\gcd(m, pq) = 1$ and $-m$ is a quadratic residue mod pq. Then d and δ determine the orbit $\bar{h}$ and the moduli space $\mathcal{V}_{n,d,\delta}$ has two connected components.

Example 6.2.32. If $\gcd(2d, 2n+2)$ is square free, then $w = 1$ (cf. [GHS10, Remark 3.13]). This is the case, for example, when $2n+2$ is square free.

Example 6.2.33. Let X be a manifold of Kum^2-type. We have that:

1. $\mathcal{V}_{n,d,\delta}$ is connected in the following cases:
   (a) $\delta = 1$;
   (b) $\delta = 2$, $\gcd(d, 2) = \gcd(d, 3) = 1$ and d is a quadratic residue mod 4;
   (c) $\delta = 2$, $d = 3\tilde{d}$, $\gcd(\tilde{d}, 2) = 1$ and $-\tilde{d}$ is a quadratic residue mod 4;
   (d) $\delta = 3$, $d = 3\tilde{d}$, $\gcd(\tilde{d}, 3) = 1$ and $-\tilde{d}$ is a quadratic residue mod 4.
2. $\mathcal{V}_{n,d,\delta}$ has 2 connected components if $\delta = 6$, $d = 3\tilde{d}$, $\gcd(\tilde{d}, 6) = 1$ and $-\tilde{d}$ is a quadratic residue mod 12.
3. $\mathcal{V}_{n,d,\delta}$ is empty for all the remaining cases.

Appendix A

Results on lattice theory

A.1 Lattices and examples

A *lattice* is a free $\mathbb{Z}$-module L together with a non-degenerate symmetric bilinear pairing $q\colon L \times L \to \mathbb{Z}$. With an abuse of notation we also use q to denote the associated quadratic form. A lattice L is called *even* if $q(a) \in 2\mathbb{Z}$ for every $a \in L$; it is called *odd* otherwise.

An isomorphism of $\mathbb{Z}$-modules $L_1 \to L_2$ which preserves the bilinear pairing is called as *isometry*. The group of isometries of a lattice L is denoted by $\mathrm{O}(L)$.

The real vector space $L \otimes \mathbb{R}$ comes equipped with a non-degenerate symmetric bilinear form and Sylvester's theorem ensures that there exists a basis of $L \otimes \mathbb{R}$ such that the bilinear form is diagonal with only ± 1 on the diagonal. If $a \geq 0$ is the number of positive eigenvalues and $b \geq 0$ the number of negative eigenvalues, then we denote by (a, b) the *signature* of L.

Example A.1.1 (Hyperbolic plane)**.** Take $L = \mathbb{Z}^2$ with the bilinear form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to the canonical basis. This is a rank 2 even lattice of signature $(1, 1)$ denoted by U.

We write $\det L$ for the determinant of the bilinear form of a lattice L. If $\det L = \pm 1$, the lattice is called *unimodular*. The hyperbolic plane in Example A.1.1 is unimodular.

Unimodular indefinite lattices have the nice property of being uniquely determined (up to isometry) by their parity and signature.

Theorem A.1.2 ([Nik79, Theorem 0.2.1, Theorem 1.1.1])**.** *Let $a, b > 0$ be two positive integers.*

1. *There exists a unique (up to isometry) odd unimodular lattice of signature (a, b).*

2. *There exists a unique (up to isometry) even unimodular lattice of signature* (a, b) *if and only if* $a - b \equiv 0 \pmod 8$.

In particular we can talk about the hyperbolic plane U in Example A.1.1 without any further specification.

The non-degeneracy of q gives a canonical embedding $L \subset L^\vee := \mathrm{Hom}(L, \mathbb{Z}) \subset L \otimes \mathbb{Q}$ and the quotient $A_L = L^\vee / L$ is called the *discriminant group* of the lattice. A_L is a finite abelian group endowed with a quadratic form $\overline{q}\colon A_L \to \mathbb{Q}/\mathbb{Z}$ if L is odd and $\overline{q}\colon A_L \to \mathbb{Q}/2\mathbb{Z}$ if L is even. Notice that when L is unimodular we have an isomorphism $L \cong L^\vee$.

If L is a lattice and $n \in \mathbb{Z}$, we denote by $L(n)$ the lattice obtained from L by multiplying its bilinear pairing by n.

Before continuing, let us give some examples.

Example A.1.3. On $L = \mathbb{Z}h$ a bilinear form is simply given by the value $q(h) = k$. We write $\langle h \rangle$ for such a lattice and $\langle k \rangle$ for its isometry class.

Example A.1.4 (Lattices of type A_n)**.** Let $V = \mathbb{R}^{n+1}$ with the usual Euclidean product. Choose simple roots $\Delta = \{\alpha_i = e_i - e_{i+1} \mid i = 1, \cdots, n\}$, where $e_i = (0, \cdots, 0, 1, 0 \cdots, 0)$ is the canonical basis of V. The lattice generated by Δ is called the A_n-lattice. It is a positive even rank n lattice with Gram matrix

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

The discriminant group $A_n^\vee / A_n$ is the cyclic group $\mathbb{Z}\frac{1}{n+1}(e_1 + \cdots e_n - n e_{n+1})$ of order $n + 1$.

Example A.1.5. With the notation of Example A.1.4 and $n = 2$, let us take a different choice of simple roots, namely $\Delta = \{\alpha_1, \alpha_2 - \alpha_1\}$. The lattice generated by this root system is the even positive rank 2 lattice

$$\begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix},$$

called the G_2-lattice. This lattice is isometric to the A_2-lattice.

Example A.1.6 (Lattices of type E_8)**.** Take $V = \mathbb{R}^8$ with Euclidean product and canonical basis e_i. The E_8-lattice is the lattice generated by the root system generated by the simple roots $\Delta = \{e_i - e_{i+1}, e_6 + e_7, -\frac{1}{2}\sum_1^8 e_j \mid i = 1, \cdots, 6\}$. The intersection matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Notice that this is a unimodular lattice and it is, in fact, the unique unimodular even positive lattice of rank 8.

Example A.1.7 ($K3$ lattice)**.** Let S be a $K3$ surface. The second integral cohomology group $H^2(S,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 22. The intersection product is a non-degenerate symmetric bilinear form on $H^2(S,\mathbb{Z})$ of signature $(3,19)$ by the Hodge Index Theorem ([Har77, Theorem V.1.9]). Moreover it is even by the Riemann-Roch formula and unimodular by Poincaré duality. By Theorem A.1.2 and Example A.1.6, it is then isometric to the abstract lattice

$$U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1).$$

Example A.1.8 (Abelian surface lattice)**.** Let S be an abelian surface. In complete analogy with Example A.1.7, the second integral cohomology group $H^2(S,\mathbb{Z})$ is an even lattice of signature $(3,3)$ and it is abstractly isometric to the unimodular lattice

$$U \oplus U \oplus U.$$

Example A.1.9 (Mukai lattice)**.** Let S be a $K3$ or abelian surface. The even cohomology ring $H^{\text{even}}(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$ is a free $\mathbb{Z}$-module. For $\alpha = (\alpha_0,\alpha_2,\alpha_4)$ and $\beta = (\beta_0,\beta_2,\beta_4)$ in $H^{\text{even}}(S,\mathbb{Z})$, we can define a symmetric bilinear form

$$\langle \alpha, \beta \rangle = \int_S (-\alpha_0\beta_4 + \alpha_2\beta_2 - \alpha_4\beta_0).$$

It follows from the definition that this product restricts to the usual intersection product on $H^2(S,\mathbb{Z})$ and that the decomposition $\big(H^0(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})\big) \oplus H^2(S,\mathbb{Z})$ is orthogonal. Moreover, the summand $\big(H^0(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})\big)$ is isometric to a hyperbolic plane. $H^{\text{even}}(S,\mathbb{Z})$ is isometric to the abstract lattice

$$\widetilde{H}(S) := \begin{cases} U \oplus U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1) & \text{if } S \text{ is } K3 \\ \\ U \oplus U \oplus U \oplus U & \text{if } S \text{ is abelian} \end{cases}$$

which is then unimodular in both cases.

