# Moduli of deformation generalised Kummer manifolds

submitted by

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# Contents

Chapter 1

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## Summary

We study orthogonal modular varieties associated with the moduli of generalised Kummer manifolds. We are particularly interested in understanding the singularities that arise in certain toroidal compactifications. Throughout, we place particular emphasis on the application of these results to problems involving the birational classification of moduli spaces.

# Chapter 2

# Index of Notation

| | |
|---|---|
| $A_n$ | The positive definite even lattice corresponding to the $A_n$ root system (see also [CS99]). |
| $C_n$ | The cyclic group of order $n$. |
| $D(L)$ | The discriminant group $L^\vee/L$ of $L$. |
| $\mathcal{D}_L$ | A Hermitian symmetric domain of type IV defined as the component of $\Omega_L$ fixed by the group $\mathrm{O}^+(L)$. |
| $\mathcal{D}_L^v$ | The rational quadratic divisor $\mathcal{D}_L^v = \{[x] \in \mathcal{D}_L \mid (x,v) = 0\}$ for $v \in L$. |
| $E_8$ | The positive definite even unimodular lattice of rank 8 corresponding to the $E_8$ root system (see also [CS99]). |
| $\mathcal{F}_L(\Gamma)$ | The orthogonal modular variety $\Gamma\backslash\mathcal{D}_L$ where $L$ is a lattice of signature $(2,n)$ and $\Gamma \leq \mathrm{O}^+(L)$. |
| $\mathcal{F}_{L_{6,2p^2}}$ | The orthogonal modular variety $\mathcal{F}_L(\Gamma)$ where $L = L_{6,2p^2}$ and $\Gamma$ is the group $\mathrm{O}^+(L_6, h_{2p^2}^s)$ defined in Theorem 4.0.6. |
| $L^\vee$ | The dual lattice of the lattice $L$. |
| $L(m)$ | The lattice whose Gram matrix is equal to that of $L$ multiplied by $m$. |
| $L_{2n,2d}$ | The lattice $2U \oplus \langle -2n \rangle \oplus \langle -2d \rangle$. |
| $M_k(\Gamma, \chi)$ | The space of weight-$k$ modular forms with character $\chi$ for the group $\Gamma$. |
| $nL$ | The direct sum $L \oplus L \oplus \ldots \oplus L$ ($n$ times). |
| $\mathrm{O}(L)$ | The orthogonal group of the lattice $L$. |

| | |
|---|---|
| $\mathrm{O}^+(L)$ | The spinor kernel of the group $\mathrm{O}(L)$. |
| $\widetilde{\mathrm{O}}(L)$ | The stable orthogonal group of the lattice $L$. (See Equation 3.2). |
| $\mathrm{O}(m,n)$ | The indefinite orthogonal group of type $(m,n)$. |
| $\Omega_L$ | The space defined by $\Omega_L = \{[x] \in \mathbb{P}(L \otimes C) \mid (x,x) = 0, (x,\overline{x}) > 0\}$. |
| $\mathbb{Q}_p$ | The $p$-adic numbers. |
| $U$ | The hyperbolic plane: the even unimodular lattice of signature $(1,1)$. By a standard basis of $U$, we mean one for which the Gram matrix has the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. |
| $\mathbb{Z}_p$ | The $p$-adic integers. |

# Chapter 3

# Introduction

A classical problem in moduli theory is to determine the Kodaira dimension of a moduli space $\mathcal{M}$. For many moduli spaces, such as the moduli of abelian varieties and the moduli of K3 surfaces, strong results exist [O'G89], [San97] [Tai82] [GHS07] [Kon93]. However, for many other moduli spaces, less is known. In particular, there are few specific results for families of irreducible symplectic manifolds.

Families of irreducible symplectic manifolds are especially appealing classes of objects to work with. Because of the existence of a period map from $\mathcal{M}$ to an orthogonal modular variety $\mathcal{F}$ (see Chapter 4 of [GHS13]), one may prove results about the Kodaira dimension of $\mathcal{M}$ by studying the modular variety $\mathcal{F}$, as in [GHS10]. This frequently results in interesting problems involving lattices and modular forms.

One typically expects the orthogonal modular variety $\mathcal{F}$ to be of general type and this can often be proved by using a technique known as the *the low-weight cusp form trick* (see §3.7). This approach involves understanding the growth behaviour of spaces of modular forms satisfying certain conditions determined by the geometry of a compactification $\overline{\mathcal{F}}$ of $\mathcal{F}$. In particular, one needs to pay attention to the boundary of $\overline{\mathcal{F}}$, the branch locus of $\mathcal{F}$ and, in many cases, the singular locus of $\overline{\mathcal{F}}$.

This approach was used in [GHS07] to show that almost all of the components of the moduli of K3 surfaces are of general type. It has also been used to study the moduli of two of the known families of irreducible symplectic manifolds: in particular, the moduli of deformation $K3^{[2]}$ manifolds in [GHS10] and the moduli of O'Grady's 10-dimensional irreducible symplectic manifold in [GHS11].

All of the general type results in [Kon93], [GHS07], [GHS10] and [GHS11] are for orthogonal modular varieties of high dimension. In high dimension, the low-weight cusp form trick can be applied without having to consider the singularities. This is because there exists a compactification $\overline{\mathcal{F}}$ with only canonical singularities (Theorem 5.26 of [GHS13]). In lower dimensions, such compactifications might not exist and one therefore needs a more detailed understanding of the singularities in $\overline{\mathcal{F}}$ and the conditions that they impose.

Here we study such a low dimensional example: a toroidal compactification of the orthogonal modular variety associated with the moduli of deformation generalised Kummer 4-folds of split polarisation of degree $2p^2$ where $p$ is an odd prime.

We pay particular attention to the singularities in these spaces and describe a set of divisors whose union contains the non-canonical part of the singular locus in the interior, as well as the branch divisor. We also discuss the problem of extending pluricanonical forms to a resolution of singularities and give some information about the types of singularities that may occur.

We also study the boundary. In particular, we study the 1-dimensional boundary components. We give some bounds on the number of such boundary components and we provide bounds for the number of components of the singular locus in such a boundary component. We also give some information about the non-canonical singularities that may occur.

## 3.1 Irreducible symplectic manifolds

A generalised Kummer manifold is an example of an irreducible symplectic manifold. Irreducible symplectic manifolds arise naturally in a number of settings: they generalise K3 surfaces and are one of the three building blocks of compact Kähler manifolds with trivial canonical bundle. Indeed, up to a finite cover, all such manifolds can be decomposed as a product of abelian varieties, Calabi-Yau manifolds, and Irreducible symplectic manifolds [Bog74]. We outline some of the theory below, paying particular attention to their moduli. More detailed surveys can be found in [GHS13] and [GHJ03]. Our approach mostly follows [GHS13].

**Definition 3.1.1.** *A compact complex Kähler manifold $X$ is called an* irreducible symplectic manifold *if*

1. *$X$ is simply connected*

2. *$H^0(X, \Omega_X^2) \cong \mathbb{C}\omega$ where $\omega$ is an everywhere non-degenerate holomorphic 2-form.*

Note that, in particular, all irreducible symplectic manifolds have even complex dimension $2n$. The irreducible symplectic manifolds have not been classified, but all currently known examples are deformation equivalent to one of four types:

1. $\mathrm{K3}^{[n]}$ type which are given by the length $n$ Hilbert scheme $S^{[n]} = \mathrm{Hilb}^n(S)$ parametrising $n$ points on a K3 surface $S$ [Bea83].

2. Generalised Kummer varieties, which are defined as follows: if $A$ is an abelian surface and $A^{[n+1]}$ is the length $n+1$ Hilbert scheme $\mathrm{Hilb}^{n+1}(A)$ with the morphism $p : A^{[n+1]} \to A$ given by addition on $A$, the associated generalised Kummer variety is the fibre $p^{-1}(0)$ [Bea83].

3. O'Grady's 6-dimensional example, which is given in terms of a certain moduli space of sheaves on an abelian surface and depends on 6 parameters [O'G03].

4. O'Grady's 10-dimensional example, which is given in terms of a certain moduli space of sheaves on a K3 surface and depends on 22 parameters [O'G99].

A great deal of information is encoded in the cohomology group $H^2(X, \mathbb{Z})$. As for K3 surfaces, $H^2(X, \mathbb{Z})$ comes with the structure of a lattice. That is, an integral symmetric bilinear form. For irreducible symplectic manifolds, this lattice structure is given by the Beauville-Bogomolov form [Bea83]. We define the Beauville-Bogomolov form below. Suppose that $X$ is an irreducible symplectic manifold of complex dimension

$2n$ and let the Hodge decomposition of $H^2(X,\mathbb{C})$ be given by

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,1}(X).$$

If $\omega \in H^{2,0}(X)$ is normalised so that $\int_X (\omega\overline{\omega})^n = 1$, we define $q'$ by

$$q'_X(\alpha) = \frac{n}{2}\int_X \alpha^2(\omega\overline{\omega})^{n-1} + (1-n)\left(\int_X \alpha\omega^{n-1}\overline{\omega}^n\right)\left(\int_X \overline{\alpha}\omega^n\overline{\omega}^{n-1}\right).$$

One can show that after possibly normalising $q'$ by a suitable positive constant, one obtains a lattice on $H^2(X,\mathbb{Z})$ with quadratic form $q$. By a result of Fujiki [Fuj87], there exists $c \in \mathbb{Q}_{>0}$ (the *Fujiki invariant*) such that if $\alpha \in H^2(X,\mathbb{Z})$ then

$$\alpha^{2n} = cq_X(\alpha)^n$$

where $\alpha^{2n}$ is given by the intersection product on $H^2(X,\mathbb{Z})$.

The Beauville lattices of the known irreducible symplectic manifolds were computed in [Rap07] [Rap08] and are given by:

1. Deformation K3$^{[n]}$: $3U \oplus 2E_8(-1) \oplus \langle -2(n-1)\rangle$
2. Generalised Kummer: $3U \oplus \langle -2(n+1)\rangle$
3. O'Grady's 6 dimensional example: $3U \oplus \langle -2\rangle \oplus \langle -2\rangle$
4. O'Grady's 10 dimensional example: $3U \oplus 2E_8(-1) \oplus A_2(-1)$.

(For lattice theoretic notation, see Chapter 2.)

## 3.2 Moduli of irreducible symplectic manifolds

We now show that moduli spaces parametrising polarised irreducible symplectic manifolds exist, and we explain how they are related to orthogonal modular varieties via the period map. Our treatment broadly follows [GHS13].

Let $X$ be an irreducible symplectic manifold with Beauville lattice $L = H^2(X,\mathbb{Z})$. A *polarisation* on $X$ is defined as a choice of ample line bundle $\mathcal{L}$ on $X$. We shall call a pair $(X, \mathcal{L})$ consisting of an irreducible symplectic manifold $X$ and a polariation $\mathcal{L}$ for $X$ a *polarised* irreducible symplectic manifold. Once we have selected a polarisation $\mathcal{L}$ for $X$, we can identify it with its first Chern class $h := c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$. Our polarisations will be assumed to be *primitive.* That is, $h \in H^2(X, \mathbb{Z})$ will be assumed to be a primitive lattice vector. The *polarisation type* of $\mathcal{L}$ is defined as the $\mathrm{O}(L)$-orbit of $h$. The *degree* of $\mathcal{L}$ is the length $h^2 = 2d$ of $h$ in $L$. The *numerical type* of the polarised irreducible symplectic manifold $(X, \mathcal{L})$ is the tuple consisting of the dimension $2n$ of $X$, the Beauville lattice $L$, the Fujiki invariant $c$ and the polarisation type $h$. The numerical type of $(X, \mathcal{L})$ will be denoted by $N$.

In order to define the period map, we need to define marked families of irreducible symplectic manifolds.

**Definition 3.2.1.** *Let $X$ be an irreducible symplectic manifold with Beauville lattice $L$. Suppose that the polarisation type of $(X, \mathcal{L})$ is represented by $h \in L$ which will be taken as fixed. A marking on $X$ is a isomorphism*

$$\phi : H^2(X, \mathbb{Z}) \to L.$$

*If $(X, \mathcal{L})$ is a polarised irreducible symplectic manifold with $c_1(\mathcal{L}) = h \in L$, then a marking $\phi$ on $X$ is said to be a polarised marking if $\phi(c_1(\mathcal{L})) = h$.*

If $X$ is marked by $\phi$ then we can define its period point. We define the domain

$$\Omega_L = \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x, x) = 0, (x, \overline{x}) > 0\}$$

and consider the Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

The marking $\phi$ defines an obvious map from $H^2(X, \mathbb{C})$ to $\Omega_L$. Any symplectic form $\omega$ generating $H^{2,0}(X)$ satisfies the property that $(\omega, \omega) = 0$ and $(\omega, \overline{\omega}) > 0$ and so $[\phi(\omega)] \in \Omega_L$. We call $[\phi(\omega)]$ the period point of $(X, \phi)$.

For a flat family $\rho : \mathcal{X} \to U$, a marking $\phi$ on the central fibre $\mathcal{X}_0$ of $\rho$ can be extended to the whole family by defining

$$\phi_U : R^2\rho_*\mathbb{Z}_U \to L_U$$

where $L_U$ is the constant sheaf with fibre $L$ on $U$. One then obtains a holomorphic map $\pi_U : U \to \Omega_L$ by sending each point in $U$ to its period. We call this map the *period map* defined by the family $\rho : \mathcal{X} \to U$ and the marking $\phi$. We wish to define a period map for the moduli $\mathcal{M}_N$ of irreducible symplectic manifolds of fixed numeric type $N$. We begin by stating some facts about $\mathcal{M}_N$.

By Viehweg's results [Vie95], there is a moduli space $\mathcal{M}_N$ parametrising irreducible symplectic manifolds of fixed numerical type $N$. The space $\mathcal{M}_N$ is quasi-projective and exists as a group quotient in the sense of GIT.

In general, if $S$ is the set of polarized irreducible symplectic manifolds of fixed numeric type $N$ then, by Matsusaka's big theorem and a result of Kollár and Matsusaka [Mat72] [KM83], there exists $N_0 \in \mathbb{Z}$ so that $\mathcal{L}^{\otimes N_0}$ is very ample for all $(X, \mathcal{L}) \in S$. Therefore, for all $(X, \mathcal{L}) \in \mathcal{S}$, the linear system $\mathcal{L}^{\otimes N_0}$ embeds $X$ into $\mathbb{P}^{m-1}$ where $m = h^0(X, \mathcal{L}^{\otimes N})$. From the Hilbert scheme $\mathrm{Hilb}_p(\mathbb{P}^{m-1})$, where $p$ is the Hilbert polynomial of the line bundle $\mathcal{L}$, we select an irreducible component $H$ that contains at least one smooth irreducible symplectic manifold $X$, and from $H$ we take the open part $H_{sm}$ parametrising smooth manifolds. One can show that there is universal family $\mathcal{S}_{sm} \to H_{sm}$ and that the group $\mathrm{SL}(m, \mathbb{Z})$ acts on $H_{sm}$. Crucially, each component $\mathcal{M}'_N$ of $\mathcal{M}_N$ is of the form $H_{sm}/\,\mathrm{SL}(m, \mathbb{Z})$, and so one can define a period map on $\mathcal{M}_N$ by defining a polarised marking on each universal family $\mathcal{S}_{sm} \to H_{sm}$. Any two markings differ by an element in the group $\mathrm{O}(L, h) = \{g \in \mathrm{O}(L) \mid g.h = h\}$ and so the period map descends to a map from $H_{sm} \to \Omega_L \backslash \mathrm{O}(L, h)$. Furthermore, one can show that this map factors through the action of $\mathrm{SL}(N, \mathbb{C})$ on $H_{sm}$. By noting that $\omega$ and $h$ are such that $(\omega, h) = 0$, it is easy to see that the image lies in the set $\Omega_{L,h} = \{[x] \in h^\perp\} \cap \Omega_L$ and so one obtains a holomorphic map $\pi' : \mathcal{M}_N \to \Omega_{L,h} \backslash \mathrm{O}(L, h)$. The domain $\Omega_{L,h}$ has two components that are interchanged by elements in $\mathrm{O}(L, h)$ of negative spinor norm (see Definition 3.5.10) and so $\Omega_{L,h}\backslash \mathrm{O}(L,h)$ is isomorphic to $\mathcal{D}_{L_h}\backslash \mathrm{O}^+(L,h)$ where $\mathcal{D}_{L_h}$ is one of the connected components of $\Omega_{L,h}$ and $\mathrm{O}^+(L,h)$ is the kernel of the spinor norm on $\mathrm{O}(L,h)$. We shall instead consider the map

$$\pi : \mathcal{M}_N \to \mathcal{D}_{L_h} \to \mathrm{O}^+(L,h)\backslash\mathcal{D}_L.$$

The varieties $\Omega_{L,h}\backslash \mathrm{O}(L,h)$ and $\mathcal{D}_{L_h}\backslash \mathrm{O}^+(L,h)$ are examples of *orthogonal modular varieties* (see also Section 3.4). By a result of Baily and Borel [BB66] they are quasi-projective and, therefore, by a result of Borel [Bor72], the map $\pi$ is a morphism of quasi-projective varieties.

## 3.3 The Torelli theorems

As in the case of K3 surfaces, one can prove a number of Torelli theorems for irreducible symplectic manifolds.

**Theorem 3.3.1.** *(The Local Torelli Theorem) [Bea83] [Bog74] If $X$ is an irreducible symplectic manifold and $p : \mathcal{X} \to U$ is a representative of the Kuranishi family of deformations of $X$ with sufficiently small contractible base, then the differential of the period map $p_U$ is an isomorphism. Therefore, the period map is a local isomorphism.*

If $\mathcal{M}_L$ is the moduli of marked irreducible symplectic manifolds with Beauville lattice $L$ then (as in Section 3.2) one can define a map

$$p : \mathcal{M}'_L \to \Omega_L$$

from each component $\mathcal{M}'_L$ of $\mathcal{M}_L$ by mapping each manifold to its period. By the following theorem of Huybrechts, the map $p$ is surjective.

**Theorem 3.3.2.** *[Huy99] If $L$ is the Beauville lattice of an irreducible symplectic manifold and $\mathcal{M}'_L$ is non-empty then the period map*

$$p : \mathcal{M}'_L \to \Omega_L$$

*is surjective.*

As for K3 surfaces, one also has a Hodge theoretic Torelli theorem. It should be noted, however, that this theorem is somewhat weaker than the K3 case. In order to state it, we must firstly define Markman's monodromy operators [Mar08] [Mar] [Mar10]. We follow [GHS13].

Let $X_1$ and $X_2$ be irreducible symplectic manifolds that are isomorphic to the fibres over $b_1, b_2 \in B$ of a smooth, proper flat family

$$\pi : \mathcal{X} \to B$$

under the isomorphisms $\alpha_1$ and $\alpha_2$, respectively. The map

$$f : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$$

is said to be a parallel transport operator if there exists a continuous path

$$\gamma : [0, 1] \to B$$

such that $\gamma(0) = b_1$, $\gamma(1) = b_2$ and the parallel transport in the local system $R\pi_*\mathbb{Z}$ along $\gamma$ induces an isomorphism

$$(\alpha_2^{-1})^* \circ f \circ \alpha_1^* : H^*(\mathcal{X}_{b_1}, \mathbb{Z}) \to H^*(\mathcal{X}_{b_2}, \mathbb{Z}).$$

If $X$ is an irreducible symplectic manifold, then an element

$$g \in \mathrm{Aut}(H^*(X, \mathbb{Z}))$$

is called a monodromy operator if it is a parallel transport operator for $X_1 = X_2 = X$. The group of monodromy operators is denoted by $\mathrm{Mon}(X)$ and the image in $\mathrm{O}(L)$ is denoted by $\mathrm{Mon}^2(X)$. The group $\mathrm{Mon}^2(X)$ has been characterised by Giovanni Mongardi for deformation generalised Kummer manifolds and O'Grady's 10 dimensional example in [Mon14]. We can now state the Hodge theoretic Torelli theorem, which is due to Markman [Mar11] and uses the results of Verbitsky [Ver13].

**Theorem 3.3.3.** *(Hodge Theoretic Torelli) Suppose that $X_1$ and $X_2$ are irreducible symplectic manifolds*

1. *If*

$$f : H^2(X_2, \mathbb{Z}) \to H^2(X_1, \mathbb{Z})$$

*is an isomorphism of integral Hodge structures which is a parallel transport operator, then $X_1$ and $X_2$ are bimeromorphic.*

2. *If, in addition, $f$ maps a Kähler class of $X_2$ to a Kähler class of $X_1$, then $X_1$ and $X_2$ are isomorphic.*

## 3.4 Orthogonal modular varieties

Let $L$ be a lattice of signature $(2, n)$ with $n \geq 3$ and let $\mathrm{O}(L)$ be the orthogonal group of $L$. The group $\mathrm{O}(L)$ acts naturally on symmetric space $\Omega_L$ where

$$\Omega_L = \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x, x) = 0,\ (x, \overline{x}) > 0\}.$$

The space $\Omega_L$ has two connected components. These components are interchanged by elements in $\mathrm{O}(L)$ of negative spinor norm. To simplify matters, we pick one of the components and call it $\mathcal{D}_L$. It is fixed by the kernel of the spinor norm $\mathrm{O}^+(L)$.

**Definition 3.4.1.** *If $\Gamma \leq \mathrm{O}^+(L)$ is a subgroup of finite index, then we call quotients of the form*

$$\mathcal{F}_L(\Gamma) = \Gamma \backslash \mathcal{D}_L$$

*orthogonal modular varieties. We may sometimes broaden this definition to arithmetic subgroups $\Gamma \leq \mathrm{O}^+(L)$.*

Orthogonal modular varieties can be studied from a number of different angles. They are examples of locally symmetric varieties, i.e. they are quotients of a symmetric space by a discrete group of automorphisms; they are complex analytic spaces; by the results of Baily and Borel [BB66], they are quasi-projective; and, if $\Gamma$ is torsion free, they are complex manifolds.

The elliptic elements of $\Gamma$ are especially important to us as they determine the branch locus of the cover

$$\pi : \mathcal{D}_L \to \mathcal{F}_L(\Gamma)$$

and careful attention needs to be paid to the branch locus if one tries to prove general type results using the low-weight cusp form trick (see Section 3.7). The branch locus of $\pi$ is precisely the image of points in $\mathcal{D}_L$ that are fixed by elliptic elements (elements of finite order) in $\Gamma$. The branch locus can have both a smooth and a singular part. The stabiliser in $\Gamma$ of any fixed point in $\mathcal{D}_L(\Gamma)$ is a finite subgroup of $\Gamma$ and, by a theorem of Cartan [Car57], the action can be locally linearised. Therefore, the singularities of $\mathcal{F}_L(\Gamma)$ are finite quotient singularities, i.e. they are locally isomorphic to quotients of the form $\mathbb{C}^n/G$ where $G < \mathrm{GL}(n, \mathbb{C})$ is a finite subgroup. By a theorem of [Che55], $\mathbb{C}^n/G$ is smooth if and only if $G$ is generated by quasi-reflections. We recall that a *quasi-reflection* $g$ is an elliptic element of $\mathrm{GL}(n, \mathbb{C})$ with 1 as an eigenvalue $\xi$ of multiplicity $n-1$. If $\xi = -1$, then $g$ is called a *reflection*. By a result of [GHS07] (Corollary 2.13), if $n > 2$ the elements of $\Gamma$ that act as quasi-reflections correspond precisely to $\pm\sigma \in \Gamma$ where $\sigma$ is a reflection. Therefore, the smooth part of the branch locus of $\mathcal{F}_L(\Gamma)$ corresponds precisely to the image of points in $\mathcal{D}_L$ that a are fixed only by $\pm\sigma \in \Gamma$ where $\sigma$ is a reflection.

