On the degenerations of (1,7)-polarised abelian surfaces

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To my sister Fawzia

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Summary

La morte non è nel non poter comunicare ma nel non poter più essere compresi.
(Death is not when you cannot communicate, but when you can no longer be understood.)

Pier Paolo Pasolini [1]

The problem

The starting point for this thesis is the paper by Manolache and Schreyer [MS01], where the authors find a birational model of the moduli space of (1,7)-polarised abelian surfaces with canonical level structure, namely a Fano 3-fold of genus 12 called V_{22}. The results in [MS01] are found by studying the locally free resolution of a (1,7)-polarised abelian surface $A \subset \mathbb{P}^6(V)$, where $V \cong H^0(A, (O)(1))^\vee$. As we shall see, $\mathrm{SL}_2(\mathbb{Z}_7)$ acts on V, and therefore on these surfaces and on their resolutions. Every such resolution determines a twisted cubic curve in $\mathbb{P}^3(U)$, where U is a certain 4-dimensional irreducible $\mathrm{SL}_2(\mathbb{Z}_7)$-module. The class of such curves consists of those annihilated by the unique $\mathrm{SL}_2(\mathbb{Z}_7)$-invariant net of quadrics $\Delta \subset S^2U$ and the moduli space $H(\Delta)$ of such curves is isomorphic to V_{22}.

The other fundamental paper on which this thesis is based is [GP01] by Gross and Popescu, where the authors also show that the moduli space of (1,7)-polarised abelian surfaces is birational to V_{22}, but using a different approach and consequently a different model. Their model is the variety of sum of powers $\mathrm{VSP}(Q, 6)$

[1] From www.pasolini.net

that parameterises polar hexagons to the Klein quartic $Q \subset \mathbb{P}^2(W)$, where W is a 3-dimensional irreducible $\mathrm{SL}_2(\mathbb{Z}_7)$-module and Q is the unique invariant curve of minimal degree 4 (see Appendix B). Such a polar hexagon is a configuration of six points in $\mathbb{P}^2(W')$, where W' is the dual of W, and $\mathbb{P}^2(W')$ is naturally contained in $\mathbb{P}^6(V)$ as the projectivisation of the $+1$-eigenspace of a certain involution $\iota : V \to V$. What Gross and Popescu show in [GP01] is that, for a general A, these six points are precisely the odd 2-torsion points of A, and that each point determines a Calabi-Yau 3-fold containing A. More precisely A is defined as the intersection of those six 3-folds. Notice that Q is a model of $X(7)$, the modular curve of level 7, and that in [GP01] the latter notation is used in place of Q. However the curve $Q' \subset \mathbb{P}^2(W')$ (also the unique invariant quartic and also isomorphic to $X(7)$) will also play a role in this thesis and we will be careful to distinguish the two.

Now, for the case of (1,5)-polarised abelian surfaces Horrocks and Mumford [HM73] showed that a birational model of the moduli space is $\mathbb{P}^3$. In a more detailed analysis, Barth, Hulek and Moore [BHM87] showed that in fact every point of $\mathbb{P}^3$ determines a surface and they also described the subvariety of $\mathbb{P}^3$ that parameterises degenerate abelian surfaces, that is the boundary of the moduli space. Furthermore they gave a precise description of all the degenerations. The natural question raised by the work of [MS01] and [GP01] is, can a similar analysis be made for V_{22}?

This thesis takes a step toward answering this question by a detailed study of the subvariety $B \subset H(\Delta)$ which parameterises degenerate twisted cubic curves. We show that all points in B determine surfaces in $\mathbb{P}^6(V)$ and we find the corresponding elements of $\mathrm{VSP}(Q, 6)$. Our original expectation was that B would also be the subvariety which parameterises degenerate abelian surfaces. The first piece of positive evidence that we find for this is the fact that B is birational to the Kummer surface that parameterises translation scrolls. Unfortunately, all the additional evidence is negative and points to the fact that B does not in fact parameterises the translation scrolls and so cannot be the whole boundary. Indeed, we can not in the end determine whether all points of B do actually parameterise degenerate surfaces. However, in this thesis we will still speak of B as the "boundary" in the sense that it parameterises degenerate twisted cubic curves and will consider that the corresponding abelian surfaces are still, in some sense, degenerate. Indeed we find that they are not general in the sense of [GP01].

The structure of this thesis

Our thesis is divided into four chapters. In the first one we introduce the subject of abelian varieties, then the crucial action of the Heisenberg group, which is the key to handling these varieties. Then we talk about compactifications of moduli spaces of $(1, p)$-polarised abelian varieties in general, and specifically about the toroidal compactification. Finally we report all the key results we need from [MS01] and [GP01] that are going to define the starting point of the following chapters. Most of the notations are thereafter consistent with those papers. Among others, especially important are the 4-dimensional -1-eigenspace $U' := V_-$ of the involution ι, which is dual to U. Notice that $S^2U' = L \oplus W'$ where $L = \Delta^\perp$. But also $S^3W' = L \oplus W$ and hence we have an incomplete linear system of cubics on $\mathbb{P}^2(W')$ which gives an embedding $\mathbb{P}^2(W') \hookrightarrow \mathbb{P}^6(L)$.

In the second chapter we view $H(\Delta)$ as an orbit space, specifically the space M_Δ of 3×2 matrices with entries in the 4-dimensional space U' and with the condition Δ on the minors (that is, the minors are in L), and obvious actions of $\mathrm{GL}(3, \mathbb{C})$ and $\mathrm{GL}(2, \mathbb{C})$. Via geometric invariant theory we find the semi-stable points of this space, and in doing so in Section 2.2 we briefly introduce the theory of Kronecker modules, which will provide a short and alternative description of the semi-stable points. We find that this leads to Proposition 2.5, which says that the quotient space $M^s_\Delta / \sim$ and $H(\Delta)$ are isomorphic.

Furthermore we show that the boundary B may be also described as the locus of orbits which contain a representative matrix with at least one zero entry. This leads to Proposition 2.9, which says that the boundary B of $H(\Delta)$ is birational to $C \times \mathbb{P}^1 \cong Q \times \mathbb{P}^1$, where $C \subset \mathbb{P}^3(U')$ is a curve naturally isomorphic to Q.

Note that the modular curve of level 7 has 24 cusps corresponding to the 24 flexes in the models Q and Q'. Throughout this thesis we are going to use the word “cusp” also for the images of the cusps of $X(7)$ in Q, Q' and C.

At the end of the chapter we also report several results from [Sch01], describing in more detail three different models of the Fano 3-fold V_{22}, two of which ($H(\Delta)$ and $\mathrm{VSP}(Q, 6)$) we have introduced already.

We start the third chapter presenting several results from [HM73] that are going to be used for comparisons in the sequel. In this way we can recover more information about the features of surfaces parametrised by boundary points of $H(\Delta)$.

Then we report facts on the toroidal compactification, in particular that there is a relationship between boundary points of this compactification and degenerations of abelian surfaces, as shown in [HKW93]. We also report specific results on the toroidal compactification of (1,7)-polarised abelian surfaces and a comparison with the work by [BHM87] on the $(1, 5)$ case.

The fourth chapter is divided into four sections. In the first one we prove several results, among them Proposition 4.4: that to every point of the boundary B of $H(\Delta)$ it is possible to associate a surface.

The next section, devoted to general degenerations, contains several side results and culminates in Proposition 4.13, which relates degenerate twisted cubic curves and the first degeneration of a set of six points in $\mathrm{VSP}(Q, 6)$: where three of the six points lie on the Hessian of the Klein quartic $\mathrm{Hes}(Q') \subset \mathbb{P}^2(W')$. Furthermore, the images of these three points under the embedding $\mathbb{P}^2(W') \hookrightarrow \mathbb{P}^6(L)$ are collinear and the intersection of the three associated Calabi-Yaus is a 3-fold $\mathcal{U}_a$ of degree 7. Thus the configuration of six points is not general in the sense of [GP01] (see Proposition 1.9.2). The most general surface related to such a degeneration is then $\mathcal{U}_a$ intersected with any Calabi-Yau defined by any of the other three points.

The surfaces in the next section (parameterised by a subvariety $B' \subset B$) are relatively simple to describe, because they are defined by the intersection of two 3-folds $\mathcal{U}_a$ and $\mathcal{U}_e$ of degree 7 as above. There are two sets of three points as above with one point in common, and the degenerate twisted cubic curve associated is the connected union of three lines. This is summarised in Proposition 4.15

The last degeneration of (1,7)-polarised abelian surfaces arises over the cusps of C, and Proposition 4.17 tells that one gets three types of reducible surfaces: 7 quadric surfaces, each contained in some $\mathbb{P}^3 \subset \mathbb{P}^6(V)$, or 7 double planes in $\mathbb{P}^6(V)$, or 14 planes in $\mathbb{P}^6(V)$. We work out the related degenerations of $H(\Delta)$ and $\mathrm{VSP}(Q, 6)$ as well. Notice that in the $(1, 5)$ case there is no configuration analogous to the 14 planes (see Remark 4.20).

All the results proved in this thesis are presented in the next comprehensive main theorem which lists and classifies the surfaces (possibly degenerations of a smooth $(1, 7)$-polarised abelian surfaces) parameterised by $B \subset H(\Delta)$:

Theorem A. *Let* $[\alpha] \in B \subset H(\Delta) \cong \mathrm{VSP}(Q, 6)$*. Then* $[\alpha]$ *determines:*

i. a singular twisted cubic curve in $\mathbb{P}^3(U)$*,*

ii. six points in $\mathbb{P}^2(W')$ *and* *iii. a surface* $A_\alpha \subset \mathbb{P}^6(V)$,

as follows:

1. $[\alpha] \in B \setminus B'$ *and* $[\alpha]$ *is not over a cusp of* C, *then it determines*

    i. *a smooth conic in a general plane of* $\mathbb{P}^3(U)$ *union a line,*

    ii. *six points of* $\mathbb{P}^2(W')$, *three of which lie on* $\mathrm{Hes}(Q')$, *and whose images in* $\mathbb{P}^6(L)$ *are collinear,*

    iii. A_α *is the intersection of a* 3*-fold* $\mathcal{U}_a$ *of degree* 7 *determined by the collinear points, and any Calabi-Yau* 3*-fold, determined by any of the remaining ones.*

2. $[\alpha] \in B'$ *and* $[\alpha]$ *is not over a cusp of* C, *then it determines*

    i. *the connected union of three lines in* $\mathbb{P}^3(U)$,

    ii. *six points of* $\mathbb{P}^2(W')$ *with* $\{p_1, p_2, p_3\}$ *and* $\{p_1, p_4, p_5\}$ *lying on* $\mathrm{Hes}(Q')$, *and whose images in* $\mathbb{P}^6(L)$ *form two sets of three collinear points of* $\mathbb{P}^6(L)$,

    iii. A_α *is the intersection of two* 3*-folds of degree* 7*:* $\mathcal{U}_a$ *determined by the first set of collinear points, and* $\mathcal{U}_e$ *determined by the second set of collinear points,*

3. $[\alpha]$ *is general over a cusp of* C, *then it determines*

    i. *a smooth conic in a special plane (defined by the cusp) of* $\mathbb{P}^3(U)$ *union a line,*

    ii. *a double point and two single points on a line in* $\mathbb{P}^2(W')$, *plus a second double point. Both the double points are in* $Q \cap \mathrm{Hes}(Q')$, *i.e. at cusps,*

    iii. A_α *is the union of* 7 *quadric surfaces, each contained in some* $\mathbb{P}^3 \subset \mathbb{P}^6(V)$,

4. $[\alpha]$ *is special over a cusp of* C *like in Proposition 4.17, part (2a), then it determines*

    i. *a smooth special conic union a line,*

    ii. *a quadruple point plus a double point in* $\mathbb{P}^2(W')$, *both at cusps,*

iii. *A_α is the union of 7 double planes in $\mathbb{P}^6(V)$,*

5. *$[\alpha] \in B'$ is over a cusp of C, then it determines*

   i. *three lines through a point in $\mathbb{P}^3(U)$,*

   ii. *three double points in $\mathbb{P}^2(W')$, all at cusps,*

   iii. *A_α is the union of 14 planes in $\mathbb{P}^6(V)$.*

Proof. See Proposition 1.6, Proposition 1.9, Proposition 4.13, Proposition 4.15 and Proposition 4.17. □

We also illustrate the results of the main theorem in Table 1

A partial conclusion

The research in this thesis has been conducted because $B \subset H(\Delta)$ promised to be a good candidate for the boundary (in $H(\Delta)$) of the moduli space of (1,7)-polarised abelian surfaces. This is because B is birational to the appropriate Kummer surface (whose base curve is indeed $X(7)$) that makes up the central boundary component of the toroidal compactification and parameterises translation scrolls. On the other hand each of these scrolls should be contained in the secant scroll over the elliptic curve on which it is defined, see Proposition 1.8. Therefore at least one point among the six defining the element of $\mathrm{VSP}(Q,6)$ related to such a scroll should lie on Q', but a fact like this does not appear to be true. Furthermore in the Kummer surface there is a trisection that parameterises elliptic scrolls, and in B there is the trisection B' that could correspond to it. But if $p \geq 5$ each elliptic scroll is defined over three elliptic curves (see [CH98]), whereas the general point of B' is only contained in two fibres.

In the spirit of our parallel strategy, also the model $\mathrm{VSP}(Q,6)$ seems to provide an argument against B as the subvariety of $H(\Delta)$ parameterising translation scrolls. More precisely we think it is possible to achieve this result using Proposition 5.3 in [GP01], since it carries information on the configurations of six points in $\mathbb{P}^2(W')$. Moreover it could also help to understand the Kummer surface of translation scrolls in $\mathrm{VSP}(Q,6)$ and indeed $H(\Delta)$. Unfortunately we did not have enough time to complete the argument, but it appears to be a promising line of research.

Table 1: Hierarchy of the degenerations

1.

conic in a general plane of $\mathbb{P}^3(U)$ + line

six points of $\mathbb{P}^2(W')$,
three on $\mathrm{Hes}(Q')$,
collinear images in $\mathbb{P}^6(L)$

intersection 3-fold $\mathcal{U}_a$ of degree 7 and Calabi-Yau 3-fold

$\downarrow$ $\downarrow$

2.

connected union
of three lines in $\mathbb{P}^3(U)$

$\{p_1, p_2, p_3\} + \{p_1, p_4, p_5\}$
lying on $\mathrm{Hes}(Q')$,
collinear images in $\mathbb{P}^6(L)$

intersection of two 3-folds of degree 7: $\mathcal{U}_a$, $\mathcal{U}_e$

3.

conic in special plane
of $\mathbb{P}^3(U)$ + a line

double point and
two single points
on line in $\mathbb{P}^2(W')$,
+ double point

union of 7 quadric surfaces

$\downarrow$ $\swarrow$ $\downarrow$

5.

three lines
through a point in $\mathbb{P}^3(U)$

three double points in $\mathbb{P}^2(W')$

14 planes in $\mathbb{P}^6(V)$

4.

special conic union a line

quadruple point +
double point in $\mathbb{P}^2(W')$

7 double planes in $\mathbb{P}^6(V)$

Chapter 1

Moduli spaces

In this chapter we introduce the topic of moduli spaces of abelian varieties. In particular we are interested in the case of abelian surfaces with canonical level structure of type (1,7). So we include an overview of the known results about this case, due to Manolache and Schreyer in [MS01].

1.1 Moduli spaces of abelian varieties

In this section we follow the survey article by Hulek and Sankaran [HS02].

An *abelian variety* (over the complex numbers $\mathbb{C}$) is a g-dimensional torus $A = \mathbb{C}^g/L$, where $L \subset \mathbb{C}^g$ is a maximal lattice and A is a projective variety, i.e. can be embedded into some projective space $\mathbb{P}^n$. This is the case if and only if A admits a *polarisation*. Here below we give two definitions of polarisation. The most common one involves *Riemann forms*. A Riemann form on $\mathbb{C}^g$ with respect to the lattice L is a positive definite hermitian form H on $\mathbb{C}^g$ whose imaginary part $H' = \operatorname{Im}(H)$ is integer-valued on L, i.e. defines an alternating bilinear form

$$H' : L \otimes L \to \mathbb{Z}.$$

The $\mathbb{R}$-linear extension of H' to $\mathbb{C}^g$ satisfies $H'(x, y) = H'(ix, iy)$ and determines H by the relation

$$H(x, y) = H'(ix, y) + iH'(x, y).$$

H is positive definite if and only if H' is non-degenerate. In this case H (or equivalently H') is called a polarisation. By the elementary divisor theorem there exists then a basis of L with respect to which H' is given by the form

$$\Lambda = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, E = \operatorname{diag}(e_1, \dots, e_g),$$

where the $e_1, \dots, e_g$ are positive integers such that $e_1 | e_2 | \dots | e_g$. The g-tuple $(e_1, \dots, e_g)$ is uniquely determined by H and is called the *type of the polarisation.* A *polarised abelian variety* is a pair (A, H) consisting of a torus A and a polarisation H.

If we choose a basis of the lattice L and write each basis vector of L in terms of the standard basis of $\mathbb{C}^g$ we obtain a matrix $\Omega \in M(2g \times g, \mathbb{C})$ called a *period matrix* of A. The fact that H is hermitian and positive definite is equivalent to

$${}^t\Omega\Lambda^{-1}\Omega = 0, \text{ and } \operatorname{Im} {}^t\Omega\Lambda^{-1}\bar{\Omega} > 0.$$

These are the *Riemann bilinear relations.* We consider vectors of $\mathbb{C}^g$ as row vectors. Using the action of $\mathrm{GL}(g, \mathbb{C})$ on row vectors by right multiplication we can transform the last g vectors of the chosen basis of L to be $(e_1, 0 \dots, 0)$, $(0, e_2, \dots, 0)$,…,$(0, \dots, 0, e_g)$. Then Ω takes on the form

$$\Omega = \Omega_\tau = \begin{pmatrix} \tau \\ E \end{pmatrix}$$

and the Riemann bilinear relations translate into

$$\tau = {}^t\tau, \ \operatorname{Im} \tau > 0.$$

Therefore τ is an element of the *Siegel space of degree g*

$$\mathbb{H}_g = \{\tau \in M(g \times g, \mathbb{C}); \ \tau = {}^t\tau, \ \operatorname{Im} \tau > 0\}.$$

Conversely, given an element $\tau \in \mathbb{H}_g$ we can associate to it the period matrix Ω_τ and the lattice $L = L_\tau$ spanned by the rows of Ω_τ. The complex torus $A = \mathbb{C}^g / L_\tau$ carries a Riemann form given by

$$H(x, y) = x \operatorname{Im}(\tau)^{-1}\, {}^t\bar{y}.$$

This defines a polarisation of type $(e_1, \dots, e_g)$. Hence for every given type of polarisation we have a surjection

$$\mathbb{H}_g \to \{(A, H) |\ (A, H) \text{ an } (e_1, \dots, e_g)\text{-polarised abelian variety}\}/\text{isomorphism}.$$

To describe this set of isomorphic classes we have to see what happens when we change the basis of L. Consider the *symplectic group*

$$\mathrm{Sp}(\Lambda,\mathbb{Z}) = \{g \in \mathrm{GL}(2g,\mathbb{Z});\ g\Lambda\,{}^t g = \Lambda\}.$$

We write elements $g \in \mathrm{Sp}(\Lambda,\mathbb{Z})$ in the form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix};\ A, B, C, D \in M(g \times g, \mathbb{Z}).$$

It is useful to work with the "right projective space P of $\mathrm{GL}(g,\mathbb{C})$", i.e. the set of all $(2g \times g)$-matrices of rank g divided out by the equivalence relation

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \sim \begin{pmatrix} M_1 M \\ M_2 M \end{pmatrix} \text{ for any } M \in \mathrm{GL}(g,\mathbb{C}).$$

Notice that P is isomorphic to the Grassmannian $G = \mathrm{Gr}(g, \mathbb{C}^{2g})$. The group $\mathrm{Sp}(\Lambda,\mathbb{Z})$ acts on P by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} AM_1 + BM_2 \\ CM_1 + DM_2 \end{bmatrix}$$

where $[\,]$ denotes equivalence classes in P. One can embed $\mathbb{H}_g$ into P by $\tau \mapsto \begin{bmatrix} \tau \\ E \end{bmatrix}$
Then the action of $\mathrm{Sp}(\Lambda,\mathbb{Z})$ restricts to an action on the image of $\mathbb{H}_g$ and is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} \tau \\ E \end{bmatrix} = \begin{bmatrix} A\tau + BE \\ C\tau + DE \end{bmatrix} = \begin{bmatrix} (A\tau + BE)(C\tau + DE)^{-1}E \\ E \end{bmatrix}.$$

Therefore $\mathrm{Sp}(\Lambda,\mathbb{Z})$ acts on $\mathbb{H}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \mapsto (A\tau + BE)(C\tau + DE)^{-1}E.$$

From what we have seen here above we conclude that for a given type $(e_1, \ldots, e_g)$ of a polarisation the quotient

$$\mathcal{A}_{(e_1,\ldots,e_g)} = \mathrm{Sp}(\Lambda,\mathbb{Z})\backslash\mathbb{H}_g$$

parametrises all the isomorphism classes of $(e_1, \ldots, e_g)$-polarised abelian varieties; that is $\mathcal{A}_{(e_1,\ldots,e_g)}$ is the coarse moduli space of $(e_1, \ldots, e_g)$-polarised abelian varieties. As a variety $\mathcal{A}_{(e_1,\ldots,e_g)}$ is an orbifold.