A.2 Primitive embeddings

Suppose that $L \subset M$ is an embedding of lattices and suppose that the quotient $H_M = M/L$ is a finite group. M is called an *overlattice* of L in [Nik79]. We have a chain of inclusions

$$L \subset M \subset M^\vee \subset L^\vee$$

which gives an inclusion $H_M \subset A_L$. Notice that $(M^\vee/L)/H_M = A_M$ and that $\overline{q}_L(H_M) = 0$, i.e. H_M is an isotropic subgroup of A_L.

Two overlattices $L \subset M_1$ and $L \subset M_2$ are said to be isometric if there exists an isometry $M_1 \to M_2$ which restricts to an isometry of L. In this case the two isotropic groups H_{M_1} and H_{M_2} are conjugate subgroups of A_L.

The converse is given by the following proposition.

Proposition A.2.1 ([Nik79, Proposition 1.4.1, Proposition 1.4.2]). *There exists a bijective correpondence between overlattices of L and isotropic subgroups of A_L and one has $H_M^\perp = M^\vee/L \subset A_L$ and $\overline{q}|_{H_M^\perp/H_M} = \overline{q}_M$. Moreover, two overlattices are isometric if and only if the corresponding isotropic groups are conjugate subgroups of A_L.*

Let now L and M be two even lattices. An isometric embedding $L \hookrightarrow M$ is called *primitive* if the cokernel is a free $\mathbb{Z}$-module. Two primitive embeddings $L \subset M_1$ and $L \subset M_2$ are said to be *isometric* if there exists an isometry $M_1 \to M_2$ which restricts to the identity on L.

Il $L \subset M$ is a primitive embedding, we denote by K the orthogonal complement of L in M. It turns out that M is an overlattice of the direct sum $L \oplus K$ and so we can apply the arguments in Proposition A.2.1. In particular we get an isotropic subgroup $H_M \subset A_L \oplus A_K$. The two projections $p_L \colon H_M \to A_L$ and $p_K \colon H_M \to A_K$ are injective because of the primitivity of L (and K) inside M. Let us denote by $H_{M,L}$, respectively $H_{M,K}$, the image of H_M inside A_L, respectively A_K. The composition $\gamma^M_{L,K} = p_K \circ p_L^{-1} \colon H_{M,L} \to H_{M,K}$ is then an isomorphism and $q_K \circ \gamma^M_{L,K} = -q_L$.

Proposition A.2.2 ([Nik79, Proposition 1.5.1]). *The primitive embedding $L \subset M$ (with orthogonal complement K) is determined by the pair (H_M, γ), where $H_M \subset A_L$ is an isotropic subgroup and $\gamma \colon H_M \to A_K$ is injective. These data must satisfy the conditions:*

1. $q_K \circ \gamma = -q_L$;
2. $(\overline{q}_L \oplus \overline{q}_K)|_{\Gamma_\gamma^\perp/\Gamma_\gamma} = \overline{q}_M$.

In item 2, Γ_γ is the pushout of the monomorphism γ.

Moreover, two pairs (H_1, γ_1) and (H_2, γ_2) determine isometric primitive sublattices if and only if $H_1 = H_2$ and there exist $\varphi \in \mathrm{O}(L)$ and $\psi \in \mathrm{O}(K)$ such that $\gamma_1 \circ \overline{\varphi} = \overline{\psi} \circ \gamma_2$.

Here and everywhere else we write $\overline{\varphi} \in \mathrm{O}(A_L)$ for the isometry induced by $\varphi \in \mathrm{O}(L)$ on the discriminant group.

A non-trivial corollary of this result when M is unimodular gives the existence part of the following proposition (see [PŠŠ71, Appendix to Section 6] or [Jam68] for a proof of the uniqueness part).

Proposition A.2.3 ([Nik79, Theorem 1.1.2]). *Let L and M be two (even) lattices of signature, respectively, (r_L, s_L) and (r_M, s_M) and suppose that M is unimodular. If $r_L+s_L \leq \min\{r_M, s_M\}$, then there exists a primitive embedding of L in M. If moreover $r_L + s_L \leq \min\{r_M, s_M\} - 1$, then this embedding is unique up to an isometry of M.*

A.3 Orthogonal groups

Let $L \subset M$ be a primitive embedding of even lattices. We donote by $\mathrm{O}(M, L)$ the subgroup of isometries of M which preserve the sublattice L. In particular, such an isometry restricts to an isometry of the orthogonal complement K of L in M. In this section we study the restriction map

$$r\colon\ \mathrm{O}(M, L) \longrightarrow \mathrm{O}(K). \tag{A.3.1}$$

Let us denote by i the embedding $K \subset M$. Then any isometry $\psi \in \mathrm{O}(K)$ induces an embedding $i \circ \psi$ and Proposition A.2.2 says that the obstructions to these two primitive embeddings being isometric are encoded in the action of ψ on the isotropic subgroup $H_M \subset A_L \oplus A_K$ (or better on its projection into A_K). Notice that, by definition, this amounts to saying that ψ belongs to the image of the restriction map r.

Proposition A.3.1. *The image of r consists of isometries $\psi \in \mathrm{O}(K)$ such that*

$$\overline{\psi}\,(p_K(H_M)) = p_K(H_M)$$

and there exists $\varphi \in \mathrm{O}(L)$ such that $\gamma^M_{K,L} \circ \overline{\psi} = \overline{\varphi} \circ \gamma^M_{K,L}$.

Corollary A.3.2. *If $M = L \oplus K$, then r is surjective.*

Example A.3.3. Let L be a lattice and $h \in L$ a primitive element. We denote by $\mathrm{O}(L)_h$ the group of isometries ψ such that $\psi(h) = h$. If K is the orthogonal complement of $\langle h\rangle$, then the image of the restriction map $\mathrm{O}(L)_h \longrightarrow \mathrm{O}(K)$ is contained in the subgroup of $\mathrm{O}(K)$ that acts as the identity on the discriminant group (cf. [GHS10, Lemma 3.2]).

A.4 Eichler criterion

In this section we report a result, known as the *Eichler criterion*, which allows us to understand when, given two vectors $v, w \in L$, we can find an isometry $\varphi \in \mathrm{O}(L)$ moving the first into the second. As we will see, the Eichler criterion gives a strong description of the shape of φ, provided some hypothesis are satisfied.