## 3.5 Lattices

In this section, we collect some of the lattice theoretic results and definitions that will be needed later. Particular emphasis will be placed on their classification as this plays a significant role in many of our later results. More detailed treatments can be found in [Kit93], [Cas78] and [CS99].

**Definition 3.5.1.** *A lattice $L$ is an integral symmetric bilinear form. Equivalently, $L$ is a finitely generated $\mathbb{Z}$-module in an $\mathbb{Q}$-vector space $V$ so that $V$ is endowed with a symmetric bilinear form $(-, -)$ that is integral on $L$.*

If $x^2 = (x, x)$ is even for all $x \in L$, we say that $L$ is an even lattice. By the rank of

$L$, we mean the rank of $L$ as a $\mathbb{Z}$-module.

**Definition 3.5.2.** *If $L$ is a lattice in the vector space $V$, the orthogonal group* $\mathrm{O}(L)$ *of $L$ is defined by*

$$\mathrm{O}(L) = \{g \in \mathrm{GL}(V) \mid (gx, gy) = (x, y)\ \forall x, y \in L\}.$$

For many purposes, the above definition of a lattice is sufficiently general. However, when we discuss the classification of lattices it becomes necessary to work with lattices defined over the $p$-adic numbers and for these purposes it becomes convenient to introduce a broader definition.

**Definition 3.5.3.** *For a prime $p$, a $\mathbb{Z}_p$-lattice is a finitely generated $\mathbb{Z}_p$-module in a $\mathbb{Q}_p$-vector space. We also permit $p$ to formally assume $p = -1$ and, in such a case, we let $\mathbb{Z}_p = \mathbb{Z}$ and $\mathbb{Q}_p = \mathbb{Q}$.*

### 3.5.1 The classification problem

We start by introducing some invariants. If $L$ is a lattice with bilinear form $B$ then, by Sylvester's law of inertia, there exists $M \in \mathrm{GL}(n, \mathbb{R})$ so that ${}^tMBM = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1)$. If $n_+$ and $n_-$ denote the number of positive and negative terms in this decomposition then the pair $(n_+, n_-)$ is called the *signature* of the lattice. If $n_+n_- < 0$, then $L$ is said to be *indefinite*; otherwise, $L$ is said to be *definite*. If $L$ is definite and $n_+ > 0$ ($n_- > 0$), $L$ is said to be *positive* (*negative*) definite. We define the determinant of $L$ $\det(L)$ by $\det B$.

By the classification of lattices, we mean a classification up to *class* or *integral equivalence*.

**Definition 3.5.4.** *If $L_1$ and $L_2$ are lattices in the vector space $V$, then we say that $L_1$ and $L_2$ belong to the same class (or that $L_1$ and $L_2$ are integrally equivalent, or isomorphic) if there exists $\sigma \in \mathrm{O}(V)$ such that*

$$\sigma(L_1) = L_2.$$

*The set of all lattices in the same class as the lattice $L$ is denoted by* $\mathrm{cls}(L)$.

It turns out that the method of classification depends strongly upon the signature of the lattice. In the indefinite case, the classification is given in terms of $p$-adic invariants: either the the genus or a refinement of the genus called the spinor genus. Subject to minor restrictions, one can show that the notions of spinor genus, genus, and class coincide. The classification can then be given in terms of the genus (which is usually easy to express). Indeed, we shall establish some simple notation for the genus later. The full details of the classification of indefinite lattices are lengthy and lie outside the scope of this introduction. Two good references on the subject, with an emphasis on the arithmetic aspects, are [Kit93] and [Cas78]. Our intention here is mostly to introduce the main results of the classification, and introduce Conway's genus notation. Much of our approach will follow [CS99] §15.

In the definite case, the genus and spinor genus are far weaker invariants and the classification is instead given in terms of combinatorial algorithms: either the reduction algorithms of Gauss and Minkowski, or the glueing theory of Kneser and Niemeier. We shall not say much about the classification of definite lattices other than to say that tables have been produced for lattices of low rank and small determinant and that the classification of higher rank lattices tends to be impractical due to the complexity of the algorithms involved. More details may be found in [CS99].

### 3.5.2 The genus

We now introduce the first $p$-adic invariant: the genus. If $L$ is a lattice in a $\mathbb{Q}$-vector space $V$, we define $L_p := L \otimes \mathbb{Z}_p$ and $V_p := V \otimes \mathbb{Q}_p$.

**Definition 3.5.5.** *If $L_1$ and $L_2$ are lattices in the $\mathbb{Q}$-vector space $V$, we say that $L_1$ and $L_2$ belong to the same genus if for every prime $p$ and $p = \infty$ there exists $\sigma_p \in \mathrm{O}(V_p)$ so that*

$$(L_1)_p = \sigma_p(L_2).$$

*The set of all lattices in the same genus as $L_1$ is denoted by* $\mathrm{gen}(L_1)$.

The genus arises naturally when one tries to prove statements via local-global arguments (that is, understanding the 'global' $\mathbb{Z}$-lattice $L$ by studying the 'local' properties of $L_p$ for all primes $p$). Such results can be quite strong. For example, one can consider the classical problem of representability of an integer by a lattice. We say that an integer $a$ is *representable* by a lattice $L$ if there exists $x \in L$ such that $(x, x) = a$. One can prove the following statements.

**Theorem 3.5.6.** *[Cas78] Let $L$ be a regular lattice and let $a \in \mathbb{Z}$ be non-zero. If $a$ is represented by $L_p$ for all primes $p$ (and $p = -1$), then $a$ is represented (over $\mathbb{Z}$) by some $H \in \operatorname{gen}(L)$.*

**Theorem 3.5.7.** *[Cas78] Let $L$ be a regular lattice of rank $n \geq 4$ and let $a \in \mathbb{Z}$ be non-zero. If $a$ is represented by $L_p$ for all primes $p$ (and $p = -1$) then $a$ is represented by $L$ over $\mathbb{Z}$.*

When studying the classification of lattices one finds that, in the most general setting, the genus of a lattice defines a strictly weaker equivalence relation than the class. For example, $\langle 1 \rangle \oplus \langle 82 \rangle$ and $\langle 2 \rangle \oplus \langle 41 \rangle$ belong to the same genus but they do not belong to the same class [Cas78].

In order to obtain stronger statements, one can consider a refinement of the genus: the spinor genus. In particular, the spinor genus usually contains at most one class. Moreover, one can show that subject to minor restrictions, the class, genus and spinor of a lattice all coincide. Following [CS99] §15, we define some notation to describe the genus of a lattice. The first step is to introduce the $p$-adic *Jordan decomposition.*

**Theorem 3.5.8.** *[CS99] If $L$ is a lattice and $p \neq 2$ then $L_p$ can be diagonalised over $\mathbb{Z}_p$. If $p = 2$, then $L_p$ can be written as an orthogonal product of $\mathbb{Z}_p$ lattices whose forms are given by*

$$(qx) \qquad \textit{and} \qquad \begin{pmatrix} qa & qb \\ qb & qc \end{pmatrix}$$

*where $q$ is a power of 2, $a$ and $c$ are divisible by 2 but 2 divides neither of $x$ nor $b$ nor $d = ac - b^2$.*

We can therefore express $L_p$ as

$$L_p = L^1 \oplus pL^p \oplus p^2 L^{p^2} \oplus \ldots \oplus qL^q \oplus \ldots \tag{3.1}$$

where each $L^q$ is a $p$-adic unit form. That is, a $\mathbb{Z}_q$-lattice whose determinant is coprime to $p$ (if $p \geq 2$) or a positive definite form $p = -1$. The factors $qL^q$ are called *Jordan constituents* of $L$ and the decomposition given in Equation (3.1) is called the *Jordan decomposition* of $L$. The number $q$ is called the scale of the factor $qL^q$.

If $p \neq 2$ then, from the decomposition given in Equation (3.1), we define the dimensions $n_q = \dim L^q$ and the signs

$$\epsilon_q = \left( \frac{\det L^q}{p} \right)$$

where the left-hand is the Legendre symbol of $\det L^q$ for the prime $p$.

We now define the $p$-adic symbol. For $p \neq 2$, we can define the $p$-adic symbol of the lattice $L$ from the Jordan decomposition given in Equation (3.1). If $p = -1$, this is defined as the formal product

$$+^{n_+} -^{n_-}$$

where $(n_+, n_-)$ is the signature of $L$.

If $p > 2$, the $p$-adic symbol of $L$ is defined as the formal product of the terms

$$q^{\epsilon_q n_q}.$$

For $p \neq 2$, two lattices are equivalent over $\mathbb{Z}_p$ if and only if they have the same $p$-adic symbol ( [CS99] §15.7). In order to define a complete set of invariants for the genus of a lattice, one also has to consider $p = 2$. In this case, there are slightly more invariants to consider. If $L$ has a 2-adic decomposition given by

$$L = L^1 \oplus 2L^2 \oplus 4L^4 \oplus \ldots \oplus qL^q \oplus \ldots$$

The term $qL^q$ has invariants consisting of

(i) The *scale* $q$ of $qL^q$

(ii) The *type* $S_q$ of $L^q$ which assumes the value $I$ or $II$ (see below)

(iii) The *dimension* $n_q = \dim L^q$

(iv) The *sign*

$$\epsilon_q = \left(\frac{\det L^q}{2}\right)$$

(v) The *oddity* $t_q$ of $L^q$ (see below).

The *type* $S_q$ of $L^q$ is defined to be $I$ if $qL^q$ represents an odd multiple of $q$; otherwise, $S_q$ is defined to be $II$. One can also show that $S_q = I$ if and only if there is an odd entry on the main diagonal of the matrix representing $L^q$; otherwise, $II$. If $S_q = I$, the *oddity* $t_q$ is defined as the trace of $L^q$ read modulo 8; otherwise, $t_q = 0$.

We can now define the 2-adic symbol of the Jordan decomposition. If $p = 2$, the 2-adic symbol of the Jordan decomposition given by (3.1) is a formal product of terms of the form

$$q_{t_q}^{\epsilon_q n_q}$$

if $L^q$ is of type $I$; or

$$q^{\epsilon_q n_q}$$

if $L^q$ is of type $II$.

Neither the $p$-adic Jordan decomposition of $L$ nor its associated $p$-adic symbol are unique. Therefore, there is an associated equivalence relation on all the possible $p$-adic symbols of a lattice. This equivalence can be given in combinatorial terms, but is a little lengthy to state. For details see [CS99] (§15 7.5).

Given the $p$-adic symbols of $L$, we write

$$I_{r,s}(\ldots q_t^{\pm m} \ldots) \qquad \text{or} \qquad II_{r,s}(\ldots q_t^{\pm m} \ldots)$$

where $I$ or $II$ correspond to the type of the 2-adic form $L^1$ (sometimes called the *parity* of $L$); the subscripts $r, s$ is the $-1$-adic symbol $+^r-^s$ (i.e. the signature of the lattice); and the terms $q_t^{\pm m}$ run over all of the factors $p$-adic symbols for $p \geq 2$. It can be shown that the above notation expresses the genus of $L$ [CS99].

### 3.5.3 The spinor genus and the spinor norm

One can also study lattices in terms of a refinement of the genus: the *spinor genus.* The spinor genus takes a little more work to define, but one is rewarded with significantly stronger results; in particular, one obtains strong general results on integral equivalence. If $V$ is a regular quadratic space of dimension $n > 2$ over a field $k$ where $\operatorname{char} k \neq 2$ then, for all $v \in V$ is such that $v^2 \neq 0$ then the reflection $\sigma_v \in \mathrm{O}(L)$ in $v$ is defined as the map

$$\sigma_v : x \mapsto x - 2\frac{(x,v)}{(v,v)}v$$

for all $x \in V$. We define $\mathrm{O}(V) = \{g \in \mathrm{GL}(V) \mid (gx, gx) = (x,x)\ \forall x \in V\}$.

**Theorem 3.5.9.** *[Cas78] For $V$ as above, $\mathrm{O}(V)$ is generated by reflections.*

**Definition 3.5.10.** *If $g \in \mathrm{O}(V)$ is such that $g = \sigma_{v_1} \ldots \sigma_{v_s}$ then the spinor norm $\mathrm{sn}_V(g)$ of $g$ is defined by*

$$sn_V(g) = -\frac{(v_1,v_1)}{2} \cdots \frac{(v_s,v_s)}{2} \qquad\qquad k^*/(k^*)^2.$$

One can show (see [Cas78], §10, for example) that this definition is a well defined group homomorphism. (We remark that our definition of the spinor norm has a different sign convention than many other sources.) If $L$ is a lattice in $V$, the spinor norm on

O$(L)$ will be taken to mean the restriction to $\mathrm{O}(L) \leq \mathrm{O}(V)$ of the spinor norm on $\mathrm{O}(V)$. The kernel of the spinor norm on $\mathrm{O}(V)$ is denoted by $\mathrm{O}^+(V)$. We define the groups $\mathrm{O}^+(L) := \mathrm{O}^+(V) \cap \mathrm{O}(L)$ and $\mathrm{SO}^+(L) := \mathrm{O}^+(L) \cap \mathrm{SO}(L)$ etc.

**Definition 3.5.11.** *If $L_1$ and $L_2$ are lattices in the $\mathbb{Q}$-vector space $V$, we say that $L_1$ and $L_2$ belong to the same spinor genus if there exists $\eta \in \mathrm{O}(V)$ such that for all primes $p$ there exists $\delta_p \in \mathrm{O}(L_1)$ such that*

$$\eta(L_2) = \delta_p((L_1)_p)$$

*for all $p$.*

We denote the spinor genus of a lattice $L$ by $\mathrm{sg}(L)$. It is clear that

$$\mathrm{sg}(L) \subset \mathrm{gen}(L) \subset \mathrm{cl}(L).$$

The number of spinor genera contained in a genus can be determined effectively and is always finite and a power of 2 ( [Cas78] §11). It is clear that classifying spinor genera is somewhat more involved than classifying genera. It is therefore desirable to know when the two notions conincide. In fact, this happens quite often.

**Theorem 3.5.12.** *( [Cas78] §11, Theorem 1.3 ) Let $L$ be a lattice of determinant $d$ in the quadratic space $V$. If $\mathrm{gen}\, L$ contains more than one spinor genus then at least one of the following occur:*

1. *There is an odd prime $p$ such that $p^{n(n-1)/2} \mid d$*
2. *$2^{n(n-3)/2+[(n+1)/2]} \mid d$.*

*(where $[(n+1)/2]$ denotes the integral part of $(n+1)/2$.)*

Remarkably, the notions of spinor genus, genus and class all coincide for indefinite lattices of rank greater than or equal to three.

**Theorem 3.5.13.** *( [Cas78] §11, Theorem 1.4) If $L$ is an indefinite lattice of dimension $n \geq 3$, then $\mathrm{cls}(L) = \mathrm{sg}(L)$.*

This is a particularly useful result as it allows one to work at the level of the genus and still obtain strong classification results.

### 3.5.4 The Discriminant form

The *discriminant group* $D(L)$ of an even lattice $L$ is the abelian group is defined by

$$D(L) = L^\vee / L$$

(where $L^\vee$ is the dual lattice of $L$). The discriminant group comes with a $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form *(the discriminant form)* inherited from $L$. We shall often denote this form by $q$. Many lattice theoretic results can be succinctly expressed in terms of the discriminant form (which is due to Nikulin). For more details, the reader is referred to [Nik79b]. A particularly useful fact, that we use often, is that the signature and discriminant form of a lattice form a set of invariants for the genus ( [Nik79b] Corollary 1.9.4). From the discriminant form on $L$, one can can also define a natural subgroup *(the stable orthogonal group)* $\widetilde{\mathrm{O}}(L)$ of $\mathrm{O}(L)$ by

$$\widetilde{\mathrm{O}}(L) := \{g \in \mathrm{O}(L) \mid \overline{g} = id\} \tag{3.2}$$

where $\overline{g}$ denotes the natural action of $g$ on $D(L)$. The group $\widetilde{\mathrm{O}}(L)$ is particularly important in moduli theory and the theory of orthogonal modular forms and has the useful property that if $S \leq L$ then $\widetilde{\mathrm{O}}(S) \leq \widetilde{\mathrm{O}}(L)$ (see [GHS13] Lemma 7.1, cf. [Nik79b] Proposition 1.15.1).

### 3.5.5 The two dimensional space groups

For later applications, we need to know about the orthogonal group $\mathrm{O}(B)$ of a definite lattice $B$ of rank 2. The group $\mathrm{O}(B)$ is, of course, finite and by the crystallographic restriction theorem ( [Sen95] p. 50), if $g \in \mathrm{O}(B)$ then $g$ has order 1, 2, 3, 4 or 6 and $B$

admits a basis such that $g$ is given by $\pm I_2$ or by

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} && \text{if} && \chi_g(x) = \phi_1(x)\phi_2(x) \\
&\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}^{\pm 1} && \text{if} && \chi_g(x) = \phi_3(x) \\
&\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\pm 1} && \text{if} && \chi_g(x) = \phi_4(x) \\
&\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{\pm 1} && \text{if} && \chi_g(x) = \phi_6(x).
\end{aligned}$$

## 3.6 Orthogonal modular forms

We start with a definition. More details can be found in [GHS13].

**Definition 3.6.1.** *If $L$ is a lattice of signature $(2, n)$ where $n \geq 3$ and $\Gamma \leq \mathrm{O}^+(L)$ is of finite index with character $\chi : \Gamma \to \mathbb{C}^*$ then a weight $k$ ($k \in \mathbb{Z}$) modular form $F$ for $\Gamma$ with character $\chi$ is defined as a holomorphic function $F : \mathcal{D}_L^\bullet \to \mathbb{C}$ (where $\mathcal{D}_L^\bullet$ is the affine cone of $\mathcal{D}$) such that*

$$\begin{aligned}
&F(tZ) = t^{-k}F(Z) && \forall t \in \mathbb{C}^* \\
&F(gZ) = \chi(g)F(Z) && \forall g \in \Gamma
\end{aligned}$$

*and $F$ is called a cusp form if it vanishes at each cusp of $\mathcal{D}_L$.*

**Definition 3.6.2.** *If $\Gamma < \mathrm{O}(L)$ is of finite index and $\chi : \Gamma \to \mathbb{C}^*$ is a character of $\Gamma$ we denote the space of weight $k$ modular forms for $\Gamma$ with character $\chi$ by*

$$M_k(\Gamma, \chi)$$

*and the subspace of cusp forms by*

$$S_k(\Gamma,\chi)$$

*respectively.*

We shall often omit the character $\chi$ and refer simply to $M_k(\Gamma)$ and $S_k(\Gamma)$.

## 3.7 Kodaira dimension of Orthogonal modular varieties

Recall that the Kodaira dimension of a smooth projective variety $X$ is defined by

$$\kappa(X) = \mathrm{trdeg} \bigoplus_{k\geq 0} H^i(X, kK_X) - 1$$

or by $-\infty$ if $H^i(X, kK_X) = 0$ for all $k > 0$. Equivalently, as $h^0(X, kK_X) \sim k^{\kappa(X)}$ for sufficiently divisible $k$, one can define $\kappa(X)$ in terms of the growth of the $k^{th}$ plurigenus $h^0(X, kK_X)$ of $X$. In the case $\kappa(X) = \dim(X)$, we say that $X$ is of *general type.*

Many orthogonal modular varieties $\mathcal{F}$ are of general type. This can often be proved by using special modular forms to produce pluricanonical forms. This method is based on the fundamental observation *(the Hirzebruch-Mumford proportionality principle)* that if $\Gamma$ is a discrete subgroup of $\mathrm{O}(2,n)$ then the dimension of the space of weight-$k$ modular forms for $\Gamma$ grows like $k^n$. Therefore, if one can produce pluricanonical forms from a sufficiently large subspace of $M_k(\Gamma)$, then one should arrive at general type results. We discuss Hirzebruch-Mumford proportionality in more detail in Section 3.9. It is straightforward to construct a pluricanonical form on the regular part of a modular variety from a modular form. However, in order to prove general type results, one must also check that the forms constructed extend to a smooth projective model $\widetilde{\mathcal{F}}$ of the modular variety $\mathcal{F}$. In order to solve this extension problem, one needs to understand the geometry of the branch locus, the singularities and the boundary components of a suitable compactification in order to determine which modular forms define pluricanonical forms on $\widetilde{\mathcal{F}}$. From this point, we will take $\overline{\mathcal{F}}$ to mean a toroidal compactification of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ to mean a desingularisation of $\overline{\mathcal{F}}$.

One method of producing pluricanonical forms on $\widetilde{\mathcal{F}}$ from special modular forms is the *low-weight cusp form trick* (see [GHS13] pp. 497-499). Here one starts with a cusp form $F_a$ of weight $a < n$ and a modular form $G_{(n-a)k}$ of weight $(n-a)k$ and one defines the modular form

$$F_{nk} := F_a^k G_{(n-a)k}$$

and the differential form

$$\Omega_{nk} := F_{nk}(dZ)^k$$

where $dZ = dz_1 \wedge \ldots \wedge dz_n$ is a volume form on the regular part of $\mathcal{D}$. The form $\Omega_{nk}$ is $\Gamma$-invariant and therefore descends to a differential form $\omega_{nk}$ on $\mathcal{F}$ that defines a section of the pluricanonical bundle of $\mathcal{F}$ away from the cusps and away from the branch locus. The form $\Omega_{nk}$ has zeros of order $k$ along along the boundary of $\mathcal{F}$ and therefore, by the results of [AMRT10], $\omega_{nk}$ defines a section of the pluricanonical bundle of $\overline{F}$ away from the branch locus. We must therefore understand the conditions imposed by the branch locus of $\overline{\mathcal{F}}$ on $F_{nk}$ so that $\omega_{nk}$ extends to $\widetilde{\mathcal{F}}$. As explained in Section 3.4, the smooth part of the branch locus corresponds precisely to the fixed locus of elements in $\Gamma$ that act as quasi-reflections. These elements are given by $\pm\sigma \in \Gamma$ where $\sigma$ is a reflection. Therefore, as can be shown by direct calculation in the spirit of Chapter I of [Rei87], the form $\omega_{nk}$ extends over the smooth part of the branch locus if $F_{nk}$ vanishes to order $k$ along the fixed loci of all $\pm\sigma \in \Gamma$.

One still has to consider the singular part of the branch locus but, in some cases, the singularities of $\overline{\mathcal{F}}$ do not impose any conditions on $F_{nk}$. This is the case if all the singularities of $\overline{\mathcal{F}}$ are canonical.

**Definition 3.7.1.** *If $X$ is a normal complex variety, we say that $X$ has canonical singularities if it is $\mathbb{Q}$-Gorenstein and for some resolution of singularities*

$$f : \widetilde{X} \to X$$

*the discrepancy $\Delta = K_{\widetilde{X}} - f^*K_X$ is an effective Weil $\mathbb{Q}$-divisor.*

We recall that $X$ is $\mathbb{Q}$-Gorenstein if there exists $r \in \mathbb{N}$ so that if $K_X$ is the canonical (Weil) divisor $K_X$ is then $rK_X$ is Cartier. Equivalently, $X$ has canonical singularities if for all open $U \subset X$, any pluricanonical form on the smooth part of $U$ extends holomorphically to a desingularisation $\widetilde{U}$.

## 3.8 Canonical singularities in orthogonal modular varieties

In order to apply the low-weight cusp form trick, we wish to be able to decide whether or not a singularity is canonical. As the singularities in an orthogonal modular variety are all finite quotient singularities, one can use the *Reid-Tai criterion*. An excellent introduction to canonical singularities is [Rei87]. Many of the following results can be found in [GHS07] or [GHS13].