One is often interested in polarised abelian varieties with extra structures, the so-called *level structure*. If L is a lattice equipped with a non-degenerate form Λ the *dual lattice* $L^\vee$ of L is defined by

$$L^\vee = \{y \in L \otimes \mathbb{Q};\ \Lambda(x,y) \in \mathbb{Z} \text{ for all } x \in L\}.$$

Then $L^\vee/L$ is non-canonically isomorphic to $(\mathbb{Z}_{e_1} \times \cdots \times \mathbb{Z}_{e_g})^2$. The group $L^\vee/L$ carries a skew form induced by Λ and the group $(\mathbb{Z}_{e_1} \times \cdots \times \mathbb{Z}_{e_g})^2$ has a $\mathbb{Q}/\mathbb{Z}$-valued skew form which with respect to the canonical generators is given by

$$\begin{pmatrix} 0 & E^{-1} \\ -E^{-1} & 0 \end{pmatrix}.$$

If (A,H) is a polarised abelian variety, then a *canonical level structure* on (A,H) is a symplectic isomorphism

$$\alpha : L^\vee/L \to (\mathbb{Z}_{e_1} \times \cdots \times \mathbb{Z}_{e_g})^2$$

where the two groups are equipped with the forms described above. Given Λ we can define the group

$$\mathrm{Sp}^{lev}(\Lambda,\mathbb{Z}) := \{g \in \mathrm{Sp}(\Lambda,\mathbb{Z});\ g|_{L^\vee/L} = \mathrm{id}|_{L^\vee/L}\}.$$

The quotient space

$$\mathcal{A}^{lev}_{(e_1,\dots,e_g)} := \mathrm{Sp}^{lev}(\Lambda,\mathbb{Z})\backslash\mathbb{H}_g$$

has the interpretation

$$\begin{aligned}\mathcal{A}^{lev}_{(e_1,\dots,e_g)} = \{(A,H,\alpha);\ (A,H) \text{ is an } (e_1,\dots,e_g)\text{-polarised abelian variety,}\\ \alpha \text{ is a canonical level structure}\}/\text{isomorphism.}\end{aligned}$$

A torus $A = \mathbb{C}^g/L$ is projective if and only if there exists an ample line bundle $\mathcal{L}$ on it. By the Lefschetz theorem the first Chern class defines an isomorphism

$$c_1 : \mathrm{NS}(A) \cong H^2(A,\mathbb{Z}) \cap H^{1,1}(A,\mathbb{C}).$$

The natural identification $H_1(A,\mathbb{Z}) \cong L$ induces isomorphisms

$$H^2(A,\mathbb{Z}) \cong \mathrm{Hom}(\wedge^2 H_1(A,\mathbb{Z}),\mathbb{Z}) \cong \mathrm{Hom}(\wedge^2 L,\mathbb{Z}).$$

Hence given a line bundle $\mathcal{L}$ the first Chern class $c_1(\mathcal{L})$ can be interpreted as a skew form on the lattice L. Let $H' := -c_1(\mathcal{L}) \in \mathrm{Hom}(\wedge^2 L,\mathbb{Z})$. Since $c_1(\mathcal{L})$ is a

$(1,1)$-form it follows that $H'(x,y) = H'(ix,iy)$ and hence the associated form H is Hermitian. The ampleness of $\mathcal{L}$ is equivalent to positive definiteness of H. In this way an ample line bundle defines, via its first Chern class, a Hermitian form H. Reversing this process one can also associate to a Riemann form an element in $H^2(A,\mathbb{Z})$ which is the first Chern class of an ample line bundle $\mathcal{L}$. The line bundle $\mathcal{L}$ itself is only defined up to translation. One can also view level structures from this point of view. Consider an ample line bundle $\mathcal{L}$ representing a polarisation H. This defines a map

$$\begin{aligned} \lambda: \quad A &\to \hat{A} = \operatorname{Pic}^0 A \\ x &\mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1} \end{aligned}$$

where t_x is translation by x. The map λ depends only on the polarisation, not on the choice of the line bundle $\mathcal{L}$. If we write $A = \mathbb{C}^g/L$ then we have $K(\mathcal{L}) := \ker\lambda \cong L^\vee/L$ and this defines a skew form on $K(\mathcal{L})$, the *Weil pairing*. This also shows that $K(\mathcal{L})$ and the group $(\mathbb{Z}_{e_1}\times\cdots\times\mathbb{Z}_{e_g})^2$ are (non-canonically) isomorphic. We have already equipped the latter group with a skew form. From this point of view a canonical level structure is nothing but a symplectic isomorphism

$$\alpha : K(\mathcal{L}) \cong (\mathbb{Z}_{e_1} \times \cdots \times \mathbb{Z}_{e_g})^2.$$

1.2 The Heisenberg group

In this section we follow [GP98] and [GP01].

Let $x,\ y \in K(\mathcal{L})$, then the Weil pairing induced by H' is given by

$$e^{\mathcal{L}}(x,y) = \exp(2\pi i H'(x,y)).$$

By definition we have that if $x \in K(\mathcal{L})$, then there is an isomorphism $t_x^*\mathcal{L} \cong \mathcal{L}$. Therefore x induces a projective automorphism on $\mathbb{P}(H^0(\mathcal{L}))$ and so this leads to a representation $K(\mathcal{L}) \to \operatorname{PGL}(H^0(\mathcal{L}))$. This representation does not lift to a linear representation of $K(\mathcal{L})$, but it does after taking a central extension of $K(\mathcal{L})$,

$$1 \to \mathbb{C}^* \to \mathcal{G}(\mathcal{L}) \to K(\mathcal{L}) \to 0,$$

whose Schur commutator map is the previously defined pairing $e^{\mathcal{L}}$. $\mathcal{G}(\mathcal{L})$ is called the *theta group* of $\mathcal{L}$. In practice a finite version of $\mathcal{G}(\mathcal{L})$ is used: since $\mathcal{K}(\mathcal{L})$ is finite, for our purposes it is enough to work with the finite group $\mu_{\mathcal{L}}$ (in place of $\mathbb{C}^*$) generated by the image of $e^{\mathcal{L}}$. Notice that $|\mu_{\mathcal{L}}|$ is a divisor of the exponent of $K(\mathcal{L})$.

$\mathcal{G}(\mathcal{L})$ is isomorphic to the (infinite) Heisenberg group $\mathcal{H}(D) := \mathcal{H}(e_1, \dots, e_g)$, which can be described as follows: as a set it is $\mathbb{C}^* \times K(D)$, where $K(D) \cong (\mathbb{Z}_{e_1} \times \cdots \times \mathbb{Z}_{e_g})^2$. Let $f_1, \dots, f_{2g}$ be the standard basis of $K(D)$, and define an alternating multiplicative form $e^D : K(D) \times K(D) \to \mathbb{C}^*$ by

$$e^D(f_\nu, f_\mu) := \begin{cases} \exp(-2\pi i/e_\nu) & \text{if } \mu = g + \nu \\ \exp(2\pi i/e_\nu) & \text{if } \nu = g + \mu \\ 1 & \text{otherwise.} \end{cases}$$

To define the group structure on $\mathcal{H}(D)$, we take for any $(\alpha, x_1, x_2), (\beta, y_1, y_2) \in \mathcal{H}(D)$

$$(\alpha, x_1, x_2)(\beta, y_1, y_2) := (\alpha\beta e^D(x_1, y_2), x_1 + y_1, x_2 + y_2).$$

A *theta structure* for $\mathcal{L}$ is an isomorphism between $\mathcal{G}(\mathcal{L})$ and $\mathcal{H}(D)$ which restricts to the identity on $\mathbb{C}^*$. Any such isomorphism preserves the alternating pairings e^D and $e^{\mathcal{L}}$ and induces a *canonical level structure* on $(A, \mathcal{L})$ (or (A, H)). The natural representation $\mathcal{G}(\mathcal{L}) \to \mathrm{GL}(H^0(\mathcal{L}))$ for the theta group lifts uniquely the representation $K(\mathcal{L}) \to \mathrm{PGL}(H^0(\mathcal{L}))$. Then, if a theta structure has been chosen, we have that the last representation is isomorphic to the Schrödinger representation of $\mathcal{H}(D)$, which we are going to introduce. If $V = \mathbb{C}(\mathbb{Z}_{e_1} \times \cdots \times \mathbb{Z}_{e_g})$ is the vector space of complex-valued functions on the set $\mathbb{Z}_{e_1} \times \cdots \times \mathbb{Z}_{e_g}$, the Schrödinger representation $\rho : \mathcal{H}(D) \to \mathrm{GL}(V)$ is given by

$$\rho(\alpha, x_1, x_2)(\gamma) = \alpha e^D(\,\cdot\,, x_2)\gamma(\,\cdot\, + x_1).$$

This representation is irreducible, and since the centre $\mathbb{C}^*$ acts by scalar multiplication, one gets a projective representation of $K(D)$.

We restrict our attention to surfaces, then the Schrödinger representation on projective space is as follows. Let $D = (e_1, e_2)$ and fix a basis $\{\delta_\gamma | \gamma \in \mathbb{Z}_{e_1} \times \mathbb{Z}_{e_2}\}$ of V, where δ_γ is the delta function

$$\delta_\gamma(\gamma') = \begin{cases} 0 & \text{if } \gamma \neq \gamma' \\ 1 & \text{if } \gamma = \gamma'. \end{cases}$$

We denote by H_D the subgroup of $\mathcal{H}(D)$ generated by $\sigma_1 = (1, 1, 0, 0, 0)$, $\sigma_2 = (1, 0, 1, 0, 0)$, $\tau_1 = (1, 0, 0, 1, 0)$ and $\tau_2 = (1, 0, 0, 0, 1)$, and these act on V via

$$\sigma_1(\delta_{i,j}) = \delta_{i-1,j}, \quad \sigma_2(\delta_{i,j}) = \delta_{i,j-1},$$

$$\tau_1(\delta_{i,j}) = \xi_1^{-i}\delta_{i,j}, \quad \tau_2(\delta_{i,j}) = \xi_2^{-j}\delta_{i,j}.$$

where $\xi_k := \exp(2\pi i/e_k)$. In the case $e_1 = 1$, both σ_1 and τ_1 are just the identity, and we shall denote by σ and τ the generators σ_2 and τ_2, and leave off the first index on the variables.

Let $K(\mathcal{L}) = K_1(\mathcal{L}) \oplus K_2(\mathcal{L})$ be a decomposition for $\mathcal{L}$ with $K_1(\mathcal{L}) \cong K_2(\mathcal{L}) \cong \mathbb{Z}_{e_1} \times \mathbb{Z}_{e_2}$, both subgroups being isotropic with respect to the Weil pairing. A basis of canonical theta functions (see [LB92]) $\{\theta_\gamma | \gamma \in K_1(\mathcal{L})\}$ for $H^0(\mathcal{L})$ yields an identification of $H^0(\mathcal{L})$ and V via $\theta_\gamma \mapsto \delta_\gamma$, such that the representations $\mathcal{G}(\mathcal{L}) \to \mathrm{GL}(H^0(\mathcal{L}))$ and $\mathcal{H}(D) \to \mathrm{GL}(V)$ coincide. Thus if we map A into $\mathbb{P}(H^0(\mathcal{L})^\vee)$ using as coordinates $x_\gamma = \theta_\gamma$, $\gamma \in \mathbb{Z}_{e_1} \times \mathbb{Z}_{e_2}$, the image of A will be invariant under the action of the Heisenberg group via the Schrödinger representation. In particular, if A is embedded this way in $\mathbb{P}(H^0(\mathcal{L})^\vee)$, then $H^0(\mathcal{I}_A(n))$ is also a representation of the Heisenberg group.

In general one works with $e_1 = 1$, then the action of the Heisenberg group $H_e := H_{1,e}$ on the coordinates of $\mathbb{P}(H^0(\mathcal{L})^\vee)$ is

$$\sigma(x_i) = x_{i-1},$$

$$\tau(x_i) = \xi^{-i}(x_i).$$

As mentioned before, we will only consider the action of H_e, the finite subgroup of $\mathcal{H}(D) \to \mathrm{GL}(V)$ generated in the Schrödinger representation by σ and τ. Notice that $[\sigma, \tau] = \xi$; thus H_e is a central extension

$$1 \to \mu_e \to H_e \to \mathbb{Z}_e \times \mathbb{Z}_e \to 0.$$

$K(\mathcal{L})$ can be viewed as a subgroup of the automorphism group of A via translations, and we get that the order 2 subgroup $\langle(-1_A)\rangle$ acts on $K(\mathcal{L})$ by inner automorphisms. From this fact it is possible to define $K^e(\mathcal{L})$ as $K(\mathcal{L}) \rtimes \langle(-1_A)\rangle$, and then define the extended theta group $\mathcal{G}^e(\mathcal{L})$ to be a central extension of $K^e(\mathcal{L})$ by $\mathbb{C}^*$. In fact $\mathcal{G}^e(\mathcal{L}) = \mathcal{G}(\mathcal{L}) \rtimes \langle(-1_A)\rangle$. The extended Heisenberg group is defined by

$$\mathcal{H}^e(D) := \mathcal{H}(D) \rtimes \langle\iota\rangle,$$

where ι acts on $\mathcal{H}(D)$ via $\iota(\alpha, x_1, x_2) = (\alpha, -x_1, -x_2)$. An extended theta structure is an isomorphism between $\mathcal{G}^e(\mathcal{L})$ and $\mathcal{H}^e(D)$ inducing the identity on $\mathbb{C}^*$. Each extended theta structure restricts to a theta structure, but a theta structure does not always come from an extended theta structure. In fact, a theta structure

$b : \mathcal{G}(\mathcal{L}) \to \mathcal{H}(D)$ can be extended to an extended theta structure if and only if it is a symmetric theta structure, that is if the diagram

$$\begin{array}{ccc} \mathcal{G}(\mathcal{L}) & \xrightarrow{(-1)_{\mathcal{L}}} & \mathcal{G}(\mathcal{L}) \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b} \\ \mathcal{H}(D) & \xrightarrow{\iota} & \mathcal{H}(D) \end{array}$$

commutes. In order for a symmetric theta structure to exist, $\mathcal{L}$ must be a symmetric line bundle, that is $(-1_A)^*\mathcal{L} \cong \mathcal{L}$. There always exist a finite number of symmetric line bundles, each admitting a finite number of symmetric theta structures.

The Schrödinger representation ρ of $\mathcal{H}(D)$ extends to a representation ρ^e of $\mathcal{H}^e(D)$, with $\rho^e(\iota) \in \mathrm{SL}^{\pm}(V) = \{M \in \mathrm{GL}(V) | \det(M) = \pm 1\}$. We denote by G_D (with $G_{1,e} := G_e$) the subgroup of $\mathcal{H}^e(D)$ generated by H_D and ι. In the case that $D = (1, e)$, ι acts on V by $\iota(x_i) = -x_{-i}$. Note that our ι here is $-\iota$ in [GP98] and [GP01]. In fact

$$G_D := H_D \rtimes \langle \iota \rangle.$$

ι, acting as an involution on V, has two eigenspaces $V_{\pm}$, with eigenvalues ± 1. We will refer to the projectivization of the positive eigenspace as $\mathbb{P}_+ \subseteq \mathbb{P}(V)$, and the negative eigenspace as $\mathbb{P}_- \subseteq \mathbb{P}(V)$.

In particular, if $D = (1, e)$, then $\mathbb{P}_-$ is given by the equation

$$\{x_i = x_{-i} | \; i \in \mathbb{Z}/e\mathbb{Z}\},$$

and $\mathbb{P}_+$ is given by the equation

$$\{x_i = -x_{-i} | \; i \in \mathbb{Z}/e\mathbb{Z}\}.$$

In our case, namely $(e_1, e_2) = (1, 7)$, let $N(H_7)$ be the normaliser of the Heisenberg group H_7 inside $\mathrm{SL}(V)$, where the inclusion $H_7 \hookrightarrow \mathrm{SL}(V)$ is via the Schrödinger representation. We have a sequence of inclusions

$$Z(H_7) = \mu_7 \subseteq H_7 \subseteq N(H_7),$$

and it is easy to see that $N(H_7)/H_7 = \mathrm{SL}_2(\mathbb{Z}_7)$, in fact $N(H_7)$ is a semi-direct product $H_7 \rtimes \mathrm{SL}_2(\mathbb{Z}_7)$. Therefore the Schrödinger representation of H_7 induces a 7-dimensional representation

$$\rho_7 : \mathrm{SL}_2(\mathbb{Z}_7) \to \mathrm{SL}(V).$$

$\mathrm{PSL}_2(\mathbb{Z}_7)$ can be defined by the following generators and relations:

$$\mathrm{PSL}_2(\mathbb{Z}_7) = \langle S, T| \quad S^7 = 1, (ST)^3 = T^2 = 1, (S^2TS^4T)^3 = 1\rangle$$

where S and T are the projective classes of, respectively,

$$S' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

The representation ρ_7 is given projectively by

$$\rho_7(S') = (\xi^{-\frac{ij}{2}}\delta_{ij})_{i,j\in\mathbb{Z}_7}, \quad \rho_7(T') = \frac{1}{\sqrt{-7}}(\xi^{ij})_{i,j\in\mathbb{Z}_7},$$

where ξ is a fixed primitive 7-th root of unity.

The centre of $\mathrm{SL}_2(\mathbb{Z}_7)$ is generated by T'^2, and $\rho_7(T'^2) = \iota$. Thus the representation ρ_7 is reducible. If we introduce V_+ and V_- as the positive and negative eigenspaces respectively of the Heisenberg involution ι acting on V, then V_+ and V_- are both invariant under ρ_7 , and ρ_7 splits as $\rho_+ \oplus \rho_-$, where $\rho_\pm$ is the representation of $\mathrm{SL}_2(\mathbb{Z}_7)$ acting on $V_\pm$. Note that ρ_+ is trivial on the centre of $\mathrm{SL}_2(\mathbb{Z}_7)$, so in fact it descends to give an irreducible representation

$$\rho_+ : \mathrm{PSL}_2(\mathbb{Z}_7) \to \mathrm{GL}(V_+).$$

In the beautiful treatise [Kle79] by Felix Klein, the polynomial invariants of this representation are computed and the quartic[1]

$$f'_{\mathrm{Klein}} = y_1^3y_2 + y_2^3y_3 + y_3^3y_1$$

is the unique invariant of minimal degree 4. The smooth quartic curve defined by this invariant

$$Q' = \{f'_{\mathrm{Klein}} = 0\} \subset \mathbb{P}^2_+$$

is an isomorphic image of the modular curve of level 7, and has $\mathrm{PSL}_2(\mathbb{Z}_7)$ as its full automorphism group.

Throughout these pages we are going to use the following notation: $W' := V_+$ and $U' := V_-$, and we are going to work with their dual spaces as well, that as $\mathrm{SL}_2(\mathbb{Z}_7)$-modules are not isomorphic. More information about the representations of $\mathrm{SL}_2(\mathbb{Z}_7)$ and these spaces can be found in A.3.

[1]The notation f'_{Klein} is chosen because in Appendix B we work with a quartic given by the same equation, but embedded in the dual space $\mathbb{P}^{2*}_+$, non isomorphic to $\mathbb{P}_+$ as a $\mathrm{SL}_2(\mathbb{Z}_7)$-module.

1.3 Compactification of moduli spaces of abelian surfaces

It is a fact that there is not a unique compactification of a quotient

$$\mathcal{A}^{lev}_{(e_1,\ldots,e_g)} := \mathrm{Sp}^{lev}(\Lambda, \mathbb{Z})\backslash\mathbb{H}_g.$$

Since we are interested in surfaces, we restrict our work to the case

$$(e_1, \ldots, e_g) = (1, p).$$

In this section we are going to report the main facts one gets from the most advanced compactification on the market, that is the toroidal compactification. Practically all these results can be found in [HKW93].

Then, in the sequel we are going to draw several comparisons between our work and the facts of this section.

We think this is not the place to introduce the subject of toric geometry. Thus we just say that via toric methods, and after making choices, often natural, which have to fulfill certain condition of admissibility for the fans used, one gets the toroidal compactification $\mathcal{A}^*_{(e_1,\ldots,e_g)}$ of $\mathcal{A}^{lev}_{(e_1,\ldots,e_g)}$.

We list the main features of the geometry of $\mathcal{A}^*_{(1,p)}$, and in Chapter 3 we will describe the degenerations parametrised by the different kind of boundary points.

A combinatorial object called the *Tits building* enumerates the various boundary components, as well as containing important information about their intersection behaviour. From [HKW93], Theorem I.3.40 and Definition I.3.105, we get that there are

1. a corank 1 (open) central boundary component $D^\circ(l_0)$.

2. $(p^2 - 1)/2$ corank 1 (open) peripheral boundary components $D^\circ(l)$.

3. $p + 1$ corank 2 boundary components.

Then, from [HKW93] Theorem I.3.151 and I.4.8 we get

1. Let $D(l)$ be the closure of $D^\circ(l)$ in $\mathcal{A}^{lev}_{(e_1,\ldots,e_g)}$, then there is an isomorphic map from a Kummer modular surface $K(1)$:

$$f : K(1) \to D(l).$$

2. There is a map from the Kummer modular surface $K(p)$ to the closure $D(l_0)$ of $D^\circ(l_0)$ in $\mathcal{A}^{lev}_{(e_1,\ldots,e_g)}$, which is an isomorphism around each singular fibre, but for $p > 3$ is not an isomorphism.

Notice that by [HKW93] Theorem 2.31(ii) the Kummer modular surface $K(p)$ is a fibre space over the modular curve $X(p)$ of level p, whose singular fibres over the cusps are folded p-gons, i.e. strings of $[p/2]+1$ smooth rational curves intersecting transversally.

The geometry of the $p+1$ corank 2 boundary components will not be treated in generality here, but we will describe the picture for our $p = 7$ case in Chapter 3. We just say here that every corank 2 boundary component lies in the closure of the central boundary component $D^\circ(l_0)$, and intersects $(p-1)/2$ peripheral boundary components $D(l)$, each one at one of their fibres. There are no other intersections (apart from those ones involving corank 2 boundary components) between the central and the peripheral boundary components.

1.4 The moduli space of (1,7)-polarised abelian surfaces

In this section we briefly review the main results by Manolache and Schreyer [MS01] about the moduli space of (1,7)-polarised abelian surfaces. A central result is the following

Theorem 1.1 ([MS01], Theorem 3.2). *An abelian surface A, G_7-invariantly embedded in $\mathbb{P}^6(V)$, has a G_7-invariant resolution of the form:*

$$0 \leftarrow \mathcal{O}_A \leftarrow \mathcal{O} \xleftarrow{\beta} 3V_4\mathcal{O}(-3) \xleftarrow{\alpha} 2S\Omega^3 \xleftarrow{\alpha'} 3V_1\mathcal{O}(-4) \xleftarrow{\beta'} \mathcal{O}(-7) \leftarrow 0$$

with $\alpha' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t\alpha$ *and* $\beta' = {}^t\beta$.

Note that all the modules in the above sequence are G_7-modules, and S is the non-trivial character of $\langle\iota\rangle$. See appendix for the table of characters of G_7.

We give here below more results we are going to use in the next chapter:

Proposition 1.2 ([MS01], Proposition 3.3).

$$\mathrm{Hom}_{G_7}(S\Omega^3, V_4\mathcal{O}(-3)) = 4I,$$

i.e. α has entries in a 4-dimensional vector space (see appendix).

A key observation is the following

Remark 1.3 ([MS01], Remark 3.4). Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two G_7-sheaves, then $\mathrm{Hom}_{G_7}(\mathcal{F}_1, \mathcal{F}_2)$ is a N-module, because $G_7 = H_7 \rtimes \mathbb{Z}_2$ is a normal subgroup of $N \cong H_7 \rtimes \mathrm{SL}_2(\mathbb{Z}_7)$, ι being central in $\mathrm{SL}_2(\mathbb{Z}_7)$.

As pointed out by Manolache and Schreyer, using the character table of $\mathrm{SL}_2(\mathbb{Z}_7)$ and with the notation from the appendix, we see that

$$\mathrm{Hom}(\Omega^3, \mathcal{O}(-3)) \cong \mathrm{Hom}(\mathcal{O}(-4), \Omega^3) = \wedge^3 V = V_1 \oplus 4V_1^{\#}, \text{ and}$$
$$\mathrm{Hom}_{G_7}(S\Omega^3, V_4\mathcal{O}(-3)) \cong \mathrm{Hom}_{G_7}(V_1\mathcal{O}(-4), S\Omega^3) = U' := \langle u_0, u_1, u_2, u_3 \rangle.$$

If we equip V with the canonical basis $\langle e_0 \dots e_6 \rangle$, then we get that the elements of $\mathrm{Hom}_{G_7}(V_1\mathcal{O}(-4), S\Omega^3)$ are given by the following 1×7 matrices with entries in $\wedge^3 V$

$$u_0 = (e_{k+1} \wedge e_{k+4} \wedge e_{k+2} - e_{k+6} \wedge e_{k+3} \wedge e_{k+5})_{k \in \mathbb{Z}_7}$$

and similarly:

$$u_1 = (e_k \wedge e_{k+1} \wedge e_{k+6})_{k \in \mathbb{Z}_7},\ u_2 = (e_k \wedge e_{k+2} \wedge e_{k+5})_{k \in \mathbb{Z}_7},\ u_3 = (e_k \wedge e_{k+4} \wedge e_{k+3})_{k \in \mathbb{Z}_7}.$$

Finally $\mathrm{Hom}_{G_7}(V_1^{\#}\mathcal{O}(-4), S\Omega^3) = \langle u_0^{\#} \rangle$, where $u_0^{\#} = (e_{k+1} \wedge e_{k+4} \wedge e_{k+2} + e_{k+6} \wedge e_{k+3} \wedge e_{k+5})_{k \in \mathbb{Z}_7}$.

