On the regularity of Cylindrical Algebraic Decompositions

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Cylindrical algebraic decomposition is a powerful algorithmic technique in semi-algebraic geometry. Nevertheless, there is a disparity between what algorithms output and what the abstract definition of a cylindrical algebraic decomposition allows. Some work has been done in trying to understand what the ideal class of cylindrical algebraic decompositions should be — especially from a topological point of view.

We prove a special case of a conjecture proposed by Lazard in [22]; the conjecture relates a special class of cylindrical algebraic decompositions to regular cell complexes. Moreover, we study the properties that define this special class of cell decompositions, as well as their implications for the actual topology of the cells that make up the cell decompositions.

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The work presented in this thesis is a summation of my interactions with all the people mentioned directly, or indirectly, above; I take responsibility for all its short-comings.

Let R denote a real closed field.

Unless explicitly mentioned otherwise, we endow $\mathbb{R}^n$, for $n \in \mathbb{N}$, with the "euclidean topology".

Let $B^d(x, \epsilon)$ denoted the open ball of $\mathbb{R}^d$ centred at x with radius ϵ; moreover, let $B^d = B^d(0, 1)$. We denote the boundary of $B^d(x, \epsilon)$, the $(d-1)$-sphere centred at x with radius ϵ, by $S(x, \epsilon)$. Similarly to above, let $\mathbb{S}^d$ denoted the unit sphere in $\mathbb{R}^{d+1}$. When the dimension of B^d is clear, we drop the superscripts. Lastly, let $\mathbb{H}^n$ denote $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$; this is sometimes referred to as the Euclidean half-space.

We use the standard notation for intervals: $(a, b) \subset \mathbb{R}$ denotes the open-intervals, $[a, b] \subset \mathbb{R}$ denotes the closed intervals.

Unless mentioned otherwise, X and Y will denote a topological space; S a semi-algebraic set; and C, D and E cells of some cell decomposition – see Definition 2.1.1 and Definition 2.1.2.

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CHAPTER 1

INTRODUCTION

Tarski proved in [37] that the theory of real closed fields admits quantifier elimination. Collins introduced the concept of a cylindrical algebraic decomposition, c.a.d. for short, in [8], as an effective method of eliminating quantifiers in the theory of real closed fields.

Loosely speaking, a cylindrical algebraic decomposition $\mathcal{P}$ is a partition of R^n into cells, defined inductively, that satisfies the following condition: if π denotes the standard projection, then, for all cells C and D in $\mathcal{P}$, the projections $\pi(C)$ and $\pi(D)$ are either disjoint or the same. Moreover, given a semi-algebraic set S, we can construct $\mathcal{P}$ so that S is the union of cells of $\mathcal{P}$.

While cylindrical algebraic decompositions show that the theory of real closed fields is model complete – that is, the theory of real close fields admits quantifier elimination – we will focus on its use to study semi-algebraic sets.

Cylindrical algebraic decomposition is a fundamental tool in the study of semi-algebraic geometry. For example, we can determine whether a semi-algebraic set is open, connected, or bounded; we can determine its dimension, closure or interior. Furthermore, we can study motion planning problems — sometimes referred to as Piano mover's problem. See [30], [31], and more recently, [42]. Another application is to compute branch cuts of algebraic expressions; see [14]. A brief survey of some standard applications of cylindrical algebraic decompositions is contained in [21].

In this thesis, we view a c.a.d. not as a algorithm but as a result about the structure of semi-algebraic sets. That is, from a semi-algebraic geometry point of view, we view a c.a.d. as an object that encodes topological information about semi-algebraic sets. In particular, we are interested in finding classes of cylindrical algebraic decompositions that exhibit well-behaved properties.

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The next example serves as motivation for the questions studied in this thesis.

Example 1.0.1 (Motivating example)**.** *Suppose that we have a semi-algebraic set*

$$S = \{(x,y,z) \in D \times \mathbb{R} \mid z = \frac{x^2}{y^2}\},$$

where $D = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1, -x < y < x\}$. The set S is a subset of the Whitney umbrella; that is, a subset of the surface defined by the locus of the polynomial $x^2 - zy^2$.
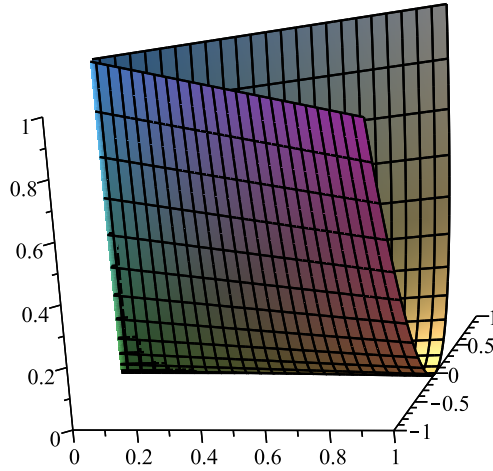
We can partition $\overline{S}$ into cells: the set S, $(0,0) \times [0,1]$, $\{(x,y,1) \in \overline{D} \times \mathbb{R} \mid y = x\}$, $\{(x,y,1) \in \overline{D} \times \mathbb{R} \mid y = -x\}$, and $\{(x,1,z) \in \overline{D} \times \mathbb{R} \mid x^2 = z\}$.

This cell decomposition yields a CW-complex of $\overline{S}$. Computing the homological information from a CW-complex is not straightforward. Nevertheless, if we could construct a partition of $\overline{S}$ where each cell is regular — that is, for each cell $C \subset \overline{S}$, there exists a homeomorphism $(\overline{B}^d, B^d) \to (\overline{C}, C)$ — then our CW-complex would have enough structure to compute the homology of $\overline{S}$ combinatorially. Consequently, we can ask what properties are required so that our c.a.d. is regular and whether we can construct a c.a.d. with these properties.

The first paper to study the topological properties of a cylindrical algebraic decomposition was [22]. More recently, [2] studied a specific class of cylindrical algebraic decompositions where each cell is a semi-monotone set — see [1] for the definition and basic results about semi-monotone sets.

Both [2] and [22] studied the following conjecture.

Conjecture 1.0.2. *Let $S \subset R^n$ be a closed and bounded semi-algebraic set. Then, there exists a cylindrical algebraic decompositions such that S is the union of regular cells.*

In [2], Basu et al. show that a class of cylindrical algebraic decompositions is such that its bounded cells are regular. In the case of $\dim S \leq 2$ or $n = 3$, a proof of how to construct the cylindrical algebraic decomposition is given; this gives a partial answer to Conjecture 1.0.2.

In [22], Lazard conjectures – see Conjecture 4.0.1 – that a c.a.d. satisfying certain conditions is composed of regular cells; moreover, Lazard showed that if $n = 3$, then we can construct such cylindrical algebraic decomposition – this is proved in [22, Prop. 5.11]. Lazard's approach in [22] is to find the necessary and sufficient conditions for a c.a.d. cell to be regular. In this thesis, we follow the approach of [22]. In particular, we present a proof of regularity, under one additional condition, for the case of where $R = \mathbb{R}$, and $\dim S \leq 2$ or $n = 3$ – Lemma 5.2.9 and Corollary 5.2.14. Moreover, our proof gives a possible strategy of how to prove regularity in higher dimensions. This thesis does not consider the question of how to construct such decompositions in dimension $n \geq 4$. As mentioned above, this was proved by Lazard in the case where $n = 3$. See Chapter 7 for a discussion of the properties chosen by Lazard in [22]

Note that if we allow a change of coordinates, then we can show that cylindrical algebraic decompositions are well-behaved and regularity follows – see Theorem 3.2.4; A change of coordinates is not always desirable. Firstly, we want to understand which cylindrical algebraic decompositions have good properties; in some sense, we are looking for an ideal class of cylindrical algebraic decompositions. Furthermore, from a computational point of view, implementations of the c.a.d. algorithm can exploit sparseness to improve their computational time; a change of coordinates can destroy sparseness.

The aim of this thesis is twofold: firstly, to compare the results surrounding the basic theory cylindrical algebraic decompositions and study properties that make them well behaved; secondly, to give an answer, even if partially, to Conjecture 1.0.2 – via Conjecture 4.0.1 – and to questions posed by Lazard in [22, p. 111].

1.1 Contributions of this thesis

We summarise by chapter both the topics discussed and the contributions of this thesis.

- **Chapter 2:** we define the basic terminology of cell decompositions and its properties. We pay particular attention to properties that are related to the regularity of cells. Moreover, we study the relation between these properties and their implications to the topology of the closure of some cells.

- **Chapter 3:** we define cylindrical algebraic decompositions and give a proof of a well-known result(Theorem 3.2.4) that states that a well-behaved cylindrical algebraic decomposition is a regular cell complex. We then discuss the information contained in the index of cells and define the notion of a face of a c.a.d. cell along an axis direction. The latter gives us information related to the dimension of cells containing a point in the boundary – Lemma 3.3.1. Lastly, we show – using the language of faces of a c.a.d. cell, that is, Definition 3.4.1 – that in the case of a 3-dimensional cylindrical algebraic decomposition, two of the properties discussed in Chapter 2 are equivalent – Proposition 3.5.1.

- **Chapter 4:** we formally state the main open question considered in this thesis – Conjecture 4.0.1. Moreover, we briefly discuss the homotopy of unbounded semi-algebraic sets.

- **Chapter 5:** we outline how we can prove Conjecture 4.0.1 in the case of $R = \mathbb{R}$. We then compute the homotopy type of the closure of a c.a.d. cell and use this to prove Conjecture 4.0.1 in the case where $\dim S \leq 2$ or $n = 3$.

- **Chapter 6:** we briefly discuss a question posed by Lazard in [22] that relates cylindrical algebraic decompositions to partially ordered sets. Firstly, We modify it so it is well-defined. Secondly, we give an affirmative answer to the question.

- **Chapter 7:** we discuss how the work of this thesis can be continued.

The appendix chapters briefly discuss the background in semi-algebraic geometry and topology, respectively, used in this thesis.

The purpose of this chapter is to establish the basic language and framework which we work with. We have two goals: to collect results scattered across different areas of the literature, and to standardise the language on which these results are presented.

2.1 Cell decompositions

The most basic object we consider is a cell.

Definition 2.1.1 (Cell). *A subset C of $\mathbb{R}^n$ is a d-**cell** if there exists a homeomorphism $B^d \to C$, for some $d \in \mathbb{N}$. Moreover, we say that C has **dimension** d, denoted $\dim C$. The **boundary of** C, denoted ∂C, is the set $\overline{C} \setminus C$. If C is a point, then C is a cell of dimension 0.*

First note that the dimension of a cell is well-defined; by [18, Thm. 2.26], if a cell is homeomorphic to B^d and B^e, then $d = e$.

The boundary of a cell C does not, in general, coincide with the topological boundary of C. Consider the cell

$$C = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 < 1\},$$

endowed with the standard subspace topology. Recall that the topological boundary of C is $\overline{C} \setminus \operatorname{int}(C)$. Note that $\operatorname{int}(C)$ is empty and thus its topological boundary is $\overline{C}$; its boundary is

$$\partial C = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}.$$

Other authors – for example, see [4] or [38] – refer to the boundary of C as the **frontier of** C.

We can study a set $X \subset \mathbb{R}^n$ by partitioning X into cells. For example, [6] considers triangulations of real algebraic sets. We will study semi-algebraic sets of $\mathbb{R}^n$ by considering an appropriate partition.

Definition 2.1.2 (Cell decomposition of $\mathbb{R}^n$)**.** *Let X be a subset of $\mathbb{R}^n$. We say that a* **cell decomposition of** X *is a partition of X into cells.*

Cell decompositions of $\mathbb{R}$ are straightforward.

Example 2.1.3 (Cell decomposition)**.** *Any partition of $\mathbb{R}$ into open intervals and points is a cell decomposition of $\mathbb{R}$. In fact, partitions of this form are the only possible cell decompositions of $\mathbb{R}$.*

In general, a cell decomposition $\{C_i\}$ of X is not enough to study the space X; we can say little about X as there is little structure on the cell decomposition. For example, we can define the dimension of a semi-algebraic set S by defining a suitable cell decomposition but little about the topological properties of S. We want to understand what additional structures on a cell decomposition are required so that we can study a space X.

2.2   Properties of cells and cell decompositions

As discussed above, we want to study decompositions of a space with additional structure; we impose these additional conditions to capture the properties of the class of subsets we want to study. For example, in [38], van den Dries studies definable subsets by considering a special cell decomposition of definable subsets. A central theme in this thesis is to understand how we can combine cells with a well-behaved boundary to give topological information about semi-algebraic subsets of $\mathbb{R}^n$.

2.2.1   Standard terminology

While a cell C is a simple object in a topological sense, ∂C does not have to be. For example, ∂C can have arbitrarily many connected components: consider the cell $C = \{(x, y) \in \mathbb{R}^2 \mid y < |\csc(x)|\}$; the boundary of C has infinitely many connected components in $\mathbb{R}^2$. Thus, we need to specify what we mean by a well behaved boundary.

More specifically, some of these conditions will ensure that the cell has a well-behaved boundary in the intrinsic sense; that is, not depending in the cell decomposition. Other conditions will guarantee that the boundary of the cell is compatible, in some sense defined below, with a cell decomposition.

The archetypal well-behaved cell, in the intrinsic sense, is a regular cell.

Definition 2.2.1 (Regular cells). *Let C be a d-cell of $\mathbb{R}^n$. We say that C is a* **regular cell** *if there exists a homeomorphism $(\overline{B^d}, B^d) \to (\overline{C}, C)$.*

Example 2.2.2 (Cells of $\mathbb{R}$ are regular). *Any cell of $\mathbb{R}$ is a regular cell.*

More generally:

Example 2.2.3 (Hyper-cubes are regular cells). *The hyper-cube $(0,1)^d \subset \mathbb{R}^n$ is a regular cell via a map $[0,1]^d \to \overline{B}^d$, defined by $v \mapsto \frac{v}{\|v\|}$; this is a standard result from topology.*

This definition says that there exists a homeomorphism $B^d \to C$ that extends to a homeomorphism $\overline{B}^d \to \overline{C}$. This is equivalent to the existence of a relative map(Definition B.0.1) $(\overline{B^d}, \partial B^d) \to (\overline{C}, \partial C)$ that is a homeomorphism. Consequently, if C is a regular cell, then $\overline{C} \cong \overline{B}^d$ and $\partial C \cong \mathbb{S}^{d-1}$, for some $d \in \mathbb{N}$.

Consider the following example from [23, p. 243].

Example 2.2.4 (Closure does not characterise regularity). *The closure of the cell $C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \setminus \{(0,y) \in \mathbb{R}^2 \mid y \geq 0\}$ is equal to $\overline{B}^2$. However, C is not regular; if p is any point in $\{(0,y) \in \mathbb{R}^2 \mid 0 < y \leq 1\} \subset \partial C$, then $C \cup \{p\}$ is not simply-connected.*

This shows that we cannot determine whether or not a cell C is regular by considering the topology of $\overline{C}$ only. We need to understand how the cell relates to its boundary.

Another difficulty in showing a cell is regular is that the defining map may be unsuitable. Given a d-cell $C \subset \mathbb{R}^n$ we have a homeomorphism $f \colon C \to (0,1)^d$, but f does not need to extend to a homeomorphism $\overline{f} \colon [0,1]^d \to \overline{C}$. Consider, for example, the homeomorphism

$$
\begin{aligned}
(0,1)^2 &\to C, \\
(x,y) &\mapsto (x, xy),
\end{aligned}
$$

where $C = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1, 0 < y < x\}$. This map does not extend to $[0,1]^2$. Nevertheless, C is a regular cell.

2.2.2 Additional terminology

The following definition is useful when determining whether a cell is regular.

Definition 2.2.5 (Equiregularity)**.** *We say that $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are* **equiregular** *if there exists a homeomorphism $(\overline{X}, X) \to (\overline{Y}, Y)$. In particular, a regular cell is a cell equiregular to $(\overline{B}^d, \overline{B})$.*

As a consequence of Example 2.2.3, to show a cell is regular, it is enough to show the existence of a homeomorphism $([0,1]^d, (0,1)^d) \to (\overline{C}, C)$.

We establish a convention for naming cell decompositions where all cells have a certain property.

Remark 2.2.6 (Naming convention for cell decompositions)**.** *In general, if all cells of a cell decomposition $\mathcal{P}$ have property P, we say that $\mathcal{P}$ is a P cell decomposition.*

Before we study properties related to the boundary of a cell and how it fits in a cell decomposition, we need to describe what it means for two cells to be close to each other.

Definition 2.2.7 (c.f. [22, Def. 2.6])**.** *Suppose that C and D are cells of $\mathbb{R}^n$. We say that C and D are* **adjacent** *if $C \cap \overline{D} = \emptyset$ or $\overline{C} \cap D = \emptyset$. Moreover, if $C \cap \overline{D} \neq \emptyset$ we say that C is* **sub-adjacent** *to D.*

The notion of sub-adjacency is necessary to capture the asymmetric behaviour of the adjacency relation.

Example 2.2.8 (Sub-adjacency is asymmetric)**.** *Consider the cells*

$$
\begin{aligned}
C &= \{(x,y) \in \mathbb{R}^2 \mid y = 0\} \ and \\
D &= \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}.
\end{aligned}
$$

Then C and D are adjacent, C is sub-adjacent to D, but D is not sub-adjacent to C.

The (sub-)adjacency relation is, in some sense, an analogue to the (proper) faces relation of a simplex.

2.2.3 Closure finite and well-bordered

The next two properties we consider are finiteness conditions on the boundary of the cell. They tell us how a cell fits in with the cell decomposition.

We discuss these properties together as, under the right assumptions, we can relate them.

Definition 2.2.9 (Closure finite)**.** *Let C be a cell of a cell decomposition $\mathcal{P}$. We say that C is* **closure finite in** $\mathcal{P}$ *if $\overline{C}$ is the union of finitely many cells of $\mathcal{P}$.*

The closure finite condition is found in the literature under different names. In [22, Def. 2.7], Lazard refers to closure finite as *boundary coherent.* Both [4, Def. 2.4.1] and [33, p. 4] refer to a cell $C \in \mathcal{P}$ satisfying the *frontier condition* if ∂C is the, not necessarily finite, union of cells in a cell decomposition $\mathcal{P}$. This language is used in the study of topological stratifications. We use the term *closure finite* as it is more descriptive than the terms above.

Note that C is closure finite in $\mathcal{P}$ if and only if ∂C is the union of finitely many cells of $\mathcal{P}$. Moreover, if a cell $D \in \mathcal{P}$ intersects ∂C, then, by [3, Thm. 5.42], $\dim D \leq \dim C - 1$. As Example 2.2.14 demonstrates below, we do not necessarily have equality.

The closure finiteness property is not intrinsic to the cell; it depends on the cell decomposition C lies in. When it is clear from context which cell decomposition C is part of, we say that C is closure finite.