**Definition 3.8.1.** *If $g \in \mathrm{GL}(n, \mathbb{C})$ is of finite order $m > 1$ with eigenvalues $\zeta^{a_1}, \ldots, \zeta^{a_m}$ for $\zeta = e^{2\pi i/m}$, the Reid-Tai sum $\Sigma(g)$ is defined by*

$$\Sigma(g) = \sum_{i=1}^{m} \left\{ \frac{a_i}{m} \right\}$$

*where $0 \leq \{x\} < 1$ denotes the fractional part of $x$. We define $\Sigma(1) := 1$.*

The Reid-Tai criterion is given by the following:

**Theorem 3.8.2.** *[GHS13] If $G \leq \mathrm{GL}(n, \mathbb{C})$ is a finite subgroup not containing quasi-reflections, then $\mathbb{C}^n/G$ is non-canonical if and only if*

$$\Sigma(g) \geq 1$$

*for all $g \in G$.*

If $G$ contains quasi-reflections, we have a modified version of the Reid-Tai sum due to Katharina Ludwig:

**Definition 3.8.3.** *If $g \in \mathrm{GL}(n, \mathbb{C})$ is of finite order $m > 1$, let $k \in \mathbb{N}_0$ be minimal with the property that $g^k$ is a quasi-reflection or the identity. Let $s$ be such that $m = sk$*

*and let $g$ have eigenvalues $\zeta^{a_1}, \dots, \zeta^{a_m}$ for $\zeta = e^{2\pi i/m}$ where $\{a_i\}$ are ordered so that $\zeta^{ka_1} = \zeta^{ka_{n-1}} = 1$. The modified Reid-Tai sum $\Sigma'(g)$ is defined by*

$$\Sigma'(g) = \left\{\frac{sa_n}{m}\right\} + \sum \left\{\frac{a_i}{m}\right\}$$

*and $\Sigma'(1) := 1$. (Note that $\Sigma'(g) = \Sigma(g)$ if no power of $g$ is a quasi-reflection.)*

For applications, one needs to be able to apply the criteria to each element of $G$ in turn. In such a case, one can use the following proposition:

**Proposition 3.8.4.** *[GHS13] If $G \leq \mathrm{GL}(n, \mathbb{C})$ is a finite group, then $\mathbb{C}^n/G$ has canonical singularities if $\mathbb{C}^n/\langle g \rangle$ has canonical singularities for all $g \in G$.*

And so,

**Theorem 3.8.5.** *[GHS13] If $G \leq \mathrm{GL}(n, \mathbb{C})$ is a finite subgroup, then $\mathbb{C}^n/G$ has canonical singularities if*

$$\Sigma'(g) \geq 1$$

*for all $g \in G$.*

In order to apply the above results to $\mathcal{F}_L(\Gamma)$, one needs to understand the local action of an isotropy subgroup $G \leq \Gamma$ around a point $[w]$ in its fixed locus. Around $[w]$, the tangent space $T_{[w]}\mathcal{D}_L$ is locally isomorphic to

$$T_{[w]}\mathcal{D}_L \cong \mathrm{Hom}(\mathbb{W}, \mathbb{W}^{\perp}/\mathbb{W}) =: V$$

where $\mathbb{W} = \mathbb{C}.w \leq L \otimes \mathbb{C}$ and the group $G$ acts on $\mathbb{W} \leq L \otimes \mathbb{C}$ as a character $\alpha : G \to \mathbb{C}^*$. In [GHS07], bounds for $\Sigma(g)$ were produced by carefully studying the rational representations of $g \in G$ on the $g$-modules

$$S = (\mathbb{W} \oplus \overline{\mathbb{W}})^{\perp} \cap L$$

and

$$T = S^{\perp} \leq L.$$

By using this method, (and a similar approach at the boundary) they proved that a toroidal compactification of $\mathcal{F}_L(\Gamma)$ exists with at most canonical singularities whenever $n \geq 9$. In smaller dimensions, however, compactifications with only canonical singularities may not exist. However, some details about these compactifications are known. A fact that we shall use later (established in the proof of Theorem 2.10 of [GHS07]) is that the non-canonical singularities in an orthogonal modular variety of dimension $n \leq 5$, are fixed by quasi-reflections or elements of order 3, 4 or 6.

## 3.9 The Hirzebruch-Mumford volume

In order to prove general type results by using the low-weight cusp form trick, we need to understand the growth of the spaces $M_k(\Gamma, \chi)$. The growth of such spaces is governed by the *Hirzebruch-Mumford proportionality principle.* As proved in [GHS08], the principle implies that

$$\dim M_k(\Gamma) = \frac{2}{n!} \operatorname{vol}_{HM}(\Gamma) k^n + O(k^{n-1}).$$

The constant $\operatorname{vol}_{HM}(\Gamma)$ is known as the *Hirzebruch-Mumford volume* of the group $\Gamma$. The Hirzebruch-Mumford volume essentially compares the volume of $\Gamma \backslash \mathcal{D}_L$ with the volume of the compact dual $\mathcal{D}_L^{(c)}$. Each of these volumes may be expressed in terms of the Tamagawa measure of $\mathrm{O}(L)$ and, due to a result of Siegel, one can compute these volumes by local methods in terms of the local densities $\alpha_p(L)$ of $L$. Here,

$$\alpha_p(S) = \frac{1}{2} \lim_{r \to \infty} p^{-\frac{rn(n-1)}{2}} |\{X \in \mathrm{Mat}_n(\mathbb{Z}_p) \bmod p^r, {}^tXSX \cong S \bmod p^r\}|$$

for a quadratic form $S$ defined by the matrix $S \in M_n(K)$ over a number field $K$. Such local densities can be computed explicitly (as in [Kit93]). In [GHS08], it is proved that

**Theorem 3.9.1.** *If $L$ is an indefinite lattice of rank $\rho \geq 3$, then the Hirzebruch-Mumford volume of* $\mathrm{O}(L)$ *is equal to*

$$\operatorname{Vol}_{HM}(\mathrm{O}(L)) = \frac{2}{g_{sp}^{+}(L)} |\det L|^{(\rho+1)/2} \prod_{k=1}^{\rho} \pi^{-k/2} \Gamma(k/2) \prod_p \alpha_p(L)^{-1}$$

*where $\alpha_p(L)$ are the local densities of $L$, $g^+_{sp}(L)$ is the number of spinor genera in the genus of $L$, and $\Gamma$ is the gamma function.*

They also calculate a number of explicit examples. Moreover, they show (in the proof of Proposition 4.1) that if $M_{2b}(-\eta\mathcal{D}_K)$ is the subspace of weight $2b$ modular forms vanishing on the rational quadratic divisor

$$\mathcal{D}_K = \{[x] \in \mathcal{D}_L \mid (x, k) = 0\} \qquad\qquad \text{for } k \in L,\ k^2 < 0$$

then

$$0 \to M_{2b}(\Gamma)(-(2+2\eta)\mathcal{D}_K) \to M_{2b}(\Gamma)(-2\eta\mathcal{D}_K) \to M_{2(b+\eta)}(\Gamma \cap \widetilde{\mathrm{O}}^+(K))$$

where $K = k^\perp \subset L$. Therefore, if given a list of rational quadratic divisors containing the singular locus and formulae for their associated Hirzebruch-Mumford volumes, one can establish results on the growth of the space of modular forms vanishing along the divisors.

## 3.10 Statement of results

In this thesis, we study the geometry of a toroidal compactification of the orthogonal modular variety $\mathcal{F}_{2d}$ associated with deformation generalised Kummer 4-folds with a degree $2d$ polarisation of split type. If $L_{6,2d} = 2U \oplus \langle -6 \rangle \oplus \langle -2d \rangle$, then $\mathcal{F}_{2d}$ is the orthogonal modular variety given by

$$\mathcal{F}_{2d} = \Gamma_{6,2d} \backslash \mathcal{D}_{L_{6,2d}}$$

where $\Gamma_{6,2d} = \mathrm{O}^+(L_6, h^s_{2d}) \leq \mathrm{O}(L_{6,2d})$ and $\mathrm{O}(L_6, h^s_{2d})$ is the group that we determine in Theorem 4.0.6. Where no confusion is likely, we shall also denote $\Gamma_{6,2d}$ by $\Gamma$.

As explained in Section 3.7, if one is interested in proving general type results for

$\mathcal{F}_{2d}$, then it is important to understand the branch locus of

$$\mathcal{D}_{L_{6,2p^2}} \to \mathcal{F}_L(\Gamma_{2p^2}).$$

If one seeks an exact solution to this problem, the question of determining the obstruction in the interior of $\mathcal{F}_{2d}$ is mostly a question of determining the conjugacy classes of finite subgroups in $\Gamma$. This, it turns out, is a hard problem. Nevertheless, an estimate will suffice if one is only interested in proving general type results. Such an estimate is given by Theorem 5.3.4.

Our intention throughout has been to provide results that are as exact as possible. In order to use methods that yield good bounds, we have made certain arithmetic restrictions. The first restriction we make is that we only consider the orthogonal modular varieties $\mathcal{F}_{2d}$ for $2d = 2p^2$ where $p$ is an odd prime. By doing so, we obtain better results than we would expect to obtain for arbitrary $d$. We shall make some comparisons with the general case in the introduction of Chapters 5.

1. The starting point for our most of our bounds is Theorem 4.0.11.

   **Theorem 4.0.11.** *The group* $\mathrm{O}^+(L_6, h^s_{2p^2})$ *is of finite index in* $\mathrm{O}^+(L_6, h^s_2)$ *and*

   $$|\,\mathrm{O}^+(L_6, h^s_2) : \mathrm{O}^+(L_6, h^s_{2p^2})| \leq 16(p^5 + p^2).$$

   Here we show that $\Gamma_{2p^2}$ is of finite index in $\Gamma_2$ and provide a bound on the index. The index estimate comes from studying the action of the orthogonal group on a finite quadratic space, and using a classical result on the order of orthogonal groups of finite type in order to arrive at a final sharp bound. Such a problem was studied in [Kon93] and [Sca87] for the moduli of K3 surfaces, but their results were not effective. Effective bounds for the number of boundary components in the moduli of certain abelian surfaces were produced in [HKW93] but by using very different methods, which do not appear to generalise to our setting.

2. As explained in Section 3.7, if one is interested in proving general type statements

then only the non-canonical singularities are significant. By a result of [GHS07], the non-canonical part of the singular locus in the interior is contained in the fixed loci of certain involutions and 3-torsion elements. In Theorem 5.3.4, we determine the rational quadratic divisors containing these singularities by adapting a recent result of Boissière, Nieper-Wißkirchen, and Sarti [BNWS13].

**Theorem 5.3.4.** *If* $[w] \in \mathcal{F}_{L_{6,2p^2}}$ *is a non-canonical singularity,*

$$[w] \in \mathcal{D}^v_{L_{6,2p^2}} \subset \mathcal{D}_{L_{6,2p^2}}$$

*where* $\mathcal{D}^v_{L_{6,2p^2}}$ *is one of, at most,* $8(p^2+1)$ *rational quadratic divisors. The vector* $v$ *can be chosen to be of length* $\pm 2$ *or* $\pm 2p^2$*.*

3. In order to produce general type results by using the low-weight cusp form trick, one needs to understand when the forms constructed extend to a smooth model of $\mathcal{F}_{2p^2}$. We provide effective criteria for establishing whether or not a pluricanonical form extends over the interior obstructions in Theorem 5.4.3.

   **Theorem 5.4.3.** *If* $\Omega$ *is a* $\Gamma$*-invariant pluricanonical form on* $\mathcal{D}_{L_{6,2p^2}}$*, then* $\Omega$ *defines a pluricanonical form on a smooth model of* $\mathcal{F}_{L_{6,2p^2}}$ *if* $\Omega$ *vanishes to suitably high order over the pre-image of the obstructions under the map*

   $$\pi : \mathcal{D}_{L_{6,2p^2}} \to \mathcal{F}_{L_{6,2p^2}}.$$

   *Moreover, the order of vanishing required can be determined effectively.*

   This involves lengthy computer calculations (the results of which are given in Appendix B) involving the Reid-Tai criterion. We only consider the interior obstructions here, but these results could be extended to the singularities in the boundary.

4. In Theorem 5.5.3 we classify the possible singularities that can occur in the interior by using representation theoretic methods.

**Theorem 5.5.3.** *Around $[w] \in \mathcal{F}_{L_{6,2p^2}}$, the space $\mathcal{F}_{L_{6,2p^2}}$ is locally isomorphic to $\mathbb{C}^4/G$ where $G \leq \mathrm{GL}(4,\mathbb{C})$ and $G \cong G_1 \times G_2 \times G_3$ where $G_1$ is cyclic, and $G_2$ and $G_3$ are binary polyhedral groups. Every element in $G$ has order not exceeding 56 and the action of $G$ on $\mathbb{C}^4$ is given precisely by the degree 4 representations of $G$, which can be deduced from Appendix A.*

Lastly, we study the geometry and combinatorics of the boundary.

5. In Theorem 6.4.3 we count number of rank 2 boundary components in $\mathcal{F}(\Gamma_{2p^2})$.

   **Theorem 6.4.3.** *The modular variety $\mathcal{F}_\Gamma$ has at most $320(p^5+p^2)$ rank 2 boundary components.*

   The problem of counting the number of boundary components in a modular variety was studied for the moduli of abelian surfaces in [HKW93] and the moduli of K3 surfaces in [Sca87]. We restrict our attention to the rank 2 boundary components as, by Theorem 6.2.1, these are the important for the purposes of proving general type results. Besides being intrinsically interesting, the boundary components can all impose conditions on the space of extensible modular forms. The number of conditions imposed by the boundary depends on the number of boundary components, and so one may need an estimate of these in order to provide dimension formulae. Our approach involves counting isotropic planes in $L_{6,2}$ before using the index estimate of Theorem 4.0.11.

6. In Theorem 6.5.3, we provide bounds on the number of components of the singular locus in a rank 2 boundary component.

   **Theorem 6.5.3.** *If $(a_1, a_1a_2) = (1,1)$ the singular locus of a boundary component contains of the order of $p^6$ points and $p^5$ lines. The number of surfaces in the boundary component does not depend on $p$. If $(a_1, a_1a_2) = (1, 2p)$ the singular locus of a boundary component contains of the order of $p^{14}$ points, $p^{12}$ lines, and $p^9$ surfaces.*

(The pair of integers $(a_1, a_1a_2)$ depends on the choice of boundary component and is explained in Lemma 6.3.1.) Here, we take a very different approach to the study of the singularities in the interior. We describe the neighbourhood of each rank 2 boundary component explicitly, in terms of coordinates, as the quotient of a toric variety by the action of an arithmetic group, and we obtain equations for the fixed points. These equations are solved by studying two classical objects: the congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ and the automorphisms groups of definite integral binary quadratic forms. We find that the components of the fixed locus correspond to points in a lattice. We let the group act on the lattice and show that the solutions can be taken to lie inside a box. The final step involves counting the number of points inside the box.

We think that it is worthwhile to end by comparing these results to the moduli of abelian surfaces, in which most of the above problems have been solved in a pleasingly exact way. (For example, [Bra95] [HKW93] [San97].) The moduli space of polarised $(1,t)$ abelian surfaces is the quotient

$$\mathcal{A}_t = \Gamma_t \backslash \mathbb{H}_2$$

where

$$\Gamma_t = \left\{ \gamma \in \mathrm{Sp}(2,\mathbb{Q}) \mid \gamma \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t\mathbb{Z} \\ t\mathbb{Z} & \mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t\mathbb{Z} \\ \mathbb{Z} & \frac{1}{t}\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}.$$

But because of the 2:1 morphism

$$\mathrm{Sp}(2,\mathbb{R}) \to \mathrm{SO}(2,3)$$

there is no harm in thinking of $\mathcal{A}_t$ as an orthogonal modular variety (see [GH98] for an explict example of this sort of construction). With this picture in mind, the framework we use to study the orthogonal modular varieties associated with deformation generalised Kummer varieties can at once be applied to the moduli of abelian surfaces. The essential difference between the two is that all of the lattice classification problems that we encounter for the orthogonal modular varieties associated with deformation generalised Kummer varieties are completely trivial for the moduli of abelian surfaces.

Chapter 4

# Moduli of generalised Kummer varieties

In this chapter, we study the modular group $\Gamma = \mathrm{O}^+(L_{2n}, h_d) \leq \mathrm{O}^+(h_d^\perp)$ of a family of deformation generalised Kummer manifolds. In order to do so, we must first classify polarisation types. The classification of polarisation types for deformation generalised Kummer manifolds is essentially identical to the classification for manifolds of $K3^{[2n]}$ type as the two Beauville lattices differ only by a factor of $2E_8(-1)$. The classification for the $K3^{[n]}$ case is given in [GHS10], and so we omit the details here.

**Proposition 4.0.1.** *If $h_d \in L_{2n}$ is primitive of length $2d > 0$ with $\mathrm{div}(h_d) = f$. Let $g = \left(\frac{2n}{f}, \frac{2d}{f}\right)$, $w = (g, f)$, $g = wg_1$, $f = wf_1$. Then $2n = fgn_1 = w^2 f_1 g_1 n_1$ and $2d = fgd_1 = w^2 f_1 g_1 d_1$ where $(n_1, d_1) = (f_1, g_1) = 1$.*

1. *If $g_1$ is even then $h_d$ exists if and only if $(d_1, f_1) = (f_1, n_1) = 1$ and $d_1/n_1$ is a quadratic residue modulo $f_1$. Moreover, the number of $\widetilde{\mathrm{O}}(L_{2(n-1)})$-orbits of $h_d$ with fixed $f$ is equal to $w_+(f_1)\phi(w_-(f_1)).2^{\rho(f_1)}$ where $w = w_+(f_1)w_-(f_1)$ and $w_+(f_1)$ is the product of all powers of primes dividing $(w, f_1)$, $\rho(n)$ is the number of prime factors of $n$ and $\phi(n)$ is the Euler function.*

2. *if $g_1$ is odd and $f_1$ is even or $f_1$ and $d_1$ are both odd, then such an $h_d$ exists if and only if $(d_1, f_1) = (t_1, 2f_1) = 1$ and $-d_1/n_1$ is a quadratic residue modulo $2f_1$. The number of $\widetilde{\mathrm{O}}(L_{2n})$ orbits is equal to $w_+(f_1)\phi(w_-(f_1)).2^{\rho(f_1/2)}$ if $f_1$ is even. and $w_+(f_1)\phi(w_-(f_1)).2^{\rho(f_1)}$ if $f_1$ and $d_1$ are both odd.*

3. *If $g_1$ and $f_1$ are both odd and $d_1$ is even, then such an $h_d$ exists if and only if $(d_1, f_1) = (n_1, 2f_1) = 1$, $-d_1/(4t_1)$ is a quadratic residue modulo $f_1$ and $w$ is odd. The number of $\widetilde{\mathrm{O}}(L_{2n})$-orbits of such an $h_d$ is equal to $w_+(f_1)\phi(w_-(f_1)).2^{\rho(f_1)}$.*

4. *If $c \in \mathbb{Z}$, determined modulo $f$ satisfies $(c, f) = 1$ and $b = (d + c^2 n)/f^2$ then*

$$(h_d)^{\perp}_{L_{2n}} \cong 2U \oplus B$$

*where* $B = \begin{pmatrix} -2b & c\frac{2n}{f} \\ c\frac{2n}{f} & -2t \end{pmatrix}$.

*Proof.* See [GHS10]. □

**Corollary 4.0.2.** *If $w = 1$ and if there exists a primitive vector $h_d \in L_{2n}$ such that $h_d^2 = 2d$ and $\mathrm{div}(h_d) = f$, then all vectors belong to the same $\widetilde{\mathrm{O}}(L_{2n})$-orbit.*

**Corollary 4.0.3.** *If $f = 1$, then for any $n$ and $d$, there is only one $\widetilde{\mathrm{O}}(L_{2n})$ orbit of primitive vectors $h_d$ with $\mathrm{div}(h_d) = 1$. Moreover, $c = 0$ and so*

$$(h_d)^{\perp}_{L_{2n}} \cong 2U \oplus \langle -2(n+1) \rangle \oplus \langle -2d \rangle$$

We define the lattice $L_{2n,2d}$ by

$$L_{2n,2d} = 2U \oplus \langle -2n \rangle \oplus \langle -2d \rangle.$$

**Definition 4.0.4.** *A polarisation determined by a primitive vector $h_d \in L_{2n}$ is called split if $\mathrm{div}(h_d) = 1$ and non-split otherwise. If a primitive vector $h_d \in L_{2n}$ is split, we indicate this by writing $h_d^s$ instead of $h_d$.*

We shall consider a family of deformation generalised Kummer 4-folds with split polarisation of degree $2p^2$ where $p > 3$ is prime. Because of Corollary 4.0.3, this choice of split polarisation is an extremely natural one to study.

Later on, we shall be interested in determining the singular locus of $\mathcal{F}_{2d}$ and this involves studying lattice embeddings. In the general case, we are led to arithmetic problems regarding the classification of definite lattices, and these make the locus difficult to describe. However with the assumption that $d = 2p^2$, we can use an idea of Kondō [Kon93] and regard

$$2U \oplus \langle -6 \rangle \oplus \langle -2p^2 \rangle \leq 2U \oplus \langle -6 \rangle \oplus \langle -2 \rangle$$

and

$$\mathrm{O}^+(L_6, h_{2p^2}) \leq \mathrm{O}^+(L_6, h_2).$$

This approach allows us to replace problems involving the classification of lattices with problems of a more combinatorial flavour, which admit a more exact solution. Geometrically, we can think of the inclusion $\mathrm{O}^+(L_6, h_{2p^2}) \leq \mathrm{O}^+(L_6, h_2)$ as corresponding to a finite cover $\mathcal{F}_{2p^2} \to \mathcal{F}_2$.

We now characterise the inclusion $\mathrm{O}(L_6, h^s_{2d}) \leq \mathrm{O}(L_{6,2d})$. We start by outlining the general theory for $\mathrm{O}(L, S) = \{g \in \mathrm{O}(L) \mid g_{|S} \in \widetilde{\mathrm{O}}(S)\}$ where $S \leq L$ is primitive. As explained in [Nik79b], the inclusion $S \subset L$ defines the series of overlattices

$$S^\perp \oplus S < L < L^\vee < (S^\perp)^\vee \oplus S^\vee.$$

The overlattice $S^\perp \oplus S$ is defined by the isotropic subgroup $H = L/(S^\perp \oplus S)$ and because

$$H = L/(S^\perp \oplus S) < (S^\perp)^\vee/S^\perp \oplus S^\vee/S = D(S^\perp) \oplus D(S),$$

$H$ can be regarded as a subgroup of $D(S^\perp) \oplus D(S)$. We can then define the projections $p_S : H \to D(S)$ and $p_{S^\perp} : H \to D(S^\perp)$. Because of Lemma 4.0.5

**Lemma 4.0.5.** *[Nik79b] [GHS10] Let $S$ be a primitive sublattice in $L$. Then $g \in \mathrm{O}(L, S)$ if and only if $g(S) = S$, $\overline{g}|_{D(S)} = \mathrm{id}$ and $\overline{g}|_{p_{S^\perp}(H)} = \mathrm{id}$.*

we can prove the following theorem.

**Theorem 4.0.6.** *If $d > 2$, the group $\mathrm{O}(L_6, h^s_d) \leq \mathrm{O}(L_{6,2d})$ and*

$$\mathrm{O}(L_6, h^s_{2d}) \cong \{g \in \mathrm{O}(L_{6,2d}) \mid \overline{g} = \left(\begin{smallmatrix} * & 0 \\ * & 1 \end{smallmatrix}\right) \in \mathrm{O}(D(L_{6,2d}))\}$$

*Moreover, if $p$ is an odd prime,* $\mathrm{O}(L_6, h^s_{2p^2}) \le \mathrm{O}(L_{6,2})$.

*Proof.* The first part of the argument is essentially a specialisation of Part (i) of Proposition 3.12 in [GHS10].