Furthermore, the elements of $\mathrm{Hom}_{G_7}(S\Omega^3, V_4\mathcal{O}(-3))$ are given by the transposes of the above matrices. Notice that the composition of $\mathrm{Hom}(\mathcal{O}(-4), \Omega^3)$ and $\mathrm{Hom}(\Omega^3, \mathcal{O}(-3))$ is given by the wedge product, if we identify canonically $\wedge^6 V$ with $V^* = V_3 = H^0(\mathcal{O}(1))$. A direct computation gives us the non-zero compositions:

$$\nu := u_0^t u_1 = u_1^t u_0 = -u_2^t u_2 = \begin{pmatrix} 0 & x_4 & 0 & 0 & 0 & 0 & -x_3 \\ -x_4 & 0 & x_5 & 0 & 0 & 0 & 0 \\ 0 & -x_5 & 0 & x_6 & 0 & 0 & 0 \\ 0 & 0 & -x_6 & 0 & x_0 & 0 & 0 \\ 0 & 0 & 0 & -x_0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & -x_1 & 0 & x_2 \\ x_3 & 0 & 0 & 0 & 0 & -x_2 & 0 \end{pmatrix}$$

$$\lambda := u_0^t u_2 = u_2^t u_0 = -u_3^t u_3 = \begin{pmatrix} 0 & 0 & x_1 & 0 & 0 & -x_6 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 & -x_0 \\ -x_1 & 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & -x_2 & 0 & 0 & 0 & x_4 & 0 \\ 0 & 0 & -x_3 & 0 & 0 & 0 & x_5 \\ x_6 & 0 & 0 & -x_4 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & -x_5 & 0 & 0 \end{pmatrix}$$

$$\mu := u_0^t u_3 = u_3^t u_0 = -u_1^t u_1 = \begin{pmatrix} 0 & 0 & 0 & -x_5 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_6 & x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_0 & x_4 \\ x_5 & 0 & 0 & 0 & 0 & 0 & -x_1 \\ -x_2 & x_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_3 & x_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_4 & x_1 & 0 & 0 & 0 \end{pmatrix}$$

Furthermore $u_0^t u_0^{\#} = 2 \cdot \operatorname{diag}(x_0 \dots x_6) = \omega$ and $u_1^t u_0^{\#} = \nu^*$, where ν^* is like ν, but with all signs positive. (The same for $u_2^t u_0^{\#} = \lambda^*$, $u_3^t u_0^{\#} = \mu^*$.) In the sequel we will omit to write $(\cdot)^t$, and we will identify loosely the elements of $\operatorname{Hom}(\Omega^3, \mathcal{O}(-3))$ and $\operatorname{Hom}(\mathcal{O}(-4), \Omega^3)$.

In other words, once we choose a basis $u_0, \dots, u_3$ of $\operatorname{Hom}_{G_7}(S\Omega^3, V_4\mathcal{O}(-3))$, we can view the matrix α associated to A as an element of $M := M(3 \times 2, U')$. That is, the entries of this matrix are linear forms over $\mathbb{C}$ in 4 variables. We are going to give more information about this fact in the next chapters.

Proposition 1.4 ([MS01], Proposition 3.5). *A matrix α as in Theorem 1.1 satisfies*
$\alpha\alpha' = 0$ if and only if the three quadrics in $\mathbb{P}^3(U)$ given by its 2×2 minors (in S^2U') are annihilated by each of the three operators

$$\Delta_1 = 2a_0a_1 - a_2^2, \quad \Delta_2 = 2a_0a_2 - a_3^2, \quad \Delta_3 = 2a_0a_3 - a_1^2.$$

We denote by Δ the linear span of the operators $\Delta_1, \Delta_2, \Delta_3$, which is also the unique $\mathrm{SL}_2(\mathbb{Z}_7)$-invariant 3-dimensional subspace of S^2U.

From the appendix we see that

$$W \cong \Delta \subset S^2U,$$

so this property simply tells us that the minors of the matrix α lie in the irreducible G_7-module L, where $S^2U' = L \oplus W'$ (see appendix).

Remark 1.5. A resolution like in Theorem 1.1 is determined up to action from the left and right respectively of $\mathrm{GL}(3,\mathbb{C})$ and $\mathrm{GL}(2,\mathbb{C})$.

Let $C_\alpha \subset \mathbb{P}^3(U)$ be the curve defined by the 2×2 minors of a matrix α as in the resolution of Theorem 1.1. Then one can prove the following

Proposition 1.6 ([MS01], Proposition 4.1). *The curve C_α is a projectively Cohen-Macaulay curve of degree three and arithmetic genus 0.*

This means that the minors of α determine a twisted cubic curve C_α. In particular these minors are independent, so define a 3-dimensional linear subspace of $S^2U' = H^0(\mathbb{P}^3(U), \mathcal{O}(2))$. Furthermore it is possible to prove that if A is smooth, then it is uniquely determined by the curve C_α associated to its resolution.

The Hilbert scheme $\mathrm{Hilb}_{3t+1}(\mathbb{P}^3(U))$ has two components of dimension 12 and 15:

$$\mathrm{Hilb}_{3t+1}(\mathbb{P}^3(U)) = H_1 \cup H_2$$

with general points of H_1, H_2 and $H_1 \cap H_2$ corresponding respectively to a twisted cubic, a plane cubic union a point or a plane nodal cubic with an embedded point at the node. For all $C \in H_1$, $H^0(\mathbb{P}^3(U), \mathcal{I}_C(2)) = 3$. The morphism

$$\begin{aligned} f : H_1 &\to \mathbb{G}(3, H^0(\mathbb{P}^3(U), \mathcal{O}(2))) \\ C &\mapsto H^0(\mathbb{P}^3(U), \mathcal{I}_C(2)) \end{aligned}$$

is birational onto its image H, regular precisely on $H_1 \setminus H_1 \cap H_2$, cf. [EPS87]. Consider

$$H(\Delta) := H \cap \mathbb{G}(3, L) \subset \mathbb{G}(3, H^0(\mathbb{P}^3(U), \mathcal{O}(2))),$$

where $\mathbb{G}(3, L)$ is the Grassmannian of 3-dimensional vector subspaces of L.

Since it does not intersect $f(H_1 \cap H_2)$, we can regard $H(\Delta)$ as a subvariety of H_1 as well (cf. [MS01]).

Theorem 1.7 ([MS01], Proposition 4.4). *$H(\Delta)$ is a smooth prime Fano 3-fold of genus three.*

By the results of [MS01] the moduli space of $(1,7)$-polarised abelian surfaces with canonical level structure is birational to $H(\Delta)$. For more information about Fano 3-folds, see [Muk92], and about Fano 3-folds of genus 12, see [Sch01].

1.5 A different model of the moduli space

We report few crucial facts from [GP98] and [GP01]: the 7×7 matrix

$$M_7'(x,y) = (x_{\frac{i+j}{2}} y_{\frac{i-j}{2}})_{i,j \in \mathbb{Z}_7}$$

has rank at most 4 on an embedded H_7-invariant $(1,7)$-polarised abelian surface in $\mathbb{P}^6(V)$. On the other hand, for any parameter point $y = (0 : y_1 : y_2 : y_3 : -y_3 : -y_2 : -y_1) \in \mathbb{P}^2(W')$, the matrix M_7' is alternating. We will denote in the sequel by $I_3(y) \subset \mathbb{C}[x_0, \dots, x_6]$ the homogeneous ideal generated by the 6×6 Pfaffians of the alternating matrix $M_7'(x,y)$ and by $V_{7,y} \subset \mathbb{P}^6(V)$ the closed subscheme defined by this ideal. Notice that, in our notation, $M_7'(x,(1:0:0)) = \lambda$, $M_7'(x,(0:1:0)) = \mu$ and $M_7'(x,(0:0:1)) = \nu$.

Now we quote two main propositions that we are going to need for our research:

Proposition 1.8 ([GP98], Proposition 5.2). *Let $y \in \mathbb{P}^2(W')$.*

1. *For $y \in Q' = \{y_1^3 y_2 + y_2^3 y_3 + y_3^3 y_1 = 0\} \subset \mathbb{P}^2(W')$, the scheme $V_{7,y}$ is the secant variety of an elliptic normal curve in $\mathbb{P}^6(V)$ (the level 7 structure elliptic curve corresponding to the point y on the modular curve Q').*

2. *For a general $y \in \mathbb{P}^2(W')$, the scheme $V_{7,y}$ is a projectively Gorenstein irreducible threefold of degree 14 and sectional genus 15.*

Proposition 1.9 ([GP98], Proposition 5.4). *Let $A \in \mathbb{P}^6(V)$be a general Heisenberg invariant $(1,7)$-polarised abelian surface, and let $A \cap \mathbb{P}^2(W') = \{p_1, \dots, p_6\}$ be the odd 2-torsion points of A. Then:*

1. *The points p_i form a polar hexagon to the Klein quartic curve $Q \subset \mathbb{P}^2(W)$.*

2. *The surface A is contained in V_{7,p_i}, for all $i = 1, \dots, 6$. Moreover, 21 cubic Pfaffians defining any three of the six V_{7,p_i}'s generate the homogeneous ideal I_A of A.*

It follows that the moduli space of (1,7)-polarised abelian surfaces with canonical level structure is birational to the space $\mathrm{VSP}(Q,6)$ of polar hexagons to the Klein quartic curve $Q \subset \mathbb{P}^2(W)$.

Remark 1.10. By direct computation we see that the seven Pfaffians of a matrix $M_7'(x,y)$ associated to an element $y = (0 : y_1 : y_2 : y_3 : -y_3 : -y_2 : -y_1) \in \mathbb{P}^2(W')$

are given by $y^3|_L \otimes V_4$, where from the appendix we have $S^3W' = L \oplus W$ and $S^3V_3 = (I \oplus U' \oplus L) \otimes V_4$: i.e. the i-th Pfaffian is

$$\begin{aligned}
&\mathrm{Pfaff}(y_1\lambda + y_2\mu + y_3\nu)_i = \\
&\quad (y_2^2y_3 - y_1^3)x_ix_{i+3}x_{i+4} + (y_3^2y_1 - y_2^3)x_ix_{i+1}x_{i+6} + (y_1^2y_2 - y_3^3)x_ix_{i+2}x_{i+5} + \\
&\quad y_1y_2y_3(x_{i+1}x_{i+2}x_{i+4} + x_{i+3}x_{i+5}x_{i+6} - x_i^3)+ \\
&\quad y_1^2y_3(x_{i+2}^2x_{i+3} + x_{i+5}^2x_{i+4}) + y_2^2y_1(x_{i+3}^2x_{i+1} + x_{i+4}^2x_{i+6}) + \\
&\quad y_3^2y_2(x_{i+1}^2x_{i+5} + x_{i+6}^2x_{i+1})
\end{aligned} \tag{1.1}$$

Then the second part of Proposition 1.9 simply says that the linear space spanned by any three of the six $p_i^3|_L \in L$ is 3 dimensional.

Chapter 2

The geometry of $H(\Delta)$ and its boundary

In this chapter we study $H(\Delta)$ from the point of view of geometric invariant theory (see Section 1.4) and describe the geometry of its boundary.

2.1 The moduli space as an orbit space

In what follows we regard the moduli space of (1,7)-polarised abelian varieties as an orbit space: in fact, by Remark 1.5, $\mathrm{GL}(3,\mathbb{C})$ and $\mathrm{GL}(2,\mathbb{C})$ act on the variety $M := M(3 \times 2, U')$.

In order to apply the theory of orbit spaces, see [New78], p. 73, we restrict our attention to $\mathbb{P}(M)$, $\mathrm{SL}(3,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{C})$. Doing so, no harm has been done to the elements

$$H^0(\mathbb{P}^3(U), \mathcal{I}_\alpha(2)) \in \mathbb{G}(3, H^0(\mathbb{P}^3(U), \mathcal{O}(2))),$$

where $\mathcal{I}_\alpha$ is the ideal generated by the minors of a matrix arising from a resolution as in Section 1.4, and to the effects of the actions.

We will see in the proof of the next proposition that the actions of $\mathrm{SL}(3,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{P}(M)$ induce an action on the minors, given by linear combinations. In any case the vector space W_α generated by the minors of α is fixed by this action.

Notice that $\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ is reductive, and we consider its obvious linearisation.

The study of this problem has been done in [EPS87]. At the time we started working we were not aware of the existence of this paper. Even though the next result can be found in [EPS87], we give it with our proof, because it is nice on its own and as a self-praise for a month of hard thinking. However we present an alternative proof as well, the second one involving the topic of Kronecker modules, which we are going to introduce briefly.

Proposition 2.1. *Let $\alpha \in M$. The following are equivalent:*

1. *α is stable,*
2. *α is semi-stable,*
3. *the minors of α are linearly independent.*

Proof (GIT version). We first prove that 2) and 3) are equivalent. If $\dim(W_\alpha) < 3$, α is conjugate to a matrix whose first minor equals 0. Hence, up to this action, α equals

$$\alpha_1 = \begin{pmatrix} * & * \\ * & * \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \alpha_2 = \begin{pmatrix} * & * \\ * & 0 \\ * & 0 \end{pmatrix}. \tag{2.1}$$

To see this, choose a non-zero element e_0 of the bottom 2×2 submatrix α_b of α (if there is none, we are done). Then, if in the same row (resp. column) there is a dependent element, then we can suppose it is zero, and we are done. Otherwise, call it e_1, then α_b is conjugate to

$$\gamma = \begin{pmatrix} e_0 & e_1 \\ * & * \end{pmatrix}$$

(resp. ${}^t\gamma$) and it is easy to see that the bottom row (resp. right column) has to be dependent on the top (resp. left) one.

The following limits of 1-parameter subgroups of $\mathrm{SL}(3, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ applied to α vanish:

$$\lim_{a \to 0} \begin{pmatrix} a & & \\ & a & \\ & & a^{-2} \end{pmatrix} (\alpha_1) = 0,$$

$$\lim_{a \to 0} \begin{pmatrix} a^4 & & \\ & a^{-2} & \\ & & a^{-2} \end{pmatrix} (\alpha_2) \begin{pmatrix} a^3 & \\ & a^{-3} \end{pmatrix} = 0,$$

so α is not semi-stable.

Now we are going to prove that if $\dim(W_\alpha) = 3$, then α is semi-stable. We write q_i for the minor of α where the i-th row has been omitted, and if $A \in \mathrm{SL}(3, \mathbb{C})$, we write A_{ij} for the determinant of the 2×2 submatrix of A with the i-th row and j-th column omitted. A straightforward computation shows that the minors of α are fixed under the action of $\mathrm{SL}(2, \mathbb{C})$, whereas an element $A \in \mathrm{SL}(3, \mathbb{C})$ acts on the minors of α as follows:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \mapsto A' \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \qquad A' = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Let $\dim(W_\alpha) = 3$ and fix a basis of $H^0(\mathbb{P}^3(U), \mathcal{O}(2))$. If f is any non-vanishing Plücker function among those defining the embedding

$$\varphi : \mathbb{G}(3, H^0(\mathbb{P}^3(U), \mathcal{O}(2))) \to \mathbb{P}^{\binom{10}{3}-1}$$

then f is an invariant homogeneous polynomial of positive degree not vanishing on α. The invariance follows from the fact that

$$\det(A') = \det \begin{pmatrix} A_{11} & -A_{12} & A_{13} \\ -A_{21} & A_{22} & -A_{23} \\ A_{31} & -A_{32} & A_{33} \end{pmatrix} = \frac{1}{\det(A)} \det({}^t A^{-1}) = \det(A^{-1}) = 1,$$

so $f(A' W_\alpha B) = \det(A') f(W_\alpha B) = f(W_\alpha)$ and therefore α is semi-stable.

Now we prove that 1) and 2) are equivalent. Let $A \in \mathrm{SL}(3, \mathbb{C})$, $B \in \mathrm{SL}(2, \mathbb{C})$ and α be a semi-stable point. If $A\alpha B = \alpha \in \mathbb{P}(M)$, then

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = A' \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

so $A = \xi I_3$, $\xi^3 = 1$. Thus $A\alpha B = \alpha B$, and $B = \zeta I_2$, $\zeta^2 = 1$. It follows that the stabiliser of α is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$, and the dimension of the orbit of α is $\dim(O(\alpha)) = \dim(\mathrm{SL}(3, \mathbb{C})) + \dim(\mathrm{SL}(2, \mathbb{C})) = 8 + 3 = 11$.

Since all the elements of $\mathbb{P}(M)^{ss}$ (the set of semi-stable points, namely where some f as above does not vanish) have orbits of the same dimension, the action of $\mathrm{SL}(3, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ is closed in $\mathbb{P}(M)^{ss}$, that is the orbits are closed subsets of $\mathbb{P}(M)^{ss}$ (see [New78], Lemma 3.7). By [New78], Theorem 3.14(iv) this concludes the proof. $\square$

2.2 Kronecker modules

We define the space of Kronecker modules to be $X = W_1 \otimes W_2^* \otimes V$, where W_i is a $\mathbb{C}$-vector space of dimension k_i and V is a $\mathbb{C}$-vector space of dimension n. Therefore X can be thought as n copies of $\mathrm{Hom}(W_2, W_1)$, and one of its elements as n maps from W_2 to W_1. We represent this by means of the quiver (a directed graph)

$$\overset{W_2}{\bullet} \quad \overset{\vdots}{\rightrightarrows} \quad \overset{W_1}{\bullet}$$

We have an obvious action of $G' = \mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$ on X given by

$$(g_1, g_2)x = g_1 x g_2^{-1},$$

where $g = (g_1,\ g_2) \in G',\ x \in X$. The subgroup $D = \{(\lambda,\ \lambda) |\ \lambda \in \mathbb{C}^*\}$ of diagonal scalars, which is normal, acts trivially on X, so the effective symmetry group is $G = G'/D$. We call $f : X \to \mathbb{C}$ a semi-invariant function of weight (d_1, d_2) if

$$f(g_1 x g_2^{-1}) = \det(g_1)^{d_1} \det(g_2^{-1})^{d_2} f(x).$$

Since the action of D is trivial, $k_1 d_1 + k_2 d_2 = 0$, and also $d_1 \geq 0,\ d_2 \leq 0$, so $(d_1, d_2) = d\left(\dfrac{k_2}{m}, -\dfrac{k_1}{m}\right)$, where $m = \mathrm{hcf}(k_1, k_2)$ and $d \geq 0$.

Define an abelian character $\chi : G \to \mathbb{C}^*$ by

$$\chi(g_1,\ g_2) = \det(g_1)^{k_2/m} \det(g_2)^{-k_1/m} :$$

then we say that $f(gx) = \chi(g)^d f(x)$ is a semi-invariant function of weight d with respect to G and χ.

Define S_d the space of semi-invariant functions of weight d, and the graded $\mathbb{C}$-algebra $S = \bigoplus_{d \geq 0} S_d$. Furthermore, define the open set of semi-stable points,

$$X^{ss} = \{x \in X |\ \exists f \in S_{>0},\ f(x) \neq 0\}$$

and the open set of stable points

$$X^s = \{x \in X^{ss} |\ Gx \text{ is closed in } X^{ss} \text{ and } \dim(Gx) = \dim(G)\} \subset X^{ss},$$

exactly as in [New78]. Then the fundamental theorem of GIT says that

$$\mathrm{Proj}\, S = X^{ss}/\sim,$$

also called the moduli space for (most) modules up to isomorphism.

If $x \in X$ and $\theta \in V^*$, then $x_\theta \in \mathrm{Hom}(W_2, W_1)$. We call (U_1, U_2) a sub-module of (W_1, W_2, x) if and only if

$$x_\theta(U_2) \subset U_1$$

for all $\theta \in V^*$.

A central result is the following

Proposition 2.2 ([Dre87], Proposition 15). *Let $x \in X$, and let (U_1, U_2) be a sub-module of x such that $U_2 \neq \{0\}$ and $U_1 \neq W_1$, then*

1. *$x \in X^{ss}$ if and only if for all sub-modules (U_1, U_2) of x as above we get*
$$\dim(U_1)/\dim(U_2) \geq \frac{k_1}{k_2};$$
2. *$x \in X^s$ if and only if for all sub-modules (U_1, U_2) of x as above we get*
$$\dim(U_1)/\dim(U_2) > \frac{k_1}{k_2}.$$

The first consequence of this result is that if k_1 and k_2 are coprime, then $X^{ss} = X^s$.

Now we can supply the

Second proof of Proposition 2.1. We have that $(k_2, k_1) = (3, 2)$ are coprime, therefore $M^s = M^{ss}$ is the subset of M whose elements admit only sub-modules (U_1, U_2) with $\dim(U_1)/\dim(U_2) = 3$ or $2 > \frac{3}{2}$.