Example 2.2.10 (Failure of closure finiteness)**.** *Consider the following cell decomposition of $[0,1]^3$:*
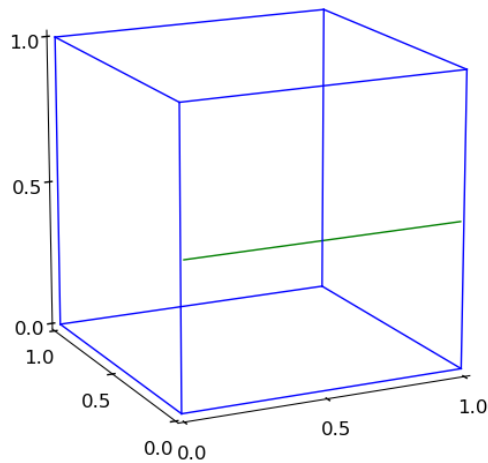
The cube is closure finite but the 1-cell $[0,1] \times \{0\} \times \{\frac{1}{2}\}$ sub-adjacent to the cube is not.

This example illustrates that the closure finite property does not permeate to the cells in the boundary.

The closure finite property affects the adjacency relation.

Example 2.2.11 (Closure finiteness controls sub-adjacency). *Suppose we have two cells* $C = \{(x,0) \in \mathbb{R}^2 \mid -1 < x < 2\}$ *and* $D = \S^2 \setminus \{(x,y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\}$ *of* $\mathbb{R}^2$.
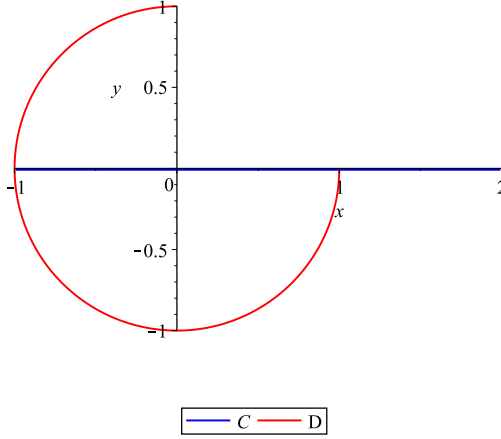
These cells are not closure finite; moreover, they are sub-adjacent to each other.

This behaviour is undesirable; the closure finite property guarantees that this pathology cannot happen.

Lemma 2.2.12 (Closure finiteness and sub-adjacency connection). *Let C be a cell of a finite cell decomposition $\mathcal{P}$. Then, C is closure finite if and only if $D \in \mathcal{P}$ sub-adjacent to C implies that $D \subset \overline{C}$. In particular, if two cells C and D are closure finite and sub-adjacent to each other, then $C = D$.*

Proof. Suppose that C is closure finite in $\mathcal{P}$; that is, $\partial C = \bigcup_{i=1}^{k} D_i$. If $D \neq C$ is any cell sub-adjacent to C, then it must intersect some D_i, and thus $D = D_i$. In particular $D \subset \partial C$.

Conversely, if $\overline{C}$ contains all cells sub-adjacent to C, then $\overline{C} = \bigcup_{i=1}^{k} D_i$, where $\{D_i\}$ is the collection of cells sub-adjacent to C. $\qquad\square$

In other words, a cell C is closure finite if and only if ∂C contains all of its sub-adjacent cells. Note that the "only if" direction does not require a finite cell decomposition.

We examine the significance of Lemma 2.2.12 in Section 6.2.

While the closure finite condition requires that the boundary is the union of cells of $\mathcal{P}$, it does not directly imposes restrictions on the topology of the boundary.

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As mentioned in the beginning of this chapter, the epitome of a well-behaved cell is a regular cells. The next property ensures that the boundary of a cell C does not have isolated components of codimension greater than 1.

Definition 2.2.13 (well-bordered). *Let C be a cell of a cell decomposition $\mathcal{P}$. We say that C is **well-bordered in** $\mathcal{P}$ if there exists a finite collection of cells $\{C_i\}$ in $\mathcal{P}$ such that $\dim C_i = \dim C - 1$ and $\partial C = \cup_i \overline{C_i}$.*

Similarly to closure finiteness, the well-bordered property is not intrinsic to the cell; it depends on the cell decomposition. Unlike closure finite property, it does directly imposes restrictions on the topology of the boundary: for example, if C is well-bordered, then $\dim \partial C = \dim C - 1$.

Example 2.2.14 (Closure finite but not well bordered I). *Consider the cell decomposition of $\mathbb{R}^3$ consisting of the cells $C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ and } z < 1\}$, $\{(0, 0, 1)\}$, $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 1\}$, and $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 > 1\}$. Then ∂C is such that its boundary does not have a cell of dimension one.*

The following example from [22, Ex. 2.9] is an augmented version of Example 2.2.14.

Example 2.2.15 (Closure finite but not well-bordered II). *Consider the following cell decomposition of $\mathbb{R}^3$*
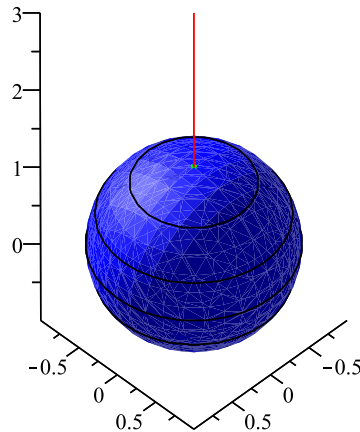
defined by the cells

$$\begin{aligned}
C_1 &= \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 1\}, \\
C_2 &= \{(x,y,z) \in \mathbb{R}^3 \mid (x^2 + y^2 + z^2 > 1) \text{ and } (z < 1 \text{ or } x^2 + y^2 > 0)\}, \\
C_3 &= \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, z < 1\}, \\
C_4 &= \{(0,0,z) \in \mathbb{R}^3 \mid z > 1\}, \text{ and} \\
C_5 &= \{(0,0,1)\}.
\end{aligned}$$

Two cells fail to be well-bordered. By Example 2.2.14, the 2-cell C_3 fails to be well-bordered as its boundary consists of a single 0-cell. The 3-cell C_2 contains one 2-cell in its boundary – that is, C_1 – but the closure of this 2-cell does not contain the 1-dimensional half-line with end-point $C_4 = (0,0,1)$. However, every cell is closure finite and thus this cell decomposition is closure finite.

Like the closure finite property, the well-bordered property does not permeate to sub-adjacent cells.

Example 2.2.16 (Well-bordered but not closure finite). *Consider the following cell decomposition*
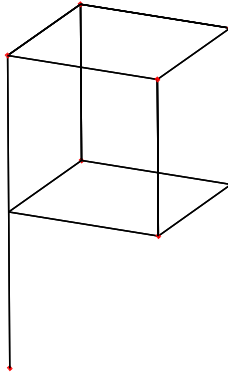
It consists of the open cube $(0,1)^3$; six 2-cells which make up the faces of the cube; eleven 1-cells which are just rotation and translations of $(0,1) \times \{0\} \times \{0\}$; the 1-cell $\{0\} \times \{0\} \times (-1,1)$; and eight 0-cells $(0,0,1)$, $(0,1,0)$, $(0,1,1)$, $(1,0,0)$, $(1,0,1)$, $(1,1,0)$, $(1,1,1)$, and $(0,0,-1)$. We observe the following:

   i. *The cell $(0,1)^3$ is well-bordered but not closure finite.*

   ii. *The faces that are adjacent to the cell $\{0\} \times \{0\} \times (-1,1)$ are neither well-bordered or closure finite. Let D denote one of these faces.*

In general, as summarised by the following table, these conditions are not related.

well-bordered	closure finite	Example
T	T	Any cell decomposition of $\mathbb{R}$.
T	F	$(0,1)^3$ in Example 2.2.16.
F	T	C_1 in Example 2.2.15.
F	F	D in Example 2.2.16.

If we require that all sub-adjacent cells to a cell C also satisfy the well-bordered or closure finite conditions, we can relate these two properties; in some circumstances, for example when studying low-dimensional cell decompositions, they are equivalent.

Lemma 2.2.17. *Let $C \subset \mathbb{R}^n$ be a cell of a cell decomposition $\mathcal{P}$. If C and all cells sub-adjacent to C are well-bordered, then C is closure finite.*

Proof. In view of Lemma 2.2.12, it suffices to show that for all cells D such that $D \cap \overline{C} \neq \emptyset$, then D is contained in ∂C; we may assume that $D \neq C$. We prove this lemma by induction on the dimension of C. For a 0-dimensional cell, the two conditions are equivalent.

As C is well-bordered, there exists a finite collection of cells $\{C_i\}$, with dimension $\dim C_i = \dim C - 1$, such that $\partial C = \cup_i \overline{C_i}$. If $D \cap C_i \neq \emptyset$, for some i, then $D = C_i$; Otherwise, we may assume that $D \cap \overline{C_i} \neq \emptyset$, where $\dim C_i = \dim C - 1$. Thus, by induction, $D \subset \partial C_i \subset \partial C$ and the result follows. $\qquad\square$

Corollary 2.2.18. *Any well-bordered cell decomposition is closure finite.*

The converse is not true, even if we assume that all sub-adjacent cells are closure finite. A d-cell might fail to have a $(d-1)$-cell in its boundary as Example 2.2.14 demonstrates.

In low dimension, these two conditions are equivalent.

Lemma 2.2.19. *Let $C \subset \mathbb{R}^2$ be a cell. The following are equivalent:*

i. C and all of its sub-adjacent cells are well-bordered.

ii. C and all of its sub-adjacent cells are closure finite.

Proof. The fact that i) implies ii) follows by Lemma 2.2.17.

Conversely, the result holds for 0- and 1-cells; thus, suppose that C is a 2-cell.

A closure finite 2-cell $C \subset \mathbb{R}^2$ fails to be well-bordered if there exists a isolated point p in ∂C. As C is open, the boundary of C and its topological boundary coincide; we use them interchangeably in this proof.

Suppose that $p \in C$ is an isolated point of ∂C; that is, there exists an $r > 0$ such that $B(p, r) \cap \partial C = \{p\}$. We can partition $\mathbb{R}^n$ as follows:

$$R^n = C \sqcup \partial C \sqcup \text{int}(R^n \setminus C),$$

By intersecting the decomposition above with $B(p, r) \setminus \{p\}$ we have:

$$B(p, r) \setminus \{p\} = ((B(p, r) \setminus \{p\}) \cap C) \cup \left[(B(p, r) \setminus \{p\}) \cap \text{int}(R^n \setminus C) \right].$$

As $B(p, r) \setminus \{p\}$ is connected, either $(B(p, r) \setminus \{p\}) \cap C$ or $B(p, r) \setminus \{p\} \cap \text{int}(\mathbb{R}^n \setminus C)$ is empty. As $p \in \partial C$ we have that $B(p, r) \setminus \{p\} \cap \text{int}(\mathbb{R}^n \setminus C) = \emptyset$ and

$$B(p, r) \setminus \{p\} = (B(p, r) \setminus \{p\}) \cap C.$$

Thus, $B(p, r) \setminus \{p\} \subset C$; this contradicts the fact that $C \subset \mathbb{R}^2$ has trivial fundamental group and completes the proof. $\qquad \square$

Corollary 2.2.20. *A cell decomposition in $\mathbb{R}^2$ is well-bordered if and only if it is closure finite.*

For $n \geq 3$, a closure finite cell decomposition of $\mathbb{R}^n$ is not necessarily well-bordered; see Example 2.2.15. If our cells happen to be regular, then there enough structure to relate the two.

Lemma 2.2.21. *Let $\mathcal{P}$ be a regular cell decomposition of a compact set $S \subset \mathbb{R}^n$. Then, $\mathcal{P}$ is closure finite if and only if it is well-bordered.*

Proof. By Corollary 2.2.18, any well-bordered cell decomposition is closure finite.

Conversely, suppose that C is a closure finite, regular d-cell. As ∂C is homeomorphic to $\S^{d-1}$, the cell C is well-bordered. $\qquad \square$

In Proposition 3.5.1, we show that in another special type of cell decomposition, the closure finite property is equivalent to well-bordered.

2.2.4 Locally boundary connected

Lastly, we consider the *locally boundary connected* property.

Definition 2.2.22 (c.f. [22, Def. 2.7]). *A cell C is **locally boundary connected**[1] – l.b.c. for short – if, for all $p \in \partial C$, there exists an $\delta > 0$ such that, for all $0 < \epsilon < \delta$, $B(p, \epsilon) \cap C$ is connected.*

---

[1]Lazard refers to this property as **boundary smooth** in [22, Def. 2.7]. We renamed this property so as to not overload the label "smooth", specially as we can consider $\mathcal{C}^\infty$ cells

Example 2.2.23 (Open cube is l.b.c.). *The open cube $(0,1)^d \subset \mathbb{R}^n$ is l.b.c.*

Example 2.2.24 (l.b.c. non-example). *Consider the following cell:*

$$C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \setminus (0,1).$$

For all $0 < \epsilon < 2$, $B((0,1), \epsilon)$ has two connected components, and thus C is not locally boundary connected.

In addition, this example illustrates that the l.b.c. property condition is not preserved under homeomorphisms. The l.b.c. property is intrinsic to C; it does not depend on how the boundary of C is decomposed in some particular cell composition.

Note that the closure of a cell does not characterise the l.b.c. property via topological type.

Example 2.2.25 (Closure does not characterise l.b.c.). *Recall Example 2.2.4. The closure of the 2-cell $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\} \setminus \{(0,y) \in \mathbb{R}^2 \mid 0 < y < 1\}$ is the closed ball. Consider the following 2-cell:*
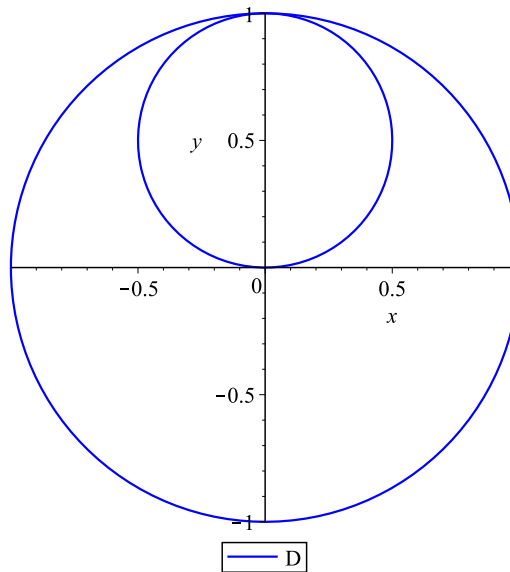
The closure of these two 2-cells have different topological type.

Any such characterisation will have to take into account the relation between the cell and its boundary. Lazard proved in [22] that we can characterise the l.b.c. property in terms of the fundamental group.

Proposition 2.2.26 ([22, Prop. 2.12]). *A cell $C \subset \mathbb{R}^n$ is l.b.c. if and only if, for all $p \in \partial C$, $\{p\} \cup C$ is simply-connected.*

Note that if a cell C is such that $\{p\} \cup C$ is simply-connected for some $p \in \partial C$, it does not imply that $\{q\} \cup C$ is simply-connected for some q in some neighbourhood of p; consider the origin in Example 2.2.25.

We defined locally boundary connectedness in such way that, for a fixed point p, the one-parameter family of semi-algebraic sets $\{B(p, \epsilon)\}$ can detect whether or not $\{p\} \cup C$ is simply-connected. We want to understand how crucial it is to use this specific family, and in particular, whether we can replace it by a different family.

The naive idea of replacing it by a family whose members are strictly contained in another and whose measure tends to zero does not work; for example, a family of open neighbourhoods of p is not enough to characterise the l.b.c. property.

Example 2.2.27 (Not all open families characterise l.b.c.). *Consider the one-parameter family $\{X_\epsilon\}$ of neighbourhoods of the origin defined as*

$$X_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < \epsilon, y + \frac{\epsilon}{4} < |x - \frac{\epsilon}{2}|\}.$$
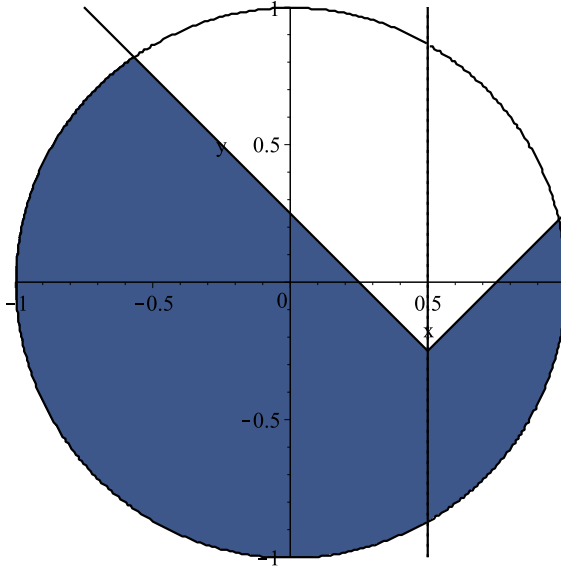
Now the cell $((0,1)^2, [0,1]^2)$ is regular but $X_\epsilon \cap C$ is disconnected for all sufficiently small $\epsilon > 0$.

For now, we assume that our cells are semi-algebraic; this is necessary as we invoke Theorem A.4.5 – the local conic structure of semi-algebraic sets – to prove the subsequent results.

Remark 2.2.28. *Let C be a d-cell of $\mathbb{R}^n$, $p \in \partial C$, and let the set $X_{p,\epsilon}$ denote the set $p * (S(p, \epsilon) \cap C) \setminus (\{p\} \cup S(p, \epsilon))$ for some $\epsilon > 0$. From the local conic structure of semi-*

algebraic sets, we know that for sufficiently small $\epsilon > 0$, there exists a homeomorphism

$$B(p, \epsilon) \cap C \to X_{p,\epsilon}.$$

From its definition, $X_{p,\epsilon} = \{p\lambda + (1 - \lambda)b \mid b \in S(p, \epsilon) \cap C, \lambda \in (0, 1)\}$. Consequently, there exists a homeomorphism

$$\begin{aligned} X_{p,\epsilon} &\longrightarrow (S(p, \epsilon) \cap C) \times (0, 1), \\ p\lambda + (1 - \lambda)b &\longmapsto (b, \lambda). \end{aligned}$$

A product $X \times Y$ is connected if and only if X and Y are connected. In particular, $B(p, \epsilon) \cap C$ is connected if and only if $S(p, \epsilon) \cap C$ is connected. This yields the following corollary.

Corollary 2.2.29. *Let $C \subset \mathbb{R}^n$ be a semi-algebraic cell. Then C is locally boundary connected if and only if, for all $p \in \partial C$, there exists a $\delta > 0$ such that, for all $0 < \epsilon < \delta$, $S(p, \epsilon) \cap C$ is connected.*

In other words, we can replace the one-parameter family $\{B(p, \epsilon)\}$ by the one-parameter family $\{S(p, \epsilon)\}$. Note that if $B(p, \epsilon) \cap C$ is connected for $0 < \epsilon < \delta$, $S(p, \epsilon) \cap C$ might not be connected. That is, we might need to choose a different δ.