We can at once consider $\mathrm{O}(L_6, h_d)$ as a subgroup of $\mathrm{O}((h_d^\perp)_{L_{6,2d}})$ because $\mathrm{O}(L_6, h_d)$ acts on both $\langle h_d \rangle$ and $h_d^\perp$ in

$$\langle h_d \rangle \oplus \langle h_d \rangle^\perp \le L_6$$

but acts trivially on $\langle h_d \rangle$ (as $D(\langle h_d \rangle) \cong C_d \ne C_2$).

If $h_d \in L_6$ is split then, by Lemma 4.0.1, we can take an $\widetilde{\mathrm{O}}(L_6)$ representative of $h_d$ to be $h_d = e_3 + bf_3 = e_3 + df_3 \in U \oplus \langle -6 \rangle$. If $k_1 = e_3 - df_3$ and $k_2 = l_6$, then a basis for $(h_d^\perp)^\vee$ is given by $\{e_1, f_1, e_2, f_2, k_1', k_2', k_3'\}$ where $k_1' = \frac{k_1}{2d}$, $k_2' = \frac{k_2}{6}$ and $k_3' = \frac{h_d}{2d}$. where $l_6$ is a generator of the $\langle -6 \rangle$ factor in $L_6 = 3U \oplus \langle -6 \rangle$. Consider

$$\langle h_d \rangle \oplus h_d^\perp < L_6 < L_6^\vee < \langle h_d^\vee \rangle \oplus (h_d^\perp)^\vee$$

where $h_d^\vee = \frac{1}{2d} h_d$ and $h_d^\perp := (h_d)^\perp \subset L_6$ is given by $h_d^\perp \cong 2U \oplus \langle -2d \rangle \oplus \langle -6 \rangle$. A simple calculation shows that the subgroup

$$H = L_6/(\langle h_d \rangle \oplus h_d^\perp) < D(\langle h_d \rangle) \oplus D(h_d^\perp)$$

is equal to $\langle k_3' - k_1', d(k_1' + k_3') \rangle \le L_6/(\langle h_d \rangle \oplus h_d^\perp)$, and so $p_{h_d^\perp}(H) = \langle k_1' \rangle$.

By Lemma 4.0.5 and because $D(h_d^\perp) = \langle k_1' \rangle \oplus \langle k_2' \rangle$,

$$\mathrm{O}(L_6, h_d) \cong \{ g \in \mathrm{O}(h_d^\perp) \mid \overline{g}|_{p(H)} = \mathrm{id} \}$$

and we obtain the first part of the claim.

For the second part of the claim, let $L_{6,2p^2}$ and $L_{6,2}$ have bases $\{e_1, f_1, e_2, f_2, v_1, v_2\}$ and $\{e_1', f_1', e_2', f_2', v_1', v_2'\}$ where $\{e_i, f_i\}$, $\{e_i', f_i'\}$ are the standard bases for $U$ and $v_1$ and $v_1'$ are generators for the copies of $\langle -6 \rangle$ in $L_{6,2p^2}$ and $L_{6,2}$, respectively and $v_2$ and $v_2'$ are generators for the copies of $\langle -2p^2 \rangle$ and $\langle -2 \rangle$ in $L_{6,2p^2}$ and $L_{6,2}$, respectively. Define the embedding $L_{6,2p^2} \le L_{6,2}$ by $(e_1, f_1, e_2, f_2, v_1, v_2) \mapsto (e_1', f_1', e_2', f_2', v_1', pv_2)$ and define the totally isotropic subspace $M$ by

$$M = L_{6,2}/L_{6,2p^2} \leq D(L_{6,2p^2}).$$

We can recover $L_{6,2}$ from $M$ by noting that

$$L_{6,2} = \{x \in L_{6,2p^2}^\vee \mid x \bmod L_{6,2p^2} \in M\}.$$

Moreover, $M$ is of the form $(0,*) \in D(L_{6,2p^2}) = \langle k_2' \rangle \oplus \langle k_1' \rangle$. The element

$$g \in \mathrm{O}(L_6, h_{2p^2}) \leq \mathrm{O}(L_{6,2p^2})$$

extends naturally to an element $\hat{g} \in \mathrm{O}(L_{6,2p^2}^\vee)$ and because $g(k_1') = k_1'$, the element $\hat{g}$ preserves $M$. Therefore, $\hat{g}(L_{6,2}) \leq L_{6,2}$ and so $g$ extends to a unique element in $\mathrm{O}(L_{6,2})$. $\square$

**Corollary 4.0.7.** *If $p$ is an odd prime and $h_{2p^2} \in L_6$ is split, then*

$$\widetilde{\mathrm{O}}^+(L_{6,2p^2}) \leq \mathrm{O}^+(L_6, h_{2p^2}).$$

We next use an idea in [Kon93] to show that $\mathrm{O}(L_6, h_{2p^2})$ is of finite index in $\mathrm{O}(L_{6,2})$ by considering the action of $\mathrm{O}(L_{6,2})$ on a finite quadratic space. We begin by outlining some classical results on the orthogonal groups of finite type. These can be found in [Die71], for example.

A non-degenerate quadratic space $V$ over a finite field $\mathbb{F}_q$ of odd characteristic is classified in terms of $\dim V$ and the discriminant $\Delta = \det B \in \mathbb{F}_q^*/(\mathbb{F}_q^*)^2$, where $B$ is the bilinear form on $V$.

If $\dim V = 2m$, then $V$ falls into one of two isomorphism classes depending on the value of $\epsilon := (-1)^m \Delta \in \mathbb{F}_q^*/(\mathbb{F}_q^*)^2$. They are:

$$V_\epsilon^{2m} = H_1 \oplus \ldots \oplus H_m \qquad \text{if } \epsilon = 1$$
$$V_\epsilon^{2m} = V_\theta \oplus H_1 \oplus H_2 \oplus \ldots \oplus H_{m-1} \qquad \text{if } \epsilon = -1.$$

Here, $H_i$ are hyperbolic planes over $\mathbb{F}_p$ and $V_\theta$ is the quadratic space $\langle u, v\rangle_{\mathbb{F}_p}$ whose bilinear form is given by $(u,u) = 1$, $(u,v) = 0$ and $(v,v) = \theta$ for some $-\theta \notin (\mathbb{F}_q^*)^2$.

If $\dim V = 2m+1$, there is only one isomorphism class for $V$, which is given by $V^{2m+1} = H_1 \oplus \ldots H_m \oplus \langle\theta\rangle$ for some $0 \neq \theta \in \mathbb{F}_q$.

We show that $\mathrm{O}(L_{6,2p^2})$ is of finite index in $\mathrm{O}(L_{6,2})$ by considering the action of $\mathrm{O}(L_{6,2p^2})$ on the finite quadratic space $Q_p$, where

$$Q_p := L_{6,2}/pL_{6,2} \leq L_{6,2p^2}/pL_{6,2}.$$

In order to do so, we need to show that $\mathrm{O}(L_{6,2})$ acts transitively on $Q_p$. We remark that this is not immediate from Witt's theorem [Asc00] as it is not clear that $\mathrm{O}(L_{6,2}) \to \mathrm{O}(Q_p)$ is surjective. In order to show transitivity, we shall use the following two lemmas to construct elements of $\mathrm{O}(L_{6,2p^2})$.

**Definition 4.0.8.** *Let $L$ be an indefinite lattice. If $e \in L$ is isotropic and $a \in e^\perp \subset L$ then the map on $L$ defined by*

$$t(e,a) : v \mapsto v - (a,v)e + (e,v)a - \frac{1}{2}(a,a)(e,v)e$$

*is called an Eichler transvection and belongs to the group $\widetilde{\mathrm{SO}}^+(L)$ (see also [GHS09]).*

**Lemma 4.0.9.** *The group $\widetilde{\mathrm{SO}}^+(2U)$ is isomorphic to* $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$.

*Proof.* Full details of the proof may be found in [GHS09]. The isomorphism is defined by mapping $(w,x,y,z) \in U \oplus U$ to $\left(\begin{smallmatrix} w & -y \\ z & x \end{smallmatrix}\right) \in M_2(\mathbb{Z})$, where $(w,x,y,z)$ is given on the standard basis for $U \oplus U$. The inner product on $U \oplus U$ is defined by the determinant on $M_2(\mathbb{Z})$. An element $(A, B) \in \mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$ acts on $U \oplus U$ by mapping

$$\begin{pmatrix} w & -y \\ z & x \end{pmatrix} \mapsto A \begin{pmatrix} w & -y \\ z & x \end{pmatrix} B.$$

□

**Lemma 4.0.10.** *The group* $\mathrm{O}(L_{6,2})$ *acts transitively on hyperplanes of the same type in* $Q_p$.

*Proof.* Let $\{e_1, f_1, e_2, f_2, v_1, v_2\}$ be a basis for $L_{6,2} = 2U \oplus \langle -6 \rangle \oplus \langle -2 \rangle$ where $v_1$ and $v_2$ generate $\langle -6 \rangle$ and $\langle -2 \rangle$, respectively and $\{e_i, f_i\}$ are the standard basis for $U$. If $w = (w_1, w_2, w_3, w_4, w_5, w_6) \in L_{6,2}$ then the Eichler transvections $t(e_1, v_1)$ and $t(e_1, v_2)$ act as

$$t(e_2, v_1)w = (w_1, w_2, w_3 + 3w_4 + 6w_5, w_4, w_5 + w_4, w_6)$$

and

$$t(e_2, v_2)w = (w_1, w_2, w_3 + w_4 + 2w_6, w_4, w_5, w_6 + w_4).$$

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in L_{6,2}/pL_{6,2}$ be non-zero. We can assume that $x_4 \neq 0$ by (if required) applying $t(e_2, v_1)$ and permuting $\{x_1, x_2, x_3, x_4\}$ by elements in $\mathrm{O}(2U)$. Rescale $x$ so that $x_4 = 1$. After repeated application of $t(e_2, v_1)$ and $t(e_2, v_2)$, we can transform $x$ to an element of the form $(*, *, *, *, 0, 0)$ and thereby identify $x$ with an element of $2U$. By using the copy of $\mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$ in $\mathrm{O}(2U)$, we can send $x$ to an element of the form $(r, s, 0, 0, 0, 0)$ and then, after rescaling, to an element of the form $(1, a, 0, 0, 0, 0)$. Now suppose that $u, v \in L_{6,2}/pL_{6,2}$ are such that $u = (1, a, 0, 0, 0, 0)$ and $v = (1, b, 0, 0, 0, 0)$. If $ab^{-1} \in (\mathbb{F}_p^*)^2$ then there exists $\mu, \lambda \in \mathbb{F}_p$ such that $(\mu u)^2 = (\lambda v)^2$. Let $\hat{u} := \mu u = (u_1, u_2, 0, 0, 0, 0)$ and $\hat{v} := \lambda v = (v_1, v_2, 0, 0, 0, 0)$ and suppose that $\hat{u} - \hat{v} = (r, s, 0, 0, 0, 0)$ is non-zero. Let $d = \gcd(r, s)$ and let $r_1, r_2, s_1, s_2$ be solutions to

$$r_2 u_1 + r_1 u_2 = d$$

$$s_2 v_1 + y_2 v_2 = d$$

and let

$$u' := (r_1, r_2, 0, 0, 0, 0) \in L_{6,2}$$
$$v' := (s_1, s_2, 0, 0, 0, 0) \in L_{6,2}$$
$$w := (\frac{r}{d}, \frac{s}{d}, 0, 0, 0, 0) \in L_{6,2}$$

be lifts to $L_{6,2}$ such that $u', v', w \in e_1^{\perp} \cap f_1^{\perp} \subset L_{6,2}$. Then, over $\mathbb{F}_p$, $(\hat{u}, u') = d$ and $(\hat{v}, v) = d$ and so the element $t(e_2, v')t(f_2, w)t(e_2, u')$ sends $\hat{u}$ to $\hat{v}$. Therefore, $\mathrm{O}(L_{6,2})$ is transitive on hyperplanes of the same type in $L_{6,2}/pL_{6,2}$. $\square$

We shall also need to know about the order of $\mathrm{O}^+(V)$ for a finite quadratic space $V$. As in [Die71], these are given by

$$|\,\mathrm{O}^+(V^{2m+1})| = (q^{2m} - 1)q^{2m-1}(q^{2m-2} - 1)\dots(q^2 - 1)q \tag{4.1}$$

and

$$|\,\mathrm{O}^+(V_{\epsilon}^{2m})| = (q^{2m-1} - \epsilon q^{m-1})(q^{2m-2} - 1)q^{2m-3}\dots(q^2 - 1)q. \tag{4.2}$$

We can now prove our main theorem.

**Theorem 4.0.11.** *The group* $\mathrm{O}^+(L_6, h^s_{2p^2})$ *is of finite index in* $\mathrm{O}^+(L_6, h^s_2)$ *and*

$$|\,\mathrm{O}^+(L_6, h^s_2) : \mathrm{O}^+(L_6, h^s_{2p^2})| \leq 16(p^5 + p^2).$$

*Proof.* There are natural homomorphisms from $\mathrm{O}(L_{6,2}) \to \mathrm{O}(L_{6,2}/pL_{6,2})$ and by Lemma 4.0.10, $\mathrm{O}(L_{6,2})$ acts transitively on $Q_p$ and so $\mathrm{O}(L_{6,2}/pL_{6,2})$ also acts transitively on $Q_p$. The group $\mathrm{O}(L_6, h_{2p^2}) \leq \mathrm{O}(L_{6,2})$ stabilises a hyperplane $\Pi \subset Q_p$ and so, by the Orbit-Stabiliser theorem,

$$|\,\mathrm{O}(L_{6,2}) : \mathrm{stab}_{\mathrm{O}(L_{6,2})}(\Pi)| = |\,\mathrm{O}(L_{6,2}/pL_{6,2}) : \mathrm{O}(L_{6,2p^2}/pL_{6,2})|$$

and

$$|\,\mathrm{O}^+(L_{6,2}) : \mathrm{stab}_{\mathrm{O}^+(L_{6,2})}(\Pi)| = |\,\mathrm{O}^+(L_{6,2}/pL_{6,2}) : \mathrm{O}^+(L_{6,2p^2}/pL_{6,2})|$$

(where we have used the fact that $\mathrm{stab}_{\mathrm{O}(L_{6,2}/pL_{6,2})}(\Pi) = \mathrm{O}(L_{6,2p^2}/pL_{6,2})$ and the fact that the spinor kernel is of index two in the full orthogonal group). By Lemma 4.0.6, $\mathrm{O}(L_6, h^s_{2p^2}) \leq \mathrm{O}(L_{6,2})$ and so

$$\widetilde{\mathrm{O}}^+(L_{6,2p^2}) \leq \mathrm{O}^+(L_6, h^s_{2p^2}) \leq \mathrm{stab}_{\mathrm{O}^+(L_{6,2})}\,\Pi \leq \mathrm{O}^+(L_{6,2p^2}).$$

As $\mathrm{O}(D(L_{6,2p^2})) \cong V_4 \oplus C_2 \oplus C_2$ where $V_4$ is the Klein 4-group,

$$|\,\mathrm{stab}_{\mathrm{O}(L_{6,2})}\,\Pi : \mathrm{O}(L_6, h^s_{2p^2})| \leq |\,\mathrm{O}(L_{6,2p^2}) : \widetilde{\mathrm{O}}(L_{6,2p^2})| = 16$$

and therefore

$$\begin{aligned}
|\,\mathrm{O}^+(L_{6,2}) : \mathrm{O}^+(L_6, h^s_{2p^2})| \qquad & \leq 16|\,\mathrm{O}^+(L_{6,2}/pL_{6,2}) : \mathrm{O}^+(L_{6,2p^2}/pL_{6,2})| \\
& \leq 16\frac{(p^5 - \epsilon p^2)(p^4-1)p^3(p^2-1)p}{(p^4-1)p^3(p^2-1)p} \\
& \leq 16(p^5 + p^2)
\end{aligned}$$

□

# Chapter 5

# Singularities in the interior

### 5.0.1 Introduction

As explained in the introduction, we need to understand the singularities of $\mathcal{F}_{L_{6,2p^2}}$ in order to prove general type results. Unfortunately, giving an exact description of the singularities of $\mathcal{F}_{L_{6,2p^2}}$ is a difficult problem. There are two ways to view the problem: as a problem in group theory, or as a problem in geometry. The group theoretic perspective takes the view that singularities of $\mathcal{F}_{L_{6,2p^2}}(\Gamma)$ correspond to finite subgroups in $\mathrm{O}(L_6, h^s_{2p^2})$ and that one should solve the problem by classifying conjugacy classes of such subgroups. The folklore approach to classifying elliptic elements is to construct a fundamental domain for the group and examine the stablisers at the boundary. This is the approach taken in [Got61a] and [Got61b] (see also [Uen72]) for the symplectic group $\mathrm{Sp}(4,\mathbb{Z})$ and finds application in the study of singularities in moduli spaces of abelian varieties [Bra95] [HKW91]. Unfortunately, it is usually difficult to exhibit a fundamental domain for an arithmetic group in a form that is suitable for calculation. Exact polyhedral fundamental domains exist for groups whose symmetric space is a cone [AMRT10], such as $\mathrm{SL}(2,\mathbb{Z})$, $\mathrm{Sp}(4,\mathbb{Z})$ or $\mathrm{O}^+(2,3)$, but outside of these cases, there are very few results.

The geometric perspective, which comes from considering the local Torelli theorem (Theorem 3.3.1), takes the view that the singularities of $\mathcal{F}_{L_{6,2p^2}}(\Gamma)$ correspond to deformation generalised Kummer varieties with a non-trivial automorphism group and that one should solve the problem by understanding automorphisms, such as in [Nik79a]. However, while the the study of automorphisms of irreducible symplectic manifolds is an active field of research, there are still many open problems.

Nevertheless, the situation improves if one is only interested in proving general type results, as results can be obtained from much weaker considerations. One only needs an estimate for the branch locus and some results on the generators of the finite quotient singularities that can occur.

We shall determine the singular locus by adapting a recent result of Boissière, Nieper-Wißkirchen, and Sarti. This involves a large amount of calculation, for which we apologise.

By results of Gritsenko, Hulek and Sankaran, the non-canonical part of the singular locus lie in the image of subvarieties of $\mathcal{D}_L$ the form

$$\mathcal{D}_L^{T_g} = \{[x] \in \mathcal{D}_L \mid (x, T_g) = 0\} = [S_g \otimes \mathbb{C}].$$

Therefore, in order to determine the singular locus, one only needs to determine the embeddings of $S_g$ in $L$. Because our intention has been to obtain results that are exact as possible, we have chosen to work with split polarisation of degree $2p^2$. Our methods, however, may be used for arbitrary polarisation if one is willing to accept weaker bounds. The classification of $S_g$ is unchanged, and one can provide a list of candidates for the genus of $T_g$ and thereby count the number of pairs $(S_g, T_g)$. One expects this to result in poorer bounds because it is more difficult to determine inclusions of the form $S_g \subset S_{g'}$.

However, in many cases $T_g$ is a negative definite binary quadratic form and one can provide bounds on the integers represented by $T_g$ (see, for example, §15.3 of [CS99]). Therefore, by using the Eichler criterion (Theorem 5.3.2), one can provide a list of rational quadratic divisors whose union contains the non-canonical locus.

We then use a result of Tai to show that the order of vanishing required to ensure extension can be effectively calculated by toric methods, and is independent of the degree of polarisation. We shall also briefly mention some results on the structure of the possible automorphisms groups of generalised Kummer varieties.

### 5.0.2 Singularities in orthogonal modular varieties

As discussed in Section 3.8, the modular varieties associated with Generalised Kummer manifolds can contain non-canonical singularities.

**Proposition 5.0.1.** *[GHS07] If $g \in G$ does not act as a quasi-reflection on $V$ and $r = 1$ or $r = 2$, then $\Sigma(g) \geq 1$.*

**Theorem 5.0.2.** *[GHS07] If $g \in G$ does not act as a quasi-reflection on $V$ and $n \geq 6$, then $\Sigma(g) \geq 1$.*

In fact, in the proof of the above theorem, a stronger result is proved: if $g$ is not a quasi-reflection, $\Sigma(g) \geq 1$ unless $\phi(r) = 2$ or 4. Moreover, if $\phi(r) = 4$, $\Sigma(g) \geq 1$ unless $n \leq 3$; and if $\phi(r) = 2$, $\Sigma(g) \geq 1$ unless $n \leq 5$. For our modular varieties, $n = 4$, and so we are left to consider $[w] \in \mathcal{D}_L$ fixed by a quasi-reflection or by an element of order 3, 4, or 6.

**Proposition 5.0.3.** *[GHS07] If $n > 2$, then the quasi-reflections on $V$ and hence the ramification divisors of*

$$\mathcal{D}_L \to \mathcal{F}_L(\Gamma)$$

*are given by elements $h \in \mathrm{O}(L)$ such that $\pm h$ is equal to a reflection $\sigma_v \in \mathrm{O}(L)$.*

## 5.1 Invariant and perp-invariant lattices in $\mathrm{O}(L_6, h_2^s)$

We modify a result of [BNWS13] to classify the invariant and perp-invariant lattices of certain elliptic elements in $\mathrm{O}(L_6, h_2^s) \leq \mathrm{O}(L_{6,2})$ in terms of $p$-elementary lattices. We are essentially following the approach taken in [BCS14] for automorphisms of Hyperkähler manifolds of $K3^{[2]}$ type. By combining this with the results of the previous section, we deduce that the non-canonical singularities are contained in certain rational quadratic divisors, which we determine.

**Definition 5.1.1.** *Let $L$ be a lattice and let $g \in \mathrm{O}(L)$ be an elliptic element (an element of finite order). We define the invariant lattice $T_g$ of $g$ to be*

$$T_g = \{x \in L \mid gx = x\}$$

*and the perp-invariant lattice $S_g$ to be*

$$S_g = T_g^{\perp} \subset L.$$

*Where no confusion is likely to arise, we drop the subscript $g$. We call the pair $S_g$ and $T_g$ the invariant lattices of $g$.*

**Lemma 5.1.2.** *Let $S$ be the perp-invariant lattice of an elliptic element $g \in \mathrm{O}(L_6, h_d^s)$. If $g'$ is the induced action on $h_d^{\perp}$, then $S_g = S_{g'}$.*

*Proof.* By definition, if $g \in \mathrm{O}(L_6, h_d^s)$ then $g(h_d) = h_d$ and so $h_d \in T_g$. Therefore, $S_g = (T_g)^{\perp} \subset h_d^{\perp} \subset L_6$. By definition, $S_{g'} = S_g \cap h_d^{\perp}$ and because $S_g$ and $S_{g'}$ are primitive and $S_g \subset h_d^{\perp}$, we conclude that $S_g = S_{g'}$. □

The following is essentially Lemma 4.3 in [BNWS13].

**Lemma 5.1.3.** *Suppose that $g \in \mathrm{O}(L)$ is of order $p$ where $2 \leq p \leq 19$ is prime. Then, for all $k$,*

$$\frac{L_6}{T_g(X) \oplus S_g(X)}$$

*is a $p-$torsion module. Moreoever, it is a trivial $g$-module.*

*Proof.* See [BNWS13]. $\square$

**Definition 5.1.4.** *By Lemma 5.1.3, there exists $a \in \mathbb{N}_0$ such that*

$$\frac{L_6}{S_g(X) \oplus T_g(X)} \cong C_p^a.$$

**Definition 5.1.5.** *Let $p$ be a prime. A lattice $L$ is called $p$-elementary if*

$$D(L) \cong C_p^a$$

*for some $a \in \mathbb{N}$.*

The $p$-elementary lattices are classified as follows.

**Theorem 5.1.6.** *[CS99] For $p \geq 3$, the distinct genera of even $p-$elementary lattices are given by*

$$II_{r,s}(p^{\pm k}) \text{ for } r - s \equiv \pm 2 - 2 - (p-1)k \text{ (mod 8)}$$

*but, when $k = n(= r + s)$, the sign must be $\left(\frac{-1}{p}\right)^s$. Moreover, if $n \geq 3$, each genus contains one spinor genus and therefore each genus contains one class.*

In the Lorentzian case, the sign consideration can be ignored.