If $v \in L$, we define the *divisor* $\operatorname{div}(v)$ to be the unique positive generator of the ideal $q(v, L) \subset \mathbb{Z}$. Identifying v with its image inside $L^\vee$, then $v^* := v/\operatorname{div}(v)$ is a primitive vector in $L^\vee$ and so v^* has order $\operatorname{div}(v)$ in A_L.

Suppose now that there exists an isotropic vector $e \in L$, i.e. $q(e) = 0$, and pick any vector $a \in e^\perp$. The *Eichler transvection* associated to e and a is the isometry $t(e, a)$ defined by

$$t(e,a)(v) = v - q(a,v)e + q(e,v)a - \frac{1}{2}q(a)q(e,v)e.$$

Assume that L contains two vectors e, f such that $q(e) = q(f) = 0$ and $q(e, f) = 1$, i.e. the sublattice $U_{e,f} = \mathbb{Z}e + \mathbb{Z}f$ is a hyperbolic plane as in Example A.1.1. This implies that $L = U_{e,f} \oplus L_1$ and then we define the group $E_{U_{e,f}}(L_1)$ to be the subgroup of $\mathrm{O}(L)$ generated by all transvections $t(e, a)$ and $t(f, a)$ where $a \in L_1$.

Proposition A.4.1 ([GHS09, Proposition 3.3])**.** *Suppose that $L = U_{e,f} \oplus L_1$ and suppose further that $L_1 = U_1 \oplus L_2$, where U_1 is another hyperbolic plane. If $v, w \in L$ are such that $q(v) = q(w)$ and $v^* \equiv w^*$ in A_L, then there exists a transvection $t \in E_{U_{e,f}}(L_1)$ such that $t(v) = w$.*

Remark A.4.2. The fact that L contains two copies of the hyperbolic plane is important here. In fact, it is a known result about hyperbolic planes (cf. [GHS09, Lemma 3.2]) that any vector $u \in U_{e,f} \oplus U_1$ can be moved into U_1 via an isometry in $E_{U_{e,f}}(U_1)$. Once both v and w can be assumed to be in L_1, they are easily moved to one another via Eichler transvections around e and f.

Remark A.4.3. The content of [GHS09, Proposition 3.3] is much richer than what is stated here. They also say, for example, that $\mathrm{O}(L) = \langle E_{U_{e,f}}(L_1), \mathrm{O}(L_1)\rangle$.

A.5 Orientation

Given a lattice L, there is an associated map

$$\text{orient}\colon\ \mathrm{O}(L) \to \{\pm 1\} \tag{A.5.1}$$

called the *orientation character*. Isometries in the kernel of this map are called *orientation preserving* and they will play an important role in the study of irreducible holomorphic symplectic manifolds.

The aim of this section is to define this character in greater generality, following [Mar11, Section 4].

Let L be of signature (a, b) with $a > 0$. First of all, we define the cone of positive classes [1]

$$\widetilde{\mathcal{C}}_L = \{x \in L \otimes \mathbb{R} \mid q(x) > 0\}. \tag{A.5.2}$$

We refer to this cone as the *big positive cone*.

[1] Note that this is *not* the positive cone as usually defined in algebraic geometry.

Lemma A.5.1. *If $W \subset L \otimes \mathbb{R}$ is a positive subspace of maximal dimension, then $\widetilde{\mathcal{C}}_L$ is deformation retract of $W \setminus \{0\}$.*

Proof. Pick a basis $\{v_1, \cdots, v_n\}$ of $L \otimes \mathbb{R}$ such that $\{v_1, \cdots, v_a\}$ is a basis of W. The retraction is then given by $F(\sum_1^n \lambda_i v_i, t) = \sum_1^a \lambda_i v_i + (1-t)\sum_{a+1}^n \lambda_i v_i$. □

If $a > 1$, notice that in this case $H^{r-1}(\widetilde{\mathcal{C}}_L, \mathbb{Z}) = H^{r-1}(W \setminus \{0\}, \mathbb{Z}) = \mathbb{Z}$. Any isometry in $\varphi \in \mathrm{O}(L)$ induces an automorphism of the cone of positive classes and hence it acts on its cohomology groups. The character (A.5.1) is then defined via the action of φ on $H^{r-1}(\widetilde{\mathcal{C}}_L, \mathbb{Z})$.

Example A.5.2. If $v \in L$, the reflection $R_v(w) = w - \frac{2q(v,w)}{q(v)} v$ takes integral values only if $q(v) = \pm 2$ and in this case $R_v \in \mathrm{O}(L)$. If $q(v) = 2$, then W can be taken containing v and the action of R_v on $H^{r-1}(\widetilde{\mathcal{C}}, \mathbb{Z})$ flips the generator. Vice versa, if $q(v) = -2$, then W can be taken orthogonal to v and hence W is invariant, so the action of R_v on $H^{r-1}(\widetilde{\mathcal{C}}, \mathbb{Z})$ is trivial.

Remark A.5.3. Any isometry $\varphi \in \mathrm{O}(L)$ can be seen as an isometry of the vector space $L \otimes \mathbb{R}$ and as such it can be decomposed as a product of reflections $\varphi = R_{v_1} \cdots R_{v_k}$, where $v_i \in L \otimes \mathbb{R}$. The (real) *spinor* norm of φ is the element

$$\left(-\frac{q(v_1)}{2}\right) \cdots \left(-\frac{q(v_k)}{2}\right) \in \mathbb{R}^\times / \left(\mathbb{R}^\times\right)^2,$$

where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. The choice of the sign above is such that the reflection R_v has spinor norm ± 1 according to $q(v) = \mp 1$. In particular, the orientation character (A.5.1) can be identified with the spinor norm (cf. [MM09, Section 10, Section 11]).

For convenience we define the integral isometries

$$\rho_v = \begin{cases} R_v & q(v) = -2 \\ -R_v & q(v) = 2 \end{cases} \tag{A.5.3}$$

which are all orientation preserving by definition. With an abuse of terminology the isometry ρ_v is still called a reflection.

Remark A.5.4. Notice that we could have decided to work with negative definite subspaces and this would lead to a different notion of orientation. This is a choice that we are making and it is suggested by the fact that the geometry of an irreducible holomorphic symplectic manifold naturally fixes a basis for a positive definite subspace W as above (cf. Section 1.3). We keep this convention everywhere except in Section 2.1.2, where the geometry naturally fixes a basis for a negative definite subspace.

Appendix B

O'Grady's 10-dimensional example

We collect and outline here the main ideas behind O'Grady's and Rapagnetta's works on irreducible holomorphic symplectic manifolds which arise as symplectic resolution of singularities of singular moduli spaces of sheaves on $K3$ surfaces (cf. Theorem 1.1.32). We will follow [O'G99] closely, mostly through the online versions [O'G97a] and [O'G98].

B.1 O'Grady's desingularisation

Let S be a $K3$ surface and M_S the moduli space parametrising S-equivalence classes of rank 2 sheaves F with $c_1(F) = 0$ and $c_2(F) = 4$ which are H-semistable with respect to a generic polarisation H. By generic polarisation we mean a polarisation such that if D is a divisor with $D.H = 0$ and $D^2 \geq -4$, then $D \sim 0$. Notice that, by the Hodge Index Theorem, $D^2 < 0$, hence the collections of hyperplanes $D^\perp$, for D as above, is locally finite in the ample cone of S.