We can adapt this proof to prove the following.

Lemma 2.2.30. *Let $C \subset \mathbb{R}^n$ be a semi-algebraic cell. Then C is l.b.c. if and only if for all $p \in \partial C$ there exists a δ such that, for all $0 < \epsilon < \delta$, $\overline{B}(p, \epsilon) \cap C$ is connected.*

These results show that we can detect whether a cell is locally boundary connected by using the one-parameter families $\{B(p, \epsilon) \cap C\}$, $\{\overline{B}(p, \epsilon) \cap C\}$ or $\{S(p, \epsilon) \cap C\}$ – with parameter $\epsilon > 0$ – or the fundamental group of $C \cup \{p\}$.

Corollary 2.2.31. *Let C be a semi-algebraic l.b.c. cell. The $\delta > 0$ that arises from the local conic structure of $C \cup \{p\}$ is such that, for $0 < \epsilon < \delta$, $B(p, \epsilon) \cap C$ is connected.*

This is not surprising as we can detect l.b.c. by the fundamental group of $C \cup \{p\}$ and the local conic structure can capture the local topology of a semi-algebraic set.

As mentioned above, regular cells are, in the intrinsic sense, the archetypal well-behaved cell. In particular, regular cells are locally boundary connected.

Lemma 2.2.32 (cf. [31, Lem. 1]). *Let C and D be equiregular. Then, C is l.b.c. if and only if D is l.b.c. In particular, a regular cell locally boundary connected.*

Here we do not need the semi-algebraic assumption.

Proof. As C and D are equiregular, we have a homeomorphism $\overline{f} \colon (\overline{D}, D) \to (\overline{C}, C)$. Then, for any $x \in \partial C$, the map

$$\overline{f}|_{D \cup \{y\}} \colon D \cup \{y\} \to C \cup \{x\}$$

is a homeomorphism, where $\overline{f}(y) = x$. Thus, as $D \cup \{y\}$ is simply-connected if and only if $C \cup \{x\}$ is simply-connected. If a cell C is regular, then $\{p\} \cup C$ is simply-connected as $\{f(p)\} \cup (0,1)^d$ is simply-connected for all $p \in \partial C$. $\qquad\square$

This was first pointed out in [31] and is the reason why the locally boundary connected property is of interest: it is a necessary condition for regularity, and in particular, a necessary condition for a cell decomposition to be a regular cell complex.

CHAPTER 3

CYLINDRICAL ALGEBRAIC DECOMPOSITION

Cylindrical algebraic decomposition, or c.a.d. for short, is a cell decomposition of $\mathbb{R}^n$ introduced by Collins in [8] to provide a practical, and thus computationally feasible, method to eliminate quantifiers in the theory of real closed fields.

In the literature, there are two main types of results:

i. Results which are true for all, or classes of, cylindrical algebraic decompositions.

ii. Results which are true for cylindrical algebraic decompositions produced by a specific algorithm. The main families of algorithms being:

   (a) "Projection/Lifting" – [8],

   (b) "Regular Chains"– [7], and very recently

   (c) "Comprehensive Gröbner Systems" – [16].

This thesis is in the first camp; we take the approach of studying cylindrical algebraic decompositions independent of the algorithm that computes them.

This is similar to the view of Benedetti and Risler in [4]. In [4, Thm. 2.2.1], Benedetti and Risler refer to a cylindrical algebraic decomposition as the first main structure theorem of semi-algebraic sets; thus, we see a c.a.d. as a result about the decomposition of semi-algebraic sets rather the output of some algorithm.

3.1 Cylindrical algebraic decomposition

We think of a cylindrical algebraic decompositions as a finite partition of $\mathbb{R}^n$ into semi-algebraic cells, built inductively, and whose projection onto the first k variables, for

$k < n$, are either disjoint or the same. These cells are not just arbitrary semi-algebraic sets homeomorphic to $(0,1)^d$, for some $d \in \mathbb{N}$, but cells that arise from graphs of semi-algebraic functions.

The definition of a cylindrical algebraic decomposition is found in [3], [4], or [8].

Definition 3.1.1 (Cylindrical algebraic decomposition of $\mathbb{R}^n$). *A **cylindrical algebraic decomposition of** $\mathbb{R}^n$ is a sequence $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_n)$ where each $\mathcal{P}_k$, for $1 \le k \le n$, is a finite partition of R^k satisfying the following conditions:*

    i. Every $C \in \mathcal{P}_1$ is either a point or an open interval.

    ii. For each $C_I \in \mathcal{P}_k$, we have continuous real-valued algebraic functions $f_{I,j} \colon C_I \to \mathbb{R}$ for $1 \le j \le u_I$. Moreover, these functions are such that $f_{I,1} < \cdots < f_{I,u_I}$ point-wise on C_I. If $u_I = 0$, the we set $C_{I,1} = C_I \times \mathbb{R}$. Otherwise, we define $C_{I,j}$ as follows:

$$
\begin{aligned}
C_{I,1} &= \{(a,b) \in C_I \times \mathbb{R} \mid b < f_{I,1}(a)\}, \\
C_{I,2j} &= \{(a,b) \in C_I \times \mathbb{R} \mid b = f_{I,j}(a)\} \text{ for } 1 \le j \le u_I, \\
C_{I,2j+1} &= \{(a,b) \in C_I \times \mathbb{R} \mid f_{I,j}(a) < b < f_{I,j+1}(a)\} \text{ for } 1 \le j < u_I, \text{ and} \\
C_{I,2u_I+1} &= \{(a,b) \in C_I \times \mathbb{R} \mid f_{I,u_I}(a) < b\}.
\end{aligned}
$$

*Note that the projection of a c.a.d. of $\mathbb{R}^n$ is a c.a.d. of $\mathbb{R}^{n-1}$. If $\pi\mathcal{P}$ is the projection of a c.a.d. $\mathcal{P}$, we say that $\pi\mathcal{P}$ is **induced from** $\mathcal{P}$.*

*We say that $C_{i_1,\ldots,i_n}$ is a **section** if i_n is even and a **sector** otherwise. Moreover, the set $C_{i_1,\ldots,i_{n-1}} \times \mathbb{R}$ is called the **cylinder above** $C_{i_1,\ldots,i_{n-1}}$ and we say that $C_{i_1,\ldots,i_{n-1}i_n}$ **lies above** $C_{i_1,\ldots,i_{n-1}}$.*

Example 3.1.2 (c.f. [3, Ex. 5.4]). *We start by defining a c.a.d. of $\mathbb{R}$ and then building a c.a.d. of $\mathbb{R}^2$ above it.*

Consider the c.a.d. of $\mathbb{R}$ consisting of five cells: $C_1 = (-\infty, -1)$, $C_2 = \{-1\}$, $C_3 = (-1,1)$, $C_4 = \{1\}$, and $C_5 = (1, \infty)$. We define the c.a.d. of $\mathbb{R}^2$ as follows:
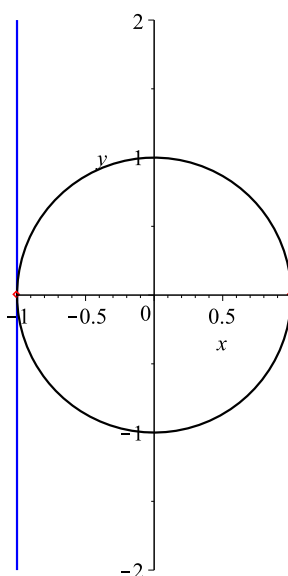
i. $C_{1,1} = C_1 \times \mathbb{R}$.

ii. *Above the cell C_2:*

    (a) $C_{2,1} = \{(x, y) \in C_2 \times \mathbb{R} \mid -\infty < y < 0\}$,

    (b) $C_{2,2} = \{(-1, 0)\}$, *and*

    (c) $C_{2,3} = \{(x, y) \in C_2 \times \mathbb{R} \mid 0 < y < \infty\}$.

iii. *Above the cell C_3:*

    (a) $C_{3,1} = \{(x, y) \in C_3 \times \mathbb{R} \mid -\infty < y < -\sqrt{1 - x^2}\}$,

    (b) $C_{3,2} = \{(x, y) \in C_3 \times \mathbb{R} \mid y = -\sqrt{1 - x^2}\}$,

    (c) $C_{3,3} = \{(x, y) \in C_3 \times \mathbb{R} \mid -\sqrt{1 - x^2} < y < \sqrt{1 - x^2}\}$,

    (d) $C_{3,4} = \{(x, y) \in C_3 \times \mathbb{R} \mid y = \sqrt{1 - x^2}\}$, *and*

    (e) $C_{3,5} = \{(x, y) \in C_3 \times \mathbb{R} \mid \sqrt{1 - x^2} < y < \infty\}$.

iv. *Above the cell C_4:*

    (a) $C_{4,1} = \{(x, y) \in C_4 \times \mathbb{R} \mid -\infty < y < 0\}$,

    (b) $C_{4,2} = \{(1, 0)\}$, *and*

    (c) $C_{4,3} = \{(x, y) \in C_4 \times \mathbb{R} \mid 0 < y < \infty\}$.

v. $C_{5,1} = C_5 \times \mathbb{R}$.

It is not a coincidence that this looks like a cell decomposition of $\S^1$.

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Remark 3.1.3. *We can deduce the following facts about a c.a.d. $\mathcal{P}$:*

    i. $\mathcal{P}$ is a finite partition of $\mathbb{R}^n$.

    *ii. If $C, D \in \mathcal{P}$, then $\pi(C)$ and $\pi(D)$ are either disjoint or equal, where π is the projection onto the first $n-1$ coordinates. We refer to this property as **cylindrical**.*

An important feature of a c.a.d. is that it yields a finite cell decomposition of $\mathbb{R}^n$. From Definition 2.1.2 and Remark 3.1.3, it is enough to show that each subset which makes up the partition is a cell.

Lemma 3.1.4 ([3, Prop. 5.3]). *Suppose $\mathcal{P} = \{C_I\}$ is a c.a.d. of $\mathbb{R}^n$. Then, every C_I in $\mathcal{P}$ is a semi-algebraic cell; that is, there exists a semi-algebraic homeomorphism between C_I and $(0,1)^d$, for some $d \in \mathbb{N}$.*

Corollary 3.1.5. *Any c.a.d. of $\mathbb{R}^n$ is a finite cylindrical semi-algebraic cell decomposition of $\mathbb{R}^n$.*

The converse is false in general.

Example 3.1.6 (cylindrical semi-algebraic cell decomposition but not c.a.d.). *Consider the cell decomposition of $\mathbb{R}^2$ consisting of seven cells:*
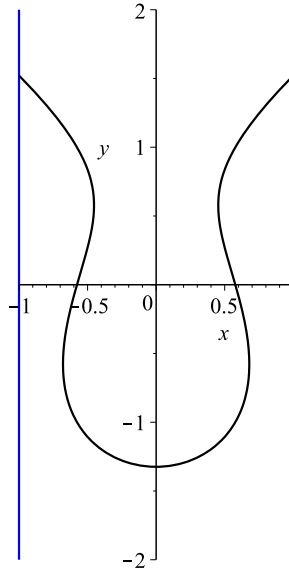
The 1-dimensional "section" is the intersection of the zero set of $3x^2 - y^3 + y - 1$ and $(-1, 1) \times \mathbb{R}$.

As we use cylindrical algebraic decompositions as a tool to study semi-algebraic sets, we want to relate a c.a.d. to a semi-algebraic set.

Definition 3.1.7 (F-invariant cad). *Let F be a finite set of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$. A c.a.d. is F-**invariant***, if for every $f \in F$, $\mathrm{sign}(f)$ is constant for each $C \in \mathcal{P}$.*

Definition 3.1.8 (c.a.d. adapted to semi-algebraic set S). *Let S be a semi-algebraic set. A c.a.d. $\mathcal{P}$ of $\mathbb{R}^n$ is **adapted**[1] **to** S if S is the union of cells of $\mathcal{P}$.*

Remark 3.1.9. *A F-invariant c.a.d. is adapted to the semi-algebraic set $\bigcup_{f_i \in F} V(f_i)$.*

An important result of semi-algebraic geometry, is that given any semi-algebraic set S, there exists a cylindrical algebraic decomposition adapted to S. We can take Collins' algorithm as the proof of such fact. See [4, Thm. 2.2.1] for a more theoretical approach.

In Example 3.1.2 above, we have a c.a.d. which is invariant with respect to the polynomial $f(x, y) = x^2 + y^2 - 1$.

Lemma 3.1.10. *A F-invariant c.a.d. $\mathcal{P}$ of $\mathbb{R}^n$ is adapted to any semi-algebraic set defined by inequalities of polynomials in F.*

Proof. Let $\{C_i\}$ be the cells of $\mathcal{P}$ where $\mathrm{sign}(f)$ take the same values that define S, for all $f \in F$. As $\mathcal{P}$ is a cell decomposition and $\mathrm{sign}(f)$ is invariant on each C_i, the C_i partition S. $\qquad\square$

In particular, we can study a semi-algebraic set S via a F-invariant cylindrical algebraic decomposition by ensuring that F includes all polynomials necessary to define S.

To determine the exactly which polynomials we need is more complicated. As presentations of semi-algebraic sets are not canonical in general, the question of a "minimal" set F is not straightforward.

There are different notions of invariance with respect to a semi-algebraic set. What we called a S-invariant c.a.d. is referred to as a sign-invariant c.a.d. in [15]. There are other notions of invariance, such as order-invariance which is defined in [24]. We will not need to make such distinctions.

Remark 3.1.11. *An application of cylindrical algebraic decomposition is to decide whether or not a semi-algebraic set S is empty. Some definitions of c.a.d. such as Collins' in [8], require that each cell comes with a **sample point**; that is, for each C in some c.a.d. we require the decomposition to specify the extra data of a real algebraic*

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[1]The c.a.d. $\mathcal{P}$ is sometimes said to be **compatible to** S

point in C. Thus, computing such a c.a.d. adapted to S allows us to decide whether $S = \emptyset$. This is particularly of interest if S is a real algebraic variety.

From [3, Ex. 5.34], given a c.a.d. adapted to a semi-algebraic set S, we cannot recover the topology of S. To elaborate further, if we are given a cell decomposition of $\mathbb{R}^n$ that was induced by an F-invariant c.a.d.– where F is unknown F – then we cannot recover F.

Given a c.a.d. adapted to a compact semi-algebraic set S, we have a cell decomposition for S; we can ask how close this is to a CW-complex of S. In general, a c.a.d. it not a CW-complex as a c.a.d. can fail to be closure finite. See Chapter 7.

As cylindrical algebraic decompositions are cell decompositions, the naming conventions of Remark 2.2.6 applies. For example, Example 3.1.2 is both well-bordered and locally boundary connected.

We will now defined an important family of cylindrical algebraic decompositions; this family is central to this thesis.

Definition 3.1.12. *We say that a c.a.d. $\mathcal{P}$ of $\mathbb{R}^n$ is a **strong c.a.d.** if $\mathcal{P}$ is well-bordered and locally boundary connected.*

Note that, a priori, this definition is weaker than the definition of a strong c.a.d. given by Lazard in [22]. By Corollary 2.2.18 these definitions are equivalent; that is, the closure finite requirement is superfluous.

We end this section by discussing a convenient extra condition we can impose on a cylindrical algebraic decomposition. First, we discuss a way on which we can relate two cylindrical algebraic decompositions.

Definition 3.1.13 (refinement of a c.a.d.)**.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be cylindrical algebraic decompositions of $\mathbb{R}^n$. We say that $\mathcal{P}$ **is a refinement** of $\mathcal{Q}$ if every cell of $\mathcal{Q}$ is a union of cells of $\mathcal{P}$.*

For example, if $F \subset F'$, then any F'-invariant c.a.d. is a refinement of a F-invariant c.a.d.

Consider the following refinement of Example 3.1.2.

Example 3.1.14 (non-reduced c.a.d.)**.** *Suppose that the cylinder $C_{1,1} \times \mathbb{R}$ in Example 3.1.2 was instead split into three cells $C_{1,1} = \{(x,y) \in \mathbb{R}^2 \mid x < 0, y < 0\}$, $C_{1,2} = \{(x,y) \in \mathbb{R}^2 \mid x < 0, y = 0\}$, and $C_{1,3} = \{(x,y) \in \mathbb{R}^2 \mid x < 0, y > 0\}$. If we wanted to study the 1-sphere via this cell decomposition, it is clear that this new augmented cylinder is unnecessary.*

As the definition of a cylindrical algebraic decomposition does not restrict the cell decomposition from having unnecessary cells, we require that any section is defined by sign conditions on F.

Definition 3.1.15. *We say that a F-invariant c.a.d. is* **reduced** *if for every section C, there exists some $f \in F$ such that $\operatorname{sign} f = 0$ in C.*

See Chapter 7 for a discussion between the relation of reduced cylindrical algebraic decompositions and minimal.

3.2 Well-based c.a.d.

We can ask whether we can impose extra conditions so that a c.a.d. adapted to a compact semi-algebraic set S yields a CW-complex of S. In [31], Schwartz and Sharir showed that a special type of c.a.d. is not only a CW-complex of S but a regular cell complex of S.

The main idea is to use the projection mapping π on the closure of C to give a homeomorphism between $(\overline{C}, C)$ and $(\overline{\pi(C)}, \pi(C))$; this homeomorphism together with an inductive hypothesis allows us to prove the regularity of C.

Consider the following example.

Example 3.2.1 (regular cell but not via π)**.** *Consider the real surface of $\mathbb{R}^3$ defined by the zero set of $f(x, y, z) = x^2 - zy^2$. We can construct a closure finite $\{f\}$-invariant c.a.d. of $\mathbb{R}^3$ which contains the following cell:*

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < x < 1, -x < y < x, z = \frac{x^2}{y^2}\}.$$

As C is not locally boundary connected, it is not regular. We can partition this cell further so that C is the union of regular cells. For example, we can partition C into:

$$\{(x, y, x) \in C \mid y < 0\},$$
$$\{(x, y, x) \in C \mid y = 0\}, \text{ and}$$
$$\{(x, y, x) \in C \mid y > 0\}.$$

While the lack of local boundary connectedness is not an issue in this refinement, we cannot prove regularity via π; the fibre of $(0, 0)$ under π is a closed segment.

We want to impose conditions on the c.a.d. so that the lack of injectivity of π is not an issue.

Definition 3.2.2 (c.f. [22, Def. 4.1]). *Let $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_n)$ be a F-invariant c.a.d. of $\mathbb{R}^n$. We say that $D \in \mathcal{P}_{n-1}$ is a **bad cell of** $(\mathcal{P}, F)$ if, for some $f \in F$, $f(a, x) = 0$ for $a \in D$.*

When the context is clear, we will just say that D is a bad cell.