These are exactly the elements with independent minors, in fact if a matrix is conjugate to one of (2.1), then there are sub-modules with $\dim(U_2)/\dim(U_1) = 1$ or less and vice versa. □

Now let $\mathbb{P}(M)^s$ be the subset of $\mathbb{P}(M)$ given by the stable points of $\mathbb{P}(M)$. $\mathbb{P}(M)^s$ is open, so quasi-projective (cf. [New78]). Let $\mathbb{P}(M)^s_\Delta$ be the subvariety of $\mathbb{P}(M)^s$ defined by the nine quadratic conditions (three each minor)

$$\Delta = \langle \Delta_1, \Delta_2, \Delta_3 \rangle.$$

Proposition 2.3. *There is a projective variety Y and an affine surjective morphism ϕ such that $\phi : \mathbb{P}(M)^s \to Y$ is a geometric quotient and $\phi(\mathbb{P}(M)^s_\Delta) := \mathcal{Y}$ is projective as well. Furthermore distinct orbits are mapped to distinct elements of Y (and so of $\mathcal{Y}$ as well).*

Proof. After noting that a stable orbit of $\mathbb{P}(M)$ is either entirely in $\mathbb{P}(M)^s_\Delta$ or entirely outside, everything follows from [New78], Theorem 3.14 and Theorem 3.5(iv). □

Lemma 2.4. *Let $A,\ B \in M(3 \times 2, U')$ be two matrices whose minors span a 3-dimensional subspace. If $\langle q_i(A)\rangle_{i=1,2,3} = \langle q_i(B)\rangle_{i=1,2,3}$, then A and B lie in a common orbit under the action of* $\mathrm{GL}(3,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})$

Proof. First of all, up to the action, we can rearrange the matrices in such a way that $q_i(A) = q_i(B) := q_i,\ i = 1,2,3$. Obviously the ideals these minors generate define the same variety $Z \in \mathbb{P}^3(U)$. The obvious resolution

$$0 \to 2\mathcal{O}(-3) \xrightarrow{A} 3\mathcal{O}(-2) \xrightarrow{(q_1,q_2,q_3)} \mathcal{O} \to \mathcal{O}_Z \to 0 \tag{2.2}$$

is unique up to action by $\mathrm{GL}(2,\mathbb{C})$, and it still works if we replace A with B. The columns of A and B are just the syzygies of $q_i,\ i = 1,2,3$ (see Theorem 4.2), thus there is some $x \in \mathrm{GL}(2,\mathbb{C})$ such that $A = Bx$. □

This lemma allows us to prove the main result of this section:

Proposition 2.5. *$\mathcal{Y}$ is isomorphic to $H(\Delta)$.*

Proof. Consider the morphism

$$\begin{aligned} \omega : \mathbb{P}(M)^s_\Delta &\to H(\Delta) \\ \alpha &\mapsto W_\alpha. \end{aligned}$$

Obviously ω is constant on orbits, and since $\mathbb{G}(3,(L))$ does not intersect $f(H_1 \cap H_2)$, any of its elements is representable by some element of $\mathbb{P}(M)^s_\Delta$, and then ω is surjective as well. By [New78], Corollary 3.5.1, we see that $(\mathcal{Y},\ \phi)$ is a categorical quotient of $\mathbb{P}(M)^s_\Delta$. By definition of categorical quotient (see [New78], definition after Proposition 2.9), there is a (unique) morphism $\psi : \mathcal{Y} \to H(\Delta)$ such that $\omega = \psi \circ \phi$:

$$\begin{tikzcd} \mathbb{P}(M)^s_\Delta \arrow[r, "\phi"] \arrow[dr, "\omega"'] & \mathcal{Y} \arrow[d, "\exists!"] \\ & H(\Delta) \end{tikzcd}$$

By Lemma 2.4 and Proposition 2.3, ψ is a bijective morphism, and by [Muk92] $H(\Delta)$ is a smooth (normal) irreducible variety. Consider the normalisation $\nu : \tilde{\mathcal{Y}} \to \mathcal{Y}$. In particular $\psi \circ \nu$ is a normalisation of $H(\Delta)$, so by [Sha74] (corollary to Theorem 5 in Chapter 2) we can conclude that $\psi \circ \nu$ is the identity and then ψ is actually an isomorphism. □

2.3 Geometry of the boundary B

In this section we study the geometry of the boundary B of the moduli space of twisted cubic curves annihilated by Δ. The key observation is that by [Har92], Proposition 9.4, if a matrix $\alpha \in M$ is not conjugate to a matrix α' having a 0-entry, then the minors of α determine a twisted cubic curve. Obviously, if the minors of $\alpha \in M$ define a twisted cubic curve, it cannot be conjugate to a matrix having a 0-entry. This means that in this section we are studying the subset B of $H(\Delta)$ defined by the condition that its elements are the images via ϕ of the set of all the matrices conjugate to some matrix having a 0-entry. In the next lemma we prove that B is actually a subvariety of $H(\Delta)$.

Lemma 2.6. *B is a subvariety of $H(\Delta)$.*

Proof. Consider the subvariety $\hat{B}$ of

$$\mathbb{P}(M)^s_\Delta \times \mathbb{G}(1,2) \times \mathbb{G}(2,3) \cong \mathbb{P}(M)^s_\Delta \times \mathbb{P}^1 \times \mathbb{P}^2$$

defined by the condition that $(m, a, b) \in \hat{B}$ if and only if

$$m_i(a) \subset b, \ i = 0, \dots, 3$$

where m_i is the matrix given by projecting the entries of m to $\langle u_i \rangle$. In other words, $(m, a, b) \in \hat{B}$ if a is a 1-dimensional subspace of $\mathbb{C}^2$ such that its image is contained in the 2-dimensional subspace b of $\mathbb{C}^3$ for every m_i; and this is the case if and only if m is conjugate to a matrix of M having a 0-entry. Clearly $\hat{B}$ is closed, and so is its projection $\pi|_{M^s_\Delta}(\hat{B})$ into $\mathbb{P}(M)^s_\Delta$, see [Har92], Theorem 3.12. Moreover it is invariant under the action of $\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. Finally, by [New78], Theorem 3.5.iv) we get that $\phi(\pi|_{\mathbb{P}(M)^s_\Delta}(\hat{B})) = B$ is closed in $H(\Delta)$. $\square$

The next step is to find a suitable representative for each $[\alpha] \in B$. Namely we want to parametrise the boundary B of $H(\Delta)$. We can suppose that $\alpha_{12} = 0$. For simplicity from now on we write

$$\alpha = \begin{pmatrix} a & 0 \\ b & d \\ c & e \end{pmatrix}. \tag{2.3}$$

If we write an entry like $\sum_{i=0}^{3} a_1 u_i$, then the minors ad and ae satisfy Δ if and only if d and e satisfy the linear system

$$a^* = \begin{pmatrix} a_1 & a_0 & -a_2 & 0 \\ a_2 & 0 & a_0 & -a_3 \\ a_3 & -a_1 & 0 & a_0 \end{pmatrix}$$

Remark 2.7. Let x and y be any two elements of U'. Via the previous matrix x^* we get a multiplication $U' \times U' \to W' \subset S^2U' = L \oplus W'$ given by $(x, y) \mapsto x^*y$. Notice that these are all $\mathrm{SL}_2(\mathbb{Z}_7)$-modules. This induces a product

$$* : U' \times U' \to W'.$$

Obviously $*$ is symmetric, and the condition Δ says that every minor of a matrix α is contained in $L \subset S^2U'$. This is a fact we are going to use heavily in the next chapter.

The rank of a^* must be at most 2, because otherwise d and e would be dependent, and so a minor of α would be zero. Thus the minors of a^* must vanish, that is

$$\begin{aligned} a_0^3 - a_1a_2a_3 &= 0 && (2.4)\\ a_0^2a_1 + a_0a_2^2 + a_2a_3^2 &= 0 && (2.5)\\ a_0^2a_2 + a_0a_3^2 + a_3a_1^2 &= 0 && (2.6)\\ a_0^2a_3 + a_0a_1^2 + a_1a_2^2 &= 0 && (2.7) \end{aligned}$$

Let us call C the curve in $\mathbb{P}^3(U')$ defined by these equations. As shown in Appendix B, C is isomorphic to the modular curve of level 7, whose standard model is the Klein quartic Q, a smooth plane curve of genus 3 given by the equation

$$f_{\mathrm{Klein}} = v_1^3v_2 + v_2^3v_3 + v_3^3v_1. \quad (2.8)$$

It is easy to check that the rank of a^* cannot be 1. So the space of solutions is 2-dimensional, and then we can fix a basis for it, namely (d, e). In other words, as long as d and e are independent, any choice we make is good, and does not affect the space spanned by the minors of the matrix.

Lemma 2.8. *Let $\alpha \in M^s$. Then none of its minors is of the form l^2, where l is a linear form.*

Proof. Let $l = l_0u_0+l_1u_1+l_2u_2+l_3u_3$ such that l^2 satisfies Δ. Then we can certainly suppose that $l_0 = 1$, and from α^* (with l in place of a), we get $2l_i = l_{i+1}^2,\ i = 1,2,3$. Thus, for example, $l_2 = 2\xi$, where $\xi^7 = 1$, but then $l_1l_2l_3 = 2\xi 2\xi^2 2\xi^4 = 8 \neq 1$, so the linear system a^* would have rank 3, contradicting the fact that $\alpha \in M^s$. $\square$

This lemma implies that, because of the Δ condition, the variety defined by the minors of α is a (possibly reducible) curve, as opposed to [EPS87], Lemma 3, where the authors encounter a surface.

In the next proposition we describe the geometry of B:

Proposition 2.9. *B is birational to $C \times \mathbb{P}^1 \cong Q \times \mathbb{P}^1$.*

Proof. If as in (2.3) α is a matrix representing the point $[\alpha] \in B$, then $a \in C$, while d and e are determined up to choice of a basis. We call V_α the variety defined by the minors of α.
For the minor $q_1(\alpha)$ to satisfy Δ, the vector (b, c) must satisfy the linear system

$$\Lambda = \begin{pmatrix} e_1 & e_0 & -e_2 & 0 & -d_1 & -d_0 & d_2 & 0 \\ e_2 & 0 & e_0 & -e_3 & -d_2 & 0 & -d_0 & d_3 \\ e_3 & -e_1 & 0 & e_0 & -d_3 & d_1 & 0 & -d_0 \end{pmatrix}.$$

If $a_0 = 0$, by (2.4),...,(2.7) two more coefficients among a_1, a_2 and a_3 must be zero. We can suppose $a_1 = 1$, and so $d, e \in \langle u_2, u_3\rangle$. Let $d = u_2$ and $e = u_3$. Then we can suppose $b_1 = b_2 = c_1 = 0$, so $b_3 = -c_0$, $b_0 = c_2 = 0$ and c_3 is free. Therefore

$$\alpha = \begin{pmatrix} u_1 & 0 \\ b_3u_3 & u_2 \\ -b_3u_0 + c_3u_3 & u_3 \end{pmatrix}.$$

These matrices are parametrised by $(b_3 : c_3) \in \mathbb{P}^1$. Unless $b_3 = 0$, their minors determine a line and a conic, non coplanar and meeting at a point. If $b_3 = 0$ the minors determine three non coplanar lines meeting at a common point.

If $a_0 \neq 0$ we can set $a_0 = 1$, and (2.4) gives us, say, $a_3 = \dfrac{1}{a_1a_2}$, while (2.5), (2.6) and (2.7) become the non-homogeneous equation $1 + a_1^2a_2^3 + a_1^3a_2 = 0$. It is easy to see that either d_0 or e_0 is nonzero, so let $e_0 = 1$, and $d_0 = 0$, which implies $d_1d_2d_3 \neq 0$. We can assume that $b_0 = b_1 = c_0 = 0$, so now the vector $(b_2, b_3, c_1, c_2, c_3)$ must satisfy the linear system

$$\Lambda' = \begin{pmatrix} -e_2 & 0 & 0 & d_2 & 0 \\ 1 & -e_3 & 0 & 0 & d_3 \\ 0 & 1 & d_1 & 0 & 0 \end{pmatrix}.$$

The rank of Λ' is three, so we get a 2-dimensional space of solutions. Obviously multiplication by a scalar on (b, c) does not affect V_α, and so we get a $\mathbb{P}^1$ of solutions. Also in this case in general V_α is a conic and a line, non-coplanar and meeting at a point. If this does not happen, then V_α can only be the union of three non coplanar lines, one of them meeting the other two, possibly at a single point. If so, by Lemma 2.4 the minors of α can be arranged in such a way that

$$\alpha = \begin{pmatrix} a & 0 \\ b & d \\ 0 & e' \end{pmatrix}. \tag{2.9}$$

So we are looking for a form $e' = xd + ye \in C$. Actually $e' = xd + e$, because $d \notin C$. Consider the linear system

$$\begin{pmatrix} e'_1 & 1 & -e'_2 & 0 \\ e'_2 & 0 & 1 & -e'_3 \\ e'_3 & -e'_1 & 0 & 1 \end{pmatrix}.$$

Obviously a satisfies it, and since we can assume that $b_0 = 0$, the minor

$$1 - (e_1 + xd_1)(e_2 + xd_2)(e_3 + xd_3) \tag{2.10}$$

must be zero.

Remark 2.10. If $e' \in \langle d,\ e\rangle$ satisfies (2.10), then as this implies $e' \in C$, we get that e' satisfies all the polynomials (2.4),. . . ,(2.7).

From $d_1d_2d_3 \neq 0$ we conclude that the degree of the previous polynomial is 3, and so for any given a (with $a_0 \neq 0$) we get in general three matrices (and then elements of B) whose minors define three lines.

So far we have proved that over every point $a \in C$ there is a $\mathbb{P}^1$ of elements of B. That is, the closure of the fibres of the following map

$$B \dashrightarrow C \cong Q$$
$$\alpha \mapsto a$$

are $\mathbb{P}^1$'s. The map is not well-defined if and only if an element of B is represented by a matrix like (2.9). Indeed a determines the plane where the conic lives, and so if the conic is smooth, a must be unique.

Therefore now we only need to show that the locus $B' \subset B$ whose elements are represented by matrices like (2.9) is closed.

This is clear after noticing that

$$B^* := \{(a,\ b) \in C \times C |\ a^* b = 0\} \subset C \times C \subset \mathbb{P}^3(U') \times \mathbb{P}^3(U')$$

is a proper closed subset of $C \times C$, and that B' is isomorphic to B^*, in fact the entries b (resp. d') of (2.9) can be chosen arbitrarily among the solutions of e^* (resp. a^*), and we are still in the same orbit of matrices representing $\xi \in B'$. Notice that the projections $p_i : B^* \to C,\ i = 1, 2$ are generically $3 : 1$. □

From (2.10) we see that in general a fibre of B over a point $a \in C$ meets three other fibres, each one at a single point. In the next lemma we work out over which fibres the intersection is non-smooth.

Lemma 2.11. *Let α be a matrix whose image is in B, and let the top-left entry be $a \in C$. If the rational fibre over a does not meet three distinct fibres, then a is the image of a cusp of Q.*

Proof. The first observation is that over the cusp $a = u_1$, corresponding to y_2, where y_i is the dual basis of v_i, the space of solutions of a^* is given by the last two columns of (B.7), that is $\langle u_2, u_3 \rangle$. Then after replacing the polynomial (2.10) with the polynomial given by (2.5), we get clearly a double solution, namely $e'_2 e'^2_3 = 0$.

Now notice that, by Remark 2.10, the condition that the polynomials like (2.10) have a double solution is invariant under the action of $\mathrm{SL}_2(\mathbb{Z}_7)$, whose action on the entries (in U') of α descends to an action on the coefficients of the polynomials. Therefore the set of points of C we are after is a union of $\mathrm{SL}_2(\mathbb{Z}_7)$-orbits of C, and it contains the orbit of 24 images of cusps of Q.

Let us assume $a_0 \neq 0$. Then we only need the polynomial (2.10) to have a double solution. Using as d and e the first two columns of (B.7), the discriminant D of (2.10), divided by a common monomial, has degree 11, and is not divisible by f_{Klein}. Then the zero locus of D intersects Q at at most 44 reduced points, including the three cusps with zero entries. By [Kle79], there is one orbit only with at most 44 points, precisely the orbit of 24 cusps. □

Remark 2.12. As mentioned in the summary, at first sight we have a good candidate for a possible subspace of $H(\Delta)$ whose points could parameterise translation scrolls. In fact we have just seen in Proposition 2.9 that B is birational to the Kummer surface that parameterises those surfaces. But things will not be as straightforward as expected, as we will see in Chapter 4. For a brief description of the translation scrolls, see Section 3.1, where everything works for $(1, p)$ in place of $(1, 5)$.

2.4 More on the isomorphic models of the moduli space

We have seen that the moduli space we are interested in is birational to $H(\Delta)$, the moduli space of twisted cubic curves annihilated by the net of quadrics Δ. By Section 1.5 it turns out that an isomorphic model to it is given by the variety of sum of powers $\mathrm{VSP}(Q, 6)$. In fact $H(\Delta)$ and $\mathrm{VSP}(Q, 6)$ are isomorphic, and in this section we are going to discuss the isomorphism in more detail and present more facts on these spaces.

First of all we need the following

Theorem 2.13 ([Sch01], Theorem 1.1).

$$H(\Delta) \cong \mathbb{G}(3, L, \eta_{\mathrm{Klein}}) \cong \mathrm{VSP}(Q, 6). \tag{2.11}$$

As explained in [Sch01], every Fano 3-fold of genus 12 has these descriptions over an algebraically closed field of characteristic 0, where

$$\mathrm{VSP}(Q, 6) = \overline{\{\{l_1 \dots l_6\} \in \mathrm{Hilb}_6(\mathbb{P}^2(W')) |\ l_1^4 + \cdots + l_6^4 = v_1^3 v_2 + v_2^3 v_3 + v_3^3 v_1\}}$$

and $\mathbb{G}(3, L, \eta_{\mathrm{Klein}})$ is as follows (notice that the spaces here correspond to those in the appendix): consider on

$$L = (f_{\mathrm{Klein}}^{\perp})_3^* =$$

$$\langle v_1 v_2 v_3,\ v_2 v_3^2,\ v_3 v_1^2,\ v_1 v_2^2,\ v_3^2 v_1 - v_2^3,\ v_1^2 v_2 - v_3^3,\ v_2^2 v_3 - v_1^3 \rangle$$

the net $\eta_{\text{Klein}} : \wedge^2 L \to W \cong \mathbb{C}^3$ of alternating forms defined by the matrix

$$\eta_{\text{Klein}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -y_2 & y_1 \\ 0 & 0 & 0 & 0 & -y_3 & 0 & y_2 \\ 0 & 0 & 0 & -y_1 & 0 & 0 & y_3 \\ 0 & 0 & y_1 & 0 & y_2 & -y_3 & 0 \\ 0 & y_3 & 0 & -y_2 & 0 & y_1 & 0 \\ y_2 & 0 & 0 & y_3 & -y_1 & 0 & 0 \\ -y_1 & -y_2 & -y_3 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here $(f_{\text{Klein}}^{\perp})_3 = \{x \in S^3W = \mathbb{C}[y_1, y_2, y_3]_3 \mid x f_{\text{Klein}} = 0\}$, the 7-dimensional vector space of differentials of the third order that annihilates f_{Klein} (see [Sch01]). We write y_i in place of ∂v_i for the dual generators of W with respect to the v_i, the elements of the basis of W' in the appendix. Then

$$\mathbb{G}(3, L, \eta_{\text{Klein}}) = \{E \in \mathbb{G}(3, L) \mid \wedge^2 E \subset \ker(\eta_{\text{Klein}} : \wedge^2 L \to W)\}.$$

We follow [Sch01], Theorem 2.6, to give a sketch of the proof of the second isomorphism in (2.11):

1. The Pfaffians of η_{Klein} are $(f_{\text{Klein}}^{\perp})_3$, and if I_{Pfaff} is the ideal they generate, then the dual socle generator (see [Eis95]) of $\mathbb{C}[y_1, y_2, y_3]/I_{\text{Pfaff}}$ is f_{Klein}.

2. Given an element $P \in \mathbb{G}(3, L, \eta_{\text{Klein}})$, choose a basis l for L with the last three generators being taken from a basis p for P. Then, with respect to this chosen basis, η_{Klein} gets a block decomposition form

$$\eta'_{\text{Klein}} = \begin{pmatrix} * & \psi \\ -\psi^t & 0 \end{pmatrix}, \tag{2.12}$$

Now ψ can be viewed as a 4×3 syzygy matrix for the exact complex

$$0 \to 3\mathcal{O}_{\mathbb{P}^2(W')}(-4) \xrightarrow{\psi} 4\mathcal{O}_{\mathbb{P}^2(W')}(-3) \xrightarrow{\text{minors}} \mathcal{O}_{\mathbb{P}^2(W')} \to \mathcal{O}_\Gamma \to 0,$$

where $\Gamma \subset \mathbb{P}^2(W')$ is a variety given by six points, simply by computing the Hilbert series:

$$P_{\text{Hilb}}(\Gamma) = (1, 3, 6, 6, 6 \dots).$$

3. Notice that if we replace p with $p' = pa$, $a \in \text{GL}_3$, then we simply get $\psi' = \psi a$, and the above resolution is not affected. Similarly if we choose differently the first four generators of l, then only the top-left block $*$ of η'_{Klein} will be affected.

4. Consider the first four Pfaffians of η'_{Klein}. They must be a linear combination of the original ones, and are given by the minors of ψ, so we see that
$$(I_\Gamma)_3 \subset (f^\perp_{\mathrm{Klein}})_3,$$
thus
$$S^3W \to A_\Gamma = \mathbb{C}[y_1, y_2, y_3]/I_\Gamma \to A_{\mathrm{Klein}} = \mathbb{C}[y_1, y_2, y_3]/(f^\perp_{\mathrm{Klein}})$$
and
$$\mathrm{Hom}\big((A_{\mathrm{Klein}})_4, \mathbb{C}\big) \subset \mathrm{Hom}\big((A_\Gamma)_4, \mathbb{C}\big).$$

5. Finally, f_{Klein} is a generator for $\mathrm{Hom}((A_{\mathrm{Klein}})_4, \mathbb{C})$, being the dual socle generator, and the fourth power of the six points γ_i of $\Gamma \subset W'$ impose independent conditions on quartics. Then $\langle \gamma_i^4 \rangle_{i=1,\dots,6} = \mathrm{Hom}((A_\Gamma)_4, \mathbb{C})$, so
$$f_{\mathrm{Klein}} = \sum_{i=1}^{6} \lambda_i \gamma_i^4.$$

6. Reversing this process one gets the isomorphism starting from $\mathrm{VSP}(Q, 6)$.

Finally, the first isomorphism in (2.11) comes from [Sch01], Theorem 5.1, and in our picture it is clear after we notice that, with the notation from the appendix, the correspondence with the data in [Sch01] is given by the net of quadrics
$$q : W \hookrightarrow S^2U, \quad q(W) = \Delta,$$
and by the 7-dimensional space annihilated by Δ
$$(\Delta^\perp)_2 = L \subset S^2U' = L \oplus W',$$
$$(\Delta^\perp)_2 = \langle u_0^2, u_2u_3, u_3u_1, u_1u_2, u_0u_3 + u_1^2, u_0u_1 + u_2^2, u_0u_2 + u_3^2 \rangle.$$

Then we get that $\eta_q = \eta_{\mathrm{Klein}}$, where η_q is a skew-symmetric matrix one can recover from the resolution of the module $SU'/(\Delta^\perp)$. Let $H(q)$ denote the variety of twisted cubics $\mathcal{C} \subset \mathbb{P}^3(U)$ whose equations $H^0(\mathbb{P}^3(U), I_{\mathcal{C}}(2)) \subset S^2U'$ are annihilated by q. Since a twisted cubic is defined by the quadrics that contains it and $h^0(\mathbb{P}^3(U), I_{\mathcal{C}}(2)) = 3$, $H(q)$ is a subset of $\mathbb{G}(3, V_q)$ in a natural way. Then one can prove that η_q is a net of alternating forms on V_q which defines $H(q) \subset \mathbb{G}(3, V_q)$, namely
$$H(\Delta) \cong \mathbb{G}(3, V_q, \eta_q) = \mathbb{G}(3, V_\Delta, \eta_{\mathrm{Klein}}).$$

In our picture we need the fact that L is self dual, then the copies of L we use are

$$(f_{\mathrm{Klein}}^{\perp})_3^* \subset S^3W' = L \oplus W$$

and

$$V_\Delta \subset S^2U' = L \oplus W'.$$

We describe the $\mathrm{SL}_2(\mathbb{Z}_7)$-isomorphism

$$S^3W = L \oplus W' = S^2U', \tag{2.13}$$

and that of their dual spaces:

$$y_1y_2y_3 = u_0^2, \quad y_2y_3^2 = u_2u_3, \quad y_3y_1^2 = u_3u_1, \quad y_1y_2^2 = u_1u_2,$$
$$y_3^2y_1 - y_2^3 = u_0u_3 + u_1^2, \quad y_1^2y_2 - y_3^3 = u_0u_1 + u_2^2, \quad y_2^2y_3 - y_1^3 = u_0u_2 + u_3^2.$$

And in terms of the duals we get again the equations (B.8):

$$v_1v_2v_3 = a_0^2, \quad v_2v_3^2 = 2a_2a_3, \quad v_3v_1^2 = 2a_3a_1, \quad v_1v_2^2 = 2a_1a_2,$$
$$v_3^2v_1 - v_2^3 = 2a_0a_3 + a_1^2, \quad v_1^2v_2 - v_3^3 = 2a_0a_1 + a_2^2, \quad v_2^2v_3 - v_1^3 = 2a_0a_2 + a_3^2,$$

where the a_i's are the duals of the u_i's.