Notice that, by [Muk84], the smooth locus of M_S coincides with the open subset M_S^s parametrising stable sheaves. Moreover, M_S^s is endowed with a symplectic structure inherited from S.

In the following, if G is a reductive group acting linearly on a smooth projective variety X, we always denote by X^s (resp. X^{ss}) the open subset containing stable (resp. semistable) points.

Kirwan's partial desingularisation. Let X be a smooth projective variety, G a reductive group acting on X and $Y \subset X$ a G-invariant closed subvariety. Consider the blow-up $\pi\colon \widetilde{X} \to X$ of X along Y. If the action of G on X is linearised by the ample line bundle L, then the induced action on $\widetilde{X}$ is linearised by the ample line bundle $\pi^* L^{\otimes l} \otimes \mathcal{O}_{\widetilde{X}}(-E)$, where E is the exceptional divisor of the blow-up and $l \geq 0$.

Theorem B.1.1 ([Kir85, Section 3])**.** *In the situation above, if $l \gg 0$, then the sets $\widetilde{X}^s$*

and $\widetilde{X}^{ss}$ are independent of the choice of l. In this case we have inclusions

$$\pi(\widetilde{X}^{ss}) \subset X^{ss} \qquad \textit{and} \qquad \pi^{-1}(X^s) \subset \widetilde{X}^s$$

and the induced map $\pi \colon \widetilde{X} /\!\!/ G \to X /\!\!/ G$ is identified with the blow-up of $X /\!\!/ G$ at $Y /\!\!/ G$.

This theorem says informally that, blowing up the right closed subscheme, we can pass from the initial quotient to a quotient with better singularities. In order to study this new situation, we need a detailed picture of the local geometry of $\widetilde{X}$ close to the exceptional divisor E. Recall that the exceptional divisor is identified with the projective normal cone $\mathbb{P}(C_Y X)$, where

$$C_Y X = \underline{\operatorname{Spec}}\left(\oplus_{k\geq 0} I^k/I^{k+1}\right)$$

where I is the ideal sheaf of the closed embedding $Y \subset X$ (cf. [Ful98, Appendix B.6]).

If $x \in X^{ss}$ has closed orbit, Luna's Étale Slice Theorem gives us a slice V, transversal to the orbit $\mathfrak{o}(x)$ and containing x, such that étale locally around x the GIT-quotient $X /\!\!/ G$ behaves like the GIT-quotient $V /\!\!/ \operatorname{Stab}(x)$. Here and in the following $\mathfrak{o}(x)$ and $\operatorname{Stab}(x)$ denote, respectively, the orbit and the stabiliser of $x \in X$. O'Grady noticed that in this situation, if $y \in Y$, there exists a $\operatorname{Stab}(y)$-equivariant isomorphism

$$(C_Y X)_y \cong (C_W V)_y$$

where $W = Y \cap V$ ([O'G97a, Corollary 1.2.2]).

O'Grady's desingularisation. First of all, let us recall how the moduli space M_S is constructed. There exists a Quot scheme Q parametrising quotients

$$\mathcal{O}_S(-k)^N \twoheadrightarrow F$$

where F has rank 2 and Chern classes as above. Here k is sufficiently big so that $F(k)$ is generated by global sections, and N is $\dim H^0(F(k))$. There is a natural linearised action of $\operatorname{PGL}(N)$ on Q and $M_S = Q /\!\!/ \operatorname{PGL}(N)$ ([HL10, Section 4]). We want to blow up Q at a certain closed subset Ω parametrising strictly semistable sheaves and study the normal cone $C_\Omega Q$. As above, the local structure of this cone at a point $x \in Q^{ss}$ can be studied by passing to a normal slice V. The germ of V at x defines a versal family of deformations of the sheaf F corresponding to $x \in Q$ ([O'G97a, Proposition 1.2.3]). Hence the problem is now reduced to understanding the local deformation theory of F. This is the main ingredient for O'Grady's computations.

Let us start by recalling O'Grady's desingularisation argument. First of all, we want to understand the shape of strictly semistable sheaves. Since the polarisation is generic, a direct computation gives the following result.

Proposition B.1.2 ([O'G97a, Lemma 1.1.5]). *$F \in Q^{ss} \setminus Q^s$ if and only if there exist $Z, W \in S^{[2]}$ such that*

$$0 \longrightarrow I_Z \longrightarrow F \longrightarrow I_W \longrightarrow 0. \tag{B.1.1}$$

Remark B.1.3. Notice that the orbit $\mathfrak{o}(F)$ is closed only if (B.1.1) splits.

Let us define

$$\Omega^0_Q = \left\{F \in Q \mid F = I_Z \oplus I_Z, \quad Z \in S^{[2]}\right\} \tag{B.1.2}$$

$$\Sigma^0_Q = \left\{F \in Q \mid F = I_Z \oplus I_W, \quad Z, W \in S^{[2]}\right\}. \tag{B.1.3}$$

and put $\Omega_Q = \overline{\Omega^0_Q} = \Omega^0_Q$ and $\Sigma_Q = \overline{\Sigma^0_Q}$. Notice that the stabilisers

$$\mathrm{Stab}(F) = \begin{cases} \mathrm{PGL}(2) & \text{if } F \in \Omega^0_Q \\ \mathbb{C}^* & \text{if } F \in \Sigma^0_Q \end{cases}$$

are reductive groups.

Remark B.1.4. In $Q^{ss} \setminus Q^s$ we can find loci, other than Ω^0_Q and Σ^0_Q, parametrising strictly semistable sheaves F such that the exact sequence (B.1.1) is not split. These loci are not closed and have either trivial stabilisers or non-reductive stabiliser.

Since we want to blow up invariant closed subschemes, the first step is to blow up Q along Ω_Q.
If $F \in \Omega_Q$, let us write $F = I_Z \otimes V$ for a two-dimensional complex vector space V. Then the automorphism group of F is identified with the group $\mathrm{PGL}(V)$ and the Lie algebra of this group is $W = \mathfrak{sl}(V)$. Let E_Z be the tangent space at $Z \in S^{[2]}$, i.e. $E_Z = \mathrm{Ext}^1(I_Z, I_Z)$. Using Yoneda product and Serre duality, E_Z has a skew-symmetric non-degenerate form q_ω, for any $\omega \in H^0(S, K_S)$. Define

$$\mathrm{Hom}^\omega(W, E_Z) = \{\varphi \in \mathrm{Hom}(W, E_Z) \mid \varphi^* q_\omega \equiv 0\}$$

and notice that $\mathrm{Stab}(F) = \mathrm{PGL}(V)$ acts naturally on $\mathrm{Hom}^\omega(W, E_Z)$ via the adjoint representation.

Proposition B.1.5 ([O'G97a, Propositions 1.5.1]). *Ω^0_Q is smooth and its normal cone $C_{\Omega^0_Q} Q$ is a locally trivial fibration over Ω^0_Q. Moreover, if $F \in \Omega^0_Q$, there exists a $\mathrm{Stab}(F)$-equivariant isomorphism*

$$(C_{\Omega^0_Q} Q)_F \cong \mathrm{Hom}^\omega(W, E_Z).$$

Let $\pi_R \colon R \to Q$ be the blow-up of Q at Ω_Q. The local structure of the exceptional divisor Ω_R is described in Proposition B.1.5. Denote by Σ_R the strict transform of Σ_Q.