Example 3.2.3 (bad cell). *In Example 3.2.1, $(0, 0)$ is a bad cell.*

The concept of bad cells is important due to the following result.

Theorem 3.2.4 (c.f. [31, Thm. 2]). *Let $\mathcal{P}$ be a F-invariant strong c.a.d. of $\mathbb{R}^n$ adapted to a compact semi-algebraic set S. If $(\mathcal{P}, F)$ has no bad cells, then $\mathcal{P}$ is a regular cell complex of S.*

This result is stated in [31, Thm. 2] in terms of a *well-based cylindrical algebraic decomposition*. A well-based c.a.d. – see [31, Def. 5] – is a *F*-invariant c.a.d. that does not have any bad cells.

Remark 3.2.5. *In [22, Thm. 4.4], Lazard showed that the output $\mathcal{P}$ of Collins' algorithm to compute a F-invariant c.a.d.– see [8] – is a strong c.a.d. if $(\mathcal{P}, F)$ does not have any bad cells.*

Before we proceed with the proof of Theorem 3.2.4 we prove a few auxiliary lemmas. The following result is a small generalisation of [6, Lem. 2.5.6].

Lemma 3.2.6. *Let $\mathcal{P}$ be a reduced F-invariant c.a.d. of $\mathbb{R}^n$, where F is closed under the $\frac{\partial}{\partial x_n}$ operator. Moreover, suppose C is a bounded section of $\mathcal{P}$. If $g \colon D \to \mathbb{R}$ is the semi-algebraic continuous bounded function that defines C and $(G, \mathcal{P})$ has no bad cells, where G is the set of polynomials that vanish at C, then g can be extended continuously to $\overline{D}$.*

Proof. The proof follows [6, Lem. 2.5.6]. Let $p \in \partial D$ and note that it suffices to extend g continuously to a function with domain $\overline{g} \colon D \cup \{p\} \to \mathbb{R}$.

By the curve selection lemma, [6, lemma 2.5.5], there exists a continuous, semi-algebraic function $h \colon [0, 1] \to \mathbb{R}^{n-1}$ such that $h(0) = p$ and $h((0, 1]) \subset D$. We then define the function $\varphi(0, 1] \to \mathbb{R}$ as

$$\varphi = g \circ (h|_{(0,1]}).$$

As g is a bounded function, φ is bounded. We use [6, Lem. 2.5.3] to extend φ continuously to 0; we define $\overline{g}(p) = \varphi(0)$ and claim that $\overline{g} \colon D \cup \{p\} \to \mathbb{R}$ is continuous.

Suppose that $\overline{g}$ is not continuous: then there exists a $\mu \in \mathbb{R}$, $\mu > 0$, such that for all $\delta \in \mathbb{R}$, $\delta > 0$, there exists an $x \in D$ with $\|x - p\| < \delta$ and $|g(x) - \varphi(0)| \geq \mu$.

Now, consider the following semi-algebraic subset of D:

$$S_\mu = \{x \in D : |g(x) - \varphi(0)| \geq \mu\}.$$

As $p \in \overline{S_\mu}$, we can apply the curve selection lemma to S_μ and p. We get a continuous, semi-algebraic function $h^*\colon [0,1] \to \mathbb{R}^{n-1}$ such that $h^*(0) = p$ and $h^*((0,1]) \subset S_\mu$. Similarly to φ, we define $\psi\colon (0,1] \to \mathbb{R}$ as

$$\psi = g \circ (h^*|_{(0,1]}).$$

As g is bounded, ψ is bounded; we use [6, Lem. 2.5.3] to extend ψ continuously to 0. As a consequence of the continuity of φ and ψ at 0, we have $|\varphi(0) - \psi(0)| \geq \mu$; in particular, $\varphi(0) \neq \psi(0)$. Moreover, $\operatorname{sign}(f_k(p, \psi(0))) \in \overline{\epsilon(k)}$ and $\operatorname{sign}(f_k(p, \varphi(0))) \in \overline{\epsilon(k)}$, for all $f_k \in F$.

Let $A_{\epsilon(k)} = \{y \in \mathbb{R} \mid \operatorname{sign}(f_k(p, y)) = \epsilon(k)\}$ — see Definition A.1.6 for notation. As F is closed under the $\frac{\partial}{\partial x_n}$ operator, by Thom's Lemma(Theorem A.1.8), $A_{\epsilon(k)}$ is empty, a point, or an interval. Since at least one $\epsilon(k)$, for $f_k \in G$, must equal 0, and there are no bad cells, $A_{\epsilon(k)}$ is not an open interval; without loss of generality say $\epsilon(1) = 0$. Invoking Thom's Lemma for a second time, $\overline{A_{\epsilon(1)}} = A_{\overline{\epsilon(1)}}$ which now is either empty or a point; this contradicts the observation that – for all $f_k \in F$, and in particular $k = 1$ – $\operatorname{sign}(f_k(p, \psi(0))) \in \overline{\epsilon(k)}$ and $\operatorname{sign}(f_k(p, \varphi(0))) \in \overline{\epsilon(k)}$. Hence $\overline{g}$ is continuous.

$\square$

Lemma 3.2.7. *Let C be a bounded, locally boundary connected section of a F-invariant c.a.d. of $\mathbb{R}^n$, where C is defined by a semi-algebraic continuous function $g\colon D \to \mathbb{R}$. Moreover, suppose that $\{C_i\}$ partitions C, where each C_i is the graph of $g_i\colon D_i \to \mathbb{R}$. If all the g_i extend continuously to $\overline{D_i}$, then g extends continuously to D.*

Proof. We define the function $\overline{g}\colon \overline{D} \to \mathbb{R}$ by

$$\overline{g}(x) = \overline{g_i}(x) \text{ if } x \in \overline{C_i}.$$

We only need to show that $\overline{g}$ is well-defined; more specifically, show that $\overline{g_i}$ and $\overline{g_j}$ agree on $\overline{C_i} \cap \overline{C_j}$ for all $i \neq j$.

As $\overline{g}$ is well-defined if $p \in C$ or $p \in \partial C \cap \partial C_i$ for a single i, we need to show that if $p \in \partial C \cap \partial C_i \cap \partial C_j$, for some $i \neq j$, then $\overline{g_i}(p) = \overline{g_j}(p)$.

Since determining whether $\overline{g_i}$ and $\overline{g_j}$ agree at p is a local question, we consider whether $\overline{g}$ is well-defined at $\overline{C} \cap B(p, \epsilon)$, for some $\epsilon > 0$. We choose ϵ so that $\overline{C} \cap B(p, \epsilon)$ satisfies the following properties:

i. The set $C \cap B(p, \epsilon)$ is connected; we can choose such $\epsilon > 0$ as C is, by assumption, locally boundary connected.

ii. If C_i and C_j are such that their intersection with $B(p, \epsilon)$ is non-empty, then the semi-algebraic set $\overline{C_i} \cap \overline{C_j} \cap B(p, \epsilon) \subset C$ consists of a single connected component whose closure contains p. We can choose a smaller $\epsilon > 0$ so that, if $p \in \overline{C_i}$, then $C_i \cap B(p, \epsilon) \neq \emptyset$. Moreover, since $\overline{C_i} \cap \overline{C_j}$ is semi-algebraic, it has finitely many connected components; we restrict ϵ further so that the second condition is satisfied.

We have two cases to consider.

Firstly, suppose that $\{p\} \subsetneq \partial C_i \cap \partial C_j$: by (ii) above, p lies in the closure of a semi-algebraic set contained in $\overline{C_i} \cap \overline{C_j} \cap C$ and thus $\overline{g_i}$ and $\overline{g_j}$ agree at p.

Secondly, suppose that $\{p\} = \partial C_i \cap \partial C_j$: we show that $\overline{g_i}(p) = \overline{g_j}(p)$ by constructing a sequence of cells

$$C_i = C_{l_1}, \cdots, C_{l_k} = C_j$$

where consecutive cells satisfy $\{p\} \subsetneq \partial C_{l_r} \cap \partial C_{l_{r+1}}$ for $1 \leq r \leq k-1$; thus, by applying the first case to consecutive cells we can show that $\overline{g_{l_1}}(p) = \overline{g_{l_2}}(p) = \cdots = \overline{g_{l_k}}(p)$ as desired.

We need to show how to construct such a sequence.

The idea is to consider a semi-algebraic path between $C_i \cap B(p, \epsilon)$ and $C_j \cap B(p, \epsilon)$; this is possible as $C \cap B(p, \epsilon)$ is semi-algebraic and connected, and thus – by [6, Prop. 2.5.13] – semi-algebraically path-connected. The path gives us a way of selecting the correct consecutive cells which satisfy the desired property.

Let $\gamma \colon [0, 1] \to C \cap B(p, \epsilon)$ be a semi-algebraic path between a point in $C_i \cap B(p, \epsilon)$ and a point in $C_j \cap B(p, \epsilon)$. As $\gamma([0, 1])$ semi-algebraic, $\gamma([0, 1])) \cap C_r$ has finitely many connected components; thus $\gamma^{-1}(C_r)$ is a finite collection of intervals contained in $[0, 1]$.

Considering the preimage of every cell C_r, we get a finite partition $[0, 1] = \bigcup_{r=1}^{k} I_s$ where $\sup I_i = \inf I_{i+1}$. Let C_{l_r} be the cell associated with I_r; that is, $I_r \subset \gamma(C_{l_r})$.

It remains to show that $\{p\}$ is strictly is contained in $\partial C_{l_r} \cap \partial C_{l_{r+1}}$ for $1 \leq r \leq k$. By assumption on ϵ, it is enough to show that $\overline{C_r} \cap \overline{C_{r+1}} \cap C \neq \emptyset$. Let $x \in \overline{I_r} \cap \overline{I_{r+1}}$, and consider any open ball around $\gamma(x)$. This open ball intersects both C_r and C_{r+1}.

This completes the proof. $\qquad\square$

We now give a proof of Theorem 3.2.4.

Proof of Theorem 3.2.4. As a strong c.a.d. $\mathcal{P}$ of $\mathbb{R}^n$ is closure finite, we just need to show that any $C \in \mathcal{P}$ contained S is a regular cell. Moreover, if we can show that the

sections of $\mathcal{P}$ are regular, then by [31, Lem. 5], so are the sectors; for a more detailed discussion about the regularity of sectors, see Section 4.1.

We will show that a section C is a regular cell by proving that $(\overline{C}, C)$ is homeomorphic to $(\overline{D}, D)$, where $D \in \mathcal{P}_{n-1}$ is the cell below C. The result then follows by induction.

As C is a section, it is the graph of some function $g \colon D \to \mathbb{R}$. Thus, it suffices to show that the g extends continuously to $\overline{D}$.

Let F' be the closure of F under $\frac{\partial}{\partial x_n}$; that is, the smallest set that contains F and is closed under partial differentiation with respect to x_n. Then a F'-invariant c.a.d. of $\mathbb{R}^n$ is a refinement of $\mathcal{P}$. This gives us the following data: a partition $\{C_i\}$ of C, and functions $g_i \colon D_i \to \mathbb{R}$ such that $g|_{D_i} = g_i$ and C_i is the graph of g_i.

Now, applying Lemma 3.2.6 to the F'-invariant c.a.d. and C with $G \subset F$ the set of polynomials that define C, using the notation from Lemma 3.2.6, each g_i can be extended continuously to $\overline{D_i}$.

Finally, we use Lemma 3.2.7 to extend g continuously to $\overline{D}$.

$\square$

Corollary 3.2.8. *Let C be a bounded cell of a F-invariant strong c.a.d. of $\mathbb{R}^n$, where C is the graph of function $g \colon D \to \mathbb{R}$. The function g extends continuously to a point $p \in \overline{D}$ away from the bad cells of $(\mathcal{P}, F)$.*

We can give an upper bound for the dimension of bad cells.

Definition 3.2.9 (Primitive w.r.t. x_n)**.** *We say a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is* **primitive with respect to** x_n *if there are no irreducible factors that are constant in x_n. We say a finite set $F \subset \mathbb{R}[x_1, \ldots, x_n]$ is primitive with respect to x_n if each $f \in F$ is primitive with respect to x_n.*

Lemma 3.2.10 ([22, Lem. 4.3])**.** *Let $\mathcal{P}$ be a F-invariant c.a.d. of $\mathbb{R}^n$. If $C \subset \mathbb{R}^{n-1}$ a $(\mathcal{P}, F)$ bad cell, then C has codimension at least 1 in $\mathbb{R}^{n-1}$. Moreover, if F is primitive with respect to x_n, then C has codimension at least 2 in $\mathbb{R}^{n-1}$*

Proof. The bad cells of $(\mathcal{P}, F)$ consist of the points of $\mathbb{R}^{n-1}$ where the coefficients of some polynomial $f \in F \subset \mathbb{R}[x_1, \ldots, x_n]$ – if we view f as an element of $\mathbb{R}[x_1, \ldots, x_{n-1}][x_n]$ – vanish simultaneously.

If f is not primitive with respect to x_n, then C has at least codimension 1 in $\mathbb{R}^{n-1}$; otherwise, the set where two polynomials with no common factor vanish has codimension at least 2 in $\mathbb{R}^{n-1}$. $\square$

The extra condition of F being primitive in x_n is not an obstruction. If $f = gh \in F$, with $g \in \mathbb{R}[x_1, \ldots, x_{n-1}]$ and $h \in \mathbb{R}[x_1, \ldots, x_n]$, is not primitive with respect to x_n, we

consider a refinement defined by $F' = (F \setminus \{f\}) \cup \{g, h\}$. In particular, if a F-invariant c.a.d. is adapted to some compact semi-algebraic set S, so is its F'-invariant refinement. Consequently, we may assume that F is primitive with respect to x_n when studying compact semi-algebraic sets.

Corollary 3.2.11. *Any F-invariant strong c.a.d. $\mathcal{P}$ of $\mathbb{R}^2$ adapted to a compact semi-algebraic set S, with F primitive in x_2, is a regular cell complex of S.*

Proof. As the cell decomposition is, by assumption, closure finite, we only need to show that every cell contained in S is a regular cell. By Lemma 3.2.10, $(\mathcal{P}, F)$ does not have any bad cells, and thus, by Theorem 3.2.4, $\mathcal{P}$ is a regular cell complex. $\square$

To compute regular cell complexes of a compact semi-algebraic set S via strong cylindrical algebraic decompositions, the main obstruction, apart from computing the strong c.a.d. adapted to S, is the presence of bad cells. We can, after a change of coordinates, ensure that this obstruction is not an issue. See [38, Ch. 2, 3.5] for a description in how to choose the change of coordinates.

More specifically, if $\mathcal{P}$ is a F-invariant strong c.a.d. of $\mathbb{R}^n$ adapted to S, then there exists a change of coordinate $\nu \colon \mathbb{R}^n \to \mathbb{R}^n$ such that, for any $\nu(F)$-invariant c.a.d. $\mathcal{P}'$ of $\mathbb{R}^n$, $(\nu(F), \mathcal{P}')$ does not have any bad cells. Consequently, any $\nu(F)$-invariant strong c.a.d. of $\mathbb{R}^n$ is a regular cell complex of $\nu(S)$. Thus, we can study $\nu(S)$, and in particular S, via cylindrical algebraic decompositions.

Proposition 3.2.12 (cf. [6, Thm. 9.1.6])**.** *Let S be a semi-algebraic set of $\mathbb{R}^n$. There exists a finite set of polynomials $F \subset \mathbb{R}[x_1, \ldots, x_n]$ and a linear automorphism $\nu \colon \mathbb{R}^n \to \mathbb{R}^n$ such that any F-invariant c.a.d. $\mathcal{P}$ is adapted to $\nu(S)$ and $(F, \mathcal{P})$ contains no bad cells.*

Example 3.2.13 (Change of coordinates)**.** *Recall that not all cells in Example 3.2.1 were regular via the projection mapping. By taking the change of coordinates ν defined by*

$$(x, y, z) \longmapsto (x + z, y, z),$$

We eliminate the bad cell obstruction.

As mentioned in Chapter 1, when studying semi-algebraic sets via cylindrical algebraic decompositions, a change of coordinates is not always desirable. In Chapter 4 we state a conjecture that states that we can obtain regularity of cells without the change of coordinates.

3.3   c.a.d. cell indexing

The index of a c.a.d. cell contains information about the dimension of cells and their adjacency relation.

Lemma 3.3.1 (index determines dimension)**.** *Let $C_{i_1,\ldots,i_n}$ be a c.a.d. cell of $\mathbb{R}^n$ and k be the number of odd i_j in the index of C. Then*

$$\dim(C) = k$$

Proof. This follows directly from the inductive definition of a c.a.d.; that is, Definition 3.1.1. $\qquad\square$

The indexing also contains partial information about the adjacency relation between cells. In dimension 1, the indexing characterises the adjacency relations. Two cells C and D are adjacent if and only if their index differs by at most one; that is, one of the cells has index k and the other $k+1$, for some positive k. Moreover, we can characterise the sub-adjacency relation by considering whether or not k is even or odd.

In higher dimension, the situation is more intricate. Suppose that $C = C_{i_1,\ldots,i_{n-1},r}$ and $D = D_{i_1,\ldots,i_{n-1},s}$ are cells of a c.a.d. of $\mathbb{R}^n$ which lie in the same cylinder. Then C is adjacent to D if and only if $|r - s| < 2$. Similarly to above, we can decide whether one is sub-adjacent to the other by considering whether r or s is even.

This yields a necessary, but crude, condition for two cells to be adjacent. Suppose that $C = C_{i_1,\ldots,i_n}$ and $D = D_{j_1,\ldots,j_n}$ are two cells of $\mathbb{R}^n$. We can project C and D enough times so that they lie in the same cylinder. We then use the argument above to determined whether two cells are not adjacent.

In [38], Van den Dries defines an alternative index of a c.a.d. cell. It encodes the dimension of the cell but does not contains any information about adjacency of the cells. Note that the n-tuples associated to a cell of $\mathbb{R}^n$ in [38] does not actually index the cells.

Definition 3.3.2 ([38, Def. 2.3, Ch. 3])**.** *We define the index of a cell inductively. For a c.a.d. of $\mathbb{R}$, we say that a cell C is a (0)-cell if it is a section. If C is a sector, we say that C is a (1)-cell. Now suppose that C is a cell of a c.a.d. of $\mathbb{R}^n$. Then C is a $(i_1,\ldots,i_{n-1},0)$-cell if it is a section and a $(i_1,\ldots,i_{n-1},1)$-cell if it is a sector.*

The value of i_k is determined by whether the projection $\pi^{n-k}(C)$ is a section or sector. A cell $C = C_{i_1,\ldots,i_n}$ is a $(i'_1,\ldots,i'_n)$-cell, where $i'_j \equiv i_j + 1 \mod 2$.