**Theorem 5.1.7.** *[RS81] If $p > 2$, an even Lorentzian $p-$elementary lattice of rank $r$ is uniquely determined by the integer $a$. Moreover, an even Lorentzian $p-$elementary lattice with invariants $a$ and $r$ exists if and only if*

1. *$a \leq r$, $r \equiv 0 \mod 2$*
2. *If $a \equiv 0$ (2), $r \equiv 2 \mod 4$*
3. *If $a \equiv 1$ (2), $p \equiv (-1)^{r/2-1} \mod 4$.*

**Theorem 5.1.8.** *[Nik79b] Let $\delta_S \in \{0,1\}$. A $2-$elementary lattice with invariants $(\delta_S; t_+, t_-, a)$ exists if and only if the following conditions are satisfied:*

1. $t_+ + t_- \geq a$

2. $t_+ + t_- + a \equiv 0 \bmod 2$

3. $t_+ - t_- \equiv 0 \bmod 4$ *if* $\delta_S = 0$

4. $\delta_S = 0$, $t_- t_- \equiv 0 \bmod 8$ *if* $a = 1$

5. $\delta_S = 0$ *if* $a = 2$ *and* $t_+ - t_- \equiv 4 \bmod 8$

6. $t_+ - t_- \equiv 0 \bmod 8$ *if* $\delta_S = 0$ *and* $a = t_- + t_-$.

**Proposition 5.1.9.** *Let $g \in \mathrm{O}(L_6)$ be of order $p$. Then $D(S)$ is $p$-elementary. Moreover,*

1. *If $p = 2$, then $D(S) = C_2^{a+1}$ and $D(T) = C_3 \oplus C_2^a$ or $D(S) = C_2^a$ and $D(T) = C_3 \oplus C_2^{a+1}$.*

2. *If $p = 3$, then $D(S) = C_3^{a+1}$ and $D(T) = C_2 \oplus C_3^a$ or $D(S) = C_3^a$ and $D(T) = C_2 \oplus C_3^{a+1}$.*

3. *If $p > 3$, then $D(S) = C_p^a$ and $D(T) = C_2 \oplus C_3 \oplus C_p^a$.*

*Proof.* It is well known that (see [BHPVdV04] Chapter I.1)

$$|L_6 : S \oplus T|^2 = \mathrm{disc}(S).\,\mathrm{disc}(T).\,\mathrm{disc}(L)^{-1}$$

and so

$$\mathrm{disc}(T).\,\mathrm{disc}(S) = 6.p^{2a}.$$

Therefore,

$$\mathrm{disc}(S) = 2^{\delta} 3^{\epsilon} p^{\alpha}$$
$$\mathrm{disc}(T) = 2^{1-\delta} 3^{1-\epsilon} p^{\beta}$$

where $\alpha + \beta = 2a$ and where $\epsilon, \delta \in \{0, 1\}$. Because both $S$ and $T$ are primitive in $L_6$, by Proposition 1.4.1 of [Nik79b]

$$M = \frac{L_6}{S \oplus T} \subset D(T) \oplus D(S)$$

and so the projections $p_T : M \to D(T)$ and $p_S : M \to D(S)$ are $g$-equivariant monomorphisms. Therefore, $a \leq \alpha$ and $a \leq \beta$ and so $\alpha = \beta = a$.

We next examine the action of $g$ on $D(S)$. The possible cases for the pair $(D(S), D(T))$ are as follows:

$$D(S) = C_2 \oplus C_3 \oplus M \qquad D(T) = M \tag{5.1}$$

$$D(S) = C_3 \oplus M \qquad D(T) = C_2 \oplus M \tag{5.2}$$

$$D(S) = C_2 \oplus M \qquad D(T) = C_3 \oplus M \tag{5.3}$$

$$D(S) = M \qquad D(T) = C_2 \oplus C_3 \oplus M. \tag{5.4}$$

Let $x_2$ and $x_3$ be generators for the $C_2$ and $C_3$ factors of $D(S)$ in the above decompositions (if present). Note that $S$ is the kernel of $\sigma = 1 + g + \ldots + g^{p-1}$. If $p = 2$, then $g$ acts trivially on $x_3$ and so $\sigma(x_3) = 2x_3$. But, by assumption, $\sigma(x_3) = 0$, which is a contradiction as $x_3$ is of order 3 in $D(S)$. Therefore cases (5.1) and (5.2) cannot occur if $p = 2$. If $p = 3$, then $g$ acts trivially on $x_2$ and so $\sigma(x_2) = 3x_2$. But, by assumption, $\sigma(x_2) = 0$, which is a contradiction as $x_2$ is of order 2 in $D(S)$. Accordingly cases (5.1) and (5.3) cannot occur if $p = 3$. If $p > 3$, then $g$ acts trivially on $x_2$ and $x_3$ and so $\sigma(x_2) = px_2$ and $\sigma(x_3) = px_3$. But, by assumption, $\sigma(x_2) = \sigma(x_3) = 0$, which is a contradiction as $(2, p) = 1$ and $(3, p) = 1$ as $x_2$ and $x_3$ are of order 2 and 3 in $D(S)$, respectively. Accordingly, cases (5.1), (5.2) and (5.3) cannot occur.

We deduce that the only cases that can occur are:

$$\begin{array}{llll} D(S) = C_2 \oplus M,\ D(T) = C_3 \oplus M & \text{or} & D(S) = M,\ D(T) = C_2 \oplus C_3 \oplus M & \text{if } p = 2 \\ D(S) = C_3 \oplus M,\ D(T) = C_2 \oplus M & \text{or} & D(S) = M,\ D(T) = C_2 \oplus C_3 \oplus M & \text{if } p = 3 \\ D(S) = M,\ D(T) = C_2 \oplus C_3 \oplus M & & & \text{if } p > 3. \end{array}$$

Note that, in particular, $D(S)$ is always $p-$elementary. $\square$

### 5.1.1 The locus of non-canonical singularities

In order to prove general type results, we need only consider the non-canonical part of the singular locus of $\mathcal{F}_{L_{6,2p^2}}(\Gamma)$ and the branch divisor. By Theorem 5.0.2, we need only to consider the fixed locus of 3 and 4 torsion and special reflections. We begin by classifying the invariant lattices of such elements in $\mathrm{O}(L_{6,2})$ before considering $\mathrm{O}(L_{6,2p^2})$.

We need to classify the lattices in Lemma 5.1.9 before deciding which embed in $L_{6,2}$. The embeddings can be dealt with by 5.1.10, below.

**Theorem 5.1.10.** *[Nik79b] The primitive embeddings of a lattice $S$ into another lattice $M$ with $\mathrm{gen}(M) = (m_+, m_-, D(M))$ are determined by the sets $(H_S, H_M, \gamma; K, \gamma_K)$ where $K$ is a lattice, $H_S \subset D(S)$ and $H_M \subset D(M)$ are subgroups, $\gamma : q_S|H_S \to q_M|H_M$ is an isomorphism of finite quadratic forms. The lattice $K$ lies in the genus $\mathrm{gen}(K) = (m_+ - t_+, m_- - t_-, -\delta)$ where $\delta \cong (q_S \oplus (-q_M))|\Gamma_\gamma^\perp/\Gamma_\gamma$. The group $\Gamma_\gamma$ is the pushout of $\gamma$ in $D(S) \oplus D(M)$ and the map $\gamma_K : q_K \to (-\delta)$ is an isomorphism of finite quadratic forms.*

*Two sets $(H_S, H_M, \gamma; K, \gamma_K)$ and $(H'_S, H'_M, \gamma'; K', \gamma')$ determine isomorphism primitive sublattices if and only if $H_S$ and $H'_S$ are conjugate under some element in $\mathrm{O}(S)$.*

*For the primitive embeddings of $S$ determined by $(H_S, H_M, \gamma; K, \gamma_K)$, the lattice $K$ is isomorphic to the orthogonal complement of $S$.*

We note that Theorem 5.1.10 only provides the genus of the orthogonal complement $K$. In our applications $K$ is often of definite signature and so $\mathrm{gen}(K)$ need not coincide with $\mathrm{cls}(K)$. However, because of the following theorem, we can usually argue that $\mathrm{gen}(K)$ contains only one class.

**Theorem 5.1.11.** *[CS99] If $L$ is an indefinite lattice of rank $n$ and determinant $d$ then, if $L$ has more than one class in its genus, $|d| \geq d_0$ where $d_0$ is given by the following*

| | | | | |
|---|---|---|---|---|
| $n$ | 2 | 3 | $4,6,8\ldots$ | $5,7,9\ldots$ |
| $d_0$ | 17 | 128 | $5^{\binom{5}{2}}$ | $2.5^{\binom{n}{2}}$ |

### 5.1.2 Invariant lattices of 3-torsion

We classify $S_g$ for $g$ of order 3.

**Lemma 5.1.12.** *If $g \in \mathrm{O}(L_{6,2})$ is 3-torsion then the perp-invariant lattice $S$ is one of the following lattices: $A_2(\pm 1)$, $2A_2(-1)$, $U$, $U \oplus A_2(-1)$, $U(3) \oplus A_2(-1)$, $2U$, $U \oplus U(3)$, $A_2 \oplus A_2(-1)$.*

*Proof.* Suppose that $S$ is of signature $(r,s)$ and that $D(S) \cong C_3^a$. As $S \leq L_{6,2}$ we have $r \leq 2$ and $s \leq 4$. By Lemma 5.1.6, $r$ and $s$ must be solutions to $r-s \equiv \pm 2-2-2k \bmod 8$. We solve each for $k \leq r+s$ (as $k$ is the rank of the discriminant group). Moreover, because $T \leq M$ is of signature $(3-r, 4-s)$ and, by Proposition 5.1.9, has discriminant group $C_3^{a+1} \oplus C_2$ or $C_3^a \oplus C_2$ we have that $a \leq 7-(r+s)$, which allows us to exclude more cases. We note also that the case $(r,s)=(2,4)$ can be ignored because $S \leq L_{6,2}$ is primitive, $\operatorname{rank} S = \operatorname{rank} L_{6,2}$ and so $L_{6,2} = S$, but $L_{6,2}$ is not 3-elementary. We find that the only possibilities are

$$(r,s,a) \in \{(0,2,1),(0,4,2),(0,4,0),(1,1,0),(1,3,1),(1,3,3),(2,0,1),(2,2,0),(2,2,2)\}.$$

For the definite cases, we refer to tables in [CS99] and [Nip91].

For case $(0,2,1)$ there is precisely one such lattice: $A_2(-1)$. For case $(2,0,1)$ there is precisely one such lattice: $A_2$. For case $(0,4,0)$ there is no such lattice by the classification of unimodular lattices. For case $(0,4,2)$, we examine all integral quaternary quadratic forms of discriminant 9 and we find that there is precisely one: $2A_2$.

For the Lorentzian cases, because of Lemma 5.1.7 it suffices to find a representative for each $a$. For case $(1,1,0)$ there is precisely one such lattice: $U$. For case $(1,3,1)$

there is precisely one such lattice: $U \oplus A_2(-1)$. For case $(1,3,3)$ there is precisely one such lattice: $U(3) \oplus A_2(-1)$. For the signature $(2,-)$ cases, by Lemma 5.1.6 there are two genera for each $a > 0$.

For case $(2,2,0)$, there is precisely one such lattice: $2U$ (by the classification of unimodular lattices). For case $(2,2,2)$, $S$ is either $U \oplus U(3)$ or $A_2 \oplus A_2(-1)$. It is easy to see that $U \oplus U(3)$ and $A_2 \oplus A_2(-1)$ are inequivalent by considering their discriminant forms. $\square$

**Proposition 5.1.13.** *If $g \in \mathrm{O}(L_{6,2})$ is 3-torsion, then the invariant lattices of $g$ in $L_{6,2}$ are given by one of the following pairs:*

$$\begin{array}{ll}
S = U \oplus U & T = \langle -2 \rangle \oplus \langle -6 \rangle \\
S = A_2(-1) & T = A_2(-1) \oplus \langle -2 \rangle \oplus \langle -6 \rangle \\
S = 2A_2(-1) & T = \langle 2 \rangle \oplus \langle 6 \rangle \\
S = U \oplus A_2(-1) & T = \langle -2 \rangle \oplus \langle -6 \rangle \\
S = U \oplus U(3) & T = \langle -2 \rangle \oplus \langle -6 \rangle \\
S = A_2 \oplus A_2(-1) & T = \langle -2 \rangle \oplus \langle -6 \rangle.
\end{array}$$

*Proof.* By Lemma 5.1.12, $S \in \{A_2(\pm 1),\ 2A_2(\pm 1),\ U \oplus A_2(-1),\ U(3) \oplus A_2(-1),\ U \oplus U(3),\ A_2 \oplus A_2(-1),\ U(3) \oplus U(3)\}$. We calculate the embeddings by using Theorem 5.1.10. Throughout, we shall assume that $D(L_{6,2}) = ((1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3)$. Throughout, we shall refer to the canonical basis of an abelian group $C_{i_1} \oplus \ldots \oplus C_{i_k}$ by $\{e_1, \ldots, e_k\}$, $\{f_1, \ldots, f_k\}$ or $\{g_1, \ldots, g_k\}$.

### 5.1.3 $S = A_2$

If $S = A_2$, then $D(S) = ((-1/3), C_3)$ and so $H_S \cong \{0\}$ or $C_3$.

If $H_S \cong \{0\}$, then $\gamma : \underline{0} \mapsto \underline{0}$ and

$\Gamma_\gamma^\perp / \Gamma_\gamma = (q_S) \oplus (-q) = \delta = \Gamma_\gamma^\perp = ((1/2)^{\oplus 2} \oplus (-1/3) \oplus (1/3), C_2^{\oplus 2} \oplus C_3^{\oplus 2})$. Therefore, $\mathrm{gen}(T) = (0, 4, -\delta) = (0, 4, (1/2)^{\oplus 2} \oplus (1/3) \oplus (-1/3), C_2^{\oplus 2} \oplus C_3^{\oplus 2})$ and so $T$ is a quaternary quadratic form of determinant 36. By using tables in [Nip91], there are five negative definite quaternary quadratic forms of determinant 36:

$$T_1 := -\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 10 \end{pmatrix},$$
$$T_2 := -\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 10 \end{pmatrix},$$
$$T_3 := -\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix},$$
$$T_4 := -\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{pmatrix},$$
$$T_5 := -\begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}.$$

Of these, only $T_3$, $T_4$ and $T_5$ have discriminant group equal to $C_2^{\oplus 2} \oplus C_3^{\oplus 2}$. We find that

$$D(T_3) = ((-1/6)^{\oplus 2}, C_6^{\oplus 2})$$

which is clearly equivalent to $-\delta$; and

$$D(T_4) = (\begin{pmatrix} 1/3 & 1/2 \\ 1/2 & -1/3 \end{pmatrix}, C_6^{\oplus 2})$$

which is inequivalent to $-\delta$ because all order 2 elements are isotropic; and

$$D(T_5) = (\begin{pmatrix} -1/3 & 1/6 \\ 1/6 & -2/3 \end{pmatrix}, C_6^{\oplus 2})$$

which is also inequivalent to $-\delta$ because all order 2 elements are isotropic.

We conclude that $T = T_3 = A_2(-1) \oplus \langle -2 \rangle \oplus \langle -6 \rangle$.

If $H_S \cong C_3$, then $H_S$ is generated by $\pm e_1 \in D(S)$ and $\gamma : \pm e_1 \mapsto \pm f_3 \in D(L_{6,2})$. It is clear that all such $\gamma$ yield isomorphic $T$, so suppose $\gamma : e_1 \mapsto f_3$.

We find that $\Gamma_\gamma = \langle g_1 + g_4 \rangle$ and $\Gamma_\gamma^\perp = \langle x_1, x_2, x_3 \rangle$ where $x_1 = g_1 + g_4$, $x_2 = g_2$, $x_3 = g_3$ and so

$$\Gamma_\gamma^\perp / \Gamma_\gamma = \langle x_2, x_3 \rangle \cong C_2^{\oplus 2}$$

with form $((1/2)^{\oplus 2}, C_2^{\oplus 2}) = \delta$. Therefore $\mathrm{gen}(T) = (0, 4, (1/2)^{\oplus 2}, C_2^{\oplus 2})$. By using tables in [Nip91], we find that there is exactly one negative definite quaternary quadratic form of determinant 4:

$$T_6 := \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

but

$$D(T_6) = (\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, C_2^{\oplus 2})$$

which is totally isotropic, and therefore inequivalent to $-\delta$. We conclude that no such $T$ exists.

### 5.1.4 $S = A_2(-1)$

If $S = A_2(-1)$ then $D(S) = ((1/3), C_3)$ and $H_S \cong \{0\}$ or $C_3$.

If $H_S \cong \{0\}$, then $\gamma : \underline{0} \mapsto \underline{0}$ and

$\Gamma_\gamma^\perp/\Gamma_\gamma = (q_S) \oplus (-q) = \Gamma_\gamma^\perp = ((1/3)^{\oplus 2} \oplus (1/2)^{\oplus 2}, C_3^{\oplus 2} \oplus C_2^{\oplus 2}) = \delta$. Therefore $\mathrm{gen}(T) = (2, 2; -\delta) = ((-1/3)^{\oplus 2} \oplus (1/2)^{\oplus 2}, C_3^{\oplus 2} \oplus C_2^{\oplus 2})$. By Lemma 5.1.11, $T$ is unique in its genus and one checks that a representative is given by $A_2 \oplus \langle -2 \rangle \oplus \langle -6 \rangle$.

If $H_S \cong C_3$, then $H_S$ is generated by $\pm e_1$. Both are of length $1/3$ in $D(S)$ but $D(L_{6,2})$ has no order 3 element of length $1/3$, and so there is no embedding.

### 5.1.5 $S = 2A_2(-1)$

If $S = 2A_2(-1)$ then $D(S) = ((1/3)^{\oplus 2}, C_3^{\oplus 2})$ and $H_S \cong \{0\}$ or $C_3$.

If $H_S = \{0\}$, then $\gamma : \underline{0} \mapsto \underline{0}$ and

$\Gamma_\gamma^\perp/\Gamma_\gamma = (q_S) \oplus (-q) = \Gamma_\gamma^\perp = ((1/3)^{\oplus 3} \oplus (1/2)^{\oplus 2}, C_3^{\oplus 3} \oplus C_2^{\oplus 2}) = \delta$. Therefore $\mathrm{gen}(T) = (2, 0; -\delta)$ but a minimal generating set for $-\delta$ contains at least 3 generators, and so no such $T$ exists.

If $H_S \cong C_3$ then, up to $\mathrm{O}(S)$ equivalence, $H_S$ is generated by $e_1 + e_2 \in D(S)$ and so $\gamma : e_1 + e_2 \mapsto \pm f_3$.

If $H_S = \langle e_1 + e_2 \rangle \cong C_3$ and if

$\gamma : e_1 + e_2 \mapsto f_3$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4 \rangle$ where $x_1 = g_1 - g_2$, $x_2 = g_1 - g_5$, $x_3 = g_3$, $x_4 = g_4$ and $\Gamma_\gamma = \langle x_1 + x_2 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_1, x_3, x_4 \rangle \cong C_2^{\oplus 2} \oplus C_3 = \delta$$

with form $(-1/3) \oplus (1/2)^{\oplus 2}$.

If $\gamma : e_1 + e_2 \mapsto f_3$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4 \rangle$ where $x_1 = g_1 - g_2$, $x_2 = g_1 + g_5$, $x_3 = g_3$, $x_4 = g_4$ and $\Gamma_\gamma = \langle x_1 + x_2 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma = \langle x_1, x_3, x_4 \rangle \cong C_2 \oplus C_2^{\oplus 2} = \delta$$

with form $(-1/3) \oplus (1/2)^{\oplus 2}$.

In each case, $\text{gen}(T) = (2, 0, -\delta)$. By using tables in [CS99], we find that there are two even forms of determinant 12: $T_7 = \langle 2 \rangle \oplus \langle 6 \rangle$ and $T_8 = \left(\begin{smallmatrix} 4 & 2 \\ 2 & 4 \end{smallmatrix}\right)$. It is clear that $D(T_7) = -\delta$. However,

$$D(T_8) = (\begin{pmatrix} 0 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}, C_2 \oplus C_6)$$

which cannot be equivalent to $-\delta$ as all order 2 elements are isotropic.

Therefore, $T = \langle 2 \rangle \oplus \langle 6 \rangle$.

### 5.1.6 $S = U \oplus A_2(-1)$

If $S = U \oplus A_2(-1)$, $D(S) = ((1/3), C_3)$ and the maps $\gamma$, the groups $\Gamma_\gamma^\perp / \Gamma_\gamma$ and the finite quadratic forms on $\Gamma_\gamma^\perp / \Gamma_\gamma$ are the same as for the case $S = A_2(-1)$ and so $\text{gen}(T) = (1, 1; (-1/3)^{\oplus 2} \oplus (1/2)^{\oplus 2}, C_3^{\oplus 2} \oplus C_2^{\oplus 2})$. By referring to tables in [CS99], we find that there are four indefinite even rank 2 lattices with determinant 36: $T_9 = \left(\begin{smallmatrix} 0 & 6 \\ 6 & 0 \end{smallmatrix}\right)$, $T_{10} = \left(\begin{smallmatrix} 0 & 6 \\ 6 & 2 \end{smallmatrix}\right)$, $T_{11} = \left(\begin{smallmatrix} 0 & 6 \\ 6 & 4 \end{smallmatrix}\right)$ $T_{12} = \left(\begin{smallmatrix} 0 & 6 \\ 6 & 6 \end{smallmatrix}\right)$. Of these, only $T_9$ and $T_{12}$ have discriminant group equal to $C_2^{\oplus 2} \oplus C_3^{\oplus 2}$. One checks that

$$D(T_{12}) = (\begin{pmatrix} 0 & 1/6 \\ 1/6 & -1/6 \end{pmatrix}, C_6 \oplus C_6)$$

which, by examining the generators given by $3e_2$, $3e_1$, $2e_2$, $2e_1 + 2e_2$, is equivalent to

$$((1/2)^{\oplus 2} \oplus (1/3) \oplus (-1/3), C_2^{\oplus 2} \oplus C_3^{\oplus 2})$$

and therefore inequivalent to $-\delta$. One also checks that

$$D(T_9) = (\begin{pmatrix} 0 & 1/6 \\ 1/6 & 0 \end{pmatrix}, C_6^{\oplus 2})$$

which is inequivalent to $-\delta = ((-1/3)^{\oplus 2} \oplus (1/2)^{\oplus 2}, C_3^{\oplus 2} \oplus C_2^{\oplus 2}))$ because all order 2 elements in $(\begin{pmatrix} 0 & 1/6 \\ 1/6 & 0 \end{pmatrix}, C_6 \oplus C_6)$ are isotropic, which is not the case for $-\delta$. Therefore no such $T$ exists.

### 5.1.7 $S = U(3) \oplus A_2(-1)$

If $S = U(3) \oplus A_2(-1)$ then $D(S) = (\begin{pmatrix} 0 & 1/3 \\ 1/3 & 0 \end{pmatrix} \oplus (1/3), C_3^{\oplus 3})$ and $H_S \cong \{0\}$ or $C_3$.

If $H_S = \{0\}$, then $\gamma : \underline{0} \mapsto \underline{0}$ and

$\Gamma_\gamma^\perp/\Gamma_\gamma = (-q_S) \oplus (-q) = (\begin{pmatrix} 0 & 1/3 \\ 1/3 & 0 \end{pmatrix} \oplus (1/3) \oplus (1/2)^{\oplus 2} \oplus (1/3), C_3^{\oplus 3} \oplus C_2^{\oplus 2} \oplus C_3) = \delta$. Therefore, $\mathrm{gen}(T) = (1, 1, -\delta)$, but a minimal generating set for $-\delta$ contains at least 4 generators, and so no such $T$ exists.