The second step is to blow up R along Σ^{ss}_R.

Proposition B.1.6 ([O'G97a, Proposition 1.7.1]). *Σ_R^{ss} is smooth and its normal cone $C_{\Sigma_R^{ss}}R$ is a locally trivial fibration over Σ_R^{ss}, whose fibre is a cone over a smooth quadric in $\mathbb{P}^3$.*

Remark B.1.7. If $F = I_Z \oplus I_W \in \Sigma_Q^0$, the $\mathbb{P}^3$ in Proposition B.1.6 is the projective space

$$\mathbb{P}\left(\mathrm{Ext}^1(I_Z, I_W) \oplus \mathrm{Ext}^1(I_W, I_Z)\right).$$

Let $\pi_T \colon T \to R$ be the blow-up of R at Σ_R^{ss}. Denote by Σ_T the exceptional divisor and by Ω_T the strict transform of Ω_R. The precise description of Σ_T is given by Proposition B.1.6. Using Proposition B.1.5, one can give an exact description of Ω_T in the following way. Let $F = I_Z \otimes V \in \Omega_Q$ and consider the vector space $\mathrm{Hom}^\omega(W, E_Z)$ as before. Define

$$\mathrm{Hom}_r^\omega(W, E_Z) = \{\varphi \in \mathrm{Hom}^\omega(W, E_Z) \mid \mathrm{rk}\,\varphi \leq r\},$$

then ([O'G97a, Equation 1.8.2])

$$\Omega_T = \overline{\bigcup_{F \in \Omega_Q^0} \mathrm{Bl}_{\mathbb{P}\,\mathrm{Hom}_1^\omega(W,E_Z)} \mathbb{P}\,\mathrm{Hom}_2^\omega(W, E_Z)}. \tag{B.1.4}$$

Proposition B.1.8 ([O'G97a, Propositions 1.8.4, 1.8.7, 1.8.10]).

1. *$\Omega_T^{ss} = \Omega_T^s$ and it is smooth.*
2. *$\Sigma_T^{ss} = \Sigma_T^s$ and it is smooth.*
3. *$T^{ss} = T^s$ and it is smooth.*

Let us pass to the quotient and put

$$\Omega = \Omega_Q /\!\!/ \mathrm{PGL}(N), \qquad \Sigma = \Sigma_Q /\!\!/ \mathrm{PGL}(N), \qquad M_S = Q /\!\!/ \mathrm{PGL}(N),$$

$$\widehat{\Omega} = \Omega_T /\!\!/ \mathrm{PGL}(N), \qquad \widehat{\Sigma} = \Sigma_T /\!\!/ \mathrm{PGL}(N), \qquad \widehat{M}_S = T /\!\!/ \mathrm{PGL}(N).$$

By Theorem B.1.1 the map

$$\widehat{\pi} \colon \widehat{M}_S \longrightarrow M_S$$

is the composition of two blow-ups, the first being along Ω and the second along the strict transform of Σ. As a corollary of Proposition B.1.8 we eventually get the smoothness of $\widehat{M}_S$.

Corollary B.1.9 ([O'G97a, Proposition 1.8.3]). *$\widehat{\pi} \colon \widehat{M}_S \to M_S$ is a smooth desingularisation.*

The symplectic form on M_S^s extends to a symplectic form $\omega_{\widehat{M}_S}$ on $\widehat{M}_S$, which is non-degenerate outside $\widehat{\Omega}$ ([O'G97a, Formula 2.4.2]). Hence $\widehat{\pi}$ is not a symplectic desingularisation.

The idea is then to contract $\widehat{M}_S$ along a ray contained in $\widehat{\Omega}$ and hope that the contracted variety is still smooth and projective. The last issue is solved using Mori theory and the smoothness is checked by hand.

Notice that $\Omega \cong S^{[2]}$. Let $F = [I_Z \otimes V] \in \Omega$ and let E_Z be the tangent space of $S^{[2]}$ at $Z \in S^{[2]}$. Let $\widehat{\Omega}_Z$ be the fibre of $\widehat{\pi}$ over F, i.e. $\widehat{\Omega}_Z = \widehat{\pi}^{-1}([I_Z \otimes V])$. By the identification (B.1.4)

$$\widehat{\Omega}_Z = \mathrm{Bl}_{\mathbb{P}\,\mathrm{Hom}_1^{\omega}(W,E_Z)}\,\mathbb{P}\,\mathrm{Hom}_2^{\omega}(W,E_Z) /\!\!/ \mathrm{PGL}(V)$$

which admits a natural map to the symplectic Grassmannian $\mathrm{Gr}^{\omega}(2,E_Z)$. It can be proved (cf. proof of [O'G97a, Proposition 2.0.1]) that this map coincides with the projection of a tautological bundle on $\mathrm{Gr}^{\omega}(2,E_Z)$. More precisely, if $\mathcal{A}_Z$ is the tautological bundle of subspaces on $\mathrm{Gr}^{\omega}(2,E_Z)$, then

$$\widehat{\Omega}_Z \cong \mathbb{P}\left(\mathrm{Sym}^2 \mathcal{A}_Z\right).$$

Running this argument relatively as F varies in Ω, we eventually get the following global description of $\widehat{\Omega}$.

Proposition B.1.10 ([O'G97a, Proposition 2.0.1])**.** *Let $\mathrm{Gr}^{\omega}\left(2,T_{S^{[2]}}\right)$ be the relative symplectic Grassmannian on $S^{[2]}$ and $\mathcal{A}$ its relative tautological bundle.*
Then $\widehat{\Omega} \cong \mathbb{P}\left(\mathrm{Sym}^2 \mathcal{A}\right)$ and the map

$$\widehat{\pi}\colon \widehat{\Omega} \longrightarrow \Omega$$

coincides with the natural projection map.

If $F = [I_Z \otimes V] \in \Omega$ and $i\colon \widehat{\Omega}_Z \to \widehat{M}_S$ is the inclusion, let us define the class

$$\epsilon \in \overline{\mathrm{NE}}(\widehat{M}_S)$$

as the pushforward $i_*\epsilon_Z$ of the class of a line ϵ_Z in a fibre of $\widehat{\Omega}_Z \cong \mathbb{P}\left(\mathrm{Sym}^2 \mathcal{A}_Z\right)$.

Theorem B.1.11 ([O'G97a, Proposition 2.0.2])**.** *ϵ defines a $K_{\widehat{M}_S}$-negative extremal ray in $\overline{\mathrm{NE}}(\widehat{M}_S)$. Moreover, if $\widetilde{M}_S$ is the variety obtained by contracting ϵ, we have that $\widetilde{M}_S$ is a smooth projective symplectic variety and the contraction map $\widehat{M}_S \to \widetilde{M}_S$ is a regular morphism.*

Let us consider the composition

$$\widetilde{\pi}\colon \widetilde{M}_S \longrightarrow M_S$$

and the strict transforms $\widetilde{\Omega}$ and $\widetilde{\Sigma}$ of Ω and Σ respectively.