This alternative index yields a more straightforward formula for the dimension of a cell.

Lemma 3.3.3. *Let C be a $(i_1, \ldots, i_n)$-cell of a cylindrical algebraic decomposition of $\mathbb{R}^n$. Then*

$$\dim(C) = \sum_{j=1}^{n} i_j.$$

The proof follows directly from the definition of a $(i_1, \ldots, i_n)$-cell and Lemma 3.3.1.

3.4 Cell crafting

We want to study the additional structure on the boundary of a cell that arises from a cylindrical algebraic decomposition. In particular, we can partition the boundary of a c.a.d. cell by taking into consideration where the boundary is relative to the coordinate axis; in some sense, we generalise the notion of faces of the cube.

We first consider the part of the boundary directly above and below, called top and bottom respectively, of a cell. The idea of the top, bottom and sidewalls of a cell is defined by Basu, Gabrielov and Vorobjov in [2, Def. 3.1]. We will use a modified version of top and bottom compared to [2].

Definition 3.4.1 (Top and bottom of cells)**.** *Let $C = C_{i_1, \ldots, i_n}$ be a cell of a c.a.d. of $\mathbb{R}^n$. We define the **top of** C, denoted C_T, to be the semi-algebraic set*

$$C_T = \begin{cases} C_{i_1, \ldots, i_n + 1} & \text{if } i_n \text{ is odd and } C_{i_1, \ldots, i_n + 1} \text{ exists,} \\ \emptyset & \text{otherwise.} \end{cases}$$

*We can define the **bottom of** C, denoted C_B, to be $C_{i_1, \ldots, i_n - 1}$ if i_n is odd and $C_{i_1, \ldots, i_n - 1}$ exists.*

First note that if C_T and C_B exist, they are cells and their dimension is clear:

$$\dim(C_T) = \dim(C_B) = \dim(C) - 1.$$

The top and bottom, C_T and C_B are the portions of the boundary that lie above and below C respectively, along the x_n-coordinate. We generalise this notion to all directions along a coordinate x_k.

Definition 3.4.2 (Face of cell in the x_k direction)**.** *Let C be a c.a.d. cell of $\mathbb{R}^n$. We define the **face above** C **in the** x_k**-direction**, denoted $C_{x_k}^+$ as follows:*
If $\pi^{n-k}(C)$ is a section or $(\pi^{n-k}(C))_{x_k}^+$ is empty, then

$$C_{x_k}^+ = \emptyset.$$

Otherwise, the face above C is the x_k-direction is the semi-algebraic set

$$C_{x_k}^+ = \left[(\pi^{n-k}(C))_{x_k}^+ \times \mathbb{R}^{n-k}\right] \cap \overline{C},$$

*where $(\pi^{n-k}(C))_{x_k}^+$ consists of the cells lying above $\pi^{n-k}(C)$ in the x_k-direction; that is, $(\pi^{n-k}(C))_{x_k}^+ = (\pi^{n-k}(C))_T$. Mutatis mutandis, we define $C_{x_k}^-$, the **face below C in the x_k-direction**.*

Note that $C_{x_k}^\pm$ is, in general, not a cell but the union of cells.

Example 3.4.3 (Faces of 2-sphere)**.** *Let C be the open 3-ball in [3, Ex. 5.4] — this example contains the standard c.a.d. of the 2-sphere. Then*

$$\begin{aligned} C_z^+ &= \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, z > 1\} \\ C_z^- &= \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, z < 1\} \end{aligned}$$

That is, both the top and bottom are the upper and lower half-spheres. The faces in the y-direction, C_y^+ and C_y^-, are the half-circles $\{(x,y,0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, y > 1\}$ and $\{(x,y,0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, y < 1\}$, respectively. Lastly, C_x^+ and C_x^- are the points $(1,0,0)$ and $(-1,0,0)$, respectively.

As mentioned above, this is different from the definition in [2]. We make the convention that a section has empty top or bottom; we think of a section as the top and bottom of a sector instead. We can recover the definition of C_T and C_B given in [2, Def. 3.1].

Remark 3.4.4. *In [2, Def. 3.1], the C_T of a section is defined in the following way: let C be a $(i_1, \ldots, i_{n-1}, 0)$-cell defined by a function $g \colon D \to \mathbb{R}^{n-k}$ where $k \in \{0, \ldots, n-1\}$ is the largest number such that $i_k = 1$. The top of C, according to [2], is $\pi^{-1}(D_T)$ where $\pi \colon \overline{C} \to \overline{D}$ is the projection mapping. This is equivalent to defining the top of a section C to be $C_{x_k}^+$ where $k \in \{0, \ldots, n-1\}$ is the largest k such that $\pi^{n-k}(C)$ is a sector and $C_{x_k}^+$ is non-empty.*

We can think of our C_T and C_B as the absolute top and bottom of a cell and the definition given in [2] as the relative top and bottom.

Proposition 3.4.5. *Let C be a c.a.d. cell of $\mathbb{R}^n$. For all $1 \leq k \leq n$, $C_{x_k}^\pm$ is connected.*

Before we prove this result, we need a auxiliary lemma.

Lemma 3.4.6. *Let $\mathcal{P}$ be a c.a.d. of $\mathbb{R}^n$ and C a cell in $\mathcal{P}$. For all $k \leq n-1$,*

$$\pi(C_{x_k}^\pm) = (\pi(C))_{x_k}^\pm$$

Proof. From its definition,

$$C_{x_k}^{\pm} = \left[(\pi^{n-k}(C))_{x_k}^{\pm} \times \mathbb{R}^{n-k}\right] \cap \overline{C} \subset \mathbb{R}^n;$$

applying π to it, we get

$$\pi(C_{x_k}^{\pm}) = \left[(\pi^{n-k}(C))_{x_k}^{\pm} \times \mathbb{R}^{n-k-1}\right] \cap \overline{\pi(C)} \subset \mathbb{R}^{n-1}.$$

Applying the definition of faces along x_k to the c.a.d. cell $\pi(C)$ of $\mathbb{R}^{n-1}$ we get

$$
\begin{aligned}
(\pi(C))_{x_k}^{\pm} &= \left[(\pi^{n-k-1}(\pi(C)))_{x_k}^{\pm} \times \mathbb{R}^{n-k-1}\right] \cap \overline{\pi(C)} \subset \mathbb{R}^{n-1}, \\
&= \left[(\pi^{n-k}(C))_{x_k}^{\pm} \times \mathbb{R}^{n-k-1}\right] \cap \overline{\pi(C)} \subset \mathbb{R}^{n-1},
\end{aligned}
$$

as required.

$\qquad\square$

Proof of Proposition 3.4.5. The result holds for $n = 1$. Suppose $n > 1$ and that $C_{x_k}^{\pm}$ is not connected; that is, there exists closed X and Y, such that, $C_{x_k}^{\pm} = X \sqcup Y$. Consider the open surjection

$$\pi \colon C_{x_k}^{\pm} \to (\pi(C))_{x_k}^{\pm}.$$

By [22, Prop. 5.2], $\pi^{-1}(q)$ is connected, for all $q \in (\pi(C))_{x_k}^{\pm}$. Consequently, X and Y cannot intersect the same fibre. In particular, $\pi(X) \cap \pi(Y) = \emptyset$ and $(\pi(C))_{x_k}^{\pm}$ is not connected. This contradicts the inductive assumption. $\qquad\square$

The faces of C partition ∂C.

Lemma 3.4.7 (Faces partition boundary of cell)**.** *Suppose that C is a c.a.d. cell of $\mathbb{R}^n$. Then*

$$\partial C = (\cup_{k=1}^n C_{x_k}^+) \cup (\cup_{k=1}^n C_{x_k}^-)$$

is a partition of ∂C. Moreover, if C is closure finite, then, for all $k \in \{0, \ldots, n\}$, $C_{x_k}^{\pm}$ is the union of cells.

Proof. It is straightforward to see the equality between ∂C and $(\cup_{i=1}^n C_{x_i}^+) \cup (\cup_{i=1}^n C_{x_i}^-)$; consequently, we only need to show that these are disjoint.

Firstly, $C_{x_i}^+ \cap C_{x_i}^- = \emptyset$ for all i. Let $\clubsuit$ and $\spadesuit \in \{+, -\}$, and consider $C_{x_i}^{\clubsuit}$ and $C_{x_j}^{\spadesuit}$ with $i < j$. Then

$$\pi^{n-i}(C)_{x_{n-i}}^{\clubsuit} \xrightarrow{\pi} \cdots \xrightarrow{\pi} \pi^{n-j}(C).$$

Therefore $C_{x_i}^{\clubsuit}$ lies above $\pi^{n-j}(C)$ but $C_{x_j}^{\spadesuit}$ lies above $\pi^{n-j}(C)_{x_{n-j}}^{\spadesuit}$.

Lastly, if C is closure finite, any cell D sub-adjacent to C is contained in ∂C. If $D \cap C_{x_k}^{\pm}$, then it projects to $(\pi^{n-k}(C))_{x_k}^{\pm}$ and thus, is contained in $C_{x_k}^{\pm}$. $\qquad\square$

An implication of Lemma 3.4.7 is that if D intersects $C^{\pm}_{x_m}$, then we can give a lower bound to the dimension of D.

The case of $C^{\pm}_{x_n} (= C_T$ or $C_B)$ is discussed above. Suppose that $C^{\pm}_{x_{n-1}}$ exists, then $\pi(C)$ is a $(i_0, \ldots, i_{n-2}, 1)$-cell and, in particular, $\pi(C)^{\pm}_{x_{n-1}}$ is a $(i_0, \ldots, i_{n-2}, 0)$-cell. The cylinder above $\pi(C)^{\pm}_{x_{n-1}}$ will have cells of dimension $\dim(C) - 1$ or $\dim(C) - 2$. Similarly, we give a proof of the general case.

Lemma 3.4.8. *Let C be a $(i_1, \ldots, i_n)$-cell of a c.a.d. of $\mathbb{R}^n$. For any cell D such that $D \cap C^{\pm}_{x_k} \neq \emptyset$, we have the following lower bound for $\dim D$:*

$$\sum_{j=1}^{n-k-1} i_j \leq \dim(D).$$

Proof. If D intersects $C^{\pm}_{x_k}$, then by cylindricity it must project to $(\pi^{n-k}(C))^{+}_{x_k}$. Thus, the first $n - k - 1$ entries of the indices of C and D agree. $\qquad\square$

Corollary 3.4.9. *Suppose $\mathcal{P}$ is a c.a.d. of $\mathbb{R}^n$, C is a $(i_1, \ldots, i_n)$-cell of $\mathcal{P}$, and $D \in \mathcal{P}$ is a d-cell contained in boundary of C. Then, D is not contained in any $C^{\pm}_k$ such that $\sum_{j=1}^{k} i_j > \dim(D)$*

By Lemma 3.4.8, the combinatorial structure of a cylindrical algebraic decomposition yields a lower bound for the dimension of cells which are contained in $C^{\pm}_{x_k}$. If we try to compute an upper bound for the cells in $C^{\pm}_{x_k}$, the combinatorial structure yields an upper bound no better than an upper bound coming from simple considerations in semi-algebraic topology; that is, the dimension of cells in $C^{\pm}_{x_k}$ is less than or equal to $\dim C - 1$.

We instead need to consider how the closures of $C^{\pm}_{x_k}$ interact with each other.

Remark 3.4.10. *If C is a n-cell of a c.a.d. of $\mathbb{R}^n$ and $C_T \cap C_B \cap C^{\pm}_{x_{n-1}} \neq \emptyset$, then $\dim C^{\pm}_{x_{n-1}} = n - 2$.*

While we might be able to understand how this affects the dimension of $C^{\pm}_{x_k}$ in low-dimensional $\mathbb{R}^n$, it is unclear how to proceed in the cases where $n > 3$. See Chapter 7.

3.5 Closure finite c.a.d.

We have seen in Corollary 2.2.18 that any well-bordered cell decomposition is closure finite; moreover, we have seen that any cell decomposition of $\mathbb{R}^2$ is well-bordered if and only if it is closure finite. We show that, with the added structure of a cylindrical algebraic decomposition, we can prove the equivalence of these two properties in dimension 3.

Proposition 3.5.1. *Any closure finite c.a.d. of $\mathbb{R}^3$ is well-bordered.*

Proof. The cases of 0- and 1-cells are trivial.

For any 2-dimensional cell C, we have to show that if D is a 0-cell in the boundary of C, there exists a 1-cell $E \subset \partial C$, such that $D \subset \overline{E}$.

Suppose that C is a $(0, 1, 1)$-cell. By Lemma 3.4.8, $D \subset C_y^{\pm}$. As D is a section, then either there exists a 1-cell E above or below D such that $D \subset \overline{E}$, or $E \subset C_T$. If C is a $(1, 0, 1)$-cell, then we can proceed similarly, with the adjustment that $D \subset C_x^{\pm}$.

Suppose that C is a $(1, 1, 0)$-cell. Away from any bad cells, by Corollary 3.2.8, π is a homeomorphism. Consequently, suppose that $\{p\}$ is bad cell in the boundary of $\pi(C)$. Now, by [22, Prop. 5.2], $\pi^{-1}(p)$ is a closed segment; thus, any cell in this closed segment is either a 1-cell or in the closure of a 1-cell.

Lastly, we consider 3-dimensional cells $C \subset \mathbb{R}^3$. Suppose that D is a cell contained in ∂C. If D is a 2-cell, there is nothing to prove. Moreover, if D is a section, then either there exists a sector above or below D, or $D \subset \overline{C_T}$. This leaves us with the case where D is a 1-dimensional sector not contained in the closure of C_T or C_B. By Lemma 3.4.8, $D \subset C_x^{\pm}$.

Suppose that there does not exists a 2-dimensional $E \subset \partial C$ in $C_x^{\pm}$ such that $D \subset \overline{E}$. Then there does not exists a 1-dimensional cell E' in $(\pi(C)_x^{\pm}$ such that $\pi(D) \subset \overline{E'}$. Thus, $\pi(D) \subset \overline{(\pi(C))_T}$, and in particular, there exists some 2-cell in $C_y^{\pm}$ that contains D in its closure.

This completes the proof. $\qquad\qquad\square$

If we try to extend this proof to $\mathbb{R}^4$, and subsequently to $\mathbb{R}^n$, we run into difficulties. Suppose that C is a $(1, 1, 1, 0)$-cell of a c.a.d. of $\mathbb{R}^4$ and C a single 0-dimensional bad cell D. We cannot use the argument above to show that a 1-cell E above D is contained in the closure of a 3-cell.

In any case, it seems from the structure on the boundary of a c.a.d. C cell – the faces C along the axis directions – that well-bordered is equivalent to closure finite in this setting. This is discussed in Chapter 7; more specifically, in Conjecture 7.2.1.

REGULARITY VIA STRONG CYLINDRICAL ALGEBRAIC DECOMPOSITION

Recall Definition 3.1.12: a strong c.a.d. of $\mathbb{R}^n$ is a cylindrical algebraic decomposition that is locally boundary connected and well-bordered.

In Section 3.2, we showed that if $\mathcal{P}$ is a F-invariant strong c.a.d. of $\mathbb{R}^n$ adapted to closed and bounded S and $(\mathcal{P}, F)$ does not have any bad cells, then $\mathcal{P}$ is a regular cell complex of S. Lazard conjectured in [22] that we can drop the "no bad cells" requirement and still have regularity.

Conjecture 4.0.1. *Let $\mathcal{P}$ be a strong c.a.d. of $\mathbb{R}^n$ adapted to a closed and bounded semi-algebraic set S. Then, $\mathcal{P}$ is a regular cell complex of S.*

The main aim of this thesis is to prove Conjecture 4.0.1.

As our cell decomposition may have bad cells, we need to show regularity by means other than standard projection mappings. In Chapter 5, using theory which arises from the study of manifolds with boundary, we give a proof for the case where $\dim S = 2$; Moreover, we show that if we strengthen the assumption, we can give a proof of the case where $n = 3$. Furthermore, we discuss why the choice of assumptions by Lazard in [22] might not be sufficient for regularity in higher dimensions.

In Chapter 7, we discuss how we might be able to use blow-ups and semi-algebraic partitions of unity to give a proof of Conjecture 4.0.1.

4.1 Sector bounded by regular cells

Suppose that C is a sector of c.a.d. of $\mathbb{R}^n$. If the functions that define C_T and C_B are well-behaved, it is a well-known result that C is a regular cell.

Lemma 4.1.1 (c.f. [31, Lem. 5]). *Let C be a sector above D and suppose that C_T and C_B are the graph of the semi-algebraic functions f_T, $f_B\colon D \to \mathbb{R}$, respectively. If we extend f_T and f_B continuously to $\overline{D}$, then C is a regular cell.*

In particular, we use lemma in Theorem 3.2.4 to show a F-invariant c.a.d. $\mathcal{P}$ with no bad cells is a regular cell complex by extending all the functions defining the sections of $\mathcal{P}$.

Let E be a sector bounded below and above by regular cell. Suppose further that E lies above a regular cell D. We want to show that E is regular. In [31, Lem. 5], we assume that the functions $g_T\colon D \to \mathbb{R}$ and $g_B\colon D \to \mathbb{R}$ that define C_T and C_B, respectively, are well-defined continuous functions in $\overline{D}$. Consequently, we prove that E is a regular cell. If we look at the case of a F-invariant strong c.a.d. $\mathcal{P}$ of $\mathbb{R}^n$, g_T and g_B are only well-defined on $\overline{D}$ when $\mathcal{P}$ has no bad cells. As our ultimate aim is to prove Conjecture 4.0.1, it would be desirable to show that E is a regular cell regardless of the bad cells. We conjecture – Conjecture 7.2.2 – that this the case.

4.2 Homotopy of non-bounded S via c.a.d.s

In the statement of Conjecture 4.0.1, we required that the semi-algebraic set S is closed and bounded. We might ask whether we can compute the homology of unbounded a semi-algebraic sets S via strong cylindrical algebraic decompositions of $\mathbb{R}^n$.

Proposition 4.2.1 ([6, Cor. 9.3.7]). *If $S \subset \mathbb{R}^n$ is semi-algebraic, then, there exists an $r > 0$, such that $S \cap \overline{B}(0, r)$ is a semi-algebraic deformation retract of S.*

In particular, if S is a locally closed subset of $\mathbb{R}^n$, we can use Conjecture 4.0.1 to compute the homology of S.

According to [6, Rem. 9.3.8], [10] shows that every semi-algebraic set $S \subset \mathbb{R}^n$ contains a closed and bounded subset K which is a semi-algebraic deformation retract of S.