If $H_S \cong C_3$, $H_S$ is generated by one of $\pm(e_1+e_2), \pm(e_1-e_2+e_3), \pm(e_1-e_2-e_3) \in D(S)$. The elements $e_1-e_2+e_3$ and $(e_1-e_2-e_3)$ are equivalent under $\mathrm{O}(S)$ and so we consider only $\pm(e_1+e_2), \pm(e_1-e_2+e_3)$.

If $H_S = \langle (e_1+e_2) \rangle \cong C_3$ and if

$\gamma : e_1 + e_2 \mapsto f_1$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5 \rangle$ where $x_1 = g_1 - g_2$, $x_2 = g_1 - g_6$, $x_3 = g_4$, $x_4 = g_5$, $x_5 = g_3$ and $\Gamma_\gamma = \langle x_1 + x_2 \rangle$. Therefore,

$$\Gamma_\gamma^\perp/\Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5 \rangle \cong C_3 \oplus C_2^{\oplus 2} \oplus C_3$$

with form $\delta = (1/3) \oplus (1/2)^{\oplus 2} \oplus (1/3)$. Therefore $\mathrm{gen}(T) = (1, 1, -\delta)$ and as for the case $S = U \oplus A_2(-1)$, no such $T$ can exist.

If $\gamma : e_1 + e_2 \mapsto -f_1$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5 \rangle$ where $x_1 = g_1 - g_2$, $x_2 = g_1 + g_6$, $x_3 = g_4$, $x_4 = g_5$, $x_5 = g_3$ and $\Gamma_\gamma = \langle x_1 + x_2 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5 \rangle \cong C_3 \oplus C_2^{\oplus 2} \oplus C_3 = \delta$$

with form $(1/3) \oplus (1/2)^{\oplus 2} \oplus (1/3)$. Therefore $\mathrm{gen}(T) = (1, 1, -\delta)$, and no such $T$ exists as in the case $\gamma : e_1 + e_2 \mapsto f_1$.

If $\gamma : e_1 - e_2 + e_3 \mapsto f_1$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5 \rangle$ where $x_1 = g_1 + g_2$, $x_2 = g_1 + g_3$, $x_3 = g_1 + g_6$, $x_4 = g_4$, $x_5 = g_5$ and $\Gamma_\gamma = \langle x_2 - x_1 + x_3 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5 \rangle \cong C_3^{\oplus 2} \oplus C_2^{\oplus 2} = \delta$$

with form $(1/3)^{\oplus 2} \oplus (1/2)^{\oplus 2}$. Therefore, $\mathrm{gen}(T) = (1, 1, -\delta)$, and no such $T$ exists as in the case $\gamma : e_1 + e_2 \mapsto f_1$.

If $\gamma : (e_1 - e_2 + e_3) \mapsto -f_1$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5 \rangle$, where $x_1 = g_1 + g_2$, $x_2 = g_1 + g_3$, $x_3 = g_1 - g_6$, $x_4 = g_4$, $x_5 = g_5$ and $\Gamma_\gamma = \langle x_2 - x_1 + x_3 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5 \rangle \cong C_3^{\oplus 2} \oplus C_2^{\oplus 2} = \delta$$

with form $(1/3)^{\oplus 2} \oplus (1/2)^{\oplus 2}$. Therefore, $\mathrm{gen}(T) = (1, 1, -\delta)$, and no such $T$ exists as in the case $\gamma : e_1 + e_2 \mapsto f_1$.

### 5.1.8 $S = U \oplus U(3)$

If $S = U \oplus U(3)$ then $D(S) = (\begin{pmatrix} 0 & 1/3 \\ 1/3 & 0 \end{pmatrix}, C_3^{\oplus 2})$ and $H_S \cong \{0\}$ or $C_3$.

If $H_S = \{0\}$, then $\gamma : \underline{0} \mapsto \underline{0}$ and

$\Gamma_\gamma^\perp = (q_S) \oplus (-q) = \delta = \left(\begin{smallmatrix} 0 & 1/3 \\ 1/3 & 0 \end{smallmatrix}\right) \oplus (1/2)^{\oplus 2} \oplus (1/3), C_3^{\oplus 2} \oplus C_2^{\oplus 2} \oplus C_3)$. Therefore $\mathrm{gen}(T) = (0, 2, -\delta)$ but a minimal generating set for $-\delta$ contains at least 3 generators, and so no such $T$ exists.

If $H_S \cong C_3$, then, up to $\mathrm{O}(S)$ equivalence, $H_S$ must be generated by $e_1 + e_2 \in D(S)$.

If $H_S = \langle e_1 + e_2 \rangle \cong C_3$

$\gamma : e_1 + e_2 \mapsto f_3$ then

$\Gamma_\gamma^\perp := \langle x_1, x_2, x_3, x_4 \rangle$ where $x_1 = g_1 - g_2$, $x_2 = g_1 - g_5$, $x_3 = g_3$, $x_4 = g_4$ and $\Gamma_\gamma = \langle x_1 + x_2$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4 \rangle \cong C_3 \oplus C_2^{\oplus 2} = \delta$$

with form $(1/3) \oplus (1/2)^{\oplus 2}$. Therefore, $\mathrm{gen}(T) = (0, 2, -\delta)$. By using tables in [CS99], $T$ must be one of $T_{13} = \langle -2 \rangle \oplus \langle -6 \rangle$ or $T_{14} = \left(\begin{smallmatrix} -4 & -2 \\ -2 & -4 \end{smallmatrix}\right)$. One checks that

$$D(T_{14}) = (\begin{pmatrix} 0 & -1/2 \\ -1/2 & -1/3 \end{pmatrix}, C_2 \oplus C_6)$$

which is inequivalent to $-\delta$ as all order 2 elements are isotropic. Therefore, $T = T_{13}$, which has discriminant form equal to $-\delta$.

If $\gamma : e_1 + e_2 \mapsto -f_3$ then

$\Gamma_\gamma^\perp := \langle x_1, x_2, x_3, x_4 \rangle$ where $x_1 = g_1 - g_2$, $x_2 = g_1 + g_5$, $x_3 = g_3$, $x_4 = g_4$ and $\Gamma_\gamma = \langle x_1 + x_2$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4 \rangle \cong C_3 \oplus C_2^{\oplus 2} = \delta$$

with form $(1/3) \oplus (1/2)^{\oplus 2}$. Therefore, $\mathrm{gen}(T) = (0, 2, -\delta)$. As in the case $\gamma : e_1 + e_2 \mapsto f_3$, we conclude that $T = \langle -2 \rangle \oplus \langle -6 \rangle$.

### 5.1.9 $S = A_2 \oplus A_2(-1)$

If $S = A_2 \oplus A_2(-1)$ then $D(S) = ((-1/3) \oplus (1/3), C_3^{\oplus 2})$ and $H_S \cong \{0\}$ or $C_3$.

If $H_S = \{0\}$ then $\gamma : \underline{0} \mapsto \underline{0}$ and

$\Gamma_\gamma^\perp = (q_S) \oplus (-q) = \delta = ((-1/3) \oplus (1/3) \oplus (1/2)^{\oplus 2} \oplus (1/3), C_3^{\oplus 2} \oplus C_2^{\oplus 2} \oplus C_3)$. Therefore $\mathrm{gen}(T) = (0, 2, -\delta)$ but a minimal generating set for $-\delta$ contains at least 3 generators, and so no such $T$ exists.

If $H_S \cong C_3$, $H_S = \langle \pm e_1 \rangle$. Both are equivalent under $\mathrm{O}(S)$.

If $H_S\langle e_1 \rangle \cong C_3$ and if

$\gamma : e_1 \mapsto f_3$, then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4 \rangle$ $x_1 = g_1 + g_5$, $x_2 = g_2$, $x_3 = g_3$, $x_4 = g_4$ and $\Gamma_\gamma = \langle x_1 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4 \rangle \cong C_3 \oplus C_2^{\oplus 2},$$

with form $(1/3) \oplus (1/2)^{\oplus 2}$. Therefore, $\mathrm{gen}(T) = (0, 2, -\delta)$. As in the case $S = U \oplus U(3)$, $H_S \cong C_3$, $\gamma : e_1 + e_2 \mapsto f_3$, we conclude that $T = \langle -2 \rangle \oplus \langle -6 \rangle$.

If $\gamma : e_1 \mapsto -f_3$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4 \rangle$ $x_1 = g_1 - g_5$, $x_2 = g_2$, $x_3 = g_3$, $x_4 = g_4$ and $\Gamma_\gamma = \langle x_1 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4 \rangle \cong C_3 \oplus C_2^{\oplus 2},$$

with form $(1/3) \oplus (1/2)^{\oplus 2}$. Therefore, $\mathrm{gen}(T) = (0, 2, -\delta)$. As in the case $S = U \oplus U(3)$, $H_S \cong C_3$, $\gamma : e_1 + e_2 \mapsto f_3$, we conclude that $T = \langle -2 \rangle \oplus \langle -6 \rangle$.

### 5.1.10 $S = 2U$

If $S = 2U$ then $D(S) = \{0\}$ and $H_S = \{0\}$. Therefore $\gamma : \underline{0} \mapsto \underline{0}$ and

$$\Gamma_\gamma^\perp / \Gamma_\gamma = (q_S) \oplus (-q) = \delta = ((1/2) \oplus (1/6), C_2 \oplus C_6).$$

Therefore, $\mathrm{gen}(T) = (0, 2; -\delta)$. By referring to tables in [CS99], $T$ is either $T_{15} := \langle -2\rangle \oplus \langle -6\rangle$ or $T_{16} := \left(\begin{smallmatrix} -4 & -2 \\ -2 & -4 \end{smallmatrix}\right)$. The lattice $T_{15}$ has discriminant form

$$D(T_{15}) = ((1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3)$$

and $T_{16}$ has discriminant form

$$D(T_{16}) = (\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3).$$

It is clear that $D(T_{16})$ is inequivalent to $((1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3)$ and so $T = T_{15}$.

### 5.1.11 $S = 2U(3)$

If $S = 2U(3)$, $D(S) = (\begin{pmatrix} 0 & 1/3 \\ 1/3 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1/3 \\ 1/3 & 0 \end{pmatrix}, C_3^{\oplus 4})$. Then $H_S \cong \{0\}$ or $C_3$.

If $H_S = \{0\}$ then $\gamma : \underline{0} \mapsto \underline{0}$ and

$\Gamma_\gamma^\perp/\Gamma_\gamma = (q_S) \oplus (-q) = \Gamma_\gamma^\perp/\Gamma_\gamma = C_3^{\oplus 4} \oplus C_2^{\oplus 2} \oplus C_3 = \delta$ with form $\begin{pmatrix} 0 & 1/3 \\ 1/3 & 0 \end{pmatrix}^{\oplus 2} \oplus (1/2)^{\oplus 2} \oplus (1/3)$ and $\mathrm{gen}(T) = (0, 2, -\delta)$ but a minimal generating set for $-\delta$ contains at least 5 generators, and so no such $T$ exists.

If $H_S \cong C_3$ then, up to $\mathrm{O}(S)$ equivalence, $H_S$ is generated by one of $e_1 + e_2$, $e_1 - e_2 + e_3 - e_4$, $e_1 + e_2 + e_3$.

If $H_S = \langle e_1 + e_2\rangle \cong C_3$ and if

$\gamma : e_1 + e_2 \mapsto e_3$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle$ where $x_1 = g_1 - g_2$, $x_2 = g_1 - g_7$, $x_3 = g_3$, $x_4 = g_4$, $x_5 = g_5$, $x_6 = g_6$ and $\Gamma_\gamma = \langle x_1 + x_2\rangle$. Therefore,

$$\Gamma_\gamma^\perp/\Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5, x_5\rangle \cong C_3^{\oplus 3} \oplus C_2 = \delta$$

with form $(1/3) \oplus \begin{pmatrix} 0 & 1/3 \\ 1/3 & 0 \end{pmatrix} \oplus (1/2)^{\oplus 2}$. Therefore, $\mathrm{gen}(T) = (0, 2, -\delta)$ but a minimal generating set for $-\delta$ contains at least 3 generators, and so no such $T$ exists.

If $\gamma : e_1 + e_2 \mapsto -f_3$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle$ where $x_1 = g_1 - g_2$, $x_2 = g_1 + g_7$, $x_3 = g_3$, $x_4 = g_4$, $x_5 = g_5$, $x_6 = g_6$ and $\Gamma_\gamma = \langle x_1 + x_2\rangle$. Therefore,

$$\Gamma_\gamma^\perp/\Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5, x_5\rangle \cong C_3^{\oplus 3} \oplus C_2 = \delta$$

with form $(1/3)\oplus\begin{pmatrix} 0 & 1/3 \\ 1/3 & 0\end{pmatrix}\oplus(1/2)^{\oplus 2}$. Therefore, $\mathrm{gen}(T) = (0, 2, -\delta)$ but a minimal generating set for $-\delta$ contains at least 3 generators, and so no such $T$ exists.

If $H_S = \langle e_1 + e_2 + e_3\rangle \cong C_3$ and if

$\gamma : e_1 + e_2 + e_3 \mapsto f_3$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle$ where $x_1 = g_1 - g_2$, $x_2 = g_1 - g_4$, $x_3 = g_1 - g_7$, $x_4 = g_3$, $x_5 = g_5$, $x_6 = g_6$ and $\Gamma_\gamma = \langle x_1 + x_3 - x_4\rangle$. Therefore,

$$\Gamma_\gamma^\perp/\Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5, x_6\rangle \cong C_3^{\oplus 3} \oplus C_2^{\oplus 2} = \delta.$$

Therefore, $\mathrm{gen}(T) = (0, 2, -\delta)$ but a minimal generating set for $-\delta$ contains at least 3 generators, and so no such $T$ exists.

If $\gamma : e_1 + e_2 + e_3 \mapsto -f_3$ then,

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle$ where $x_1 = g_1 - g_2$, $x_2 = g_1 - g_4$, $x_3 = g_1 + g_7$, $x_4 = g_3$, $x_5 = g_5$, $x_6 = g_6$ and $\Gamma_\gamma = \langle x_1 + x_3 - x_4\rangle$. Therefore,

$$\Gamma_\gamma^\perp/\Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5, x_6\rangle \cong C_3^{\oplus 3} \oplus C_2^{\oplus 2} = \delta.$$

Therefore, $\mathrm{gen}(T) = (0, 2, -\delta)$ but a minimal generating set for $-\delta$ contains at least 3 generators, and so no such $T$ exists.

If $H_S = \langle (1, -1, 1, -1)\rangle \cong C_3$ and if

$\gamma : e_1 - e_2 + e_3 - e_4 \mapsto f_3$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle$ where $x_1 = g_1 + g_2$, $x_2 = g_1 - g_3$, $x_3 =$

$g_1+g_4$, $x_4 = g_5$, $x_5 = g_6$, $x_6 = g_1+g_7$ and $\Gamma_\gamma^\perp = \langle x_1+x_2+x_3-x_6\rangle$. Therefore,

$$\Gamma_\gamma^\perp/\Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5, x_6\rangle \cong C_3^2 \oplus C_2^{\oplus 2} \oplus C_3 = \delta.$$

Therefore, $\mathrm{gen}(T) = (0, 2, -\delta)$ but a minimal generating set for $-\delta$ contains at least 3 generators, and so no such $T$ exists.

If $\gamma : e_1 - e_2 + e_3 - e_4 \mapsto -f_3$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle$ where $x_1 = g_1 + g_2$, $x_2 = g_1 - g_3$, $x_3 = g_1+g_4$, $x_4 = g_5$, $x_5 = g_6$, $x_6 = g_1-g_7$ and $\Gamma_\gamma^\perp = \langle x_1+x_2+x_3-x_6\rangle$. Therefore,

$$\Gamma_\gamma^\perp/\Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5, x_6\rangle \cong C_3^2 \oplus C_2^{\oplus 2} \oplus C_3 = \delta.$$

Therefore, $\mathrm{gen}(T) = (0, 2, -\delta)$ but a minimal generating set for $-\delta$ contains at least 3 generators, and so no such $T$ exists.

□

### 5.1.12 Invariant lattices of 4-torsion

We next turn out attention to 4-torsion. As we are only interested in determining the non-canonical part of the singular locus of $\mathcal{F}_L(\Gamma)$, because of the following lemma it suffices to examine only the perp-invariant lattice $S$ of $g^2$.

**Lemma 5.1.14.** *Suppose that $g \in \mathrm{O}(L_{6,2p^2})$ is 4-torsion and $[P] \in \mathrm{Fix}(g) \subset \mathcal{D}_{L_{6,2p^2}}$ is non-canonical. Then, $P \in S_{g^2} \otimes \mathbb{C} \subset L_{6,2p^2} \otimes \mathbb{C}$ or $[P]$ lies in the branch divisor.*

*Proof.* If $g \in \mathrm{O}(L_{6,2p^2})$ then, over $\mathbb{C}$, the action of $g$ on $L_{6,2p^2} \otimes \mathbb{C}$ decomposes into $V_1 \oplus V_2 \oplus V_3 \oplus V_4$ where $V_i$ is a (possibly empty) $\zeta^i$ eigenspace where $\zeta = e^{\pi i/2}$. Because of Lemma 5.0.1, if $[P] \in \mathrm{Fix}(g)$ is non-canonical, then $P \in V_1 \cup V_3$ or $[P]$ lies inside the brach divisor. □

We now classify the invariant and perp-invariant lattices $S_{g^2}$ and $T_{g^2}$ for 4-torsion

$g$.

**Lemma 5.1.15.** *If $g \in \mathrm{O}(L_{6,2p^2})$ is 4-torsion then $S_{g^2}$ is one of the following: $2\langle 2\rangle$, $U$, $U(2)$, $\langle 2\rangle \oplus \langle -2\rangle$, $U \oplus 2\langle -2\rangle$, $3\langle -2\rangle \oplus \langle 2\rangle$, $2U$, $U \oplus U(2)$, $2U(2)$, $2\langle -2\rangle \oplus 2\langle 2\rangle$, $2\langle -2\rangle$.*

*Proof.* If $g \in \mathrm{O}(L_{6,2p^2})$ is 4-torsion then, as a $g$-module,

$$L_{6,2p^2} \otimes \mathbb{Q} = \bigoplus_{i=0}^{2} \bigoplus_{j=0}^{a_i} \mathcal{V}_{2^i}$$

as $\dim \mathcal{V}_4 = 2$ and $\dim \mathcal{V}_2 = \dim \mathcal{V}_1 = 1$, the rank of $S_{g^2}$ is even (consider the $\mathcal{V}_4$ part). By Proposition 5.1.9, where $S_{g^2}$ and $T_{g^2}$ are taken inside $L_6$, $D(S) \cong C_2^a$ and $D(T) \cong C_3 \oplus C_2^{a\pm 1}$.

It is immediate that $a \leq \operatorname{rank} S \leq 6$ (similar considerations for $T$ do not yield any further a priori constraints), and so if $\operatorname{gen}(S) = (s_+, s_-, a, \delta)$ (in the notation of Theorem 5.1.8) then, by Theorem 5.1.8, only the following cases can occur:

$(2,0,2,-)$ which corresponds to $2\langle 2\rangle$

$(1,1,0,-)$ which corresponds to $U$

$(1,1,2,-)$ which corresponds to $U(2)$ or $\langle 2\rangle \oplus \langle -2\rangle$

$(1,3,2,-)$ which corresponds to $U \oplus 2\langle -2\rangle$

$(1,3,4,-)$ which corresponds to $3\langle -2\rangle \oplus \langle 2\rangle$

$(2,2,0,-)$ which corresponds to $2U$

$(2,2,2,-)$ which corresponds to $U \oplus U(2)$

$(2,2,4,-)$ which corresponds to $2U(2)$ or $2\langle 2\rangle \oplus 2\langle -2\rangle$

$(0,2,2,-)$ which corresponds to $2\langle -2\rangle$

$(0,4,2,-)$ which corresponds to $-\begin{pmatrix} 2&0&0&1\\ 0&2&0&1\\ 0&0&2&1\\ 1&1&1&2 \end{pmatrix}$ by considering the tables of [Nip91]

$(0,4,4,-)$ which corresponds to $4\langle -2\rangle$.

$\square$

**Proposition 5.1.16.** *If $g \in \mathrm{O}(L_{6,2})$ is 4-torsion, then the invariant lattices of $g^2$ in $L_{6,2}$ are given by one of the following pairs:*

| | |
|---|---|
| $S_{g^2} = \langle 2\rangle^{\oplus 2}$ | $T_{g^2} = \langle -2\rangle^{\oplus 3} \oplus \langle -6\rangle$ |
| | $T_{g^2} = \begin{pmatrix} -2 & -1 & -1 & -1 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ -1 & 0 & 0 & 4 \end{pmatrix}$ |
| $S_{g^2} = \langle -2\rangle^{\oplus 2}$ | $T_{g^2} = \langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle \oplus \langle -6\rangle$ |
| | $T_{g^2} = A_2 \oplus \langle -2\rangle^{\oplus 2}$ |
| | $T_{g^2} = K_1$ |
| $S_{g^2} = U$ | $T_{g^2} = U \oplus \langle -2\rangle \oplus \langle -6\rangle$ |
| $S_{g^2} = U^{\oplus 2}$ | $T_{g^2} = \langle -2\rangle \oplus \langle -6\rangle$ |
| $S_{g^2} = U(2)$ | $T_{g^2} = U \oplus \langle -2\rangle \oplus \langle -6\rangle$ |
| | $T_{g^2} = \langle 2\rangle \oplus \begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix} \oplus \langle -6\rangle$ |
| $S_{g^2} = U \oplus U(2)$ | $T_{g^2} = \langle -2\rangle \oplus \langle -6\rangle$ |
| $S_{g^2} = U \oplus \langle -2\rangle^{\oplus 2}$ | $T_{g^2} = \langle -2\rangle \oplus \langle -6\rangle$ |
| $S_{g^2} = \langle -2\rangle \oplus \langle 2\rangle$ | $T_{g^2} = \langle 2\rangle \oplus \langle -2\rangle^{\oplus 2} \oplus \langle -6\rangle$ |
| | $T_{g^2} = U \oplus \langle -2\rangle \oplus \langle -6\rangle$ |
| | $T_{g^2} = K_2$ |

*where, if either exist,*

$$\mathrm{gen}(K_1) = (2, 2; \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \oplus (-1/3), C_2^{\oplus 3} \oplus C_3)$$

*and*

$$\mathrm{gen}(K_2) = (1, 3; \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \oplus (-1/3), C_2^{\oplus 3} \oplus C_3).$$

*Proof.* As in Proposition 5.1.13, we shall refer to the canonical basis of an abelian group $C_{i_1} \oplus \ldots \oplus C_{i_k}$ by $\{e_1, \ldots, e_k\}$, $\{f_1, \ldots, f_k\}$ or $\{g_1, \ldots, g_k\}$.

### 5.1.13 $S = 2\langle 2\rangle$

If $S = 2\langle 2\rangle$ then $D(S) = ((1/2)^{\oplus 2}, C_2^{\oplus 2})$ and $H_S = \{0\}$ or $C_2$ or $C_2^{\oplus 2}$.