Corollary B.1.12. *$\widetilde{\pi}\colon \widetilde{M}_S \longrightarrow M_S$ is a symplectic resolution of singularities.*

Remark B.1.13. The restriction map $\widetilde{\Omega} \to \Omega$ is a $\mathbb{P}^1$-bundle by Proposition B.1.10.

Finally, let us describe the restriction map $\widetilde{\Sigma} \to \Sigma$. Since $\widetilde{\Sigma}$ is isomorphic to $\widehat{\Sigma}$ outside $\widetilde{\Omega}$, it is enough to describe $\widehat{\Sigma}$. If $F = [I_Z \oplus I_W] \in \Sigma \setminus \Omega$ (so that $Z \neq W$), we denote by $\widehat{\Sigma}_{Z,W}$ the fibre of $\widehat{\pi}$ over F. Then, by [O'G97a, Proposition 2.3.1], there is a natural isomorphism $\widehat{\Sigma}_{Z,W} \cong \mathbb{P}^1$.

Corollary B.1.14. *The restriction map $\widetilde{\Sigma} \to \Sigma$ is generically a $\mathbb{P}^1$-fibration. Moreover, the intersection of $\widetilde{\Sigma}$ with this line is -2.*

B.2 Relation with anti-self-dual principal bundles and Uhlenbeck compactification

One of the main tools to study smooth and symplectic moduli spaces of sheaves on $K3$ surfaces is the direct relation which exists between the cohomology of the $K3$ surface itself and the cohomology of the moduli space (cf. Section 1.1.1). We would like to use similar tools to study the variety $\widetilde{M}_S$, but it is not clear a priori how to do that.

The answer comes from the mathematical interpretation of SU(2)-gauge theories in physics, mostly thanks to the work of Atiyah and Donaldson. A detailed description and account of this theory is far from the intentions of this discussion, but we need to recall some facts to state some of the main properties of $\widetilde{M}_S$. We refer to [FM94] for the general theory.

Let S be a projective $K3$ surface. Let us denote by N the smooth compact oriented and simply connected 4-manifold underlying S and let $P \to N$ be a fixed SU(2)-principal bundle such that $c_2(P) = 4$.

Remark B.2.1. What we will say holds more generally for SU(2)-principal bundles P with $c_2(P) = c > 0$.

We will deal with the space $\widehat{\mathcal{N}}_\infty(P)$, which parametrises isomorphism classes of irreducible C^∞-connections on P. Recall that a connection is called *irreducible* if its associated holonomy group is SU(2) (and not a proper subgroup).

Remark B.2.2. For fixed c, there exists a unique principal bundle with $c_2 = c$ so we can safely drop the reference to P in the notation.

$\widehat{\mathcal{N}}_\infty$ has the structure of a Hilbert manifold ([FM94, Section III.3.1.3]), and moreover there exists a universal SO(3)-bundle on $\widehat{\mathcal{N}}_\infty \times N$ with first Pontryagin class

$$\widetilde{p}_1 \in H^4(\widehat{\mathcal{N}}_\infty \times N, \mathbb{Z}).$$

The slant product associated to this distinguished class defines a map

$$\mu_i \colon H_{4-i}(N, \mathbb{Z}) \longrightarrow H^i(\widehat{\mathcal{N}}_\infty, \mathbb{Z})$$

by sending α to $-\frac{1}{4}\alpha/\widetilde{p}_1$. We usually refer to this map as the *Donaldson morphism.*

Proposition B.2.3 ([FM94, Theorem III.3.10])**.** *The map μ induces an isomorphism of graded algebras*

$$\mathrm{Sym}^*\left(H_2(N,\mathbb{Q}) \oplus H_0(N,\mathbb{Q})\right) \xrightarrow{\sim} H^*(\widehat{\mathcal{N}}_\infty, \mathbb{Q}).$$

The projectivity of S endows N with a Hodge metric g. Let us consider the open subset

$$\mathcal{N}_\infty(g) \subset \widehat{\mathcal{N}}_\infty$$

consisting of *anti-self-dual* connections, i.e. those connections A such that

$$F_A + \star F_A = 0$$

where $\star$ is the Hodge operator associated to g and F_A denotes the curvature of A.

Proposition B.2.4 ([FM94, Theorem III.2.6, Corollary III.2.14])**.** *There exists an open dense subset of Riemannian metrics such that for any metric g in this set the space $\mathcal{N}_\infty(g)$ is a smooth oriented manifold of (real) dimension* 20.

We will call a metric generic if it belongs to the open subset of the theorem above. From now on any polarisation is assumed to be taken generic. $\mathcal{N}_\infty(g)$ can be compactified inside $\widehat{\mathcal{N}}_\infty$ thanks to the Weak Compactness Theorem for anti-self-dual connections due to Uhlenbeck.

Theorem B.2.5 ([FM94, Theorem III.3.15])**.** *There exists a compact Hausdorff space*

$$\overline{\mathcal{N}}_\infty(g) \subset \mathcal{N}_\infty(g) \coprod N^{(4)}$$

which contains $\mathcal{N}_\infty(g)$ as an open subset.

As always, $N^{(4)}$ stands for the fourth symmetric product of N. We call $\overline{\mathcal{N}}_\infty(g)$ the Donaldson-Uhlenbeck-Yau moduli space of anti-self-dual connections. Notice that this space can be quite singular and in general $\mathcal{N}_\infty(g)$ is not dense inside it.

The map μ does not extend directly to $\overline{\mathcal{N}}_\infty(g)$, but rather it does extend to a thickening $\overline{\mathcal{N}}_\infty(g)_\delta$ of $\overline{\mathcal{N}}_\infty(g)$ ([FM94, Theorem III.6.1])

$$\mu_i \colon H_{4-i}(N,\mathbb{Z}) \longrightarrow H^i(\overline{\mathcal{N}}_\infty(g)_\delta, \mathbb{Z}). \tag{B.2.1}$$

We do not go into the details of the definition of $\overline{\mathcal{N}}_\infty(g)_\delta$, but we just remark that the need to work with this thickening is purely technical: it has the advantage of making it easy to define a canonical fundamental class $\sigma = [\overline{\mathcal{N}}_\infty(g)_\delta] \in H^{20}(\overline{\mathcal{N}}_\infty(g)_\delta, \mathbb{Z})$. Thanks to this class we can then define the *Donaldson polynomial* γ on $H_2(N,\mathbb{Z})$ by

$$\gamma(\alpha) := \int_\sigma \mu_2(\alpha)^{10}. \tag{B.2.2}$$

In the case we are interested in, the Donaldson polynomial satisfies the nice and useful equation ([FM94, Proposition VII.2.17])

$$\gamma(\alpha) = \frac{10!}{5!2^5}\left(\int_N \alpha^2\right)^5 . \tag{B.2.3}$$

We now explain how this is linked to our moduli space M_S of semistable sheaves. Let us consider again S with its complex Kähler structure. Thanks to the work of Donaldson, Kobayashi, Lübke and Yau, to any irreducible anti-self-dual connection on the SU(2)-principal bundle P there corresponds a holomorphic structure turning P into a holomorphic vector bundle E with trivial determinant and $c_2(E) = 4$, which is μ-stable with respect to the generic polarisation. Notice that a generic polarisation H on S, as defined at the beginning of Appendix B, induces a generic Hodge metric on N and vice versa.