The $\mathrm{SL}_2(\mathbb{Z}_7)$-isomorphism between the above copies of W' is given by

$$3y_3^2y_1 + y_2^3 = u_0u_3 - u_1^2, \quad 3y_1^2y_2 + y_3^3 = u_0u_1 - u_2^2, \quad 3y_2^2y_3 + y_1^3 = u_0u_2 - u_3^2,$$

and this leads to two isomorphic representations of W given by

$$3v_3^2v_1 + v_2^3 = 2a_0a_3 - a_1^2, \quad 3v_1^2v_2 + v_3^3 = 2a_0a_1 - a_2^2, \quad 3v_2^2v_3 + v_1^3 = 2a_0a_2 - a_3^2.$$

This allows us to map the generators of a twisted cubic curve, namely the minors of a 3×2 matrix with linear entries in U', which are annihilated by $\Delta \cong W$, to the generators of a 3-dimensional subspace of $(f_{\mathrm{Klein}}^{\perp})_3^*$, as shown in the next diagram:

$$\begin{array}{rcccl}
\mathbb{G}(3, L, \eta_{\mathrm{Klein}}) \ni & E' & \subset & (f_{\mathrm{Klein}}^{\perp})_3^* & \cong L \subset S^3W' = L \oplus W \\
& \uparrow & & \|\wr & \\
& \Phi & & (f_{\mathrm{Klein}}^{\perp})_3 & \cong L \subset S^3W = L \oplus W' \\
& \downarrow & & \|\wr & \\
H(\Delta) \ni & E & \subset & \Delta^{\perp} & \cong L \subset S^2U' = L \oplus W'
\end{array} \tag{2.14}$$

Remark 2.14. The $\mathrm{SL}(\mathbb{Z}_7)$-isomorphism Φ can now be computed, because by A.1 and (1.1) we get

$$(v_1v_2v_3 \leftrightarrow u_0^2, v_2^2v_3 - v_1^3 \leftrightarrow u_1u_2, \ldots, v_3^2v_2 \leftrightarrow u_0u_2 + u_3^2).$$

Chapter 3

The toroidal compactification

In this brief chapter we are going to present some relevant results of [HKW93] on the toroidal compactification of the moduli space of $(1, p)$-polarised abelian surfaces $\mathcal{A}^*_{(1,p)}$, restricting our attention to the case $(1, 7)$.

We are also going to report some basic fact on the Horrocks-Mumford bundle F and Horrocks-Mumford map from $\mathcal{A}^*_{(1,5)}$ to $\mathbb{P}(\Gamma(F))$, for future comparisons with our results.

3.1 The six odd 2-torsion points in the $(1, 5)$ case

The moduli space of *Horrocks-Mumford Surfaces* is given by $\mathbb{P}(\Gamma(F))$, where F is the *Horrocks-Mumford bundle* on $\mathbb{P}^4(V)$, cf. [HM73].

The normaliser of H_5 is

$$N = H_5 \rtimes \mathrm{SL}_2(\mathbb{Z}_5),$$

and $\Gamma(F)$ is irreducible and isomorphic to χ_4, the unique 4-dimensional representation of $\mathrm{SL}_2(\mathbb{Z}_5)$ which factors through $\mathrm{PSL}_2(\mathbb{Z}_5)$ (for a list of all irreducible $\mathrm{SL}_2(\mathbb{Z}_5)$-modules see the appendix of [HM73]).

Under the involution $\iota(x_i) = x_{-i}$ we get two eigenspaces, and we are interested in $\mathbb{P}^1_+ = \mathbb{P}(V_+)$. Via a suitable action of $\mathrm{SL}_2(\mathbb{Z}_5)$ on $\mathbb{P}^1_+$ compatible with the action on $\mathbb{P}^4(V)$ one gets that the space $\Gamma(\mathcal{O}_{\mathbb{P}^1_+}(6))$ is a $\mathrm{SL}_2(\mathbb{Z}_5)$-module and as such it splits into two irreducible factors of dimension 4 and 3 respectively. More precisely

$$\Gamma(\mathcal{O}_{\mathbb{P}^1_+}(6)) \cong \chi_4 \oplus \chi_3,$$

And $\chi_4 \cong \Gamma(F)$. $\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^1_+}(6)))$ parametrises sets of six points in $\mathbb{P}^1_+$, and a surface X_s defined by a section s of the Horrocks-Mumford bundle is fully determined by the (unordered) 6-tuple $X_s \cap \mathbb{P}^1_+$. The multiplicities of this 6-tuple also describe the type of surface ([BHM87]):

Table 3.1: Multiplicities of 2-torsion points in $\mathbb{P}^1_+ \subset \mathbb{P}^4(V)$

multiplicities	type of X_s
111111	abelian surface
21111	translation scroll
3111	tangent scroll
222	double elliptic quintic scroll
2211	union of five quadrics
42	five planes with a double structure

We explain these types of surface. If E is a quintic elliptic normal curve H_5-equivariantly embedded, then for every point $e \in E$ with $2e \neq 0$ we define a translation scroll X to be the union of secants

$$X = \bigcup_{P \in E} \overline{P, P+e}.$$

The surface X has degree 10 and its singular locus is the curve E, where X has transversal A_1-singularities.

If $e = 0$ in the above construction we obtain the *tangent scroll* of E. If e is a non-zero 2-torsion point then the secants $\overline{P, P+e}$ and $\overline{P, P-e}$ coincide and set-theoretically X is a elliptic quintic scroll. Since $\deg X = 10$ the zero locus of the surface supported on X has a double structure.

If finally the elliptic curve E degenerates to a pentagon of lines, the translation scroll degenerates to a union of five quadrics. These can degenerate further to a union of five planes again with double structure.

Then the multiplicities listed before are just the multiplicities with which the elliptic curve E (as a singular locus in the scrolls) intersects $\mathbb{P}^1_+$.

Remark 3.1. It seems reasonable to bear in mind this strategy in our case $p = 7$. Indeed we have just seen that $H(\Delta) \cong \mathrm{VSP}(Q, 6)$. In what follows we will comment our results with several remarks in order to compare them with the features of this nice case.

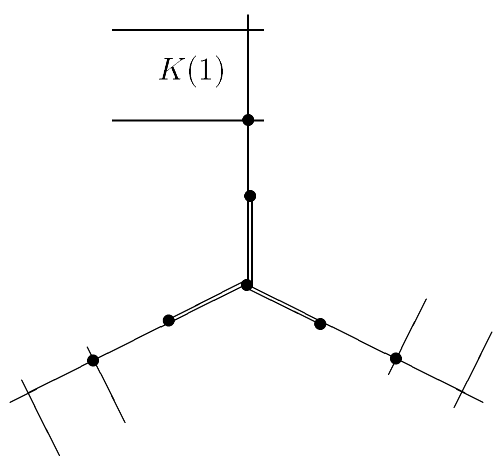
We are going to give more (known) information about $\mathbb{P}(\Gamma(F))$ and its relation with $\mathcal{A}^*_{(1,5)}$ in Section 3.3.

3.2 The toroidal compactification of $\mathcal{A}_{(1,7)}$

According to what we saw in Section 1.3, in the toroidal compactification $\mathcal{A}^*_{(1,7)}$ the open central boundary component is an open Kummer surface $K^0(7)$ and the 24 open peripheral boundary components are open Kummer surfaces $K^0(1)$, which are fibre spaces over the modular curve $X^0(7)$ (resp. $X^0(1)$) with rational fibres. Notice that as we saw in Proposition 2.9, the boundary B of $H(\Delta)$ is birational to the Kummer surface $K^0(7)$.

The closure of $X^0(7)$ is given by adding 24 cusps. Over each cusp the fibre in the closure of the central boundary component is a string of four rational curves. Inside the central boundary component the strings are divided into eight sets of three strings with six curves pairwise identified. Each set of strings with identification is the configuration of the corank 2 boundary component (cf. [HKW93], chap. 4A and the next picture). The intersections of these rational curves are called deepest points. The corank 2 boundary components lie in the closure of the central boundary component, and the closure of every peripheral boundary component meets the fibre over the corresponding cusp at the outer rational curve of the configuration shown in Figure 3.1. There is a terminology for the corank 2 boundary component, namely the outer rational curves are called cp-lines, the next rational curves are called adjacent cc-lines, and all the others are called non adjacent cc-lines.

Figure 3.1: Corank 2 boundary component, case $p = 7$



In the following list we turn our attention to the degenerations of abelian surfaces associated to points of the boundary of the toroidal compactification $\mathcal{A}^*_{(1,p)}$. This information is gathered from Definition 3.8 and Propositions 4.5, 4.7, 4.8 and 4.10 of part II in [HKW93].

Let $x \in \mathcal{A}^*_{(1,p)}$. Then

1. if x is in the open central boundary component, the corresponding surface is an elliptic ruled surface.

2. if x is in one of the open peripheral boundary components, the corresponding surface is a chain of p irreducible elliptic ruled surfaces.

3. if x is in the corank 2 boundary component, but not a deepest point, the corresponding surface is a chain of p quadrics, i.e. $\mathbb{P}^1 \times \mathbb{P}^1$.

4. if x is a deepest point, the corresponding surface consists of $2p$ copies of $\mathbb{P}^2$ and p copies of $\tilde{\mathbb{P}}^2$, where $\tilde{\mathbb{P}}^2$ denotes $\mathbb{P}^2$ blown up in three points.

3.3 The Horrocks-Mumford map

We now turn our attention to the Horrocks-Mumford map

$$\phi_5 : \mathcal{A}^0_{(1,5)} \to \mathbb{P}(\Gamma(F))_{\text{smooth}}.$$

This information and the notation is gathered from Theorems 1.5, 1.9 and 3.1 of part III in [HKW93].

Theorem 3.2 ([HKW93]). *The Horrocks-Mumford isomorphism ϕ can be extended to a morphism*

$$\phi_5^* : \mathcal{A}^*_{(1,5)} \to \mathbb{P}(\Gamma(F)).$$

Moreover the following holds:

1. *ϕ^* maps the closure of H_1^* of the Humbert surface $H_1 \subset \mathcal{A}_{(1,5)}$ onto the curve C_{12} which parametrises tangent scrolls.*

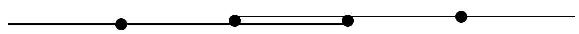
2. *ϕ_5^* maps the closure of H_2^* of the Humbert surface $H_2 \subset \mathcal{A}_{(1,5)}$ onto the curve C_6 which parametrises double elliptic scrolls.*

3. *ϕ_5^* maps the closed central boundary component birationally onto the surface $\mathbb{P}(\Gamma(F))_{\text{sing}}$ of singular HM-surfaces, namely scrolls over elliptic curves with a level 5 structure.*

4. *Under ϕ_5^* each of the* 12 *peripheral boundary components is contracted to a point corresponding to the union of five planes with double structure.*

5. *All the projective lines in each of the six corank* 2 *boundary components (see Figure 3.2), except for the non adjacent cc-lines, are contracted to points corresponding to the union of five planes with double structure. The six non-adjacent cc-lines are mapped to the six double tangents of C_6 parametrising singular HM-surfaces which are the union of five quadrics or union of five planes with double structure.*

Figure 3.2: Corank 2 boundary component, case $p = 5$



First notice that the different combinatorics of the $(1,5)$ and $(1,7)$ cases give rise to different degenerations over the cusps. In fact the two deepest points of a non adjacent line in a corank 2 boundary component in the $(1,5)$ case are in the same orbit with respect to $\mathrm{SL}_2(\mathbb{Z}_5)$, and the degenerations they parametrise are the same, see Remark 4.20. But in the $(1,7)$ case the *central* deepest point common to the three chains of rational curves is not in the same orbit of the other ones, and the degenerations they parametrise, as we shall see, are different.

And then, could it be that it is possible to find a map

$$\tilde{\phi}_7^* : \mathcal{A}_{(1,7)}^* \to H(\Delta)$$

like ϕ_5^*? In fact

$$s := \begin{pmatrix} u_1 & 0 \\ u_2 & u_2 \\ 0 & u_3 \end{pmatrix}$$

lies in the intersection of the $\mathbb{P}^1$'s over the cusps u_1, u_2, u_3 in $H(\Delta)$. Perhaps these fibres are the non-contracted (by ϕ_7^*) non adjacent cc-lines as in the $(1,5)$ case.

Chapter 4

Degenerations

In this chapter we are going to work out the surface associated to a given boundary point $[\alpha] \in B$ of the moduli space $H(\Delta)$, where α is a matrix in M.

Furthermore we will relate degenerations of twisted cubic curves (i.e. elements of $H(\Delta)$) with degenerations of six general points in the variety of sum of powers of Q.

4.1 Existence of surfaces related to degenerations

Here we are going to prove that given any element $[\alpha] \in B$ as above, that is, a map

$$3V_4\mathcal{O}(-3) \xleftarrow{\alpha} 2S\Omega^3,$$

we can actually find a map

$$\mathcal{O} \xleftarrow{\beta} 3V_4\mathcal{O}(-3), \tag{4.1}$$

and then a resolution exactly as in Theorem 1.1.

Remark 1.3 allows us to prove the following

Proposition 4.1. *For any α with $\tilde{\alpha} \in B$ we can construct a complex like*

$$0 \leftarrow \mathcal{O}_A \leftarrow \mathcal{O} \xleftarrow{\beta} 3V_4\mathcal{O}(-3) \xleftarrow{\alpha} 2S\Omega^3 \xleftarrow{\alpha'} 3V_1\mathcal{O}(-4) \xleftarrow{\beta'} \mathcal{O}(-7) \leftarrow 0. \tag{4.2}$$

Proof. In the light of the last remark, we get from the appendix that

$$\mathrm{Hom}_{G_7}(V_4\mathcal{O}(-3), \mathcal{O}) = I \oplus U' \oplus L,$$

so the map in (4.1) is naturally given by the three minors $(q_1(\alpha), q_2(\alpha), q_3(\alpha))$ of α, because by Proposition 1.4 the condition Δ implies

$$\{q_1(\alpha), q_2(\alpha), q_3(\alpha)\} \subset L \subset S^2U'.$$

We can then write a sequence exactly like (4.2), and the N-homomorphism in (A.1) between the copy of the $\mathrm{SL}_2(\mathbb{Z}_7)$-module $L \subset S^2U'$ and

$$L \otimes V_4 \subset S^3V_3 = (I \oplus U' \oplus L) \otimes V_4,$$

yields the 21 cubic generators of a variety A_α associated to a given $[\alpha] \in H(\Delta)$, when we view $H(\Delta)$ as the space of twisted cubic curves in $\mathbb{P}^3(U)$.

The sequence (4.2) is indeed a complex: $\beta\alpha = 0$ can be computed (carefully and patiently) using the exactness of (2.2), using the fact that

$$\wedge^3 V \otimes \mathcal{O}(-4) \overset{\sigma}{\twoheadrightarrow} \Omega^3$$

is surjective and so implies

$$\alpha(2S\Omega^3) = \alpha(2\sigma(\wedge^3 V \otimes \mathcal{O}(-4))),$$

and by the computations of the compositions of these maps on page 19. An example: consider the first syzygy α^1, that is, the first column of $(au_0^\#, bu_0^\#, cu_0^\#)^t = (2a_0x_0, a_1x_4, a_2x_1, a_3x_5, \dots)^t$ from the composition $\alpha\sigma$; for generality assume $a_0 \neq 0$. The composition $\beta\alpha^1$ is a polynomial of degree 4. Then the coefficient of, e.g., $x_0x_1^2x_5$ of $\beta\alpha^1$ is given by

$$\begin{aligned}
&2a_0(b_0e_2 + b_2e_0 - c_0d_2 - c_2d_0) + a_2(b_0e_0 - c_0d_0) + a_3(b_0e_3 + b_3e_0 - c_0d_3 - c_3d_0)\\
&-2b_0(a_0e_2 + a_2e_0) - b_2(a_0e_0) - b_3(a_0e_3 + a_3e_0)\\
&+2c_0(a_0d_2 + a_2d_0) + c_2(a_0d_0) + c_3(a_0d_3 + a_3d_0).
\end{aligned}$$

and by the condition Δ the coefficient of u_0u_2 equals that of u_3^2, therefore we get that the previous expression equals

$$\begin{aligned}
&a_0(b_0e_2 + b_2e_0 - c_0d_2 - c_2d_0) + a_0(b_3e_3 - c_3d_3)\\
&+a_2(b_0e_0 - c_0d_0) + a_3(b_0e_3 + b_3e_0 - c_0d_3 - c_3d_0)\\
&-b_0(a_0e_2 + a_2e_0) - b_0(a_3e_3) - b_2(a_0e_0) - b_3(a_0e_3 + a_3e_0)\\
&+c_0(a_0d_2 + a_2d_0) + c_0(a_3d_3) + c_2(a_0d_0) + c_3(a_0d_3 + a_3d_0) = 0.
\end{aligned}$$

Clearly $\beta\alpha = 0$ implies $\alpha'\beta' = 0$, and finally the Δ condition on α guarantees that the composition $\alpha\alpha'$ equals zero (on this last fact, see [MS01], Proposition 3.5). $\square$

For the next proofs we are going to use the following

Theorem 4.2 ([Eis95], Theorem 20.9). *Let R be a ring. A complex*

$$0 \to F_n \stackrel{\phi_n}{\to} F_{n-1} \to \cdots \to F_1 \stackrel{\phi_1}{\to} F_0$$

of free R modules is exact if and only if

1. $\operatorname{rank} F_k = \operatorname{rank} \phi_k + \operatorname{rank} \phi_{k+1}$, *and*

2. $\operatorname{depth} I(\phi_k) \geq k$

for k=1,...,n.

The notation $I(\phi_k)$ stands for the ideal generated by the minors of ϕ_k of dimension equal to rank ϕ_k. By [Eis95], Theorem 18.7, we can use the codimension of variety determined by $I(\phi_k)$ in place of its depth.

In order to prove the exactness of the complex (4.2) we need to prove first another interesting result. In our attempt to classify the degenerations we are after, we want to exploit the fact that a matrix

$$\alpha = \begin{pmatrix} a & 0 \\ b & d \\ c & e \end{pmatrix},$$

as in the proof of Lemma 2.6, determines a sub-morphism

$$2V_4\mathcal{O}(-3) \stackrel{\binom{d}{e}}{\leftarrow} S\Omega^3.$$

More precisely the last morphism comes from the complex (4.2) determined by a boundary point $[\alpha] \in H(\Delta)$, as shown in the next diagram:

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & V_4\mathcal{O}(-3) & \stackrel{a}{\longleftarrow} & S\Omega^3 & & \\
 & & & & & & \uparrow & & \uparrow & & \\
0 & \leftarrow & \mathcal{O}_{\mathcal{A}} & \leftarrow & \mathcal{O} & \stackrel{\beta}{\longleftarrow} & 3V_4\mathcal{O}(-3) & \stackrel{\alpha}{\longleftarrow} & 2S\Omega^3 & \leftarrow & \ldots \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & 2V_4\mathcal{O}(-3) & \stackrel{\binom{d}{e}}{\longleftarrow} & S\Omega^3 & & \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & 0 & & 0 & &
\end{array}$$

Notice that all the boundary points which are images of orbits with a matrix with a top-left entry like a will admit the sub-morphism as above.

More precisely, starting off with the ideal I_a generated by the 14 cubics defined by the minors $q_2(\alpha)$ and $q_3(\alpha)$, we can extract from the complex (4.2) the following one:

$$0 \leftarrow I_a \leftarrow 2V_4\mathcal{O}(-3) \xleftarrow{\binom{d}{e}} S\Omega^3 \xleftarrow{a} V_1\mathcal{O}(-4) \leftarrow 0. \tag{4.3}$$

We are in position to prove the following

Proposition 4.3. *Let $a \in C$. Then the complex* (4.3) *defined by the ideal I_a is exact, and therefore defines a variety $\mathcal{U}_a$ of dimension* 3 *and degree* 7*.*

Proof. First of all we need to prove the exactness of the complex. We are going to use Theorem 4.2. Notice that since I_a is not contained in any hyperplane, we can localise at $x_i \neq 0$, where Ω^3 is free.

To test the first condition, notice that $\operatorname{rank} 2V_4\mathcal{O}(-3) = 14$ and $\operatorname{rank} S\Omega^3 = 20$. Since $\alpha \in \mathbb{P}(M)^s$, all the cubics are non zero, and then the first map has rank 1, and trivially the variety the 14 cubics determine is of non-zero codimension. Now we need $\operatorname{rank} \binom{d}{e} = 13$ and $\operatorname{rank} a = 7$. Observe that from the exact complex

$$0 \to \Omega^3 \to \wedge^3 V^* \otimes \mathcal{O}(-3) \to \Omega^2 \to 0$$

there is an injective map

$$\Omega^3 \overset{\iota}{\hookrightarrow} \wedge^3 V^* \otimes \mathcal{O}(-3),$$

therefore we get

$$\operatorname{rank} a(V_1\mathcal{O}(-4)) = \operatorname{rank} \iota(a(V_1\mathcal{O}(-4)))$$

as well as

$$\begin{pmatrix} d \\ e \end{pmatrix}(\Omega^3) = \begin{pmatrix} d \\ e \end{pmatrix}(\sigma(\wedge^3 V \otimes \mathcal{O}(-4))).$$

Now let $a \neq d \in C \subset \mathbb{P}^3(U')$ (see Proposition 2.9), and let $w_d \in U'$ such that the composition $dw_d = 0$. Then we can find a submatrix η of $\binom{d}{e}(\sigma(\wedge^3 V \otimes \mathcal{O}(-4)))$ like

$$\eta := \begin{pmatrix} du_0^{\#} & 0 \\ * & ew_d \end{pmatrix}.$$

From the computations of page 19, we get that $du_0^{\#} = a_0\omega + a_1\nu^* + a_2\lambda^* + a_3\mu^*$ has maximal rank, and since the Pfaffians of ew_d are non-zero, we conclude that $\operatorname{rank} \binom{d}{e} = 13$.

The variety determined by the 13×13 minors is contained in the union of the variety determined by the Pfaffians of ew_d, which is a 3-fold, and by the one determined by the determinants of all the top left blocks of rank 7. The proof that the latter one is of dimension at most 3 is exactly as in the following test for the exactness at the map a, after replacing d with a.

We have

$$\iota(a(V_1\mathcal{O}(-4))) = (u_0^{\#}a|a_0a|u_1a|u_2a|u_3a)$$

and the block $u_0^{\#}a$ has maximal rank as before. About the second condition of exactness, notice that the Pfaffian varieties V_{u_ia} determined by u_ia, $i = 0, \ldots, 3$ are 3 dimensional. Now let $x \in \mathbb{P}^6(V)$ such that $x \notin V_{u_ia}$ for some i, but x belongs to the variety Z determined by the 7×7 minors of $\iota(a(V_1\mathcal{O}(-4)))$, which is clearly symmetric with respect to G_7. This means that there is a $j \in \{1, \ldots, 7\}$ so that $\mathrm{Pfaff}_j(u_ia)(x) \neq 0$. By G_7-symmetry we can assume that $j = 1$, so there is a linear combination of columns of $u_ia(x)$ such that it is equivalent to

$$g(x) := \begin{pmatrix} 0 & * \\ 0 & T \end{pmatrix}.$$

Where T is an invertible matrix. If a is one of the three elements with 0-entries, we can compute by hand that after rearranging the columns of $\iota(a(V_1\mathcal{O}(-4)))$ we get upper triangular matrices with the seven x_i's as entries. Taking suitable columns we see that the intersection of these determinants is contained in projective subspaces of codimension 3.