If $H_S \cong \{0\}$ then $\gamma : \underline{0} \mapsto \underline{0}$ and

$\Gamma_\gamma^\perp/\Gamma_\gamma = (q_S) \oplus (-q) = \delta = \Gamma_\gamma^\perp = (C_2^{\oplus 4} \oplus C_3; (1/2)^{\oplus 4} \oplus (1/3))$. Therefore, $\mathrm{gen}(T) = (0, 4, -\delta) = (0, 4, (1/2)^{\oplus 4} \oplus (-1/3), C_2^{\oplus 4} \oplus C_3)$. By referring to tables in [Nip91], there are 9 even, negative definite rank 4 lattices of determinant 48, but only two of these have discriminant group isomorphic to $C_2^{\oplus 4} \oplus C_3$. These are

$$T_7 := \langle -2\rangle^{\oplus 3} \oplus \langle -6\rangle,$$

which clearly has discriminant form equal to $-\delta$ and

$$T_8 := -\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

which has discriminant form

$$D(T_8) = (\begin{pmatrix} 0 & -1/2 \\ -1/2 & -1/4 \end{pmatrix}, C_2 \oplus C_6).$$

The case $T = T_8$ cannot occur as all order 2 elements in the discriminant group are isotropic. We conclude that $T = \langle -2\rangle^{\oplus 3} \oplus \langle -6\rangle$.

If $H_S \cong C_2$ then, up to $\mathrm{O}(S)$-equivalence, $H_S$ is generated by one of $(1, 0)$ or $(1, 1)$ in $D(S)$.

If $H_S = \langle e_1\rangle \cong C_2$ and if

$\gamma : e_1 \mapsto f_1$, then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4\rangle$ where $x_1 = g_1 + g_3$, $x_2 = g_2$, $x_3 = g_4$, $x_4 = g_5$ and $\Gamma_\gamma = \langle x_1\rangle$. Therefore,

$$\Gamma_\gamma^\perp/\Gamma_\gamma \cong \langle x_2, x_3, x_4\rangle \cong C_2^{\oplus 2} \oplus C_3 = \delta$$

with form $(1/2)^{\oplus 2} \oplus (1/3)$. Therefore, $\mathrm{gen}(T) = (0, 4, -\delta) = (0, 4; (1/2)^{\oplus 2} \oplus$

$(-1/3), C_2^{\oplus 2} \oplus C_3)$. By referring to tables in [Nip91], we find that there are 2 even, negative definite rank 4 lattices of determinant:

$$T_9 := -\begin{pmatrix} 2&0&0&1\\ 0&2&0&0\\ 0&0&2&0\\ 1&0&0&2 \end{pmatrix}$$

which has discriminant group

$$D(T_9) = ((1/2) \oplus (-1/6), C_2 \oplus C_6),$$

which cannot occur because all order 3 elements have length 1/3 and

$$T_{10} := -\begin{pmatrix} 2&1&1&1\\ 1&2&0&0\\ 1&0&2&0\\ 1&0&0&4 \end{pmatrix}$$

which has discriminant group

$$D(T_{10}) = (\begin{pmatrix} 0 & -1/2\\ -1/2 & 1/6 \end{pmatrix}, C_2 \oplus C_6).$$

With respect to the generators $(1,3)$, $(0,3)$ and $(0,1)$, $D(T_{10}) = ((1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3) = -\delta$, and so $T = T_{10}$.

If $\gamma : e_1 + e_2 \mapsto f_1 + f_2$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4\rangle$ where $x_1 = g_1 + g_2$, $x_2 = g_1 + g_3$, $x_3 = g_1 + g_4$, $x_4 = g_5$ and $\Gamma_\gamma = \langle x_1 + x_2 + x_3\rangle$. Therefore,

$$\Gamma_\gamma^\perp/\Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5\rangle = \delta \cong C_2^{\oplus 3} \oplus C_3$$

with form $\begin{pmatrix} 0 & 1/2 & 1/2\\ 1/2 & 0 & 1/2\\ 1/2 & 1/2 & 0 \end{pmatrix} \oplus (1/3)$. Therefore, $\mathrm{gen}(T) = (0, 4, -\delta)$.

If $H_S \cong C_2^{\oplus 2}$, then $H_S = \langle e_1, e_2\rangle$.

If $H_S = \langle e_1, e_2\rangle$, we let $\gamma : e_1 \mapsto f_1$ and $\gamma : e_2 \mapsto f_2$. (Other choices of $\gamma$ exist, but it is clear that these all yield isomorphic $T$.) Then,

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3\rangle$ where $x_1 = g_1 + g_3$, $x_2 = g_2 + g_4$, $x_3 = g_5$ and

$\Gamma_\gamma = \langle x_1, x_2\rangle$. Therefore,

$$\Gamma_\gamma^\perp/\Gamma_\gamma \cong \langle x_3\rangle = \delta \cong C_3$$

with form $(1/3)$. Therefore, $\mathrm{gen}(T) = (0, 4, -\delta)$ but by examining [Nip91] there is no negative definite rank 4 lattice of determinant 3.

### 5.1.14 $S = U$

If $S = U$ then $D(S) = \{0\}$ and $H_S = \{0\}$. Therefore $\gamma : \underline{0} \mapsto \underline{0}$ and

$$\Gamma_\gamma^\perp/\Gamma_\gamma = (q_S) \oplus (q) = \delta = ((1/2)^{\oplus 2} \oplus (1/3), C_2^{\oplus 2} \oplus C_3)$$

and so $\mathrm{gen}(T) = (1, 3, -\delta)$. By Theorem 5.1.11, $T$ is unique in its genus and a representative is given by $U \oplus \langle -2\rangle \oplus \langle -6\rangle$.

### 5.1.15 $S = U(2)$

If $S = U(2)$ then $D(S) = (\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, C_2^{\oplus 2})$ and $H_S = \{0\}, C_2$ or $C_2^{\oplus 2}$

If $H_S = \{0\}$ then $\gamma : \underline{0} \mapsto \underline{0}$ and

$\Gamma_\gamma^\perp/\Gamma_\gamma = (-q_S) \oplus (-q) = \delta = (\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \oplus (1/2)^{\oplus 2} \oplus (1/3), C_2^{\oplus 4} \oplus C_3)$. Therefore, $\mathrm{gen}(T) = (1, 3, -\delta)$. By Theorem 5.1.11, $T$ is unique in its genus and a representative is given by $\langle 2\rangle \oplus \begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix} \oplus \langle -6\rangle$

If $H_S \cong C_2$ then $H_S$ is generated by one of $e_1$ or $e_1 + e_2$ in $D(S)$.

If $H_S = \langle e_1\rangle \cong C_2$ and if

$\gamma : e_1 \mapsto f_1 + f_2$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4\rangle$ where $x_1 = g_1$, $x_2 = g_2 + g_3$, $x_3 = g_2 + g_4$, $x_4 = g_5$ and $\Gamma_\gamma = \langle x_1 + x_2 + x_3\rangle$. Therefore,

$$\Gamma_\gamma^\perp/\Gamma_\gamma \cong \langle x_2, x_3, x_4\rangle = \delta \cong C_2^{\oplus 2} \oplus C_3$$

with form $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \oplus (1/3)$ and so $\mathrm{gen}(T) = (1, 3, -\delta)$. By Theorem 5.1.11, $T$ is unique in its genus and a representative is given by $U \oplus \langle -2 \rangle \oplus \langle -6 \rangle$.

If $\gamma : e_1 + e_2 \mapsto f_1 + f_2$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4 \rangle$ where $x_1 = g_1 + g_2$, $x_2 = g_1 + g_3$, $x_3 = g_1 + g_4$, $x_4 = g_5$ and $\Gamma_\gamma = \langle x_1 + x_2 + x_3 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4 \rangle = \delta \cong C_2^{\oplus} \oplus C_3$$

with form $(1/2)^{\oplus 2} \oplus (1/3)$. Therefore, $\mathrm{gen}(T) = (1, 3, -\delta)$. By Theorem 5.1.11, $T$ is unique in its genus and a representative is given by

$U \oplus \langle -2 \rangle \oplus \langle -6 \rangle$.

If $H_S \cong C_2^{\oplus 2}$, then $H_S = \langle e_1, e_2 \rangle$ with form $\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$, but there is no embedding of $H_S$ in $D(L_{6,2p^2})$ because the only isotropic elements in $((1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3)$ are $\pm(1,1,0)$, but $H_S$ is of rank 2.

### 5.1.16 $S = 3\langle -2 \rangle \oplus \langle 2 \rangle$

If $S = 3\langle -2 \rangle \oplus \langle 2 \rangle$ then $D(S) = ((1/2)^{\oplus 4}, C_2^{\oplus 4})$ and $H_S = \{0\}, C_2$ or $C_2^{\oplus 2}$.

If $H_S = \{0\}$ then $\gamma : \underline{0} \mapsto \underline{0}$ and

$$\Gamma_\gamma^\perp / \Gamma_\gamma = (q_S) \oplus (-q) = \delta = ((1/2)^{\oplus 6} \oplus (1/3), C_2^{\oplus 6} \oplus C_3)$$

Therefore, $\mathrm{gen}(T) = (1, 1, -\delta)$ but a minimal generating set for $-\delta$ contains at least 6 generators, and so no such $T$ exists.

If $H_S \cong C_2$ then, up to $\mathrm{O}(S)$-equivalence, $H_S$ is generated by one of $e_1$ or $e_1 + e_2 + e_3$, which are of length $1/2$; or $e_1 + e_2$ or $e_1 + e_2 + e_3 + e_4$, which are of length 0.

If $H_S = \langle e_1 \rangle \cong C_2$ and if

$\gamma : e_1 \mapsto f_1$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$ where $x_1 = g_1 + g_5$, $x_2 = g_2$, $x_3 = g_3$,

$x_4 = g_4$, $x_5 = g_6$, $x_6 = g_7$ and $\Gamma_\gamma = \langle x_1 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4, x_4, x_6 \rangle = \delta \cong C_2^{\oplus 4} \oplus C_3$$

with form $(1/2)^{\oplus 4} \oplus C_3$. Therefore, $\mathrm{gen}(T) = (1, 1, -\delta)$ but a minimal generating set for $-\delta$ contains at least 4 generators, and so no such $T$ exists.

If $\gamma : e_1 + e_2 + e_3 \mapsto f_1$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$ where $x_1 = e_1 + e_2$, $x_2 = e_1 + e_3$, $x_3 = e_1 + e_5$, $x_4 = e_4$, $x_5 = e_6$, $x_6 = e_7$ and $\Gamma_\gamma = \langle x_1 + x_2 + x_3 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5, x_6 \rangle = \delta \cong C_2^{\oplus 4} \oplus C_3.$$

Therefore, $\mathrm{gen}(T) = (1, 1, -\delta)$ but a minimal generating set for $-\delta$ contains at least 4 generators, and so no such $T$ exists.

If $\gamma : e_1 + e_2 \mapsto f_1 + f_2$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$ where $x_1 = g_1 + g_2$, $x_2 = g_1 + g_5$, $x_3 = g_1 + g_6$, $x_4 = g_7$, $x_5 = g_3$, $x_6 = g_4$ and $\Gamma_\gamma = \langle x_1 + x_2 + x_3 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma = \langle x_2, x_3, x_4, x_5, x_6 \rangle = \delta \cong C_2^{\oplus 4} \oplus C_3.$$

Therefore, $\mathrm{gen}(T) = (1, 1, -\delta)$ but a minimal generating set for $-\delta$ contains at least 4 generators, and so no such $T$ exists.

If $\gamma : e_1 + e_2 + e_3 + e_4 \mapsto f_1 + f_2$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$ where $x_1 = g_1 + g_2$, $x_2 = g_1 + g_3$, $x_3 = g_1 + g_4$, $x_4 = g_1 + g_5$, $x_5 = g_1 + g_6$, $x_6 = g_7$ and $\Gamma_\gamma = \langle x_1 + x_2 + x_3 + x_4 + x_5 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5, x_6 \rangle = \delta \cong C_2^{\oplus 4} \oplus C_3.$$

Therefore, $\mathrm{gen}(T) = (1, 1, -\delta)$ but a minimal generating set for $-\delta$ contains at least 4 generators, and so no such $T$ exists.

### 5.1.17 $S = 2U(2)$

If $S = 2U(2)$ then $D(S) = (\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}^{\oplus 2}, C_2^{\oplus 4})$ and $H_S = \{0\}, C_2$ or $C_2^{\oplus 2}$.

If $H_S = \{0\}$ then $\gamma : \underline{0} \mapsto \underline{0}$ and

$\Gamma_\gamma^\perp / \Gamma_\gamma = (q_S) \oplus (-q) = (\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}^{\oplus 2} \oplus (1/2)^{\oplus 2} \oplus (1/3), C_2^{\oplus 6} \oplus C_3) = \delta$ and $\mathrm{gen}(T) = (0, 2, -\delta)$, but a minimal generating set for $-\delta$ contains at least 6 generators, and so no such $T$ exists.

If $H_S \cong C_2$ then, up to $\mathrm{O}(S)$-equivalence, $H_S$ is generated by one of $e_1$, $e_1 + e_2$, $e_1 + e_2 + e_3$ or $e_1 + e_2 + e_3 + e_4$ in $D(S)$.

If $H_S = \langle e_1 \rangle \cong C_2$ and if

$\gamma : e_1 \mapsto f_1 + f_2$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$ where $x_1 = g_1$, $x_2 = g_2 + g_5$, $x_3 = g_2 + g_6$, $x_4 = g_3$, $x_5 = g_4$, $x_6 = g_7$ and $\Gamma_\gamma = \langle x_1 + x_2 + x_3 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5, x_6 \rangle = \delta \cong C_2^{\oplus 4} \oplus C_3$$

and $\mathrm{gen}(T) = (0, 2, -\delta)$, but a minimal generating set for $-\delta$ contains at least 4 generators, and so no such $T$ exists.

If $H_S = \langle e_1 + e_2 \rangle \cong C_2$ and if

$\gamma : e_1 + e_2 \mapsto f_1 + f_2$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$ where $x_1 = g_1 + g_2$, $x_2 = g_1 + g_5$, $x_3 = g_1 + g_6$, $x_4 = g_3$, $x_5 = g_4$, $x_6 = g_7$ and $\Gamma_\gamma = \langle x_1 + x_2 + x_3 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5, x_6 \rangle = \delta \cong C_2^{\oplus 4} \oplus C_3$$

and $\mathrm{gen}(T) = (0, 2, -\delta)$, but a minimal generating set for $-\delta$

contains at least 4 generators, and so no such $T$ exists.

If $H_S = \langle e_1 + e_2 + e_3 \rangle \cong C_2$ and if

$\gamma : e_1 + e_2 + e_3 \mapsto f_1 + f_2$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$ where $x_1 = g_1 + g_2$, $x_2 = g_1 + g_4$, $x_3 = g_1 + g_5$, $x_4 = g_1 + g_6$, $x_5 = g_3$, $x_6 = g_7$ and $\Gamma_\gamma = \langle x_1 + x_3 + x_4 + x_5 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5, x_6 \rangle = \delta \cong C_2^{\oplus 4} \oplus C_3$$

and $\mathrm{gen}(T) = (0, 2, -\delta)$, but a minimal generating set for $-\delta$ contains at least 4 generators, and so no such $T$ exists.

If $H_S = \langle e_1 + e_2 + e_3 + e_4 \rangle \cong C_2$ and if

$\gamma : e_1 + e_2 + e_3 + e_4 \mapsto f_1 + f_2$ then

$\Gamma_\gamma^\perp = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$ where $x_1 = g_1 + g_2$, $x_2 = g_1 + g_3$, $x_3 = g_1 + g_4$, $x_4 = g_1 + g_5$, $x_5 = g_1 + g_6$, $x_6 = g_7$ and $\Gamma_\gamma = \langle x_1 + x_2 + x_3 + x_4 + x_5 \rangle$. Therefore,

$$\Gamma_\gamma^\perp / \Gamma_\gamma \cong \langle x_2, x_3, x_4, x_5, x_6 \rangle = \delta \cong C_2^{\oplus 4} \oplus C_3$$

and $\mathrm{gen}(T) = (0, 2, -\delta)$, but a minimal generating set for $-\delta$ contains at least 4 generators, and so no such $T$ exists.

### 5.1.18 $S = \langle 2 \rangle \oplus \langle -2 \rangle$

If $S = \langle 2 \rangle \oplus \langle -2 \rangle$ then $D(S) = ((1/2)^{\oplus 2}, C_2^{\oplus 2})$ which is the same as in the case $S = 2\langle 2 \rangle$. Therefore,

$$\mathrm{gen}(T) = \begin{cases} (1, 3, (1/2)^{\oplus 4} \oplus (-1/3), C_2^{\oplus 4} \oplus C_3) \\ (1, 3; (1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3) \\ (1, 3; \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \oplus (-1/3), C_2^{\oplus 3} \oplus C_3) \\ (1, 3; (-1/3), C_3) \end{cases}$$

1. If $\mathrm{gen}(T) = (1,3,(1/2)^{\oplus 4} \oplus (-1/3), C_2^{\oplus 4} \oplus C_3)$ then, by Theorem 5.1.11, $T$ is unique in its genus and a representative is given by $\langle 2\rangle \oplus \langle -2\rangle^{\oplus 2} \oplus \langle -6\rangle$.

2. If $\mathrm{gen}(T) = (1,3;(1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3)$ then, by Theorem 5.1.11, $T$ is unique in its genus and a representative is given by $U \oplus \langle -2\rangle \oplus \langle -6\rangle$.

3. If $\mathrm{gen}(T) = (1,3;\left(\begin{smallmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{smallmatrix}\right) \oplus (-1/3), C_2^{\oplus 3} \oplus C_3)$ then, by Theorem 5.1.11, $T$ is unique in its genus.

4. By Theorem 5.1.6, the genus $(1,3;(-1/3),C_3)$ is empty.

### 5.1.19 $S = 2U$

If $S = 2U$ then $D(S) = \{0\}$ which is the same as in the case $S = U$. Therefore, $\mathrm{gen}(T) = (0,2;(1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3)$ and is therefore a negative definite even lattice of determinant 12. By referring to tables in [CS99], $T$ is either

$$\begin{aligned} T_{11} &:= \langle -2\rangle \oplus \langle -6\rangle \\ T_{12} &:= \left(\begin{smallmatrix} -4 & -2 \\ -2 & -4 \end{smallmatrix}\right). \end{aligned}$$

The lattice $T_{11}$ has discriminant form $((1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3)$ and $T_{12}$ has discriminant form $(\left(\begin{smallmatrix} 0 & 1/2 \\ 1/2 & 0 \end{smallmatrix}\right) \oplus (-1/3), C_2^{\oplus 2} \oplus C_3)$, which is clearly inequivalent to $((1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3)$ by consider the length of order 2 elements. Therefore, $T = T_{11}$.

### 5.1.20 $S = U \oplus 2\langle -2\rangle$

If $S = U \oplus 2\langle -2\rangle$ then $D(S) = ((1/2)^{\oplus 2}, C_2^{\oplus 2})$ which is the same as in the case $S = 2\langle 2\rangle$. Therefore,

$$\mathrm{gen}(T) = \begin{cases} (0,2,(1/2)^{\oplus 4} \oplus (-1/3), C_2^{\oplus 4} \oplus C_3) \\ (0,2;(1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3) \\ (0,2;\left(\begin{smallmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{smallmatrix}\right) \oplus (-1/3), C_2^{\oplus 3} \oplus C_3) \\ (0,2;(1/3),C_3) \end{cases}$$

1. The genus $(0,2,(1/2)^{\oplus 4}\oplus(-1/3),C_2^{\oplus 4}\oplus C_3)$ is empty, because the minimum number of generators of $C_2^{\oplus 4}\oplus C_3$ is greater than 2.

2. If $\mathrm{gen}(T)=(0,2;(1/2)^{\oplus 2}\oplus(-1/3),C_2^{\oplus 2}\oplus C_3)$, then $T$ is a negative definite even rank 2 lattice of determinant 12. As in a previous case, the genus $(0,2;\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}\oplus(-1/3),C_2^{\oplus 2}\oplus C_3)$ contains one class which is given by the form $T_{11}:=\langle-2\rangle\oplus\langle-6\rangle$. Therefore, $T=\langle-2\rangle\oplus\langle-6\rangle$.

3. The genus $(0,2;\begin{pmatrix}0&1/2&1/2\\1/2&0&1/2\\1/2&1/2&0\end{pmatrix}\oplus(-1/3),C_2^{\oplus 3}\oplus C_3)$ is empty, because the minimum number of generators of $C_2^{\oplus 3}\oplus C_3$ is greater than 2.

4. The genus $(0,2;(1/3),C_3)$ is empty. This can be seen by considering Theorem 5.1.6.

### 5.1.21 $S=U\oplus U(2)$

If $S=U\oplus U(2)$ then $D(S)=(\begin{pmatrix}0&1/2\\1/2&0\end{pmatrix},C_2^{\oplus 2})$ which is the same as in the case $S=U(2)$. Therefore,

$$\mathrm{gen}(T)=\begin{cases}(0,2;\begin{pmatrix}0&1/2\\1/2&0\end{pmatrix}\oplus(1/2)^{\oplus 2}\oplus(-1/3),C_2^{\oplus 4}\oplus C_3)\\(0,2;\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}\oplus(-1/3),C_2^{\oplus 2}\oplus C_3)\\(0,2;(1/2)^{\oplus 2}\oplus(-1/3),C_2^{\oplus 2}\oplus C_3)\end{cases}$$

1. The genus $(0,2;\begin{pmatrix}0&1/2\\1/2&0\end{pmatrix}\oplus(1/2)^{\oplus 2}\oplus(-1/3),C_2^{\oplus 4}\oplus C_3)$ is empty because the minimum number of generators of $C_2^{\oplus 4}\oplus C_3$ is greater than 2.

2. As in a previous case, the genus $(0,2;\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}\oplus(-1/3),C_2^{\oplus 2}\oplus C_3)$ contains one class which is given by the form $T_{11}:=\langle-2\rangle\oplus\langle-6\rangle$. Therefore, $T=\langle-2\rangle\oplus\langle-6\rangle$.

### 5.1.22 $S = 2\langle -2\rangle$

If $S = 2\langle -2\rangle$ then $D(S) = ((1/2)^{\oplus 2}, C_2^{\oplus 2})$ which is the same as in the case $S = 2\langle 2\rangle$. Therefore,

$$\mathrm{gen}(T) = \begin{cases} (2,2,(1/2)^{\oplus 4} \oplus (-1/3), C_2^{\oplus 4} \oplus C_3) \\ (2,2;(1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3) \\ (2,2;\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \oplus (-1/3), C_2^{\oplus 3} \oplus C_3) \\ (2,2;(1/3), C_3) \end{cases}$$

1. If $T$ lies in the genus $(2,2,(1/2)^{\oplus 4} \oplus (-1/3), C_2^{\oplus 4} \oplus C_3)$ then, by Theorem 5.1.11, $T$ is unique in its genus and a representative is given by $\langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle \oplus \langle -6\rangle$.

2. If $T$ lies in the genus $(2,2;(1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3)$ then, by Theorem 5.1.11, $T$ is unique in its genus and a representative is given by $A_2 \oplus \langle -2\rangle^{\oplus 2}$.

3. By Theorem 5.1.11, the genus $(2,2;\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \oplus (-1/3), C_2^{\oplus 3} \oplus C_3)$ contains at most one class.

4. Because of Theorem 5.1.6, the genus $(2,2;(1/3), C_3)$ is empty.

### 5.1.23 $S = 2\langle 2\rangle \oplus 2\langle -2\rangle$

If $S = 2\langle 2\rangle \oplus 2\langle -2\rangle$ then $D(S) = (1/2)^{\oplus 4}, C_2^{\oplus 4})$ which is the same as in the case $S = 3\langle -2\rangle \oplus \langle 2\rangle$. Therefore, $\mathrm{gen}(T) = (1,1,-\delta)$ where $-\delta$ is a finite quadratic form on a group isomorphic to $C_2^{\oplus 4} \oplus C_3$ or $C_2^{\oplus 6} \oplus C_3$ as, in each case, the minimal number of generators exceeds 2, no such $T$ can exist. □

## 5.2 Branch divisors in $\mathcal{F}_{L_{6,2p^2}}(\Gamma)$

We determine the branch divisor of $\mathcal{F}_{L_{6,2p^2}}(\Gamma)$. The branch divisor corresponds precisely to the fixed locus of elements in $\Gamma$ that act as quasi-reflections and, by Lemma 5.0.3, these correspond to elements of the form $\pm\sigma_v \in \Gamma$ where $\sigma_v \in \mathrm{O}(L_{6,2p^2})$ is a reflection.