Remark 4.2.2. *When computing cylindrical algebraic decompositions, we can replace the hypercube $[-r, r]^n$ by $\overline{B}(0, r)$.*

CHAPTER 5

REGULARITY OF STRONG C.A.D.S OF $\mathbb{R}^N$ VIA MANIFOLDS

As mentioned in Chapter 4, we study Conjecture 4.0.1 with the use of manifold theory. Suppose we have a compact semi-algebraic set S, a strong c.a.d. $\mathcal{P}$ of $\mathbb{R}^n$ adapted to S, and let $C \subset S$ be a cell of $\mathcal{P}$; we want to show that the cell C is regular.

If we have a regular d-cell C — that is, there exists a homeomorphism between $(\overline{C}, C) \to (\overline{B}^d, B^d)$ — we can endow $\overline{C}$ with a manifold with boundary structure, where we view the boundary of the cells as the boundary of the manifold. Consequently, a necessary condition for C to be regular is that we can view $\overline{C}$ as a manifold with boundary. We make the proviso that whenever we discuss a cell with the structure of a manifold with boundary, the boundary of the manifold is the boundary of the cell.

This gives us the following heuristic to prove Conjecture 4.0.1: try to find what conditions, together with the manifold with boundary structure, are sufficient for C to be regular.

We need to determine when $\overline{C}$ is homeomorphic, as a manifold with boundary, to $\overline{B}^d$. As the closure of a regular cells can be viewed as a compact, contractible manifold with boundary, we can ask whether the converse is true. Following [32, p. 388f], we need to determine when a compact, contractible manifold with boundary X homeomorphic to $\overline{B}^d$?

Before we proceed, we introduce some auxiliary terminology.

Definition 5.0.1 (Homology and Homotopy sphere). *Let X be a closed d-manifold. We say that X is a **homology d-sphere** if X has the same homology groups as S^d. Moreover, we say that X is a **homotopy d-sphere** if X has the same homotopy type as S^d.*

Remark 5.0.2. *From Definition 5.0.1, it is clear that any homotopy sphere is a homology sphere. By Hurewicz's and Whitehead's Theorem — [40, Thm. 7.1,IV] and Theorem B.3.2, respectively — any simply-connected homology sphere is a homotopy sphere.*

Given a compact contractible manifold with boundary X, it is well known that we can compute the homology of ∂X.

Lemma 5.0.3. *Let X be a d-dimensional compact, contractible manifold with boundary. The boundary of X is a homology $(d-1)$-sphere.*

Proof. Consider the relative homology group $H_i(X, \partial X)$. An application of the Snake Lemma is the following long exact sequence:

$$\cdots \to H_i(\partial X) \to H_i(X) \to H_i(X, \partial X) \to H_{i-1}(\partial X) \to \cdots$$

By the Lefschetz-Poincaré duality, $H_k(X, \partial X) \cong H^{d-k}(X)$. In particular

$$
\begin{array}{ccccccc}
H_{i+1}(X, \partial X) & \longrightarrow & H_i(\partial X) & \longrightarrow & H_i(X) & \longrightarrow & H_i(X, \partial X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{n-i-1}(X) & \longrightarrow & H_i(\partial X) & \longrightarrow & H_i(X) & \longrightarrow & H^{n-i}(X)
\end{array}
$$

As X is contractible,

$$H_i(X) \cong H^i(X) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Applying this to the long exact sequence above yields the desired result:

$$H_i(\partial X) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, n; \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Consequently, this reduces to answering the following question: when is a homology $(d-1)$-sphere homeomorphic to $\S^{d-1}$

In general, a homology sphere is not a sphere. The Poincaré sphere – see [29] – is an example of a homology 3-sphere that is not simply-connected, and in particular, not homeomorphic to $\S^3$. If we can show that our ∂X is simply-connected, then ∂X is a homotopy $(d-1)$-sphere and, by the Generalised Poincaré Conjecture, it is homeomorphic to the $(d-1)$-sphere.

We have the following breakdown depending on the dimension.

If $d \leq 2$, then ∂X is a homology $(d-1)$-sphere for a for $d = 0, 1$. In this case, we can use the low-dimension classification of closed manifolds to conclude that ∂X is homeomorphic to $\S^{d-1}$

If $d = 3$, then an equivalent statement – see [19, Conj. 3.5']— of the Poincaré Conjecture asserts that X is homeomorphic to the 3-disk. The Poincaré Conjecture was proved by Perelman; see [28].

If $d \geq 4$, by Remark 5.0.2 and the Generalised Poincaré Conjecture, a compact, contractible manifold with boundary X is homeomorphic to the d-disk if and only if ∂X is simply-connected.

Suppose that $d \leq 3$ and S is a compact d-dimensional semi-algebraic set of $\mathbb{R}^n$, and that $\mathcal{P}$ is a strong c.a.d. of $\mathbb{R}^n$ adapted to S. To show that $\mathcal{P}$ is a regular cell complex of S, it therefore suffices to show that every cell $C \subset S$ is a compact, contractible manifold with boundary. In particular, this gives us a proof of Conjecture 4.0.1 for $n = 3$. For the case where $d \geq 4$, we need to show additionally that ∂C is simply-connected.

Consequently, the aim of this chapter is to show that if C is a bounded d-cell of a strong c.a.d. of $\mathbb{R}^n$, then:

i. If $d \leq 3$: $\overline{C}$ is a compact, contractible manifold with boundary.

ii. If $d \geq 4$: $\overline{C}$ is a compact, contractible manifold with boundary with simply-connected boundary.

5.1   Homotopy type of $\overline{C}$

The first step in proving regularity is to compute the homotopy type of the closure of a c.a.d. cell C. More specifically, show that $\overline{C}$ is contractible.

By [22, Prop. 5.2], if C is a c.a.d. cell of $\mathcal{P}$ — where $\pi(C) = D$ and $\mathcal{P}_{n-1}$ is a strong c.a.d. — then the fibres of the standard projection mapping $\pi \colon \overline{C} \to \overline{D}$ are contractible. We exploit this fact to show that $\overline{C}$ and $\overline{D}$ have the same homotopy type.

Theorem 5.1.1 (Closure of c.a.d. cell is contractible)**.** *Suppose that $C \subset \mathbb{R}^n$ is a bounded c.a.d. cell of $\mathcal{P}$ and the induced c.a.d. $\mathcal{P}'$ of $\mathbb{R}^{n-1}$ is strong. Then $\overline{C}$ is contractible.*

Proof. We show that $\overline{C}$ and $\overline{D}$ have the same homotopy type and then, by induction on n, $\overline{C}$ is contractible.

Note that C and D are bounded. Consequently, $\overline{C}$ and $\overline{D}$ are compact and thus, by [6, Thm. 9.4.1], $\overline{C}$ and $\overline{D}$ admit a CW-complex structure. By Whitehead's theorem, Theorem B.3.2, it suffices to show that $\overline{C}$ and $\overline{D}$ are weakly homotopic equivalent.

If $n = 1$, the result is true. Suppose that $n > 1$.

To show the existence of a weak homotopy between $\overline{C}$ and $\overline{D}$, we invoke Smale's theorem(Theorem B.3.3). We just need to verify that the map $\pi \colon \overline{C} \to \overline{D}$ satisfies all the assumptions required by Smale's theorem.

The spaces $\overline{C}$ and $\overline{D}$ are connected as they are the closure of connected spaces. Moreover, as they are also Hausdorff compact metric spaces, they are locally compact and separable. The set $\overline{C}$ is locally contractible, by Corollary A.4.6, as semi-algebraic sets are locally contractible. As π is a continuous map between a compact space and a Hausdorff space; thus, π is closed and proper. Moreover, continuity together with fact that π is proper implies that $\pi(\overline{C}) = \overline{\pi(C)} = \overline{D}$; thus π is surjective.

By [22, Prop. 5.2], the fibres of π are closed segments and thus contractible. Moreover, invoking Corollary A.4.6 again, the fibres are locally contractible.

Finally, by Smale's theorem, π induces a weak homotopy equivalence between $\overline{C}$ and $\overline{D}$. This completes the proof. $\qquad\square$

Note that we do not need the c.a.d. $\mathcal{P}$ to be strong. In particular, we allow cells C that are not locally boundary connected. We need the requirement that the induced c.a.d. in $\mathbb{R}^{n-1}$ is strong as we invoke [22, Prop. 5.2]. We conjecture Theorem 5.1.1 is true without this assumption.

See Chapter 7 for a conjecture on the homotopy type of ∂C.

5.2 $\overline{C}$ is a manifold with boundary

Recall the definition of a manifold with boundary.

Definition 5.2.1 (c.f. [18, p. 252f]). *We say that M is a n-**dimensional manifold with boundary** if, for each $x \in M$, there exists a neighbourhood V of x such that either:*

    i. V is homeomorphic to $\mathbb{R}^n$, or

    ii. V is homeomorphic to a open neighbourhood in $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

*The **boundary of** M, denoted ∂M, is the set of points of M with no neighbourhood homeomorphic to $\mathbb{R}^n$. In particular, any chart maps a boundary point $p \in \partial M$ to $(x_1, \ldots, x_{n-1}, 0) \in \mathbb{H}^n$.*

Tautologically, any d-cell is a d-manifold. Consequently, to show the closure of a cell is a manifold with boundary, we only need to worry about the neighbourhoods of the boundary points.

In general, a c.a.d. cell does not admit a manifold with boundary structure.

Example 5.2.2 (c.a.d. cell that cannot be a manifold with boundary). *Recall Example 3.2.1. Let C be the non-l.b.c. 2-cell and $p \in \partial C$ be a point over the bad cell $(0,0)$; that is, $\{p\} \cup C$ is not simply-connected. There does not exist a homeomorphism between a neighbourhood of p and $\mathbb{H}^n$ that sends boundary to boundary.*

The l.b.c. condition is a necessary condition for the closure of any cell to be realised as a manifold with boundary. If $\overline{C}$ is a manifold with boundary and $p \in \partial C$, then, for sufficiently small $\epsilon > 0$, $B(p, \epsilon) \cap C$ is connected.

For (low-dimensional) c.a.d. cells, the l.b.c. condition is sufficient to show that $\overline{C}$ is a manifold with boundary.

Lemma 5.2.3. *If C is a locally boundary connected 0- or 1-cell of $\mathbb{R}^n$, then $\overline{C}$ is a manifold with boundary.*

This is immediate from the definition of manifold with boundary.

Using a more involved argument, we can show that, under certain conditions, 2-cells are manifolds with boundary.

Definition 5.2.4 (Homology Manifold – [27, Def. 1.2]). *We say a locally compact space X is a **homology n-manifold** if, for all $x \in X$, its local homology groups satisfy:*

$$H_i(X, X \setminus \{p\}) = \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

We think of a homology manifold as a space whose local homology structure is similar to that of a topological manifold. In fact, every manifold is a homology manifold. The term *generalised n-manifold* is also used to refer to homology n-manifolds. Homology manifolds are not, in general, topological manifolds. In low dimension, they coincide.

Lemma 5.2.5 (c.f. [41, p. 287ff]). *Suppose that X is a homology n-manifold. If $n \leq 2$, then X is a topological manifold.*

In addition, homology manifold satisfy a property not shared with topological manifolds.

Lemma 5.2.6 ([35]). *Suppose X and Y are topological spaces. If $X \times Y$ is a topological manifold, then X and Y are homology manifolds.*

Before we proceed, we comment on the local behaviour of semi-algebraic cells.

Remark 5.2.7. *Let $p \in \partial C$ for a d-cell C. For any $\epsilon > 0$, we denote by $X_{p,\epsilon}$ the set $p * (S(p, \epsilon) \cap C) \setminus (\{p\} \cup S(p, \epsilon))$. From the local conic structure of semi-algebraic sets, we know that for sufficiently small $\epsilon > 0$, there exists a homeomorphism*

$$B(p, \epsilon) \cap C \to X_{p,\epsilon}.$$

From its definition, $X_{p,\epsilon} = \{p\lambda + (1 - \lambda)b \mid b \in S(p, \epsilon) \cap C, \lambda \in (0, 1)\}$. Consequently, there exists a homeomorphism

$$
\begin{aligned}
X_{p,\epsilon} &\longrightarrow (S(p, \epsilon) \cap C) \times (0, 1), \\
p\lambda + (1 - \lambda)b &\longmapsto (b, \lambda).
\end{aligned}
$$

Note that $B(p, \epsilon) \cap C$ is a open subset of a manifold and thus a manifold. Consider the homeomorphism

$$B(p, \epsilon) \cap C \to (S(p, \epsilon) \cap C) \times (0, 1).$$

Now, by Lemma 5.2.6, $S(p, \epsilon) \cap C$ is a homology manifold. In particular, if $d \leq 3$, then – by Lemma 5.2.5 – $S(p, \epsilon) \cap C$ is a manifold. This will be instrumental when proving the regularity of low dimensional cells.

We can determine when locally boundary connected 2-cells of a cell decomposition are manifolds with boundary.

Lemma 5.2.8. *Let C be a l.b.c. 2-cell of $\mathbb{R}^n$. If, for all $p \in \partial C$, the set $S(p, \epsilon) \cap C$ is homeomorphic to $(0, 1)$ — where $\epsilon > 0$ arises from the local conic structure of semi-algebraic sets — then $\overline{C}$ is a manifold with boundary.*

The requirement that $S(p, \epsilon) \cap C$ is homeomorphic to $(0, 1)$ is necessary – see Example 2.2.14.

Proof. Let γ denote the homeomorphism between $(0, 1)$ and $S(p, \epsilon) \cap C$.

By [6, Prop. 2.5.3], γ extends continuously to a map $\overline{\gamma}$ on $[0, 1]$. Moreover, the l.b.c. condition implies that $\overline{\gamma}(0) \neq \overline{\gamma}(1)$. In particular, we have a homeomorphism $\overline{\gamma} \colon [0, 1] \to S(p, \epsilon) \cap \overline{C}$.

Now, by the local conic structure of semi-algebraic sets, $\overline{B}(p, \epsilon) \cap \overline{C}$ is homeomorphic to a cone whose base is homeomorphic to $[0, 1]$; thus, $\overline{B}(p, \epsilon) \cap \overline{C}$ is homeomorphic to $[0, 1]^2$.

Lastly, we can send $[0, 1]^2$ to $\mathbb{H}^2$ and thus show that $\overline{C}$ is a manifold with boundary. $\square$

The key point is that 2-cells of a strong cylindrical algebraic decompositions of $\mathbb{R}^n$ satisfies all the condition of the lemma above.

Lemma 5.2.9. *Let C be a 2-cell of a strong c.a.d. of $\mathbb{R}^n$. Then $\overline{C}$ is a manifold with boundary.*

Proof. By Remark 5.2.7 and Lemma 5.2.5, $S(p, \epsilon) \cap C$ is a connected 1-manifold, and thus, homeomorphic to either S^1 or $(0, 1)$; it suffices to show that $S(p, \epsilon) \cap C$ is not homemorphic to $\S^1$.

As a strong c.a.d. is well-bordered, there exists a 1-cell $C_i \subset \partial C$, such that $p \in \overline{C}$. If $S(p, \epsilon) \cap C$ were homeomorphic to $\S^1$, then this would imply that p is an isolated point in the boundary. $\square$

We have proved the following result.

Corollary 5.2.10. *Let $S \subset \mathbb{R}^n$ be a 2-dimensional compact semi-algebraic set. If $\mathcal{P}$ is a strong c.a.d. adapted to S, then $\mathcal{P}$ yields a regular cell complex of S.*

In view of Lemma 5.2.3 and Lemma 5.2.9, if we can show that 3-cells of a strong c.a.d. of $\mathbb{R}^3$, we can give a proof of Conjecture 4.0.1 for the case $n = 3$. We give a proof with a slightly stronger assumption.

Definition 5.2.11. *We say that a cell $C \subset \mathbb{R}^n$ is **locally boundary k-connected** if, there exists a $\delta > 0$, such that, for all $0 < \epsilon < \delta$ and for all $p \in \partial C$, $B(p, \epsilon) \cap C$ is k-connected.*

Recall that X is a k-connected space if it is non-empty, path connected, such that

$$\pi_i(X) = 0 \text{ for } 0 \leq i \leq k,$$

where $\pi_i(X)$ denotes the i-th homotopy group.

For future use we make the following additional definition.

Definition 5.2.12. *We say that a cell $C \subset \mathbb{R}^n$ is **locally boundary contractible** if, there exists a $\delta > 0$, such that, for all $0 < \epsilon < \delta$ and for all $p \in \partial C$, $B(p, \epsilon) \cap C$ is contractible.*

In particular, a locally boundary contractible cell is locally boundary k-connected, for all k.

Theorem 5.2.13. *Let $\mathcal{P}$ be a strong c.a.d. of $\mathbb{R}^3$. If C is a locally boundary 1-connected 3-cell of $\mathcal{P}$, then $\overline{C}$ is a manifold with boundary.*

Proof. We will use a similar argument to Lemma 5.2.9.

By Remark 5.2.7 and Lemma 5.2.5, for all $p \in \partial C$ and sufficiently small $\epsilon > 0$, $S(p, \epsilon) \cap C$ is a 2-manifold. We claim that $S(p, \epsilon) \cap C$ is a 2-cell that satisfies the conditions of Lemma 5.2.8. Thus, $S(p, \epsilon) \cap \overline{C}$ is a regular cell and, in particular, the cone with base homeomorphic to $[0, 1]^2$ is homeomorphic to $[0, 1]^3$.

If $S(p, \epsilon) \cap C$ is a 2-cell, then it must satisfy the l.b.c. condition; otherwise, this contradicts the fact that C is locally boundary connected. If $q \in \partial(S(p, \epsilon) \cap C)$, then by invoking the local conic structure of semi-algebraic sets and the homology manifold reasoning, $(S(p, \epsilon) \cap C) \cap S(q, \epsilon')$ is a manifold; moreover, this manifold cannot be homeomorphic to $\S^1$ otherwise we have a contradiction with the fact that $S(p, \epsilon) \cap C$ is a cell.

Lastly, we prove that $S(p, \epsilon) \cap C$ is a 2-cell.

As $S(p, \epsilon) \cap C$ is a 2-manifold, it suffices to show that $S(p, \epsilon) \cap C$ is contractible. By [25, Cor. 1], it has CW-type. Thus, by Theorem B.3.2, it suffices to show that $S(p, \epsilon) \cap C$ has trivial homotopy groups. Moreover, Hurewicz's theorem states that the first non-vanishing homology group is isomorphic to the first non-vanishing homotopy group.

We have a homeomorphism between $B(p, \epsilon) \cap C$ and $(S(p, \epsilon) \cap C) \times (0, 1)$, and thus, by assumption, $S(p, \epsilon) \cap C$ is 1-connected. As $S(p, \epsilon) \cap C$ is a non-compact connected 2-manifold, by [13, Cor. VIII.3.4], $H_2(S(p, \epsilon) \cap C) = 0$. Moreover, $H_i(S(p, \epsilon) \cap C) = 0$ for all $i \geq 3$. Thus, the space $S(p, \epsilon) \cap C$ is contractible.