Theorem B.2.6 ([FM94, Theorem IV.3.9])**.** *Let M_S^{lf} be the open subset of M_S parametrising locally free sheaves. Then there exists a 1-to-1 correspondence*

$$\mathcal{N}_\infty(g) \longrightarrow M_S^{\mathrm{lf}}$$

which is an isomorphism of real analytic spaces.

The main result we want to recall in this section is the existence of a natural morphism between the respective compactifications of these spaces, which extends the correspondence in the theorem above.

First of all, Li constructs in [Li93, Section 1, Section 2] a determinantal line bundle L on M_S such that for a sufficiently positive integer $m \gg 0$, the power L^m is base-point free.

Let us recall this construction. Let H be the (generic) polarisation on S, k a positive integer and $C \in |kH|$ a smooth divisor. Let $g = g(C)$ be the genus of C and $\theta_C \in J^{g-1}(C)$ be a line bundle on C. Since the moduli space M_S has no quasi-universal family, we first construct the determinant line bundle on the Quot scheme Q_S such that $M_S = Q_S /\!\!/ \mathrm{PGL}(N)$. Let F_Q be the universal quotient sheaf on $Q \times S$, let $F_Q|_C$ be its restriction to $Q \times C$ and let q_Q and q_C be the projections from $Q \times C$. Then

$$\widetilde{L}_k(C, \theta_C) := \left(\det R^\bullet q_{Q*}(F_Q|_C \otimes q_C^*\theta_C)\right)^\vee . \tag{B.2.4}$$

This line bundle descends to M_S by a standard argument (cf. [Li93, Lemma 1.6, Proposition 1.7]). Denote by $L_k(C, \theta_C)$ the descended line bundle.

Remark B.2.7. More generally, if S is a smooth projective surface and F_T is a sheaf on $S \times T$ flat over T, i.e. F_T is a family of sheaves on S parametrised by T, then Li's construction defines a group homomorphism

$$\rho_T \colon \mathrm{NS}(S) \longrightarrow \mathrm{NS}(T)$$

which to any (algebraic equivalence class of a) divisor C on S assigns the determinant line bundle defined as in (B.2.4) (cf. [Li93, Lemma 1.1, Lemma 1.2]).

The main result is the following.

Theorem B.2.8 ([Li93, Theorem 2, Theorem 3]). *$L_k(C, \theta_C) = L_k$ is independent of the choice of both C and θ_C. Moreover, if $k \geq 5$ and $m \gg 0$, then L_k^m is generated by global sections.*

In particular, L_k can be used to defined a regular morphism

$$\phi_{k,m} \colon M_S \longrightarrow \mathbb{P}(H^0(M_S, L_k^m)).$$

From now on we fix $k \geq 5$ and $m \gg 0$ so that we can drop them from the notation.

Theorem B.2.9. *The image of ϕ is homeomorphic to the Donaldson-Uhlenbeck-Yau moduli space $\overline{\mathcal{N}}_\infty(g)$. In particular, $\overline{\mathcal{N}}_\infty(g)$ has a natural structure of reduced projective scheme.*

Moreover, $\phi \colon M_S^{\mathrm{lf}} \to \mathcal{N}_\infty(g)$ coincides with the isomorphism in Theorem B.2.6.

Consider the boundary divisor $B := M_S \backslash M_S^{\mathrm{lf}}$ parametrising non-locally free sheaves. For any $F \in B$ we can define the *singularity cycle* by

$$s(F) = \sum_{p \in S} l_F(p)\, p$$

where $l_F(p)$ is the length at p of the quotient sheaf $R = F^{\vee\vee}/F$. When F is strictly semistable, Proposition B.1.2 says that $s(F) \in S^{(4)}$. When F is stable, [O'G98, Proposition 1.1.1] says that $F^{\vee\vee} = \mathcal{O}_S \oplus \mathcal{O}_S$ and that

$$0 \longrightarrow F \longrightarrow \mathcal{O}_S \oplus \mathcal{O}_S \longrightarrow R \longrightarrow 0 \tag{B.2.5}$$

and so again $s(F) \in S^{(4)}$.

Proposition B.2.10 ([Li93, Theorem 4], [O'G98, Section 1.1]). *For any $F \in B$, we have $\phi(F) = s(F) \in S^{(4)}$.*

As noted in [O'G98, Section 1.1], this implies that (cf. Theorem B.2.5)

$$\overline{\mathcal{N}}_\infty(g) = \mathcal{N}_\infty(g) \coprod S^{(4)}. \tag{B.2.6}$$

Proposition B.2.11 ([O'G98, Lemma 1.9.1]). *There exists an open subset $B_0 \subset B$ such that the restriction $\phi \colon B_0 \to S^{(4)}$ is a $\mathbb{P}^1$-fibration. Moreover, the intersection of a general fibre of ϕ with Σ consists of three points.*

Remark B.2.12. There is a natural stratification of $S^{(4)}$ induced by partitions of 4 and B_0 is the open stratum parametrising pairwise distinct points. If $\underline{x} \in S^{(4)}$ consists of pairwise distinct points, then by the short exact sequence (B.2.5)

$$\phi^{-1}(\underline{x}) = \mathrm{Quot}(\mathcal{O}_S \oplus \mathcal{O}_S, \underline{x}) /\!\!/ \mathrm{PGL}(2) \cong (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) /\!\!/ \mathrm{PGL}(2) \cong \mathbb{P}^1,$$

where PGL(2) acts diagonally on the product of the $\mathbb{P}^1$s.

Remark B.2.13. The divisor B on M_S is not Cartier, but as noted in [Per10] it is 2-Cartier.

B.3 The Beauville-Bogomolov-Fujiki lattice of $\widetilde{M}_S$

By equation B.2.3 the Donaldson morphism (cf. equation B.2.1)

$$\widetilde{\pi}^* \circ \phi^* \circ \mu \colon H_2(S, \mathbb{Z}) \longrightarrow H^2(\widetilde{M}_S, \mathbb{Z})$$

is injective.