So let a have all the entries non-zero, and take a column k of $u_0^{\#}a$ such that the top entry $k_1(x) \neq 0$. Substituting $k(x)$ in place of the first column of $g(x)$ we get that the rank is maximal, a contradiction. Therefore

$$Z \subset \bigcup_{i=0,\ldots,3} V_{u_ia}$$

and then $\dim(W) \leq 3$ and the complex is exact.

Finally, the Hilbert polynomial of I_a: let K be the kernel of $I_a \leftarrow 2V_4\mathcal{O}(-3)$ as a map of direct sums of line bundles. Then, exactly as in [MS01], Theorem 2.5, by "*kind of adding*" a piece of the Koszul complex to the resolution (4.3), we obtain the following commutative exact diagram

$$\begin{array}{ccccccccccccc}
 & & & & 0 & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & & & 7\mathcal{O}(-4) & & & & & & & & \\
 & & & \overset{a}{\swarrow} & \downarrow & & & & & & & & \\
0 & \leftarrow & \Omega^3 & \leftarrow & 35\mathcal{O}(-4) & \leftarrow & 21\mathcal{O}(-5) & \leftarrow & 7\mathcal{O}(-6) & \leftarrow & \mathcal{O}(-7) & \leftarrow & 0 \\
 & & \binom{d}{e}\downarrow & \swarrow & \downarrow & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\
0 & \leftarrow & K & \leftarrow & 28\mathcal{O}(-4) & \leftarrow & 21\mathcal{O}(-5) & \leftarrow & 7\mathcal{O}(-6) & \leftarrow & \mathcal{O}(-7) & \leftarrow & 0 \\
 & & & & \downarrow & & & & & & & & \\
 & & & & 0 & & & & & & & &
\end{array}$$

The bottom row leads to a resolution of $\mathcal{U}_a$ involving only direct sums of line bundles, namely

$$0 \leftarrow I_{\mathcal{U}_a} \leftarrow 14\mathcal{O}(-3) \leftarrow 28\mathcal{O}(-4) \leftarrow 21\mathcal{O}(-5) \leftarrow 7\mathcal{O}(-6) \leftarrow \mathcal{O}(-7) \leftarrow 0$$

Now we can compute the Hilbert polynomial of $\mathcal{U}_a$, from which the result follows

$$P_{\mathcal{U}_a} = \frac{7}{6}n^3 + \frac{7}{2}n^2 + \frac{7}{3}n.$$

□

Thanks to this intermediate result we can prove now the main proposition of this section:

Proposition 4.4. *The complex* (4.2) *is exact. Therefore α defines a variety A_α of dimension* 2 *and degree* 14.

Remark 4.5. We are not claiming that the statement of Proposition 4.4 holds for every $[\alpha] \in H(\Delta)$.

Proof. We know by [MS01] that for a general point of $H(\Delta)$ the complex is exact. So we restrict our attention to the case when α represents an element of $B \subset H(\Delta)$, which we write as in (2.3). As before, A_α is not contained in any hyperplane, so we can localise at $x_i \neq 0$, where Ω^3 is free.

To test the first condition, notice that $\operatorname{rank} 3V_4\mathcal{O}(-3) = 21$ and $\operatorname{rank} 2S\Omega^3 = 40$. Since $\alpha \in \mathbb{P}(M)^s$, $\beta \neq 0$, so $\operatorname{rank}\beta = 1$, and thus the only thing we need is $\operatorname{rank}\alpha = 20$. Observe that $\operatorname{rank}\alpha = \operatorname{rank}\alpha'$, in fact from the injective map

$$\Omega^3 \overset{\iota}{\hookrightarrow} \wedge^3 V^* \otimes \mathcal{O}(-3),$$

using the definition of α' and the fact that the entries of α' are just transposes of the entries of α we get

$$\mathrm{rk}\alpha'(3V_1\mathcal{O}(-4)) = \mathrm{rk}2\iota(\alpha'(3V_1\mathcal{O}(-4))) = \mathrm{rk}(\alpha(2\sigma(\wedge^3 V \otimes \mathcal{O}(-4))))^t.$$

So with $\operatorname{rank}\alpha = 20$ we shall have proved the first condition for the rest of the sequence as well.

Now let $\alpha_{22} = d \in C \subset \mathbb{P}^3(U')$ (see Proposition 2.9), and let $a \neq w_d \in U'$ such that the composition $dw_d = 0$. Then we can find a submatrix $\tilde{\alpha}$ of $\alpha(2\sigma(\wedge^3 V \otimes \mathcal{O}(-4)))$ like

$$\tilde{\alpha} := \begin{pmatrix} au_0^{\#} & 0 & 0 \\ * & du_0^{\#} & 0 \\ * & * & ew_d \end{pmatrix}.$$

From the computations of page 19, we get that $a(u_0^{\#})$ and $d(u_0^{\#})$ have maximal rank, and since the Pfaffians of ew_d are non-zero, we conclude that $\operatorname{rank}\alpha = 20$.

To test the second condition, first notice that any product of two Pfaffians of an antisymmetric matrix like ew_d can be computed as the determinant of a suitable 6×6 minor.

Secondly, we need to prove two things: that $\operatorname{codim} I_{\beta'}(= \operatorname{codim} I_\beta) \geq 4$ and $\operatorname{codim} I_{\alpha'}(= \operatorname{codim} I_\alpha) \geq 3$.

For the latter part, observe that we can extract matrices like $\tilde{\alpha}$ from α, but with the top and middle diagonal blocks being given by columns as in the proof of Proposition 4.3 about the codimension of Z determined by a. Therefore the variety Z' determined by the maximal rank minors of α is contained in the union of the varieties determined by the three blocks (which we can vary), whose codim ≥ 3.

For β', with the usual matrix, notice that by Proposition 4.3 the middle and bottom V_4's determine a threefold of degree 7. Because α is stable, the 21 cubics are independent, and symmetric with respect to G_7. Therefore all the syzygies of a resolution of A_α are G_7-modules, namely 7-dimensional vector spaces in general, except when the syzygies have degree multiple of 7. But in that case $\dim S^{7n}V^* \equiv 1$ modulo 7, and because the sum (with suitable sign) of those syzygies has to be precisely -1, the Hilbert polynomial of A_α is divisible by 7. Now consider the variety determined by all the 21 cubics. Because the resolution of $\mathcal{U}_a$ has length precisely three, it does not contain a lower dimensional (possibly embedded) component. In fact, if there was one, call it L and let I_L and I be the ideals of L

and of the union of the rest of the components. Then $I_a = I_L \cdot I$, and therefore a resolution of I_a would be at least as long as the resolution of L, namely at least 4. This leads to the fact that when we add the top 7 cubics to the 14 generating $\mathcal{U}_a$, the new variety we get, that is A_α, has to drop dimension, otherwise the degree would drop to some number not a multiple of 7. Hence we have proved that $\mathrm{codim} I_{\beta'} = \mathrm{codim} I_\beta \geq 4$.

The part on the degree and the dimension is straightforward once we notice that it holds for general $(1,7)$-polarised abelian surfaces, and that the shape of the resolution determines the Hilbert polynomial of the variety, which is a constant feature on the varieties parameterised by $H(\Delta)$. $\square$

Corollary 4.6. *Let $\alpha \in \mathbb{P}(M)$. Then the complex* (4.2) *associated to it is exact if and only if α is stable.*

Proof. If α is stable the exactness follows from the previous proposition.

If α is not stable, it is conjugate to an element as in (2.1), so there are at least 7 zero-generators. Then we get that at least a block-column of α is a zero block-syzygy and the complex is not exact. $\square$

Remark 4.7. By Proposition 2.11 we know that for a general $(1,7)$-polarised abelian variety the six points in W' in Proposition 1.9 must be the pre images of

$$\langle q_\alpha(1), q_\alpha(2), q_\alpha(3)\rangle \cap (W')^3|_L \subset L.$$

But $\langle q_\alpha(1), q_\alpha(2), q_\alpha(3)\rangle$ determines the 4-dimensional subspace of $L \subset S^3W$ given by the minors of ψ (see 2.12), and clearly we get that the space spanned by the third powers of the (possibly degenerate) six points restricted to L must be contained in, and generally equal to, $\langle q_\alpha(1), q_\alpha(2), q_\alpha(3)\rangle$.

4.2 General degenerations

So far we have proved that for all the varieties $\mathcal{A}_\alpha$ parameterised over the same $a \in C \subset \mathbb{P}^3(U)$ (see Proposition 2.9) we get $\mathcal{A}_\alpha \subset \mathcal{U}_a$. Again by Proposition 2.9 we also know that this occurs when the twisted cubic curve in $\mathbb{P}^3(U)$ defined by α is degenerate. Now we want to find out what this result means in terms of the other descriptions of $H(\Delta)$, and specifically $\mathrm{VSP}(Q, 6)$.

As usual we work with

$$\alpha = \begin{pmatrix} a & 0 \\ b & d \\ c & e \end{pmatrix}.$$

Proposition 4.8. *Let A_α be the variety associated to an element $[\alpha] \in B \subset H(\Delta)$, and let $A_\alpha \cap \mathbb{P}^2(W') = \{p_1, \ldots, p_6\}$.*

Then three of the six $p_i^3|_L \in L$, say p_1, p_2, p_3, lie on a line and therefore a statement like the second of Proposition 1.9.2 does not apply, namely A_α is not generated by $V_{7,p_1}, V_{7,p_2}, V_{7,p_3}$.

Proof. Let $w \notin \langle a \rangle$ be an element of $\mathrm{Hom}_{G_7}(V_1\mathcal{O}(-4), S\Omega^3) = U'$, that is, a linear combination of $\langle u_0, u_1, u_2, u_3 \rangle$, such that $e^* w = 0$ (see Remark 2.7). This requirement is non trivial, because in general $\mathrm{rk}(e^*) = 3$, therefore such a w does not exist. Nevertheless in the proof of Proposition 2.9 we saw that for a given a we can find in general three e's such that $a^* e = 0$ and $\mathrm{rk}(e^*) = 2$ ($\Leftrightarrow e \in C$).

From the Koszul complex of Ω^3 we get that $35\mathcal{O}(-4)$ maps surjectively to $S\Omega^3$, and we can consider the syzygies generated by the composition of w and $\binom{d}{e}$. In other words we get that $\binom{d}{e}(S\Omega^3)$ contains a column like

$$\begin{pmatrix} d \\ e \end{pmatrix}(w) = \begin{pmatrix} d^*w \\ e^*w \end{pmatrix} = \begin{pmatrix} d^*w \\ 0 \end{pmatrix}. \tag{4.4}$$

If $\mathcal{A}_\alpha$ is the degenerate $(1,7)$-polarised abelian surface $\mathcal{A}_\alpha$ associated to α, then the above column is nothing but 7 of the first 49 linear syzygies of the 21 cubics that define it. Bearing in mind Proposition 1.8 and the fact that $\alpha_{12} = 0$, we get that the middle V_4 that generates the ideal of $\mathcal{A}_\alpha$ must be given by the 7 principal Pfaffians of

$$M_7'(x, d^*w) = (w_0d_3 + w_3d_0 - d_1w_1)\mu + (w_0d_1 + w_1d_0 - d_2w_2)\nu + (w_0d_2 + w_2d_0 - d_3w_3)\lambda.$$

Notice that this does not depend on w, in fact $\{x \in \mathbb{P}^3(U') | \; e^*x = 0\} = \langle a, w \rangle$, thus $d^*(\xi a + \psi w) = \xi d^* a + \psi d^* w = 0 + \psi d^* w$. Furthermore $d^* w \neq (0:0:0)$, because even if $\mathrm{rk}(d^*) = 2$, $\{x \in \mathbb{P}^3(U') | \; d^* x = 0\} = \langle a, w' \rangle \not\ni w$, otherwise $e = d$. Observe that, by Proposition 4.4, $(d^*w)^3|_L = ae$.

The crucial observation is that because in general there are three such $e \in C$, call them e, f, g, then it is possible to find 3 corresponding sets of seven cubic (Pfaffian) generators of $\mathcal{A}_\alpha$ as above. Therefore, for a map like

$$\mathcal{O} \leftarrow 2V_4\mathcal{O}(-3) \xleftarrow{\binom{d}{e}} S\Omega^3$$

to exist, one d^*w_i, $i \in \{e, f, g\}$ has to be a linear combination of the others. But this is obvious by Remark 4.7, because

$$\dim\langle (d^*w_e)^3|_L,\ (d^*w_f)^3|_L,\ (d^*w_g)^3|_L\rangle = \dim\langle ae, af, ag\rangle = 2.$$

□

Remark 4.9. After tensoring with V_4 the three collinear elements

$$(d^*w_e)^3|_L,\ (d^*w_f)^3|_L,\ (d^*w_g)^3|_L \in L$$

of the previous proposition, they clearly generate the ideal of the variety $\mathcal{U}_a$ of Proposition 4.3.

Remark 4.10. By Proposition 4.3 and the previous Remark 4.9, we see that $\mathcal{U}_a$ must be contained in

$$V_{7,d^*w_i} \subset \mathbb{P}^6(V),\ i \in \{e, f, g\}.$$

Because $\mathcal{U}_a$ and V_{7,d^*w_i}'s are all 3-dimensional, we conclude that the V_{7,d^*w_i}'s are either non-reduced, or reducible. In the former case and by Proposition 1.8 part (2) we can argue that the d^*w_i's are not general points of $\mathbb{P}^2(W')$.

Clearly at this stage information about the nature of the three points d^*w_e, d^*w_f, $d^*w_g \in W'$ must be supplied. Therefore in the next proposition we are going to give a (slightly) computational analysis of the general picture we are dealing with.

Proposition 4.11. *Let $\mathcal{U}_a$ be as in Proposition 4.3. Then the three points d^*w_e, d^*w_f and $d^*w_g \in W'$ lie on the curve $y_1^5y_3 + y_2^5y_1 + y_3^5y_2 - 5y_1^2y_2^2y_3^2 = \mathrm{Hes}(y_1^3y_2 + y_2^3y_3 + y_3^3y_1) = 0$, the Hessian of the Klein quartic f'_{Klein} in $\mathbb{P}^2(W')$.*

Proof. What follows is restricted to the open part of $\mathbb{P}^2(W)_0 := \{(v_1 : v_2 : v_3) \in \mathbb{P}^2(W)|\ v_1v_2v_3 \neq 0\}$. In this way we lose three points of Q which, however, will be completely treated in the next section.

1. Q and C are isomorphic, so let $(v_1 : v_2 : v_3) \in Q \subset \mathbb{P}^2(W)_0$; then we get $a = \left(1; \dfrac{v_1}{v_3}; \dfrac{v_2}{v_1}; \dfrac{v_3}{v_2}\right) = (v_1v_2v_3 : v_1^2v_2 : v_2^2v_3 : v_3^2v_1) \in C$, namely a top-left entry of a matrix $\alpha \in \phi^{-1}(B)$.

2. Let $\{x \in \mathbb{P}^3(U')|\ a^*x = 0\} = \langle e,\ d\rangle$. To simplify the computations we assume $d_0 = 0$; then we get $d = \left(0; \dfrac{1}{v_1}; \dfrac{1}{v_2}; \dfrac{1}{v_3}\right) = (0 : v_2v_3 : v_1v_3 : v_2v_1)$ and let $e = (1; e_1; e_2; e_3)$ (up to scalar).

3. As in the proof of Proposition reffibre, assume $e \in C$. We saw in Lemma 2.11 that in general there are 3 such e's. Notice that this means that $\langle e,\ d\rangle$ is a tri-secant of C in $\mathbb{P}^3(U')$. Let $\{x \in \mathbb{P}^3(U')|\ e^*x = 0\} = \langle a, w\rangle$ with $w_0 = 0$.

4. The point of $\mathbb{P}^2(W')_0$ which yields $ea \otimes V_4$ up to scalar, as in Proposition 1.8, is given by $d^*w = (d_3w_3 : d_1w_1 : d_2w_2) := (y_1 : y_2 : y_3)$. We are abusing the notation, because actually we should be working with elements of W', not its projectivization.

   In other words
$$ea \otimes V_4 = I_3(d_3w_3 : d_1w_1 : d_2w_2).$$
   Hence $e = \left(1; \dfrac{y_1v_3}{y_2v_1}; \dfrac{y_2v_1}{y_3v_2}; \dfrac{y_3v_2}{y_1v_3}\right)$.

5. If e is a solution of a^*, the following equations have to vanish
$$a^*e = \begin{cases} y_2y_3v_1^2 + y_3y_1v_3^2 - y_2^2v_3v_1 := \sigma_1 \\ y_3y_1v_2^2 + y_1y_2v_1^2 - y_3^2v_1v_2 := \sigma_2 \\ y_1y_2v_3^2 + y_2y_3v_2^2 - y_1^2v_2v_3 := \sigma_3. \end{cases}$$

Let S be the variety in $\mathbb{P}^2(W) \times \mathbb{P}^2(W')$ defined by $\sigma_1,\ \sigma_2$ and σ_3, and consider the projections

$$\begin{array}{ccccc} & & \mathbb{P}^2(W) \times \mathbb{P}^2(W') & \supset S & \\ & \overset{p_1}{\swarrow} & & \overset{p_2}{\searrow} & \\ \mathbb{P}^2(W) & & & & \mathbb{P}^2(W'). \end{array}$$

It is easy to see that $v_1v_3y_3\sigma_1 + v_1v_2y_1\sigma_2 + v_2v_3y_2\sigma_3 = y_1y_2y_3 \cdot (v_1^3v_2 + v_2^3v_3 + v_3^3v_1)$, thus $p_1(S \cap \mathbb{P}^2(W)_0 \times \mathbb{P}^2(W')_0) = Q \cap \mathbb{P}^2(W)_0$. Furthermore p_1 is generically $3:1$.

In the same way, but through a computation with *Maple*, we see that

$$v_1^2v_2^2v_3^2y_3^3y_2 \cdot (y_1^5y_3 + y_2^5y_1 + y_3^5y_2 - 5y_1^2y_2^2y_3^2)^2 \in I(\sigma_1, \sigma_2, \sigma_3),$$

thus $p_2(S \cap \mathbb{P}^2(W)_0 \times \mathbb{P}^2(W')_0) = \{y_1^5y_3 + y_2^5y_1 + y_3^5y_2 - 5y_1^2y_2^2y_3^2 = 0\} \cap \mathbb{P}^2(W')_0$.

Observe that the initial restriction to $\mathbb{P}^2(W)_0$ does not change the result, in fact starting with $(1:0:0) \in W$ we get the elements $(0:1:0),\ (0:0:1) \in \mathrm{Hes}(Q')$, and by the usual action of $\mathrm{PSL}_2(\mathbb{Z}_7)$ on Q this is enough. $\square$

In Table 4.1 we show this construction, and by abuse of notation we identify vectors and their projective classes.

Table 4.1: Construction of 3 points in $\mathrm{Hes}(Q')$ from $v \in Q \subset \mathbb{P}^2(W)$

$$\begin{tikzcd}
\boxed{v \in Q \subset \mathbb{P}^2(W)} \arrow[d] & & \\
\boxed{a \in C \subset \mathbb{P}^3(U')} \arrow[d] & & \\
\boxed{\begin{array}{c} S = \{x \in U' \mid a^* x = 0\} \subset \mathbb{P}^3(U') \\ \text{3-secant of } C \text{ at } e, f, g \end{array}} \arrow[d] & & \\
\boxed{\begin{array}{c} \langle a, w_k \rangle = \{x \in U' \mid k^* x = 0\} \\ \text{3-secants of } C.\ k \in \{e, f, g\} \end{array}} \arrow[rr] \arrow[dr] & & \boxed{\begin{array}{c} d^* w_k \in \mathrm{Hes}(Q') \subset \mathbb{P}^2(W') \\ d \in S,\ k \in \{e, f, g\} \end{array}} \arrow[dl] \\
& \boxed{\begin{array}{c} \boxed{\begin{array}{c} ak = (d^* w_k)^3|_L \in L \\ k \in \{e, f, g\} \end{array}} \\ \\ \boxed{\begin{array}{c} \langle ae, af, ag \rangle \subset \mathbb{P}^6(L) \\ \text{3-secant of } \mathbb{P}^2(W')^3|_L \subset \mathbb{P}^6(L) \end{array}} \end{array}} &
\end{tikzcd}$$

Remark 4.12. The construction of the last proof associates to every point of $Q \subset \mathbb{P}^2(W)$ three points of $\mathrm{Hes}(Q')$ which we know are determined by a and the trisecant of $C \subset \mathbb{P}^3(U')$ via the product $*$. Then, by Proposition 4.8, the projective classes of their third power restricted to L lie on a trisecant of $\mathbb{P}^2(W')^3|_L \subset \mathbb{P}^6(L)$.

Moreover observe that $p_2(S)$ is, by construction, a $\mathrm{PSL}_2(\mathbb{Z}_7)$-invariant curve.

The Hessian of Q' seemingly carries special information about $V_{7,y} \subset \mathbb{P}^6(V)$, for $y \in \mathbb{P}^2(W')$. And in fact for every $y \in \mathrm{Hes}(Q')$ there is some $v = p_1(p_2^{-1}(y)) \in Q$ that, by the results of this section and the previous construction, determines a variety $V_{7,y} \subset \mathbb{P}^6(V)$ which is either non-reduced, or reducible.

All the results of this section can be summarised in the next proposition.

Proposition 4.13. *Let $[\alpha] \in B$ be a general boundary point of $H(\Delta)$, i.e. $[\alpha] \in B \setminus B' \subset H(\Delta)$ and a is not a cusp.*

Then the degenerate $(1,7)$-polarised abelian variety A_α associated to α is the intersection of a 3-fold $\mathcal{U}_a$ of degree 7 and a Calabi-Yau 3-fold.

Moreover $\mathcal{U}_a$ is generated by three distinct collinear points of $\mathbb{P}^6(L)$ tensored by V_4. These are images of three points of $\mathrm{Hes}(Q') \subset \mathbb{P}^2(W')$ uniquely determined by the top-left entry $a \in C$ of α.

Proof. By Remark 4.7 we know that the ideal generated by the minors of α is also generated by the image of six points $\{p_1, \dots, p_6\} \in W'$ as in (1.9).

By Proposition 4.8 we know that the images in $\mathbb{P}^6(L)$ of three of these six points, say $\{p_1, p_2, p_3\}$ are collinear, and by Remark 4.9 we know that (once tensored by V_4) they generate the ideal of a 3-fold $\mathcal{U}_a$ of degree 7 and are determined by a.

Clearly we have that $A_\alpha = \mathcal{U}_a \cap V_{7,p_i}$ for any $i \in \{4,5,6\}$.

Finally, by Proposition 4.11 we know that $\{p_1, p_2, p_3\} \subset \mathrm{Hes}(Q') \subset \mathbb{P}^2(W')$. □

Remark 4.14. From the previous proposition we can draw a crucial comparison with the $(1,5)$-polarisation case, see [BHM87], where a general degeneration is a translation scroll over an elliptic curve embedded in a certain $\mathbb{P}^4$.

Because our boundary B is birational to the Kummer surface parameterising translation scrolls, we could expect a similar picture, maybe over an elliptic curve in $\mathbb{P}^6(V)$ somehow determined by $a \in C \cong Q \subset \mathbb{P}^2(W)$, which is however not isomorphic to the Klein curve in $\mathbb{P}^2(W') \subset \mathbb{P}^6(V)$ as a curve with $\mathrm{PSL}_2(\mathbb{Z}_7)$-action. Therefore a degeneration like A_α would be contained in some $V_{7,a'}$, where $a' \in W'$ defines a secant variety over the above elliptic curve, see Proposition 1.8 part (1).