This concludes the proof. $\square$

Corollary 5.2.14. *Suppose that $S \subset \mathbb{R}^3$ is semi-algebraic and $\mathcal{P}$ is a strong c.a.d. adapted to S such that every 3-cell of $\mathcal{P}$ is locally boundary 1-connected. Then, $\mathcal{P}$ yields a regular cell complex of S.*

The author of this thesis is working on proving Corollary 5.2.14 under Lazard's original assumption. See Chapter 7 for a discussion on this assumption and possible changes for the n-dimensional case.

CHAPTER 6

COMBINATORIAL TOPOLOGY CONNECTION

Suppose that $S \subset \mathbb{R}^n$ is a compact semi-algebraic set, then by Conjecture 4.0.1, we can compute a regular cell complex of S via cylindrical algebraic decompositions.

The advantage of a regular cell complex of S over a CW-complex is that the additional combinatorial structure. We give an affirmative answer to a question posed by Lazard in [22] which states that a poset, defined later in the chapter, built using the sub-adjacency relation between cells captures homological information.

We start by defining this dictionary between combinatorics and topology.

6.1 Posets and Topology

The material in this section follows [5, Sec. 9]. For a more detailed treatment of the relation between partially ordered sets and topology see [5, Sec. 9] or [39].

Definition 6.1.1 (Partial order)**.** *A **partial order** is a reflexive, antisymmetric, and transitive binary relation $\leq$ on a set. A **partially ordered set** is a pair $(P, \leq)$, where P is a set and $\leq$ a partial order on P.*

A totally ordered subset $\{x_0, \ldots, x_k\}$ such that $x_0 < \cdots < x_k$ is said to be a **chain of length** k. We say that supremum of lengths of all chains is called the **rank** or **length** of P. We say that a poset is **pure** if all maximal chains have the same length. A poset P is a **lattice** if every pair of elements $x, y \in P$ has a least upper bound, denoted $x \vee y$, and a greatest lower bound, denoted $x \wedge y$.

Example 6.1.2 (Poset)**.** *The following Hasse diagram of the 3 point set $\{x, y, z\}$ represents a partially ordered set.*

$$\{x, y, z\}$$

$$\{y, z\} \qquad \{x, z\} \qquad \{x, y\}$$

$$\{z\} \qquad \{y\} \qquad \{x\}$$
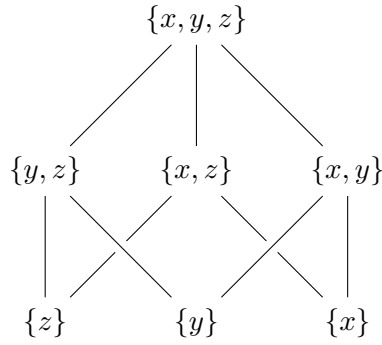
We define structure preserving functions between these objects.

Definition 6.1.3 (Arrows in the category of posets)**.** *Let P and Q be posets and $f \colon P \to Q$ be a function. We say that f is **isotone** if, for all $x, y \in P$, $x \leq y$ implies that $f(x) \leq f(y)$. In other words, f is order preserving. Moreover, a **isomorphism** between posets is an isotone map which has a two-sided isotone inverse.*

Posets arise naturally in different areas of mathematics. We focus in a specific poset that arises from topology.

Definition 6.1.4 (Face poset)**.** *Let Δ be a simplicial complex. We define the **face poset** of Δ, denoted $\mathrm{P}(\Delta)$, to be the set of faces of Δ ordered by inclusion.*

We can view a poset as a topological space via the order complex.

Definition 6.1.5 (Order complex of a poset)**.** *Let P be a poset. The **order complex** of a poset P, denoted $\Delta(P)$, is the simplicial complex whose vertex set is P and k-faces are the k-chains $x_0 < x_1 < \cdots < x_k$ in P.*
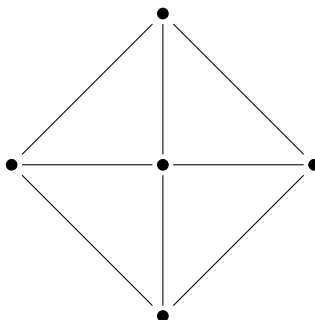
We now associate a topological space to a poset by using the geometric realisation of an order complex $\|P\| = \|\Delta(P)\|$.

Example 6.1.6 (Order-complex)**.** *Suppose that we have the following poset:*

$$x_1$$
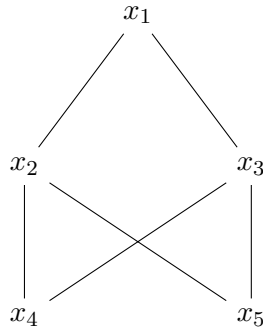
$$x_2 \qquad x_3$$

$$x_4 \qquad x_5$$

The vertex set of our order complex is $\{x_1, x_2, x_3, x_4, x_5\}$. We have five 1-faces and four 2-faces.

Definition 6.1.7 (First barycentric subdivision)**.** *The* **first barycentric subdivision** *of Δ is the simplicial complex* $\mathrm{sd}(\Delta) = \Delta(\mathrm{P}(\Delta))$.

The geometric realisation of Δ and $\mathrm{sd}(\Delta)$ are homeomorphic. In particular, to study the topology of some simplicial complex we can study the poset associated to it, and vice versa. The face poset and the order complex behave functorially; that is, if two posets are isomorphic, the geometric realisation of their order complex is homeomorphic.

6.2 Posets, Regularity, and CADs

In [22], Lazard poses the following question.

"Let us consider a compact semi-algebraic set which is the union of a subset $\mathcal{E}$ of the cells of a strong c.a.d. and the corresponding set of sample points. Consider a subset E_k of k elements of E whose dimensions are all different and such that, for any pair of element of E_k, one is adjacent to the other (i.e. contained in its closure). Let S_k be the $(k-1)$-simplex whose vertexes are the sample points of the cells in E_k. Is the set of all these simplexes S_k a triangulation (a simplicial complex)? Is the union of these simplexes homeomorphic to the given semi-algebraic set? ... "

We rephrase Lazard's question as a proposition.

Proposition 6.2.1. *Let S be a compact semi-algebraic set of $\mathbb{R}^n$ and $\mathcal{P}$ a strong c.a.d. adapted to S. Then, the order complex of the partially ordered set of cells – whose union is S – ordered by sub-adjacency is a triangulation of S. In particular, the geometric realisation of this order complex is homeomorphic to S.*

Firstly, we need to show that the sub-adjacency relation together with a set of cells of a strong c.a.d. yields a partial order set.

Note that, if C and D are cells of a closure finite cell decomposition, then, by Lemma 2.2.12, C is sub-adjacent to D if and only if $C \subset \overline{D}$. Moreover, if $C \neq D$, then $C \subset \partial D$ and $\dim C < \dim D$. Consequently, the relation defined by sub-adjacency is a partial order on the set of cells of a closure finite cell decomposition.

Lemma 6.2.2. *Let P be a set containing cells of some closure finite cell decomposition $\mathcal{P}$ of $\mathbb{R}^n$. If $\leq$ denotes the sub-adjacency between cells, then $(P, \leq)$ is a poset.*

Proof. From its definition, this relation is reflexive. To see its transitive note the following. If $C \subset \overline{D}$ and $D \subset \overline{E}$, then we can apply the closure operator to show that $\overline{C} \subset \overline{D} \subset \overline{E}$.

Lastly, we need to show this relation is anti-symmetric. Suppose that $C \subset \overline{D}$ and $D \subset \overline{C}$ but $C \neq D$. Then $C \subset \partial D$, which implies that $\dim C < \dim D$. However, as $D \subset \overline{C}$, $\dim C < \dim D \leq \dim C$, which is a contradiction.

Hence, sub-adjacency is partial order on the set of cell of $\mathcal{P}$. $\square$

Recall that we can think of a regular cell complexes as a CW-complex that is one barycentric subdivision from being a triangulation.

Lemma 6.2.3 ([5, 12.4 (ii)])**.** *Let Σ be a regular cell complex and $\mathrm{P}(\Sigma)$ be the partially ordered set of all closed cells ordered by inclusion. Then $\|\Sigma\| \cong \|\Delta(\mathrm{P}(\Sigma))\|$.*

Example 6.2.4 (Regular cell complex is not a triangulation)**.** *The poset in Example 6.1.6 is the poset associated to the regular cell complex in Example 3.1.2 of the 2-disk. The order complex gives us a triangulation of the 2-disk.*

In particular, a order complex of a poset of closed cells ordered by inclusion associated to a regular cell complex of S is a triangulation of S.

If we can prove that the partially ordered set of cell with respect to sub-adjacency is isomorphic to the partially ordered set of closed cells with respect to inclusion, we can use Conjecture 4.0.1 and Lemma 6.2.3 to give an affirmative answer to Proposition 6.2.1.

Lemma 6.2.5. *Let $\mathcal{P}$ be a strong c.a.d. of $\mathbb{R}^n$. The partially ordered set of cell with respect to sub-adjacency is isomorphic to the partially ordered set of closed cells with respect to inclusion.*

Proof. There is a natural bijection between the two as sets; that is, $C_i \mapsto \overline{C_i}$. Let $\leq_s$ denoted the sub-adjacency relation and let $\subset$ be the inclusion relation on closed cells.

We need to show that $C \leq_s D$ if and only if $\overline{C} \subset \overline{D}$. This is a corollary of Lemma 2.2.12. $\square$

We now have all the ingredients necessary to answer the question posed by Lazard.

Proof of Proposition 6.2.1. The poset of cells ordered by sub-adjacency is isomorphic to the poset of closed cells ordered by inclusion. The order complex of the latter yields a triangulation of S. □

CHAPTER 7

OPEN QUESTIONS

In this chapter we discuss possible ways the work of this thesis can be extended.

7.1 Faces of c.a.d. cells

In Chapter 3, the notion of faces along an axis direction was defined. We proved that, if we have a closure finite c.a.d. $\mathcal{P}$ and a cell $C \in \mathcal{P}$, we can give a lower bound for the dimension of the cells contained in $C_{x_i}^{\pm}$. We observed in Remark 3.4.10, for $r > s$, the behaviour of $C_{x_r}^{\pm}$ affects the dimension of $C_{x_s}^{\pm}$.

Question 7.1.1. *Let $\mathcal{P}$ be a c.a.d. of $\mathbb{R}^n$ and C a cell in $\mathcal{P}$. For $r > s$, what is the precise relation between $C_{x_r}^{\pm}$ and the dimension of $C_{x_s}^{\pm}$?*

We have proved in Proposition 3.4.5 that the $C_{x_i}^{\pm}$ are connected.

Conjecture 7.1.2. *Let C be a c.a.d. cell of $\mathbb{R}^n$. For all $1 \leq k \leq n$, $C_{x_k}^{\pm}$ and $\overline{C_{x_k}^{\pm}}$ are contractible.*

Another possible application of information about $C_{x_i}^{\pm}$ is to adjacency between cells and constructing minimal cylindrical algebraic decompositions. We discussed in Section 3.3 how we can obtain adjacency information from the index of a c.a.d. cell.

Question 7.1.3. *If the $C_{x_i}^{\pm}$ give lower, and possible upper, bound for cells sitting along the axis directions, does this help us test for the adjacency of cells?*

We say $\mathcal{P}$ is the **minimal c.a.d. adapted to** S if, for all cylindrical algebraic decompositions $\mathcal{Q}$ adapted to S, then $\mathcal{Q}$ is a refinement of $\mathcal{P}$.

Question 7.1.4. *Does minimal cylindrical algebraic decompositions always exists? if so, can we construct it effectively?*

7.2 Topology of a c.a.d. via manifolds

We proved in Lemma 2.2.19 that for $n \leq 2$, any cell decomposition is well-bordered if and only if it is closure finite. Moreover, in Proposition 3.5.1 that for 3-dimensional cylindrical algebraic decompositions, closure finite is equivalent to well-bordered. We conjecture that this is the case in all dimensions.

Conjecture 7.2.1. *A c.a.d. $\mathcal{P}$ of $\mathbb{R}^n$ is closure finite if and only if it is well-bordered.*

In Chapter 4, we discussed the question of regular sectors. We conjecture that if a sector C has regular top and bottom, then C is a regular cell.

Conjecture 7.2.2. *Let C be a sector of a c.a.d. $\mathcal{P}$ of $\mathbb{R}^n$ such that C is bounded above and below by regular sections C_T and C_B respectively. Then C is a regular cell.*

In Theorem 5.1.1, we proved that if $\mathcal{P} \subset \mathbb{R}^n$ is a c.a.d. whose induced c.a.d. $\mathcal{P}'$ of $\mathbb{R}^{n-1}$ is strong, then the closure of any cell C of $\mathcal{P}$ is contractible. We require the condition that the induced c.a.d. is strong as we use [22, Prop. 5.2]. This result states that the fibre of the projection map is a closed segment. We conjecture that Lazard's result can be extended so that it still holds without the requirement that $\mathcal{P}'$ is strong. In particular, contractibility of the fibres is intrinsic to cylindrical algebraic decompositions.

Conjecture 7.2.3. *Suppose that $\mathcal{P}$ is a c.a.d. of $\mathbb{R}^n$, C is a cell of $\mathcal{P}$, and let $\pi : \overline{C} \to \overline{D}$ denote the standard projection mapping. Then, for all $p \in \overline{D}$, the fibre $\pi^{-1}(p)$ is a closed segment. In particular, the fibre is always contractible.*

Consequently, we can extend Theorem 5.1.1.

Conjecture 7.2.4. *Suppose that $\mathcal{P}$ is a c.a.d. of $\mathbb{R}^n$. If C is a cell of $\mathcal{P}$, then $\overline{C}$ is contractible.*

Conjecture 4.0.1 states that a strong c.a.d. adapted to a closed and bounded semi-algebraic set S is a regular cell complex of S. Conjecture 7.2.5 below is a weaker version of this result.

Conjecture 7.2.5. *Suppose that $S \subset \mathbb{R}^n$ is a closed and bounded semi-algebraic set and $\mathcal{P}$ a closure finite c.a.d. adapted to S. The c.a.d. $\mathcal{P}$ is a CW-complex of S.*

We proved Conjecture 4.0.1 in the case of $n = 3$ under a stronger condition — we assumed that all 3-cells were locally boundary 1-connected. The question remains whether Conjecture 4.0.1 can be proved under Lazard's original assumptions.

If we are dealing with general cells, locally boundary connected and locally boundary 1-connected are not equivalent. Let $p = (0, 0, 1)$ in the boundary of the 2-sphere minus the north pole — which we denote by C here — in Example 2.2.14. Then, for sufficiently small $\epsilon > 0$, $B(p, \epsilon) \cap C$ is connected but not 1-connected. In fact, if we have a regular cell, then for sufficiently small $\epsilon > 0$, and any $p \in \partial C$, the set $B(p, \epsilon) \cap C$ is contractible. Thus, locally boundary 1-connected is a necessary condition for a general cell to be regular. We conjecture that this is not needed for c.a.d. cells.

Conjecture 7.2.6. *A c.a.d. cell C is locally boundary connected if and only if it is locally boundary contractible.*

If Conjecture 7.2.6 is false, the assumptions in Conjecture 4.0.1 will need to be revised.

We used the Poincaré Conjecture to give a proof of a weakened Conjecture 4.0.1. To extend this method to arbitrary dimensions, we need to understand the fundamental group of the boundary of c.a.d. cells.

Conjecture 7.2.7. *Let $\mathcal{P}$ be a locally boundary connected c.a.d. of $\mathbb{R}^n$ and C a cell of $\mathcal{P}$. If $\dim C \geq 3$, then ∂C is simply connected.*

For $\dim C < 3$, the boundary of C is not simply-connected. Thus, we need to our base case to be $\dim C = 3$.

Question 7.2.8. *Can the results of manifold theory used in Chapter 5 be extended — possibly using the theory of semi-algebraic spaces; see [12] — to work over any real closed field R?*

7.3 Regularity via Blow-ups

The approach described in this section is a result of private conversation between the author and Gregory Sankaran.

Let C be a semi-algebraic d-cell whose closure is containes in a bounded open set $U \subset \mathbb{R}^n$ (or R^n): we shall probably need to assume that C is locally boundary contractible (which should make sense over R also).

Here is a possible approach to proving that C is actually regular. There are several places where the argument is lacking in detail. This is a sketch of an idea only.

Denote the cube $(0, 1)^d$ by Δ and $[0, 1]^d$ by $\bar{\Delta}$. We say that two semialgebraic sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ are equiregular if there is a homeomorphism (not necessarily semi-algebraic) $\bar{X} \to \bar{Y}$ that restricts to a homeomorphism $X \to Y$ where $\bar{X}$ and $\bar{Y}$ denote the topological closure. X is said to be regular if X and Δ are equiregular.

By definition, there is a semi-algebraic homeomorphism $f\colon C \to \Delta$. In general this does not extend even as a continuous map to $\bar{C}$. However, since f is semialgebraic we may use Hironaka's resolution of singularities theorem to resolve the singularities of f. More precisely, we can give an algebraic morphism $\pi\colon U' \to U$ such that π is an isomorphism above C and such that f lifts to a continuous, in fact semi-algebraic, map $f'\colon \bar{C}' \to \bar{\Delta}$, where $C' = \pi^{-1}(C) \subset U'$.

In general f' will not be a homeomorphism but will contract some part of the boundary of $\bar{C}'$. However, under the assumption that C is locally boundary contractible, we expect that f' should have connected fibres and in fact should be a composite of blow-ups at boundary points of Δ. We expect that this is enough to ensure that C' is regular.

If so, it would be sufficient to prove that C' and C are equiregular. The map π will not do: it is far from injective at the boundary. It is, however, a composition of blow-ups: therefore, locally one the base U, it is a projection map. Locally on the fibres, we may replace π with a nearby generic projection, which will be a local isomorphism onto its image in U. Patching these maps together by using (semi-algebraic) partitions of unity, and the compactness of $\bar{C}$, we obtain a map $\rho\colon U' \to U$ (shrinking both U' and U if necessary, which is locally injective on the whole of U', i.e. near any point of C'.

We claim that it should be possible to choose ρ to be globally injective, again because of C being locally boundary contractible. The map ρ will be determined by finitely many points in some Grassmannian (parametrising the projection directions) and the condition that ρ is injective should be given by some non-empty open conditions associated with each choice.

Now $\rho(C')$ is equiregular with C' and close to C. By using the local conic structure of the semialgebraic sets $\rho(C)$ and C we may contract or expand $\rho(C')$ along the rays near each boundary point until we reach the boundary of C. Doing this locally near to enough points of $\partial\rho(C')$, and gluing with partitions of unity again, we would obtain a map $\rho(C') \to C'$ that extends to a homeomorphism $\rho(\bar{C}) \to \bar{C}$.

If all these steps can be carried out successfully, we have then proved that C is regular.

7.4 Miscellaneous

We conjecture that the manifold structure of $\overline{C}$ is finer than just a manifold with a boundary.