We have two natural divisors on $\widetilde{M}_S$, namely the exceptional divisor $\widetilde{\Sigma}$ of the symplectic resolution $\widetilde{\pi} \colon \widetilde{M}_S \to M_S$ and the strict transform $\widetilde{B}$ of the boundary divisor $B = M_S \setminus M_S^{\mathrm{lf}}$, so we can consider the sublattice

$$\Lambda = (\widetilde{\pi}^* \circ \phi^* \circ \mu)\,(H_2(S, \mathbb{Z})) + \mathbb{Z}\widetilde{\Sigma} + \mathbb{Z}\widetilde{B} \subset H^2(\widetilde{M}_S, \mathbb{Z}).$$

Since $H^2(S, \mathbb{Z})$ is unimodular, we can choose bases $\{\alpha_1, \cdots, \alpha_{22}\}$ and $\{\gamma_1, \cdots, \gamma_{22}\}$ of $H^2(S, \mathbb{Z})$ and $H_2(S, \mathbb{Z})$ respectively, such that the matrix $(\alpha_i \cdot \gamma_j)_{ij}$ has determinant 1. Now let δ_Σ be the class of the general fibre of $\widetilde{\Sigma} \to \Sigma$ (cf. Corollary B.1.14) and let δ_B be the class of the strict transform of the general fibre of $B \to S^{(4)}$ (cf. Proposition B.2.11). Notice that both δ_Σ and δ_B are contracted by the composition $\widetilde{\pi}^* \circ \phi^*$ and hence they have trivial intersection with the classes in $(\widetilde{\pi}^* \circ \phi^* \circ \mu)\,(H_2(S, \mathbb{Z}))$. Moreover, the intersection of γ_i with both $\widetilde{\Sigma}$ and $\widetilde{B}$ is also trivial. Finally, by a direct computation,

$$\widetilde{\Sigma} \cdot \delta_\Sigma = -2, \qquad \widetilde{B} \cdot \delta_\Sigma = 1, \qquad \widetilde{\Sigma} \cdot \delta_B = 3, \qquad \widetilde{B} \cdot \delta_B = -2.$$

From this it follows that Λ has rank 24 and that it is a saturated sublattice of $H^2(\widetilde{M}_S, \mathbb{Z})$ (cf. [O'G98, Section 3] and [Rap08, Theorem 2.0.8]). In particular, $b_2(\widetilde{M}_S) \geq 24$.

In fact, Rapagnetta proves in [Rap08, Theorem 1.0.1] that $b_2(\widetilde{M}_S) = 24$ and so we eventually get

Proposition B.3.1. $H^2(\widetilde{M}_S, \mathbb{Z}) = (\widetilde{\pi}^* \circ \phi^* \circ \mu)\,(H_2(S, \mathbb{Z})) + \mathbb{Z}\widetilde{\Sigma} + \mathbb{Z}\widetilde{B}$.

By the Fujiki relation (1.1.2), one can see that the factor $(\widetilde{\pi}^* \circ \phi^* \circ \mu)\,(H_2(S, \mathbb{Z}))$ is orthogonal to both $\widetilde{\Sigma}$ and $\widetilde{B}$. Moreover, the Beauville-Bogomolov-Fujiki form restricted to it is identified with the intersection product on S. So, in order to determine the isometry type of $H^2(\widetilde{M}_S, \mathbb{Z})$, it is enough to compute the lattice generated by $\widetilde{\Sigma}$ and $\widetilde{B}$.

The main tools to do that are again the Fujiki relation and a universal modular property satisfied by the Donaldson morphism. We refer to [Rap08, Section 3] for details.

Theorem B.3.2 ([Rap08, Theorem 3.0.11])**.** *The map*

$$\widetilde{\pi}^* \circ \phi^* \circ \mu \colon H_2(S, \mathbb{Z}) \longrightarrow H^2(\widetilde{M}_S, \mathbb{Z})$$

is an isometric embedding. Moreover,

$$H^2(\widetilde{M}_S, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$$

as abstract lattice (cf. Appendix A).

Remark B.3.3. More precisely, the Gram matrix of the lattice generated by $\widetilde{\Sigma}$ and $\widetilde{B}$ (in this order) is the $G_2(-1)$-lattice

$$\begin{pmatrix} -6 & 3 \\ 3 & -2 \end{pmatrix}$$

(cf. Example A.1.5).

Finally, to conclude this section, we want to recall the relation between $H^2(M_S, \mathbb{Z})$ and $H^2(\widetilde{M}_S, \mathbb{Z})$. First of all, we have the following important remark.

Remark B.3.4 ([Per10])**.** The group of Weil divisors of M_S is generated by $\mathrm{Pic}(S)$ and B. Moreover, B is not Cartier, but $2B$ is.

It turns out that $H^2(M_S, \mathbb{Z})$ has rank 23 and is generated by $H^2(S, \mathbb{Z})$ and $2B$. We summarise in the following proposition the main results in [Per10].

Proposition B.3.5.

1. *The pullback map $\widetilde{\pi}^* \colon H^2(M_S, \mathbb{Z}) \to H^2(\widetilde{M}_S, \mathbb{Z})$ is injective.*
2. *The Hodge and lattice structures on $H^2(\widetilde{M}_S, \mathbb{Z})$ restrict to a pure Hodge structure and a lattice structure on $H^2(M_S, \mathbb{Z})$.*
3. *If $v = (2, 0, -2)$ is the Mukai vector of M_S, then there is an isometry of lattices*

$$v^{\perp} \cong H^2(M_S, \mathbb{Z}).$$

Remark B.3.6. Compare this result with item 3 of Theorem 1.1.25. When v is primitive, the aforementioned theorem gives an isometry between $v^{\perp}$ and the second cohomology of the moduli space. When v is not primitive, but 2-divisible, then Proposition B.3.5 says that $v^{\perp}$ is not anymore isometric to the second cohomology of the moduli space, but instead it is embedded in it as a (non-saturated) sublattice.

B.4 Generalisation to other non-primitive Mukai vectors

Most of the results of the previous sections in this appendix can be generalised to moduli spaces of sheaves for other non-primitive Mukai vectors.

Let S be a projective $K3$ surface, let v be a Mukai vector and suppose that $v = mw$, where w is a primitive Mukai vector and $m \geq 2$. Fix a v-generic polarisation H on S.

Remark B.4.1. Since now v is not primitive, the definition of v-genericity is slightly different from the one we gave in Section 1.1.1. We refer to [PR13, Section 2.1] for a general discussion on v-genericity in this case.

Let $M_H(v)$ be the moduli space of H-semistable sheaves on S with invariants fixed by v.

Theorem B.4.2 ([LS06],[KL07],[KLS06],[PR13])**.** *Put* $(w, w) = 2k$*.*

1. *If* $m > 2$ *or* $m = 2$ *and* $k \geq 2$*, then* $M_H(v)$ *does not admit any symplectic resolution.*
2. *If* $m = 2$ *and* $k = 1$*, then there exists a symplectic resolution*

$$\widetilde{\pi}\colon \widetilde{M}_H(v) \to M_H(v).$$

Moreover, in this case:

   - *(a)* $\widetilde{M}_H(v)$ *is an irreducible holomorphic symplectic manifold of OG10-type;*
   - *(b)* $\widetilde{\pi}^*\colon H^2(M_H(v), \mathbb{Z}) \to H^2(\widetilde{M}_H(v), \mathbb{Z})$ *is injective and* $H^2(M_H(v), \mathbb{Z})$ *inherits a pure Hodge structure and lattice structure from* $H^2(\widetilde{M}_H(v), \mathbb{Z})$*;*
   - *(c) there is an isometry of lattices*

$$v^\perp \cong H^2(M_H(v), \mathbb{Z}).$$

Remark B.4.3. In item 1 of Theorem B.4.2, Kaledin, Lehn and Sorger prove more precisely in [KLS06] that the moduli space $M_H(v)$ has locally factorial singularities.

Remark B.4.4. In item 2 of the Theorem B.4.2, Lehn and Sorger prove more precisely in [LS06] that the symplectic variety $\widetilde{M}_H(v)$ can be obtained with a single blow-up of the reduced singular locus of $M_H(v)$, thus simplifying O'Grady's construction.

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