Certainly this does not occur so far for the general degeneration of our case, because as we saw, the points $\{p_1, p_2, p_3\}$ as before do not lie in general on the Klein quartic. The only possibility is that such an a' can be found among $\{p_4, p_5, p_6\}$, but not defined by a, otherwise the degeneration would be uniquely defined by a and we would end up with the same degeneration for every point of the fibre in B over a, which is certainly not the case, because the minors of matrices of distinct orbits in $\mathbb{P}(M)^s$ span different 3-spaces of L, and therefore once tensored with V_4 they determine different surfaces in $\mathbb{P}^6(V)$.

In our case, so far, the first degeneration of six points in $\mathbb{P}^2(W')$ is not given by the multiplicity, but by the failure of Proposition 1.9 part (2), namely by the fact that the images of three of the six points in $\mathbb{P}^6(L)$ do not span a plane, but a line only.

The obvious question now is: "are the translation scrolls in our picture? And if yes, what are they parameterised by in $H(\Delta)$?"

4.3 Degenerations arising from $B' \subset B$

At this stage degenerations of this sort are relatively simple to describe. First of all we assume that we are working with an element $[\alpha] \in B'$, therefore we can assume that

$$\alpha = \begin{pmatrix} a & 0 \\ b & d \\ 0 & e \end{pmatrix}.$$

Furthermore we assume that a is not the image of a cusp of Q; that case will be treated in the next section.

Proposition 4.15. *Let α be as above, then the degeneration of $(1,7)$-polarised abelian variety A_α it determines is the intersection of two 3-folds $\mathcal{U}_a$ and $\mathcal{U}_e$ of degree 7.*

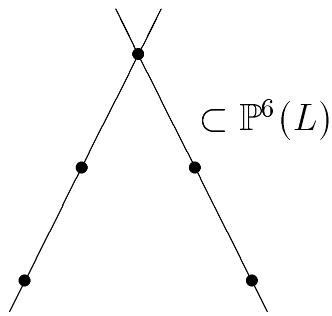
Moreover $\mathcal{U}_a$ (resp. $\mathcal{U}_e$) is generated by three distinct collinear points of $\mathbb{P}^6(L)$ tensored by V_4. These are images of three points $\{p_1, p_2, p_3\}$ (resp. $\{p_1, p_4, p_5\}$) of $\mathrm{Hes}(Q') \subset \mathbb{P}^2(W')$ uniquely determined by the top-left entry $a \in C$ (resp. bottom-right entry $e \in C$) of α.

Proof. Everything works as in the proof of Proposition 4.13, moreover the same argument holds for the sub-matrix $\binom{a}{b}$ of α and for the 3-fold $\mathcal{U}_e$ it determines, see Proposition 4.3.

Notice that $\mathcal{U}_e$ is generated by $ea \otimes V_4$ and $eb \otimes V_4$, whereas $ea \otimes V_4$ and $ad \otimes V_4$ generate $\mathcal{U}_a$. Therefore p_1 is the pre-image in W' of $ae \in L$. Or, if we set b and d to be in C, exactly as in Proposition 4.4 we get p_1 equals $d^*b = b^*d \in W'$.

Finally notice that in place of V_{7,p_4} and V_{7,p_5} of Proposition 4.13 we have used $\mathcal{U}_e$, which is clearly contained in both. $\square$

Figure 4.1: Configuration in $\mathbb{P}^6(L)$ related to B'



Remark 4.16. In the notation of this section we see that if a and e are general, namely are not cusps of C, by Lemma 2.11 then the five points of Proposition 4.15 are distinct, and so we are left again with the interpretation of the varieties parametrised by B'. If those were elliptic or tangent scrolls over an elliptic curve E, then by Table 3.1 any of them would intersect $\mathbb{P}^2(W')$ with multiplicity respectively $(2,2,2)$ and $(3,1,1,1)$, which is not the case here.

Moreover, by Proposition 1.1 in [CH98], we see that an elliptic scroll in $\mathbb{P}^6(V)$ contains three elliptic curves, and so if B' was the space parameterising elliptic scrolls, a fibre over an element like $a \in C$ should intersect three suitable distinct fibres at each of the three points of intersection with B'. This means that the degenerate twisted cubic curve defined by $[\alpha] \in B'$ should be the union of three straight lines all meeting at a point. But with a simple argument about a $\mathbb{P}^1$ of conics (like those in $\{a = 0\} \subset \mathbb{P}^3(U)$ defined by the fibre over a), we get that a case like that cannot occur.

4.4 Degenerations over cusps

In this section we study the boundary points of $H(\Delta)$ over cusps. As we will see, this can be done by hand, and it will be a good example of how we recover the 21 cubic generators, or more precisely the 3 G_7-modules V_4, from the syzygies that define a degeneration.

Here we are going to study the fibre of $H(\Delta)$ over (the image in C of) a specific cusp of Q. This is enough because the action of $\mathrm{PSL}_2(\mathbb{Z}_7)$ permutes the cusps of Q. Notice that the action of $\mathrm{PSL}_2(\mathbb{Z}_7)$ on L induces an action on $H(\Delta)$ as well, if we view an element of $H(\Delta)$ as a 3-dimensional subspace of L. Alternatively we can take the action of $\mathrm{PSL}_2(\mathbb{Z}_7)$ on the entries (in U') of a matrix α representing a point in $H(\Delta)$.

As usual let $[\alpha] \in B \subset H(\Delta)$, and

$$\alpha = \begin{pmatrix} a & 0 \\ b & d \\ c & e \end{pmatrix}.$$

We assume that $\alpha_{11} = u_1$. Then as before $\langle d, e\rangle = \langle u_2, u_3\rangle$, and after setting $d = u_2$ and $e = u_3$, we get

$$\alpha = \begin{pmatrix} u_1 & 0 \\ \xi u_3 & u_2 \\ -\xi u_0 + \tau u_3 & u_3. \end{pmatrix}. \tag{4.5}$$

Notice that if $(\xi : \tau) = (0 : 1)$, the corresponding matrix, and therefore element of B, lies on the intersection of the three $\mathbb{P}^1$'s over the cusps u_1, u_2 and u_3.

Proposition 4.17. *Let α be as in (4.5). Then the 21 cubics it determines define a variety $\mathcal{A}_\alpha$ of the following type:*

1. *If $\xi \neq 0 \neq \tau$, 7 quadric surfaces, each contained in some $\mathbb{P}^3 \subset \mathbb{P}^6(V)$. As a configuration in $\mathrm{VSP}(Q, 6)$ this corresponds to a double point and two single points on a line in $\mathbb{P}^2(W')$, plus a second double point,*
2. (a) *if $\tau = 0$, 7 double planes in $\mathbb{P}^6(V)$. Then we get that the related configuration in $\mathrm{VSP}(Q, 6)$ is a quadruple point plus a double point in $\mathbb{P}^2(W')$,*

   (b) *if $\xi = 0$, 14 planes in $\mathbb{P}^6(V)$. The related configuration in $\mathrm{VSP}(Q, 6)$ is given by three double points in $\mathbb{P}^2(W')$.*

Proof. We first compute the 3-dimensional linear subspace E of L spanned by the minors of α

$$E = \langle \xi(u_0u_2 + u_3^2) - \tau(u_2u_3),\ u_1u_3,\ u_1u_2,\ \rangle,$$

or via the SL($\mathbb{Z}_7$)-isomorphism as in Remark 2.14

$$E = \langle \xi(v_2v_3^2) - \tau(v_3^2v_1 - v_2^3),\ v_1^2v_2 - v_3^3,\ v_2^2v_3 - v_1^3\rangle.$$

This amounts to saying that, as shown in the proof of Proposition 4.4, the 21 generators of $\mathcal{A}_\alpha$, and more precisely the top, middle and bottom V_4's are

$$\begin{aligned} V_4^t &= -\tau\{x_ix_{i+1}x_{i+6}\}_{i\in\mathbb{Z}_7} + \xi\{x_ix_{i-1}^2x_{i+2} + x_{i+1}^2x_{i-2}\}_{i\in\mathbb{Z}_7},\\ V_4^m &= \{x_ix_{i+2}x_{i+5}\}_{i\in\mathbb{Z}_7},\\ V_4^b &= \{x_ix_{i+3}x_{i+4}\}_{i\in\mathbb{Z}_7}.\end{aligned}$$

By (1.1) we get the following facts:

$$\begin{aligned} \mathrm{Pfaff}(M_7'(x,(1:0:0)) &= \mathrm{Pfaff}(\lambda) = \{x_ix_{i+3}x_{i+4}\}_{i\in\mathbb{Z}_7},\\ \mathrm{Pfaff}(M_7'(x,(0:1:0)) &= \mathrm{Pfaff}(\mu) = \{x_ix_{i+1}x_{i+6}\}_{i\in\mathbb{Z}_7},\\ \mathrm{Pfaff}(M_7'(x,(0:0:1)) &= \mathrm{Pfaff}(\nu) = \{x_ix_{i+2}x_{i+5}\}_{i\in\mathbb{Z}_7}.\end{aligned}$$

and

$$\begin{aligned} V(\mathrm{Pfaff}(\lambda)) &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{x_i = x_{i+2} = x_{i+5} = 0\} \cup \bigcup_{i\in\mathbb{Z}_7}\{x_i = x_{i+1} = x_{i+6} = 0\},\\ V(\mathrm{Pfaff}(\mu)) &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{x_i = x_{i+2} = x_{i+5} = 0\} \cup \bigcup_{i\in\mathbb{Z}_7}\{x_i = x_{i+3} = x_{i+4} = 0\},\\ V(\mathrm{Pfaff}(\nu)) &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{x_i = x_{i+3} = x_{i+4} = 0\} \cup \bigcup_{i\in\mathbb{Z}_7}\{x_i = x_{i+1} = x_{i+6} = 0\}.\end{aligned}$$

Thus the middle and bottom V_4's are, respectively, Pfaff(ν) and Pfaff(λ). Therefore, the $\mathbb{P}^1$ of degenerations over the cusp $u_1 = (0:1:0:0) \in C \subset \mathbb{P}^3(U')$, corresponding to $(0:1:0) \in Q \subset \mathbb{P}^2(W)$, takes place set-theoretically in the following seven projective subspaces

$$V(\mathrm{Pfaff}(\nu)) \cap V(\mathrm{Pfaff}(\lambda)) = \bigcup_{i\in\mathbb{Z}_7}\{x_i = x_{i+1} = x_{i+6} = 0\} \subset \mathbb{P}^6(V).$$

The variety $\mathcal{A}_\alpha$ defined by the above 21 cubics are:

1. if $\xi \neq 0 \neq \tau$,
   $\mathcal{A}_\alpha = \{x_i = x_{i-1} = x_{i+1} = \xi x_{i+2}x_{i+5} - \tau x_{i+3}x_{i+4} = 0\}_{i\in\mathbb{Z}_7}$,

2. (a) if $\tau = 0$,
   $\mathcal{A}_\alpha = \bigcup_{i\in\mathbb{Z}_7}\{x_i = x_{i+1} = x_{i+2} = x_{i+3} = 0\}$ with a double structure,

   (b) if $\xi = 0$,
   $\mathcal{A}_\alpha = \bigcup_{i\in\mathbb{Z}_7}\{x_i = x_{i+1} = x_{i+2} = x_{i+4} = 0\} \cup \{x_i = x_{i+1} = x_{i+2} = x_{i+5} = 0\}$.

In terms of the configurations in $\mathrm{VSP}(Q,6)$ related to these degenerations, and then related to configurations in $H(\Delta)$, we see that E, viewed as a subset of $L \subset S^3W'$, determines via (2.13) a 4-dimensional vector space of $L^* = L \subset S^3W$, namely

$$\{E=0\} = \langle y_1y_2y_3,\ y_1y_2^2,\ y_3y_1^2,\ \tau(y_2y_3^2) + \xi(y_3^2y_1 - y_2^3)\rangle,$$

and this allows us to recover the matrix ψ in (2.12):

$$\psi = \begin{pmatrix} y_1 & 0 & 0 \\ -\xi y_2 & y_3 & 0 \\ \xi y_3 & 0 & y_2 \\ -\tau y_3 & -y_2 & -y_1 \end{pmatrix}$$

whose minors are precisely the elements in $\{E=0\}$.

Furthermore they are the generators of the ideal of the six points defining the (degenerate) element we still call ζ of $\mathrm{VSP}(Q,6)$, which is

$$\zeta = \left\{(0:0:1)\times 2,\ (1:0:0)\times 2,\ (0:1:+\sqrt{\xi/\tau}),\ (0:1:-\sqrt{\xi/\tau})\right\}.$$

We finally study the three possible cases:

1. If $\xi \neq 0 \neq \tau$, the embedded point at $(0:0:1)$ is a tangent vector along $\{+\tau y_2 + \xi y_1 = 0\}$, whereas $(1:0:0)$ points along $\{y_3=0\}$. Observe that ζ viewed in $H(\Delta)$ determines a smooth conic in $\{u_1=0\}$ union the line $\{u_2=u_3=0\}$.

2. (a) If $\tau=0$, then ζ degenerates to two points:

   the quadruple point $(0:0:1)$, whose ring of regular function is

   $$\mathbb{C}[y_1,y_2]/(y_1^2,\ y_1y_2, y_1 - y_2^3) \cong \mathbb{C} \oplus \mathbb{C}y_2 \oplus \mathbb{C}y_2^2 \oplus \mathbb{C}y_2^3,$$

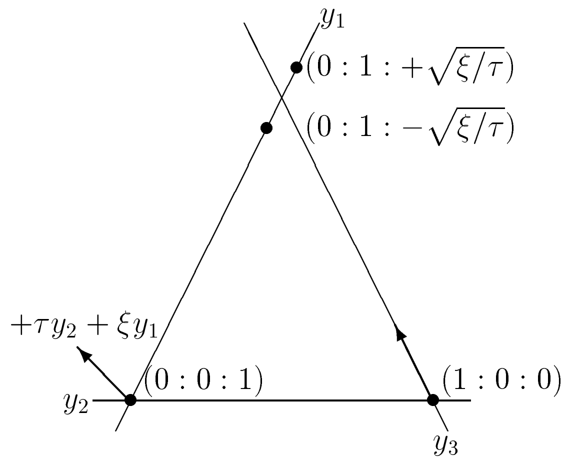
   and therefore it points along $\{y_1=0\}$ with multiplicity 4.

   The double point $(1:0:0)$ points along $\{y_3=0\}$.

   ζ viewed in $H(\Delta)$ determines the *special* smooth conic $\{u_0u_2 + u_3^2 = 0\} \subset \{u_1=0\}$ union the line $\{u_2=u_3=0\}$.

   (b) If $\xi=0$, then $(1:0:0)$ points along $\{y_3=0\}$, $(0:1:0)$ points along $\{y_1=0\}$ and $(0:0:1)$ points along $\{y_2=0\}$. ζ viewed in $H(\Delta)$ determines the union of $\{u_1=u_2=0\}$, $\{u_1=u_3=0\}$ and $\{u_3=u_2=0\}$ in $\mathbb{P}^3(U)$.

Figure 4.2: $\zeta \in \mathrm{VSP}(Q,6)$ over a cusp



□

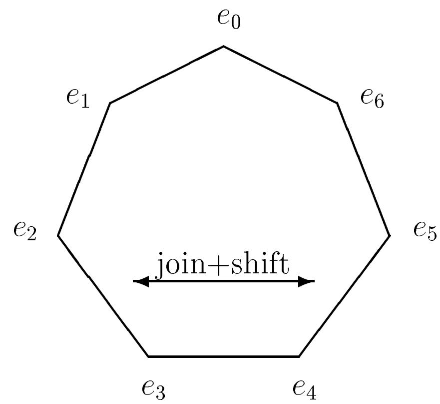
Remark 4.18. Case 2a is the only one where the surface $\mathcal{A}_\alpha$ is not generated by $V_{7,(0:0:1)}$ and $V_{7,(1:0:0)}$, in other words by $V_{7,y}$, where $y \in \zeta \in \mathrm{VSP}(Q,6)$. Notice that Remark 4.7 holds.

Remark 4.19. Figure 4.3 represents the elements of the canonical basis of $\mathbb{P}^6(V)$ with an irreducible component of a degeneration $\mathcal{A}_\alpha$ over the cusp u_1. The three degenerate elliptic curves corresponding to the cusps u_1, u_2 and u_3 are the three possible chains of seven projective lines joining two elements of the basis and invariant under the cyclic action of $\mathbb{Z}_7$ on the indexes. By the same action we can focus our attention on the projective 3-spaces $\{x_0 = x_1 = x_6 = 0\}$ where one of the irreducible components lives.

1. The general degenerate $(1,7)$-abelian surface over a cusp is given by the union of seven quadric surfaces. The surface on $\{x_0 = x_1 = x_6 = 0\}$ is given by the product of the projective lines through e_2, e_3 and e_4, e_5 shifted by $(\xi : \tau)$, namely $\xi x_2 x_5 - \tau x_3 x_4 = 0$.

2. When $(\xi : \tau) \to (1 : 0)$ the quadric surface splits in two projective planes, namely $\{x_0 = x_1 = x_2 = x_6 = 0\}$ and $\{x_0 = x_1 = x_5 = x_6 = 0\}$. The total of 14 projective planes are divided into 7 pairs of coinciding planes by the cyclic action of $\mathbb{Z}_7$, hence we get the double structure.

Figure 4.3: An irreducible component of a degeneration over a cusp



3. Finally, when $(\xi : \tau) \to (0 : 1)$, the quadric surface again splits in two projective planes, namely $\{x_0 = x_1 = x_3 = x_6 = 0\}$ and $\{x_0 = x_1 = x_4 = x_6 = 0\}$. This time the 14 projective planes are all different under the cyclic action of $\mathbb{Z}_7$. This is the most special degeneration, because it lies on the only point where the fibres over the cusps u_1, u_2 and u_3 intersect.

Remark 4.20. The result of Proposition 4.17 highlights a difference with the results by [BHM87] on the $(1, 5)$ case. The combinatorics in that case tells us that when the quadric surfaces in $\mathbb{P}^4$ split up —exactly as in Remark 4.19— into 10 planes, in both the splits one gets 5 pairs of coinciding planes. This can be seen easily with the corresponding picture and an argument as above. And in fact in the $(1, 5)$ case, the multiplicity of the six points *over cusps* is $(2, 2, 1, 1)$ or $(4, 2)$, but not $(2, 2, 2)$.

We remind the reader that all the results of this chapter are summarised in Theorem A in the summary.

Appendix A

Representation theory of G_7 and $\mathbf{SL}_2(\mathbb{Z}_7)$

Here, we follow [MS01]. As mentioned in Section 1.2, if $V = \mathbb{C}(\mathbb{Z}_7)$, then the Heisenberg group $H_7 := H_{1,7}$ is generated by

$$\sigma(x_i) = x_{i-1},$$

$$\tau(x_i) = \xi^i(x_i),$$

where $\xi := \exp(2\pi i/7)$. The Galois group Θ of $\mathbb{Q}(\xi)$ over $\mathbb{Q}$ acts on H_7: let θ be the generator given by $\theta(\xi) = \xi^3$. Then θ^3= complex conjugation.

The irreducible H_7-module V produces five more modules by the composition with the automorphisms $\theta^i \in \Theta$. Denote by V_i the representation $H_7 \xrightarrow{\theta^i} H_7 \to \mathrm{Aut}(V)$. These six representations are inequivalent, as one sees computing their characters, and together with the characters of $\mathbb{Z}_7 \times \mathbb{Z}_7$ these are all the irreducible characters of H_7.

We equip $V = \mathbb{C}(\mathbb{Z}_7)$ with the canonical basis $\{e_i\}_{i\in\mathbb{Z}_7}$, where $e_i(l) = \delta_{il}$. If $\{x_i\}_{i\in\mathbb{Z}_7}$ is the dual basis of $V^\vee = V_3$, then the action of σ and τ on V and on $V^\vee = V_3 = H^0(\mathcal{O}(1))$ is given by

$$\begin{array}{ll} \sigma(e_i) = e_{i-1} & \sigma(x_i) = x_{i-1} \\ \tau(e_i) = \xi^i(e_i) & \tau(x_i) = \xi^{-i}(x_i). \end{array}$$

We recall that $G_7 := H_7 \rtimes \langle\iota\rangle$, where $\iota \in \mathrm{SL}(V)$, $\iota x(i) = -x(-i)$.

A.1 Character table of G_7 and useful formulae

We set $\{\alpha\}$ to be the conjugacy class containing only the central element $\{\alpha\} \in \mu_7$;
$C_{m,n} = \{(\alpha, m, n),\ (\alpha, -m, -n) |\ \alpha \in \mu_7\}$ and $(m, n) \neq 0$;
$C_\alpha = \{(\alpha, m, n)\iota |\ m, n \in \mathbb{Z}_7\}$.

There are 7 classes $\{\alpha\}$, 24 classes $C_{m,n}$, each containing 14 elements, and 7 classes C_α, each containing 49 elements. We denote by Z the sum of all 24 $Z_{s,t}$.