Conjecture 7.4.1. *Let $C \subset \mathbb{R}^n$ be a l.b.c. c.a.d. cell. Then $\overline{C}$ is a manifold with corners.*

The definition and basic development of manifolds with corners can be found in [20]. Instead of having looking at open neighbourhoods inside $[0, \infty) \times \mathbb{R}^{n-1}$ we look at open neighbourhoods inside $[0, \infty)^k \times \mathbb{R}^{n-k}$.

Our current proof is limited by the local conic structure and thus cannot detect all the manifold with corners structure.

Another question posed by Lazard in [22] is about the relation of smooth cylindrical algebraic decompositions and Whitney stratifications; here, by a smooth c.a.d. we mean that every d-cell is diffeomorphic to $(0, 1)^d$.

Question 7.4.2 (c.f. [22, p. 111]). *What is the relation between smooth strong cylindrical algebraic decompositions and Whitney stratifications?*

Appendices

The material in Chapter A is discussed in standard references of semi-algebraic geometry: [3], [4], [6], or [38].

Semi-algebraic sets arise naturally in real algebraic geometry. Consider the elliptic curve X over $\mathbb{R}$ defined by the equation $y^2 = x^3 - x$. We cannot describe the connected components of X in terms of varieties over $\mathbb{R}$ only. Similarly, we can consider the one-parameter family $\{x^2 - a\}_{a \in \mathbb{R}}$, and ask for which values of a, the polynomial $x^2 - a$ has real roots. The set of values for which this holds, $[0, \infty)$, cannot be described in terms of varieties over the reals. In particular, we cannot describe these sets without the use of inequalities.

In fact, different frameworks to study real algebraic sets rely heavily on machinery that arises from the study of semi-algebraic sets. Consequently, the study of the geometric and topological properties of semi-algebraic sets are of interest. For a more in depth treatment, see one of the references above.

A.1   Semi-algebraic sets

We think of a semi-algebraic set $S \subset R^n$ as a set defined by a finite number of polynomial equations and inequalities.

Definition A.1.1 ([4, Def. 2.1.1])**.** *We say $S \subset R^n$ is semi-algebraic if it is a finite union of sets of the form*

$$\{x \in R^n \mid f_1(x) = 0, \ldots, f_r(x) = 0, g_i(x) > 0, \ldots, g_s(x) > 0\}.$$

We can define semi-algebraic sets to be the smallest family $\mathcal{F}$ of sets of R^n such that: $\mathcal{F}$ contains all the sets of the form $\{x \in R^n \mid f(x) > 0\}$, for some $f \in R[t]$; and $\mathcal{F}$ is closed under finite unions, finite intersections, and taking complements. In fact, the family of semi-algebraic sets form a boolean algebra.

Examples A.1.2 (Semi-algebraic sets)**.**

    i. All real algebraic sets are semi-algebraic.

    ii. $S = \{(x,y) \in \mathbb{R}^2 \mid y = |x|\}$ is a semi-algebraic that is not algebraic.

Example A.1.3 (Semi-algebraic non-example)**.** *The set $X = \{(x,y) \in \mathbb{R}^2 \mid y = \lfloor x \rfloor\}$ is not semi-algebraic. Semi-algebraic sets have a finite number of connected components.*

In keeping with the maxim of studying a family of objects via functions, we define what it means for a function to be semi-algebraic.

Definition A.1.4 ([4, Def. 2.3.2])**.** *Let $S \subset R^n$ and $T \subset R^m$ be semi-algebraic sets. We say that a function $f \colon S \to T$ is **semi-algebraic** if the graph of f, denoted $\Gamma(f)$, is a semi-algebraic subset R^{n+m}.*

The next result forms the foundation for a large body of results in semi-algebraic geometry.

Theorem A.1.5 (Tarski-Seidenberg principle)**.** *Let $f \colon X \to Y$ be a semi-algebraic function. The image of f is a semi-algebraic set. In particular, the projection of a semi-algebraic set is semi-algebraic.*

This is often referred to as the Geometric formulation of the Tarski-Seidenberg principle. The original equivalent formulation of the Tarski-Seidenberg principle states that theory of real closed fields admits quantifier elimination – see [37].

As a consequence of the Tarski-Seidenberg principle, we can show that $\overline{S}$, int S, and the boundary of S is semi-algebraic, for a semi-algebraic set S. Moreover, we can show that if $f \colon S \to R^m$ is a semi-algebraic function, then, for $T \subset R^m$ semi-algebraic, $f^{-1}(T)$ is semi-algebraic — see [4, p. 60f]. In particular, the fibre of a semi-algebraic function is semi-algebraic.

The Tarski-Seidenberg principle illustrates one of the differences between algebraic and semi-algebraic sets. The projection of algebraic is not necessarily algebraic. In fact, the family of semi-algebraic sets is the smallest family containing algebraic sets that is closed under projection — see [4, Prop. 2.3.10].

Often, we will study semi-algebraic sets that arise from inequality and equality relations from a given finite set of polynomials F. We will call such semi-algebraic

sets, F-semi-algebraic; see [3, Def. 5.5]. We have a convenient way of presenting such semi-algebraic sets.

Definition A.1.6. *Given a finite set of polynomials $F = \{f_1, \ldots, f_m\} \subset R[x_1, \ldots, x_n]$ and a function $\epsilon\colon \{1, \ldots, m\} \to \{-1, 0, 1\}$, we can associate a semi-algebraic set A_ϵ defined as*

$$A_\epsilon = \bigcap_{k=1}^m \{x \in R \mid \operatorname{sign}(f_k(x)) = \epsilon(k)\}.$$

The function ϵ is called a **sign condition on** *F and A_ϵ the* **realisation of** *ϵ; see [38, p. 32] or [3, Def. 2.2.5].*

Given some value of ϵ, say $\epsilon(k)$, we define $\overline{\epsilon(k)}$ to be the relaxation of the inequalities they represent; that is, $\overline{\{-1\}} = \{-1, 0\}$, $\overline{\{0\}} = \{0\}$, and $\overline{\{1\}} = \{0, 1\}$. We denote the realisation of a sign condition with relaxed inequalities, denoted by $A_{\overline{\epsilon}}$ by:

$$A_{\overline{\epsilon}} = \bigcap_{k=1}^m \{x \in R \mid \operatorname{sign}(f_k(x)) \in \overline{\epsilon(k)}\}.$$

Note that some sign conditions on F might yield empty sets; we say ϵ is **realisable** if A_ϵ is non-empty.

One nuance of semi-algebraic sets is that their presentation is not canonical. Moreover, certain presentations are better behaved, in some sense, than others. As we saw above, a corollary of the Tarski-Seidenberg principle is that if S is semi-algebraic, then so is $\overline{S}$. We might try to compute the closure of S by relaxing the inequalities in some presentation of S. This does not work in general.

Example A.1.7 (Relaxation of inequalities). *Consider the semi-algebraic set $S = \{x \in \mathbb{R} \mid x^2(x-1) > 0\}$. The closure of S is $[1, \infty)$ which is different than $\{x \in \mathbb{R} \mid x^2(x-1) \geq 0\} = \{0\} \cup [1, \infty)$.*

Another example of such occurrence is [6, p. 27]. The issue is not with a semi-algebraic set S but with the presentation of S. The next result shows that in R, certain families of polynomials of $R[x]$ are such that we can compute the closures through an easy procedure.

Theorem A.1.8 (Thom's Lemma, c.f. [6, Prop. 2.5.4]). *Let $F = \{f_1, \ldots, f_m\} \subset R[x]$ be a set of polynomials closed under differentiation; that is, if $f \in F$, then $f' \in F$ or $f' = 0$. Then*

   i. A_ϵ is either empty, a point, or an open interval.

   ii. If A_ϵ is non-empty, then $\overline{A_\epsilon} = A_{\overline{\epsilon}}$.

iii. If A_ϵ is empty, then $A_{\bar\epsilon}$ is either empty or a point.

A decomposition of a semi-algebraic set called stratifying family of polynomials generalises Thom's Lemma to R^n; see [4, Thm. 2.4.4] or [6, Chap. 9].

A.2 Semi-algebraic maps

If S and T are semi-algebraic sets and S and T are homeomorphic, it does not imply that S and T are semi-algebraic homeomorphic. In [34], Shiota and Yokoi showed the existence of two homeomorphic compact semi-algebraic sets that are not semi-algebraic homeomorphic.

An important fact, provided without proof, about semi-algebraic sets is that they admit semi-algebraic partitions of unity.

Proposition A.2.1 ([11, Prop. 1.5]). *Let $\{U_i\}_{i\in I}$ be a finite semi-algebraic open cover of an semi-algebraic S. Then, there exists a family $\{f_i\}_{i\in I}$ of semi-algebraic functions f_i on M with values in $[0,1]$ such that:*

i. $\operatorname{supp}(f_i) = \overline{\{x \in S \mid f_i(x) \neq 0\}} \subset U_i$ *for every $i \in I$.*

ii. $\sum_{i\in I} f_i(x) = 1$ *for every $x \in S$.*

A.3 Semi-algebraic connectedness

The usual definition of connectedness does not capture the desired notion when we work with real closed fields different than $\mathbb{R}$. In fact, if $R \neq \mathbb{R}$, then we can show that R is totally disconnected. To mend this pathology of real closed fields we restrict to looking ourselves solely to semi-algebraic sets.

Definition A.3.1 ([6, Def. 2.4.2]). *We say a semi-algebraic set $S \subset R^n$ is **semi-algebraically connected** if, for closed U and V in S, $U \cup V = S$ implies that $U = S$ or $V = S$.*

By [6, Thm. 2.4.5], if $R = \mathbb{R}$, the notion of connectedness agrees with the notion of semi-algebraic connectedness.

We also have an analogue of path connectedness for semi-algebraic sets.

Definition A.3.2 ([6, Def. 2.5.13]). *We say that a semi-algebraic set $S \subset R^n$ is **semi-algebraically path-connected** if, for all $x,y \in S$, there exists a continuous semi-algebraic function $\varphi\colon [0,1] \to S$ such that $\varphi(0) = x$ and $\varphi(1) = y$.*

Unlike general subsets of R^n, semi-algebraic path-connectedness and semi-algebraic connectedness coincide when dealing with semi-algebraic subsets.

Theorem A.3.3 ([6, Thm. 2.5.13])**.** *Let $S \subset R^n$ be semi-algebraic. Then S is semi-algebraically connected if and only if S is semi-algebraically path-connected.*

A.4   Semi-algebraic triviality and its applications

We think of trivial maps as functions which look like a projection mapping. More specifically, maps which factor through the standard projection map, with the appropriate domain. Such maps form a special class of continuous functions as they are, in some sense, simple. In the section, we discuss Hardt's Theorem(Theorem A.4.2), which says that locally, in some sense, continuous semi-algebraic maps are always trivial. This result has many application to the topology of semi-algebraic sets.

Hardt's theorem original proof can be found in [17]. Some standard, more current, references for Hardt's theorem and the material discussed in this section are [3, Sect. 5.8] or [6, Sect. 9.3].

Firstly, we define precisely what we mean by a trivialisation of a continuous semi-algebraic function.

Definition A.4.1 ([6, Def. 9.3.1])**.** *Let $S \subset R^n$ and $T \subset R^m$ be semi-algebraic sets and $f \colon S \to R^m$ a continuous semi-algebraic map. We say that f is* **semi-algebraically trivial over** T *if there exists a semi-algebraic set $F \subset S$ and a homeomorphism $\theta \colon f^{-1}(T) \to T \times F$ such that the diagram*

$$
\begin{array}{ccc}
f^{-1}(T) & \xrightarrow{\ \ \theta\ \ } & T \times F \\
 & \searrow_{f} \quad \swarrow_{\pi} & \\
 & T &
\end{array}
$$

commutes. As usual, π denotes the standard projection mapping. Moreover, we say that a trivialisation is **compatible to S'***, for a subset $S' \subset S$, if there exists an $F' \subset F$ such that $\theta(S' \cap f^{-1}(T)) = T \times F'$.*

For the rest of this section, let S,T and f be denoted as in definition above.

If a trivialisation exists, then for any $t \in T$, $f^{-1}(t)$ is homeomorphic to F; we can thus choose F to be equal to $f^{-1}(t)$, for some $t \in T$, with $\theta(x) = (x,t)$. In particular, we can choose F and θ to be semi-algebraic.

The following result, proved by Hardt in [17], shows that we can always partition R^m into finitely many $\{T_i\}$ in such way that f is semi-algebraically trivial over each T_i.

Theorem A.4.2 (Hardt's semi-algebraic triviality)**.** *Let $S \subset R^n$ be a semi-algebraic set and $f\colon S \to R^m$ a continuous semi-algebraic map. There exists a finite semi-algebraic partition $\{T_i\}$ of R^m such that f is semi-algebraically trivial over each T_i. Moreover, if $\{S_j\}$ is a finite collection of semi-algebraic subsets of S, we can choose this trivialisation to be compatible to each S_j.*

Given a semi-algebraic set $S \subset R^n$, a cylindrical algebraic decomposition gives us an effective, in the computational sense, trivialisation of the projection map that is compatible with respect to S.

Example A.4.3 (Trivialisation of the projection)**.** *A cylindrical algebraic decomposition(see Definition 3.1.1), gives us an effective way of constructing a semi-algebraic trivialisation of the standard projection which is compatible to any semi-algebraic set. More specifically, if we build a c.a.d.adapted to $S \subset R^n$(Definition 3.1.8), then there exists a finite partition of R^{n-1} into semi-algebraic sets $\{C_i\}$ such that $\pi\colon R^n \to R^{n-1}$ is trivial over each C_i in a way compatible to S.*

An application of Hardt's Theorem is that to the local topology of semi-algebraic sets.

Definition A.4.4. *Let $B \subset R^m$ be a semi-algebraic set and v some point R^m. The* **cone over B with vertex** *v, denoted $v * B$, is the set of points in R^n of the form*

$$[b, \lambda] = \lambda v + (1 - \lambda)b,$$

where $\lambda \in [0, 1]$ and $b \in B$.

The following result, first proved by Milnor in [26], shows that locally, semi-algebraic sets behave like cones. More precisely, it states that if we choose a non-isolated point x of a semi-algebraic set, then there exists a neighbourhood of x which looks like a cone with vertex x.

Theorem A.4.5 (c.f. [6, Thm. 9.3.6])**.** *Let T be a semi-algebraic set and $v \in T$ a non-isolated point. Then there exists a $\delta > 0$, such that for $0 < \epsilon < \delta$, there exists a semi-algebraic homeomorphism $\phi\colon \overline{B}(v, \epsilon) \cap T \to v * S(x, \epsilon) \cap T$ such that:*

i. $\|\phi(x) - v\| = \|x - v\|$ for all $x \in \overline{B}(v, \epsilon) \cap T$, and

ii. $\phi(S(v, \epsilon)) = S(v, \epsilon)$.

In particular, $\phi(v) = v$.

An immediate consequence of the local conic structure of semi-algebraic sets is the following corollary.

Corollary A.4.6. *Every semi-algebraic set is locally contractible.*

This holds as a cone is contractible via strong deformation retract

$$F \colon v * B \times [0,1] \to v * B,$$
$$F([b, \lambda], t) = [b, (1 - \lambda)t + \lambda].$$

Remark A.4.7. *Analogues to the results in the section can be proved in the framework of o-minimal structures.*

APPENDIX B

TOPOLOGY

For the remainder of the chapter, let X and Y denote topological spaces.

Suppose A and B are sub-spaces of X and Y, respectively. Then a **relative map**

$$f \colon (X, A) \to (Y, B).$$

is a map $f \colon X \to Y$ such that $f(A) \subset B$. In other words, the restriction $f|_A \colon A \to B$ is a map. We say that such f is a **homeomorphism** if $f \colon X \to Y$ is a homeomorphism which induces a homeomorphism on $f|_A \colon A \to B$.

Definition B.0.1. *We say a mapping $f \colon (X, A) \to (Y, B)$ is a **relative homeomorphism** if $f|_{X \setminus A} \colon X \setminus A \to Y \setminus B$ is a homeomorphism.*

Any homeomorphism $f \colon (X, A) \to (Y, B)$ is a relative homeomorphism; the converse does not hold. See Example 2.2.4.

B.1 CW-complexes

There are two main ways of defining CW-complex. One approach uses adjunction spaces and an inductive step to build the complex from the bottom up. As we will start with a topological space and want to build a CW-complex associated to this space, we build the cell complex by describing a defining homeomorphism for each cell. Standard references for CW-complexes are [9] or [40].

Definition B.1.1. *Suppose X is a non-empty Hausdorff topological space. A **CW-complex of** X is a finite collection of relative homeomorphisms $\varphi_i \colon (\overline{B}^{d_i}, B^{d_i}) \to (\overline{C_i}, C_i)$ such that the C_i are disjoint and $X = \cup_i C_i$.*

For a more complete treatment of CW-complexes, see [40].

B.2   Regular cell complex

Definition B.2.1. *We that a CW-complex is a **regular cell complex** if relative homeomorphism $\varphi_i \colon (\overline{B}^{d_i}, B^{d_i}) \to (\overline{C_i}, C_i)$ is a homeomorphism.*

As Whitehead explains in [40], a regular cell complex is a compromise between simplicial and CW-complexes. The reason for this remark is due to the following fact:

Lemma B.2.2 ([5, 12.4 (ii)]). *Let Σ be a regular cell complex and $\mathrm{P}(\Sigma)$ be the partially ordered set of all closed cells ordered by inclusion. Then $\|\Sigma\| \cong \|\Delta(\mathrm{P}(\Sigma))\|$.*

As per the analysis of Bjorner in [5], an important consequence of this result is that we can think of regular cell complexes as objects of combinatorial nature. They are a barycentric sub-division away from a simplicial complex. We can view them as posets without any loss of topological information. This is particularly important in Chapter 6.

B.3   Whitehead's and Smale's Theorem

The next two results will are important results from homotopy theory. We start with a useful definition.

Definition B.3.1. *We say that a space X **has CW-type** if X has the homotopy type of a CW-complex.*

Theorem B.3.2 (Whitehead's Theorem — [18, Thm. 4.5]). *Suppose that X and Y are connected and have CW-type. If $f \colon X \to Y$ induces a weak homotopy equivalent between X and Y, then X and Y are homotopy equivalent.*

The next theorem states that under certain conditions, we can deduce contractibility of a space via contractible fibres.

Theorem B.3.3 (Smale's Theorem). *Let X and Y be connected, locally compact separable metric spaces. Assume also that X is locally contractible. Consider a proper surjective continuous map $f \colon X \to Y$. Assume that for all $y \in Y$, the space $f^{-1}(y)$ is contractible and locally contractible. Then f is a weak homotopy equivalence.*

See [36] for the proof of Smale's theorem.

Note that if X and Y are of CW-type, this result implies that X and Y are in fact homotopy equivalent.

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