With this notation we get the character table of G_7 (Table A.1), where the column $\star$ gives the corresponding representation

Table A.1: Character table of G_7

$\{\alpha\}$	$C_{m,n}$	C_α	$\bigstar$
1	1	1	I
$7\theta^i(\alpha)$	0	$\theta^i(\alpha)$	V_i
1	1	-1	S
$7\theta^i(\alpha)$	0	$-\theta^i(\alpha)$	$V_i^\#$
2	$\xi^{sm+tn} + \xi^{-sm-tn}$	0	$Z_{s,t}$

We have the following formulae, with the notation $V = V_0$ and $\Omega = \Omega_{\mathbb{P}^6(V)}$:

$$\begin{aligned} V_i \otimes V_i &= 3V_{i+2} \oplus 4V_{i+2}^\# \\ V_i \otimes V_{i+1} &= 3V_{i+4} \oplus 4V_{i+4}^\# \\ V_i \otimes V_{i+2} &= 3V_{i+1} \oplus 4V_{i+1}^\# \\ V_i \otimes V_{i+3} &= I \oplus Z \end{aligned}$$

$$\begin{aligned} \wedge^2 V_i &= 3V_{i+2} & S^2 V_i &= 4V_{i+2}^\# \\ \wedge^3 V_i &= V_{i+1} \oplus 4V_{i+1}^\# & S^3 V_i &= 8V_{i+1} \oplus 4V_{i+1}^\# \\ \wedge^4 V_i &= V_{i+4} \oplus 4V_{i+4}^\# & S^4 V_i &= 10V_{i+4} \oplus 20V_{i+4}^\# \\ \wedge^5 V_i &= 3V_{i+5} & S^5 V_i &= 38V_{i+5} \oplus 28V_{i+5}^\# \\ \wedge^6 V_i &= V_{i+3} & S^6 V_i &= 56V_{i+3} \oplus 76V_{i+3}^\# \\ \wedge^7 V_i &= I & S^7 V_i &= 8I \oplus 28S \oplus 35Z \end{aligned}$$

$$\begin{aligned} H^0(\mathcal{O}_A(1)) &= V_3 & H^0(\Omega^3(3)) &= 0 \\ H^0(\mathcal{O}_A(2)) &= 4V_5^\# & H^0(\Omega^3(4)) &= \wedge^3 V = V_1 \oplus 4V_1^\# \\ H^0(\mathcal{O}_A(3)) &= 5V_4 \oplus 4V_4^\# & H^0(\Omega^3(5)) &= 16V_2 \oplus 16V_2^\# \\ H^0(\mathcal{O}_A(4)) &= 6V_1 \oplus 10V_1^\# & H^0(\Omega^3(6)) &= 56V \oplus 64V^\# \\ H^0(\mathcal{O}_A(5)) &= 13V_2 \oplus 12V_2^\# & H^0(\Omega^3(7)) &= 24I \oplus 24S \oplus 49Z \end{aligned}$$

A.2 The group $\mathrm{SL}_2(\mathbb{Z}_7)$

First of all we need some notation:

$$\begin{array}{ccc} \alpha = \sqrt{-7} & \lambda_1 = \xi - \xi^6 & \eta_1 = \xi + \xi^6 \\ \alpha^+ = (1+\alpha)/2 & \lambda_2 = \xi^4 - \xi^3 & \eta_2 = \xi^4 + \xi^3 \\ \alpha^- = (1-\alpha)/2 & \lambda_1 = \xi^2 - \xi^5 & \eta_3 = \xi^2 + \xi^5 \end{array}$$

Then we have the following equalities:

$$\begin{array}{ccc} \xi + \xi^2 + \xi^4 = -\alpha^- & \lambda_1 + \lambda_2 + \lambda_3 = \alpha & \eta_1 + \eta_2 + \eta_3 = -1 \\ \xi^3 + \xi^5 + \xi^6 = -\alpha^+ & \lambda_1\lambda_2\lambda_3 = \alpha & \eta_1\eta_2\eta_3 = 1 \end{array}$$

and

$$\begin{array}{ccc} \lambda_1^2 = \eta_3 - 2 & \lambda_1\lambda_2 = \eta_3 - \eta_2 & \alpha\eta_1 = \lambda_1 - 2\lambda_2 \\ \lambda_2^2 = \eta_1 - 2 & \lambda_2\lambda_3 = \eta_1 - \eta_3 & \alpha\eta_2 = \lambda_2 - 2\lambda_3 \\ \lambda_3^2 = \eta_2 - 2 & \lambda_3\lambda_1 = \eta_2 - \eta_1 & \alpha\eta_3 = \lambda_3 - 2\lambda_1 \end{array}$$

the general shape of an element A of $N = H_7 \rtimes \mathrm{SL}_2(\mathbb{Z}_7)$ is:
$A_{jk} = \pm\frac{1}{\sqrt{7}}\xi^{aj^2+bjk+ck^2+dj+ek+f}$ $(a, b \dots f \in \mathbb{Z}_7,\ b \neq 0)$
$A_{jk} = \pm\xi^{aj^2+bj+c}\delta_{j,dk+e}$ $(a, b \dots e \in \mathbb{Z}_7,\ d \neq 0)$ where the signs are chosen to have $\det(A) = 1$.

For convenience of computation, it is useful to identify some elements in N and their images in $\mathrm{SL}_2(\mathbb{Z}_7)$:
$\mu x(j) = x(2j)$ (resp. $\mu e_j = e_{j/2}$) with $\bar{\mu} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_7)$
$\nu x(j) = \xi^{j^2}x(j)$ (resp. $\nu e_j = \xi^{j^2}e_j$) with $\bar{\nu} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_7)$
$\delta x(j) = \sqrt{\frac{-1}{7}}\sum_k \xi^{kj}x(k)$ (resp. $\delta e_j = \sqrt{\frac{-1}{7}}\sum_k \xi^{kj}e_k$) with
$\bar{\delta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(\mathbb{Z}_7)$.

Observe that $\delta^2 = \iota$ and that the elements in $\mathrm{SL}_2(\mathbb{Z}_7)$ are given according to

$$\begin{array}{cccc} \mu\sigma\mu^{-1} = \sigma^2 & \iota\sigma\iota = \sigma^{-1} & \nu\sigma\nu^{-1} = \xi^{4\cdot1\cdot2}\sigma\tau^2 & \delta\sigma\delta^{-1} = \tau \\ \mu\tau\mu^{-1} = \tau^4 & \iota\tau\iota = \tau^{-1} & \nu\tau\nu^{-1} = \tau & \delta\tau\delta^{-1} = \sigma^{-1} \end{array}$$

We reproduce here the

Table A.2: Character table of $\mathrm{SL}_2(\mathbb{Z}_7)$

id	ι	μ	$\iota\mu$	ν	ν^3	$\iota\nu^3$	$\iota\nu$	δ			
id	$-$id	$\bar{\mu}$	$\iota\bar{\mu}$	$\bar{\nu}$	$\bar{\nu}^3$	$\iota\bar{\nu}^3$	$\iota\bar{\nu}$	$\bar{\delta}$	$\begin{pmatrix}2&2\\5&2\end{pmatrix}$	$\begin{pmatrix}5&2\\5&5\end{pmatrix}$	★
1	1	56	56	24	24	24	24	42	42	42	
1	1	1	1	1	1	1	1	1	1	1	I
8	-8	-1	1	1	1	-1	-1	0	0	0	M_1
8	8	-1	-1	1	1	1	1	0	0	0	M_2
7	7	1	1	0	0	0	0	-1	-1	-1	L
4	-4	1	-1	α^-	α^+	$-\alpha^+$	$-\alpha^-$	0	0	0	U
4	-4	1	-1	α^+	α^-	$-\alpha^-$	$-\alpha^+$	0	0	0	$U' = U^*$
6	-6	0	0	-1	-1	1	1	0	$\sqrt{2}$	$-\sqrt{2}$	T_1
6	-6	0	0	-1	-1	1	1	0	$-\sqrt{2}$	$\sqrt{2}$	T_2
6	-6	0	0	-1	-1	-1	-1	2	0	0	T
3	3	0	0	$-\alpha^+$	$-\alpha^-$	$-\alpha^-$	$-\alpha^+$	-1	1	1	W
3	3	0	0	$-\alpha^-$	$-\alpha^+$	$-\alpha^+$	$-\alpha^-$	-1	1	1	$W' = W^*$

We indicate here also the multiplication table of characters of $\mathrm{SL}_2(\mathbb{Z}_7)$:

$$
\begin{aligned}
M_1 \otimes M_1 &= I \oplus 3M_2 \oplus 3L \oplus 2T \oplus W \oplus W' \\
M_1 \otimes M_2 &= 3M_1 \oplus 2U \oplus 2U' \oplus 2T_1 \oplus 2T_2 \\
M_1 \otimes L &= 3M_1 \oplus U \oplus U' \oplus 2T_1 \oplus 2T_2 \\
M_1 \otimes U &= 2M_2 \oplus L \oplus T \oplus W \\
M_1 \otimes U' &= 2M_2 \oplus L \oplus T \oplus W' \\
M_1 \otimes T_1 &= 2M_2 \oplus 2L \oplus 2T \oplus W \oplus W' \\
M_1 \otimes T_2 &= 2M_2 \oplus 2L \oplus 2T \oplus W \oplus W' \\
M_1 \otimes T &= 2M_1 \oplus U \oplus U' \oplus 2T_1 \oplus 2T_2 \\
M_1 \otimes T &= 2M_1 \oplus U \oplus U' \oplus 2T_1 \oplus 2T_2 \\
M_1 \otimes W &= M_1 \oplus U \oplus T_1 \oplus T_2 \\
M_1 \otimes W' &= M_1 \oplus U' \oplus T_1 \oplus T_2
\end{aligned}
$$

$$\begin{aligned}
M_2 \otimes M_2 &= I \oplus 3M_2 \oplus 3L \oplus 2T \oplus W \oplus W' \\
M_2 \otimes L &= 3M_2 \oplus 2L \oplus 2T \oplus W \oplus W' \\
M_2 \otimes U &= 2M_1 \oplus U \oplus T_1 \oplus T_2 \\
M_2 \otimes U' &= 2M_1 \oplus U' \oplus T_1 \oplus T_2 \\
M_2 \otimes T_1 &= 2M_1 \oplus U \oplus U' \oplus 2T_1 \oplus 2T_2 \\
M_2 \otimes T_2 &= 2M_1 \oplus U \oplus U' \oplus 2T_1 \oplus 2T_2 \\
M_2 \otimes T &= 2M_2 \oplus 2L \oplus 2T \oplus W \oplus W' \\
M_2 \otimes W &= M_2 \oplus L \oplus T \oplus W \\
M_2 \otimes W' &= M_2 \oplus L \oplus T \oplus W'
\end{aligned}$$

$$\begin{aligned}
L \otimes L &= I \oplus 2M_2 \oplus 2L \oplus 2T \oplus W \oplus W' \\
L \otimes U &= M_1 \oplus U \oplus U' \oplus T_1 \oplus T_2 \\
L \otimes U' &= M_1 \oplus U \oplus U' \oplus T_1 \oplus T_2 \\
L \otimes T_1 &= 2M_1 \oplus U \oplus U' \oplus T_1 \oplus 2T_2 \\
L \otimes T_1 &= 2M_1 \oplus U \oplus U' \oplus 2T_1 \oplus T_2 \\
L \otimes T &= 2M_2 \oplus 2L \oplus T \oplus W \oplus W' \\
L \otimes W &= M_2 \oplus L \oplus T \\
L \otimes W' &= M_2 \oplus L \oplus T
\end{aligned}$$

$$\begin{aligned}
U \otimes U &= L \oplus T \oplus W & & \\
U \otimes U' &= I \oplus M_2 \oplus L & U' \otimes U' &= L \oplus T \oplus W' \\
U \otimes T_1 &= M_2 \oplus L \oplus T \oplus W' & U' \otimes T_1 &= M_2 \oplus L \oplus T \oplus W \\
U \otimes T_2 &= M_2 \oplus L \oplus T \oplus W' & U' \otimes T_2 &= M_2 \oplus L \oplus T \oplus W \\
U \otimes T &= M_1 \oplus U' \oplus T_1 \oplus T_2 & U' \otimes T &= M_1 \oplus U \oplus T_1 \oplus T_2 \\
U \otimes W &= T_1 \oplus T_2 & U' \otimes W &= M_1 \oplus U \\
U \otimes W' &= M_1 \oplus U' & U' \otimes W' &= T_1 \oplus T_2
\end{aligned}$$

$$\begin{aligned}
T_1 \otimes T_1 &= I \oplus 2M_2 \oplus L \oplus T \oplus W \oplus W' \\
T_1 \otimes T_2 &= 2M_2 \oplus 2L \oplus T \\
T_1 \otimes T &= 2M_1 \oplus U \oplus U' \oplus T_1 \oplus T_2 \\
T_1 \otimes W &= M_1 \oplus U' \oplus T_1 \\
T_1 \otimes W' &= M_1 \oplus U \oplus T_1
\end{aligned}$$

$$\begin{aligned}
T_2 \otimes T_2 &= I \oplus 2M_2 \oplus L \oplus T \oplus W \oplus W' \\
T_2 \otimes T &= 2M_1 \oplus U \oplus U' \oplus T_1 \oplus T_2 \\
T_2 \otimes W &= M_1 \oplus U' \oplus T_2 \\
T_2 \otimes W' &= M_1 \oplus U \oplus T_2
\end{aligned}$$

$$\begin{array}{rclrcl} T \otimes T &=& I \oplus 2M_2 \oplus L \oplus 2T & W \otimes W &=& T \oplus W' \\ T \otimes W &=& M_2 \oplus L \oplus W' & W \otimes W' &=& I \oplus M_2 \\ T \otimes W' &=& M_1 \oplus L \oplus W & W' \otimes W' &=& T \oplus W \end{array}$$

A.3 Decompositions of $\mathrm{SL}_2(\mathbb{Z}_7)$ representations

Consider now the decomposition of V into eigenspaces of ι:

$V = V_{-} \oplus V_{-}$ where:

$V_{-} = \mathrm{span}\{e_1 - e_6,\ e_4 - e_3,\ e_2 - e_5\}$

$V_{-} = \mathrm{span}\{2e_0,\ e_1 + e_6,\ e_4 + e_3,\ e_2 + e_5\}$.

Restricting μ, ν and δ to V_{-} and V_{-} respectively, one gets:

$$\mu^+ = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ \mu^- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\nu^+ = \mathrm{diag}(\xi, \xi^4, \xi^2),\ \nu^- = \mathrm{diag}(1, \xi, \xi^4, \xi^2),$$

$$\delta^+ = \sqrt{\frac{-1}{7}} \begin{pmatrix} \xi - \xi^6 & \xi 2 - \xi^5 & \xi^4 - \xi^3 \\ \xi 2 - \xi^5 & \xi^4 - \xi^3 & \xi - \xi^6 \\ \xi^4 - \xi^3 & \xi - \xi^6 & \xi 2 - \xi^5 \end{pmatrix},$$

$$\delta^- = \sqrt{\frac{-1}{7}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & \xi + \xi^6 & \xi 2 + \xi^5 & \xi^4 + \xi^3 \\ 2 & \xi 2 + \xi^5 & \xi^4 + \xi^3 & \xi + \xi^6 \\ 2 & \xi^4 + \xi^3 & \xi + \xi^6 & \xi 2 + \xi^5 \end{pmatrix}.$$

From the character table of $\mathrm{SL}_2(\mathbb{Z}_7)$ one sees that, as a $\mathrm{SL}_2(\mathbb{Z}_7)$-module, $V = W' \oplus U'$ and from the above computations one gets concrete realisations of W' and U', namely $W' = V_{-}$ and $U' = V_{-}$.

Furthermore the following computations play a crucial role:

$$\begin{array}{rclrcl} S^2W &=& T & S^2W' &=& T \\ S^3W &=& L \oplus W' & S^3W' &=& L \oplus W \\ S^4W &=& I \oplus M_2 \oplus T & S^4W' &=& I \oplus M_2 \oplus T. \end{array}$$

If we denote by

$$v_1 = e_1 - e_6,\ v_2 = e_4 - e_3 \text{ and } v_3 = e_2 - e_5$$

the chosen basis of W', then the only $\mathrm{SL}_2(\mathbb{Z}_7)$-invariant quartic is the Klein quartic:

$$f_{\mathrm{Klein}} = v_1^3 v_2 + v_2^3 v_3 + v_3^3 v_1.$$

Notice also that

$$S^2 U' = L \oplus W'.$$

We choose as basis for $L \subset S^2 U'$ the following elements:

$$f_0 = u_0^2,\ f_1 = u_2 u_3,\ f_2 = u_3 u_1,\ f_3 = u_1 u_2,$$

$$f_4 = u_0 u_3 + u_1^2,\ f_5 = u_0 u_1 + u_2^2,\ f_6 = u_0 u_2 + u_3^2,$$

and as basis for W' the elements

$$v_2 = u_0 u_3 - u_1^2,\ v_3 = u_0 u_1 - u_2^2,\ v_1 = u_0 u_2 - u_3^2.$$

Then in the decomposition

$$S^3 V_3 = (I \oplus U' \oplus L) \otimes V_4$$

the elements corresponding to $f_j e_0$ are given by

$$\begin{array}{lll} f_0 e_0 = x_1 x_2 x_4 + x_3 x_5 x_6 - x_0^3 & f_1 e_0 = x_0 x_1 x_6 & f_4 e_0 = x_2^2 x_3 + x_5^2 x_4 \\ & f_2 e_0 = x_0 x_2 x_5 & f_5 e_0 = x_4^2 x_6 + x_3^2 x_1 \\ & f_3 e_0 = x_0 x_3 x_4 & f_6 e_0 = x_1^2 x_5 + x_6^2 x_2 \end{array} \tag{A.1}$$

From here one obtains all $f_j e_k$ via cyclic permutation, in other words via the action of σ.

Appendix B

The Klein quartic Q

We think this interesting curve deserves some space in this thesis, so we give the following presentation that was suggested by Alastair King. We use the notation $(\cdot)'$ for the dual of a vector space. The interested reader can find a beautiful and classic treatise on this topic in [Kle79], where most of these material can be found. Notice that all the notation is consistent with A.2 and A.3.

Let $X(7)$ be the abstract modular curve of level 7. It is embedded in the projective plane $\mathbb{P}^2(W)$ by the canonical linear system $W' = H^0(X(7), \omega)$, where ω is the canonical line bundle. There is a choice of basis v_1, v_2, v_3 for W' so that the homogeneous coordinate vector (v_1, v_2, v_3) of a point in the concrete model $Q \subset \mathbb{P}^2(W)$ of $X(7)$ satisfies

$$f_{\text{Klein}} = v_1^3 v_2 + v_2^3 v_3 + v_3^3 v_1 \tag{B.1}$$

The simple group $G = \mathrm{PSL}_2(\mathbb{Z}_7)$ of order 168 acts on the Klein quartic as its full automorphism group.

The space W' is a 3-dimensional 'fundamental' representation of G: it is faithful, of minimal dimension and all other irreducible representations are contained in spaces of tensors over it.

Note that W' is unimodular but not self-dual, i.e. $W \cong \wedge^2 W' \ncong W'$, and so it is important to distinguish W and W', and consequently $\mathbb{P}^2(W)$ and $\mathbb{P}^2(W')$. In particular, $\mathbb{P}^2(W')$ also contains a unique G-invariant quartic Q', which is isomorphic to Q as an abstract curve, but not as a curve with G-action.

$\mathrm{PSL}_2(\mathbb{Z}_7)$ also has a 4-dimensional projective representation $U = H^0(Q, \omega^{3/2})$. In other words, U is a representation of the central extension $\widehat{G} = \mathrm{SL}_2(\mathbb{Z}_7)$ in which the centre ± 1 acts non trivially. It is necessary to pass to this central extension to lift the G-action to the chosen square root $\omega^{1/2}$ of the canonical bundle. In the case of Q there is a natural choice (see [Kle79], Section 9). The representation U is also unimodular and is also not self-dual.

We give some results we use in the thesis. Consider the kernel R of the multiplication map

$$U \otimes W' \to H^0(Q, \omega^{5/2}).$$

Taking a certain basis a_0, a_1, a_2, a_3 for U, the kernel R has a basis of 'bilinear relations'

$$\begin{aligned} v_1a_1 + v_2a_2 + v_3a_3 &= 0 \\ v_1a_0 - v_3a_1 &= 0 \\ v_2a_0 - v_1a_2 &= 0 \\ v_3a_0 - v_2a_3 &= 0 \end{aligned} \tag{B.2}$$

which can be written either as

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} 0 & v_1 & v_2 & v_3 \\ v_1 & -v_3 & 0 & 0 \\ v_2 & 0 & -v_1 & 0 \\ v_3 & 0 & 0 & -v_2 \end{pmatrix} = 0, \tag{B.3}$$

or

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} a_1 & a_0 & -a_2 & 0 \\ a_2 & 0 & a_0 & -a_3 \\ a_3 & -a_1 & 0 & a_0 \end{pmatrix} = 0 \tag{B.4}$$

Thus the consistency conditions for the bilinear system B.2 are

$$\det \begin{pmatrix} 0 & v_1 & v_2 & v_3 \\ v_1 & -v_3 & 0 & 0 \\ v_2 & 0 & -v_1 & 0 \\ v_3 & 0 & 0 & -v_2 \end{pmatrix} = -\left(v_1^3v_2 + v_2^3v_3 + v_3^3v_1\right) = 0, \tag{B.5}$$

and

$$\operatorname{rank} \begin{pmatrix} a_1 & a_0 & -a_2 & 0 \\ a_2 & 0 & a_0 & -a_3 \\ a_3 & -a_1 & 0 & a_0 \end{pmatrix} \leq 2 \tag{B.6}$$

i.e. the system of cubics (2.4),...,(2.7).

This latter system gives the equations satisfied by the embedding of $X(7)$ in the projective 3-space $\mathbb{P}^3(U')$ by the linear system U. In this thesis we have called C this concrete curve in $\mathbb{P}^3(U')$.

Note further that the bilinear system (B.2) also implies a further system of equations

$$\begin{pmatrix} a_1 & a_0 & -a_2 & 0 \\ a_2 & 0 & a_0 & -a_3 \\ a_3 & -a_1 & 0 & a_0 \end{pmatrix} \begin{pmatrix} 0 & -v_2v_3 & -v_3v_1 & -v_1v_2 \\ v_3v_2 & 0 & v_1^2 & -v_3^2 \\ v_1v_3 & -v_1^2 & 0 & v_2^2 \\ v_1v_2 & v_3^2 & -v_2^2 & 0 \end{pmatrix} = 0 \qquad \text{(B.7)}$$

These 12 equations are a basis for the kernel of the multiplication map

$$U \otimes S^2W' \to H^0(Q, \omega^{7/2}).$$

The Pfaffian of the second matrix in (B.7) is simply the Klein's quartic equation (B.1) again. This matrix also reflects the isomorphism

$$S^2W \cong \Lambda^2U' \subset \mathrm{Hom}(U, U').$$

In [Kle79] the system (B.6) and most of (B.2) are worked out via another relationship between U and W', namely the isomorphism

$$S^2U \cong H^0(Q, \omega^3) \cong S^3W'.$$

This is expressed explicitly by the following system of equations, the first seven of which make up the summand L:

$$\begin{gathered} a_2a_3 = v_2v_3^2 \qquad a_1a_2 = v_1v_2^2 \qquad a_3a_1 = v_3v_1^2 \qquad a_0^2 = v_1v_2v_3 \\ 2a_0a_1 + a_2^2 = v_1^2v_2 - v_3^3 \qquad 2a_0a_2 + a_3^2 = v_2^2v_3 - v_1^3 \qquad 2a_0a_3 + a_1^2 = v_3^2v_1 - v_2^3 \end{gathered} \qquad \text{(B.8)}$$

while the three others make up the summand W

$$2a_0a_1 - a_2^2 = 3v_1^2v_2 + v_3^3 \qquad 2a_0a_2 - a_3^2 = 3v_2^2v_3 + v_1^3 \qquad 2a_0a_3 - a_1^2 = 3v_3^2v_1 + v_2^3.$$

Note that the right hand sides are the derivatives of the Klein quartic. Combinations of the last six equations give expressions for the various other monomials:

$$\begin{gathered} a_0a_1 = v_1^2v_2 \qquad a_0a_2 = v_2^2v_3 \qquad a_0a_3 = v_3^2v_1 \\ a_2^2 = -(v_1^2v_2 + v_3^3) \qquad a_3^2 = -(v_2^2v_3 + v_1^3) \qquad a_1^2 = -(v_3^2v_1 + v_2^3) \\ v_3^3 = -(a_0a_1 + a_2^2) \qquad v_1^3 = -(a_0a_2 + a_3^2) \qquad v_2^3 = -(a_0a_3 + a_1^2) \end{gathered} \qquad \text{(B.9)}$$

These equations easily imply three of the four equations in (B.2) in the following form which Klein records in [Kle79], Equation (43).

$$\frac{a_1}{a_0} = \frac{v_1}{v_3} \qquad \frac{a_2}{a_0} = \frac{v_2}{v_1} \qquad \frac{a_3}{a_0} = \frac{v_3}{v_2}$$

The remaining equation is then effectively the equation of the quartic itself, which follows by computing e.g. $(a_1 a_2)^2$ in two different ways. Note that the system (B.9) can not follow directly from (B.2) because the later is unchanged by independent rescaling of the variables, while the system above determines (a_0, a_1, a_2, a_3) from (v_1, v_2, v_3) up to a sign. However, (B.2) does imply (B.9) up to a single overall constant of proportionality.

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