**Lemma 5.2.1.** *If $p > 3$ and $v \in L_{6,2p^2}$ is primitive so that $\sigma_v$ is a reflection, then*

$$\operatorname{div}(v) \in \{1, 2, 3, 6, p^2, 2p^2, 3p^2, 6p^2\}$$

*and*

$$\pm v^2 \in \{2, 4, 6, 12, 2p^2, 4p^2, 6p^2, 12p^2\}.$$

*Proof.* (The first part of the argument makes use of a number of observations in Chapter 3 of [GHS07].) The reflection $\sigma_v \in \mathrm{O}(L)$ is defined by

$$\sigma_v : x \mapsto x - 2\frac{(x, v)}{(v, v)}v$$

and, therefore, $\operatorname{div}(v)|v^2$ and $v^2|2\operatorname{div}(v)$. Because $\operatorname{div}(v)$ is the order of $v^* = v/\operatorname{div}(v)$ in $D(L_{6,2p^2}) = C_6 \oplus C_{2p^2}$, we conclude that $\operatorname{div}(v)|6p^2$. We can exclude the cases where $p$ properly divides $\operatorname{div}(v)$: if $p|\operatorname{div}(v)$ then, on the standard basis of $L_{6,2p^2}$, $x$ belongs to the set $(p\mathbb{Z}, p\mathbb{Z}, p\mathbb{Z}, p\mathbb{Z}, p\mathbb{Z}, \mathbb{Z})$. Therefore $p^2|x^2$ and as $\operatorname{div}(v)|v^2|2\operatorname{div}(v)$, the cases $\operatorname{div}(v) = p, 2p, 3p, 6p$ cannot occur. $\square$

Throughout, we shall identify $D(L_{6,2p^2})$ with $C_6 \oplus C_{2p^2}$. We begin by classifying the reflective primitive vectors $v \in L_{6,2p^2}$ (as in, primitive $v \in L_{6,2p^2}$ such that $\sigma_v \in \mathrm{O}(L_{6,2p^2})$) up to $\widetilde{\mathrm{O}}(L_{6,2p^2})$-equivalence, before deciding whether $\pm\sigma_v \in \mathrm{O}^+(L_6, h^s_{2p^2})$. We can immediately assume that $v^2 < 0$ as we are working with $\mathrm{O}^+(L_6, h^s_{2p^2})$.

**Lemma 5.2.2.** *Suppose that $v \in L_{6,2p^2}$ is such that $\operatorname{div}(v)|v^2|2\operatorname{div}(v)$. Then, up to $\widetilde{\mathrm{O}}(L_{6,2p^2})$-equivalence, $v$ is represented by one of the following in $L_{6,2p^2}$.*

$$v = (1, -1, 0, 0, 0, 0) \qquad \operatorname{div}(v) = 1, \qquad v^2 = -2$$

$$v = (\alpha, \beta, 0, 0, 0, 1) \qquad \operatorname{div}(v) = 2, \qquad v^2 = -2$$

*or, if $p^2|\operatorname{div}(v)$, by $v = (\alpha, \beta, 0, 0, x_5, \pm 1)$ where $\operatorname{div}(v)|\alpha, \beta$ and where $v^*$ has image*

$(\mu, \pm 1) \in D(L_{6,2p^2})$. *In such a case, the following conditions are satisfied:*

$$\begin{array}{llllllr}
\textit{if} & v^2 = -2p^2 & \textit{and} & \operatorname{div}(v) = p^2, & \textit{then} & \mu = 0 & \mod 6 \\
\textit{if} & v^2 = -2p^2 & \textit{and} & \operatorname{div}(v) = 2p^2, & \textit{then} & \mu = 0,3 & \mod 6 \\
\textit{if} & v^2 = -6p^2 & \textit{and} & \operatorname{div}(v) = 6p^2, & \textit{then} & \mu = 0,1,2 & \mod 6 \\
\textit{if} & v^2 = -6p^2 & \textit{and} & \operatorname{div}(v) = 6p^2, & \textit{then} & \mu = 1,2,4,5 & \mod 6.
\end{array}$$

*Moreover, there are no other solutions.*

*Proof.* We can restrict our attention to $v \in L_{6,2p^2}$ satisfying the conditions of Lemma 5.2.1.

If $v^2 = -2$ and $\operatorname{div}(v) = 1$, then $v^* = (0,0)$ and an $\widetilde{\mathrm{O}}^+(L_{6,2p^2})$-representative is given by $(1,-1,0,0,0,0)$.

If $v^2 = -2$ and $\operatorname{div}(v) = 2$, then $v = (2x_1, 2x_2, 2x_3, 2x_4, x_5, x_5)$ and $v^* = (0, 1/2)$, $(1/2, 0)$, or $(1/2, 1/2)$ and by considering $v^2/2$,

$$4x_1x_2 + 4x_3x_4 - 3x_5^2 - p^2x_6 = -1 \tag{5.5}$$

and the image of $v^*$ in $D(L_{6,2p^2})$ corresponds to taking $(x_5, x_6)$ modulo 2. Taking Equation (5.5) modulo 4, we have

$$x_5^2 - p^2x_6^2 = 3 \mod 4$$

which, by considering squares modulo 4, has solutions if and only if $x_6 \equiv 0$ modulo 2 and $x_5 \equiv 0$ modulo 2. Accordingly, $v^* = (0,1) \in D(L_{6,2p^2})$ and a representative is given by $v = (\alpha, \beta, 0, 0, 0, 1)$ where $4\alpha\beta = p^2 - 1$.

If $v^2 = -6$ and $\operatorname{div}(v) = 3$, $v = (3x_1, 3x_2, 3x_3, 3x_4, x_5, 3x_6)$ (with the assumption that $p \neq 3$), and $v^* = (2,0)$ or $(4,0)$. By considering $v^2/3$, we obtain

$$3x_1x_2 + 3x_3x_4 - x_5^2 - 3p^2x_6^2 = -3. \tag{5.6}$$

By considering squares modulo 3, we conclude that $3|x_5$, but as $\mathrm{div}(v) = 3$, $x_5 \equiv 1, 2$ modulo 3 and so so no solution exists.

If $v^2 = -6$ and $\mathrm{div}(v) = 6$, $v = (6x_1, 6x_2, 6x_3, 6x_4, x_5, 3x_6)$ and $v^* = (1, p^2)$ or $(5, p^2)$ and $x_6 = \pm 1$ modulo $p^2$; or $\mathrm{div}(v) = (1, 0)$, or $(5, 0)$ and $x_6 = \pm 0$ modulo 6. By considering $v^2/6$ we obtain

$$3(4x_1x_2 + 4x_3x_4 - p^2x_6^2) = -1 + x_5^2. \tag{5.7}$$

If $v^* = (\pm 1, 0)$, a representative for $v$ is given by $(0, 0, 0, 0, \pm 1, 0)$. If $v^* = (\pm 1, p^2)$, we conclude that $x_6 = 2y_6 + 1$ and

$$\begin{aligned} 12(x_1x_2 + x_3x_4 - p^2y_6 - p^2y_6) - x_5^2 - 3p^2 &= -1 \\ - 3p^2x_6^2 = -1 + x_5^2 & \qquad \mod 4 \end{aligned}$$

by consider squares modulo 4 and noting that $x_5$ is odd, we see that there is no solution.

If $v^2 = 12$ and $\mathrm{div}(v) = 6$, $v = (6x_1, 6x_2, 6x_3, 6x_4, x_5, 3x_6)$ and by considering $v^2/6$ we obtain

$$12(x_1x_2 + x_3x_4) = -2 + x_5^2 + 3p^2x_6^2$$

which, by considering squares modulo 4, has no solution.

If $v^2 = -2p^2$ and $\mathrm{div}(v) = p^2$, $v = (p^2x_1, p^2x_2, p^2x_3, p^2x_4, p^2x_5, x_6)$ and by considering $v^2/2p^2$ we obtain

$$p^2(x_1x_2 + x_3x_4) - 3p^2x_5 - x_6^2 = -1 \tag{5.8}$$

and so

$$x_6^2 = 1 \mod p^2$$

which has two solutions $x_6 \equiv \pm 1$ modulo $p$. A representative is given by $v = (\alpha, \beta, 0, 0, 0, \pm 1)$ and $v^* = (0, \pm\gamma)$.

If $v^2 = -2p^2$ and $\mathrm{div}(v) = 2p^2$, $v = (2p^2x_1, 2p^2x_2, 2p^2x_3, 2p^2x_4, p^2x_5, x_6)$ and by considering $v^2/2p^2$, we obtain

$$4p^2(x_1x_2 + x_3x_4) - 3p^2x_5^2 - x_6^2 = 1 \tag{5.9}$$

and so

$$x_6^2 = \pm 1 \mod p$$

which has two solutions $x_6 \equiv \pm 1$ modulo $p$ if $\left(\frac{-1}{p^2}\right) = 1$. In such a case, Equation 5.9 is always satisfied. A representative is given by $v = (\alpha, \beta, 0, 0, 0, \pm 1)$ and $v^* = (0, \pm 1)$ or $(3, \pm 1)$.

If $v^2 = -4p^2$ and $\operatorname{div}(v) = 2p^2$, $v = (2p^2x_1, 2p^2x_2, 2p^2x_3, 2p^2x_4, p^2x_5, x_6)$ and by considering $v^2/2p^2$, we obtain

$$4p^2(x_1x_2 + x_3x_4) - 3p^2x_5^2 - x_6^2 = -2 \tag{5.10}$$

$$x_5^2 - x_6^2 = 2 \mod 4$$

which has, by considering squares modulo 4, has no solution.

If $v^2 = -6p^2$ and $\operatorname{div}(v) = 3p^2$, $v = (3p^2x_1, 3p^2x_2, 3p^2x_3, 3p^2x_4, p^2x_5, 3x_6)$ and by considering $v^2/6p^2$ we obtain

$$3p^2(x_1x_2 + x_3x_4) - 2p^2x_5^2 - 3x_6^2 = -1 \tag{5.11}$$

and so

$$3x_6^2 - 1 = 0 \mod p^2$$

which has at most two solutions $\pm\gamma$ modulo $p^2$. By assumption, $x_5/3p^2 \equiv 0, 1, 2$ modulo 6. If such an $x_6$ exists, then Equation (5.11) clearly has a solution for any suitable $x_5$ chosen modulo 6. A representative for $v$ is given by $v = (\alpha, \beta, 0, 0, \mu, \pm\gamma)$ and $v^* = (0, \pm\gamma)$ $(1, \pm\gamma)$, $(2, \pm\gamma)$ where $3\gamma^2 - 1 \equiv 0$ $(p^2)$.

If $v^2 = -6p^2$ and $\operatorname{div}(v) = 6p^2$, $v = (6p^2x_1, 6p^2x_2, 6p^2x_3, 6p^2x_4, p^2x_5, 3x_6)$ and, by considering $v^2/6p^2$, we obtain

$$12p^2(x_1x_2 + x_3x_4) - p^2x_5^2 - 3x_6^2 = -1 \tag{5.12}$$

and so

$$3x_6^2 - 1 = 0 \mod p^2$$

which has at most two solutions $\pm\gamma$ modulo $p^2$. If such an $x_6$ exists, then Equation (5.12) clearly has a solution for any $x_5$ chosen suitably modulo 6. In order to satisfy the condition that $\mathrm{div}(v) = 6p^2$, $x_5 \equiv 1, 2, 4, 5$ (6). A representative for $v$ is given by $v = (\alpha, \beta, 0, 0, \mu, \pm\gamma)$ and $v^* = (i, \pm\gamma)$ where $i \in \{1, 2, 3, 4\}$ and $3\gamma^2 + 1 \equiv 0$ $(p^2)$.

If $v^2 = -6p^2$ and $\mathrm{div}(v) = 12p^2$, $v = (6p^2x_1, 6p^2x_2, 6p^2x_3, 6p^2x_4, p^2x_5, 3x_6)$ and, by considering $v^2/6p^2$, we obtain

$$12p^2(x_1x_2 + x_3x_4) - p^2x_5^2 - 3x_6^2 = -2 \tag{5.13}$$

and so

$$2 + p^2x_5^2 - x_6^2 = 0 \mod 4$$

which, by considering squares modulo 4, has no solution. The result then follows. □

We next determine which of the reflective vectors $v$ determine $\sigma_v \in \mathrm{O}(L_6, h^s_{2p^2})$ by using the characterisation of Theorem 4.0.6.

**Proposition 5.2.3.** *If $v \in L_{6,2p^2}$ and $\sigma_v \in \mathrm{O}^+(L_6, h^s_{2p^2})$ then, up to $\widetilde{\mathrm{O}}^+(L_{6,2p^2})$-equivalence, $v = (1, -1, 0, 0, 0, 0)$.*

*Proof.* We consider the action of $\sigma_v$ on $v_6^* \in D(L_{6,2p^2})$ where $v_6^* = (0, 0, 0, 0, 0, 1/2p^2)$ for each of the $v \in L_{6,2p^2}$ in Lemma 5.2.2. If $v = (1, 1, 0, 0, 0, 0)$, one checks that $\sigma_v(v_6^*) = v_6^* \in D(L_{6,2p^2})$. If $v = (\alpha, \beta, 0, 0, 0, 1)$, $\sigma_v(v_6^*) = -v_6^* \in D(L_{6,2p^2})$.

If $v = (p^2\alpha, p^2\beta, 0, 0, p^2x_5, x_6)$ where $v^2 = 2p^2$, $\mathrm{div}(v) = p^2$ $v^* = (0, \pm1)$. Then,

$$\sigma_v(v_6^*) = (0, 0, 0, 0, 0, \frac{1 - 2x_6^2}{2p^2}) \in D(L_{6,2p^2})$$

which is equal to $v_6^* \in D(L_{6,2p^2})$ if and only if $p^2 | x_6^2$. This is never true as $x_6^2 \equiv 1\ (p^2)$ and so $\sigma_v \notin \mathrm{O}(L_6, h^s_{2p^2})$.

If $v = (2p^2\alpha, 2p^2\beta, 0, 0, 2p^2 x_5, x_6)$ where $\mathrm{div}(v) = 2p^2$ and $v^2 = 2p^2$ and where $v^* = (\mu, \pm 1) \in D(L_{6,2p^2})$ where $\mu = 0$ or 3.

$$\sigma_v(v_6^*) = (0,0,0,0,0,\frac{1+2x_6^2}{2p^2}) \in D(L_{6,2p^2})$$

and so $\sigma_v \notin \mathrm{O}(L_6, h^s_{2p^2})$ for the same reason as the above case.

If $v = (3p^2\alpha, 3p^2\beta, 0, 0, p^2 x_5, 3x_6)$ where $\mathrm{div}(v) = 3p^2$ and $v^2 = 6p^2$ and where $v^* = (\mu, \pm 1) \in D(L_{6,2p^2})$ where $\mu =$ 0,1, or 2 and $3x_6^2 - 1 \equiv 0\ (p^2)$.

$$\sigma_v(v_6^*) = (0,0,0,0,*,\frac{3+6x_6^2}{6p^2}) \in D(L_{6,2p^2})$$

and so if $\sigma_v(v_6^*) = v_6^*$, $p^2 | x_6^2$ which is never true as $3x_6^2 + 1 \equiv 0\ (p^2)$.

If $v = (6p^2\alpha, 6p^2\beta, 0, 0, p^2 x_5, 3x_6)$ where $\mathrm{div}(v) = 6p^2$ and $v^2 = 6p^2$ and where $v^* = (\mu, \pm\gamma) \in D(L_{6,2p^2})$ where $x_5 = 1, 2, 4, 5\ (6)$ and $3x_6^2 + 1 \equiv 0\ (p^2)$.

$$\sigma_v(v_6^*) = (0,0,0,0,*,\frac{1+2x_6^2}{2p^2}) \in D(L_{6,2p^2})$$

and so if $\sigma_v(v_6^*) = v_6^*$, $p^2 | x_6^2$ which is never true as $3x_6^2 + 1 \equiv 0\ (p^2)$.

$\square$

## 5.3 Non-canonical singularities in $\mathcal{F}_{L_{6,2p^2}}(\Gamma)$

We begin with a definition.

**Definition 5.3.1.** *Let $L$ be a lattice of signature $(2, n)$ and let $v \in L \otimes \mathbb{Q}$. The subset*

$$\mathcal{D}_L^v = \{[x] \in \mathcal{D}_L \mid (x, v) = 0\} \subset \mathcal{D}_L$$

*is called a rational quadratic divisor.*

We note that rational quadratic divisors are especially important if one is interested in proving general type results, as there exists a theory of reflective orthogonal modular forms, which are modular forms which vanish along $\mathcal{D}_L^v$ for reflective $v \in L$ (see, for example, [Bor98], [Gri10]).

**Theorem 5.3.2.** *(The Eichler criterion) [Eic74] If the lattice $L$ assumes the form $L = 2U \oplus L_0$ and $v, w \in L$ are primitive such that $v^2 = u^2$ and $u^* = v^* \mod L$, then there exists $\tau \in \widetilde{\mathrm{O}}^+(L)$ such that $\tau v = w$.*

*Proof.* [GHS09] or [Eic74] §10. $\square$

**Lemma 5.3.3.** *If $g$ is of order 3 and $[x] \in \mathcal{D}_{L_{6,2}}$ and $x \in S_g \otimes \mathbb{C}$ or if $g$ is of order 4 and $[x] \in \mathcal{D}_L$ and $x \in S_{g^2} \otimes \mathbb{C}$ then*

$$[x] \in D^v_{L_{6,2p}}$$

*where $v^2 = \pm 2$.*

*Proof.* If $g$ is 3-torsion then, by Proposition 5.1.13, $T$ contains a $-2$-vector, with the exception of the case $S = 2A_2(-1)$, for which we have a 2-vector. If $g$ is 4-torsion then, by Proposition 5.1.16, all $T_{g^2}$ contains a $-2$-vector except for possibly $S_{g^2} = \langle -2 \rangle \oplus \langle 2 \rangle$ and $S_{g^2} = \langle -2 \rangle^{\oplus 2}$. In these exceptional cases, we examine the possible actions of $g$ on $S$. Let $S = \langle -2 \rangle^{\oplus 2}$ and consider $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{O}(\langle -2 \rangle^{\oplus 2})$, which is of order 4. We determine the embeddings of $\langle -2 \rangle^{\oplus 2}$ in a more explicit way, and show that the case $T = K_1$ does not occur.

By the Eichler criterion (Theorem 5.3.2), there are at most four $\mathrm{O}(L_{6,2})$ equivalence classes of $-2$-vectors in $L_{2,6}$. If $v \in L_{6,2}$ is a $-2$-vector, then $v^*$ has image $(0,0) \in D(L_{6,2})$ if $\operatorname{div} v = 1$, or $(1,0)$, $(0,3)$ or $(1,3)$ if $\operatorname{div} v = 2$. If $\operatorname{div} v = 2$, then $v$ is of the form $(2x_1, 2x_2, 2x_3, 2x_4, x_5, x_6) \in L_{6,2}$ and satisfies

$$8(x_1x_2 + x_3x_4) - 2x_5^2 - 6x_6^2 = -2.$$

By considering squares modulo 4, we conclude that $x_5$ and $x_6$ have different parities, which excludes the case $(1,3)$; and by working modulo 8, we exclude the case $(0,3)$.

We are then left with two cases represented by $(1,0,0,0,1,0)$ and $(0,0,0,0,1,0)$. One then calculates that the orthogonal complement of each case is given by $2U \oplus \langle -6 \rangle$ and $U \oplus \langle -6 \rangle \oplus \left(\begin{smallmatrix} -2 & 2 \\ 2 & 0 \end{smallmatrix}\right)$, respectively. A further calculation using Theorem 5.1.10 (and Theorem 5.1.11 to check completeness), shows that the orthogonal complement of a second -2-vector in each is given by $U \oplus \langle 2 \rangle \oplus \langle -6 \rangle$ with discriminant group $C_2^{\oplus 3} \oplus C_3$ or $\left(\begin{smallmatrix} -4 & 6 \\ 6 & 6 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} -2 & 2 \\ 2 & 0 \end{smallmatrix}\right)$ with discriminant group $C_2^{\oplus 4} \oplus C_3$. Accordingly, the case $T = K_1$ does not occur.

Because of the case $S = 2A_2(-1)$, where $g$ acts as 3-torsion on $S$, the inclusion of a 2-vector is unavoidable, as we show below.

The group $\mathrm{O}(2A_2(-1))$ can be decomposed as $G_1 \rtimes G_2$ where $G_1$ is the subgroup preserving both copies of $A_2(-1)$ and $G_2$ is the permutation group induced on the two factors in the sum $A_2(-1) \oplus A_2(-1)$ (this technique is referred to as *glue theory* in [CS99]). It is well known (see, for example, [Hum72]) that the automorphism group of a root system $\mathcal{R}$ is generated by the Weyl group $W(\mathcal{R})$ and the group of diagram automorphisms $D(\mathcal{R})$ of $\mathcal{R}$. For $A_2(-1)$, $W(A_2(-1)) \cong S_3$ and $D(A_2(-1)) \cong C_2$ and so $G_1 \cong S_3 \rtimes C_2$ and, clearly, $G_2 \cong C_2$. Therefore, $\mathrm{O}(2A_2(-1)) \cong (S_3 \rtimes C_2) \rtimes C_2$. We take generators for each of the subgroups, and compute the conjugacy classes of 3-torsion in $\mathrm{O}(2A_2(-1))$.

We did our calculations in the computer algebra system GAP. We found that $\mathrm{O}(2A_2(-1))$ is a group of order 288 and that there are two conjugacy classes of 3 torsion. These are represented by the elements

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

and first case does not fix a -2-vector.

It is clear that if $g$ is 4-torsion then $g$ also acts on $S_{g^2} = \langle -2 \rangle \oplus \langle 2 \rangle$. A direct calculation shows that $\mathrm{O}(\langle -2 \rangle \oplus \langle 2 \rangle) = \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \rangle \cong V_4$. Therefore $\mathrm{O}(\langle -2 \rangle \oplus \langle 2 \rangle)$ contains no 4-torsion. $\square$

**Theorem 5.3.4.** *If $[w] \in \mathcal{F}_{L_{6,2p^2}}$ is a non-canonical singularity,*

$$[w] \in \mathcal{D}^v_{L_{6,2p^2}} \subset \mathcal{D}_{L_{6,2p^2}}$$

*where $\mathcal{D}^v_{L_{6,2p^2}}$ is one of, at most, $8(p^2+1)$ rational quadratic divisors. The vector $v$ can be chosen to be of length $\pm 2$ or $\pm 2p^2$.*

*Proof.* By Theorem 5.0.2, if $[w] \in \mathcal{F}_{6,2p^2}$ is non-canonical, then $[w]$ lies in the fixed locus of a quasi-reflection or an element of 3 or 4 torsion. If $[w]$ lies in the fixed locus of a quasi-reflection, then the result follows by Proposition 5.2.3. If not, we consider the inclusion $\Gamma_{6,2p^2} \leq \mathrm{O}(L_{6,2})$ of Theorem 4.0.6 and denote the action of $g \in \Gamma_{6,2p^2}$ on $L_{6,2}$ by $g'$. By Lemma 5.3.3, $T_{g'} \subset L_{6,2}$ contains a $\pm 2$-vector and because of the inclusion

$$pL_{6,2} \subset L_{6,2p^2} \subset L_{6,2}$$

$T_g \subset L_{6,2p^2}$ contains a vector of length $\pm 2$ or $\pm 2p^2$, which we assume to be primitive. If $v^2 = \pm 2$, then $\operatorname{div} v = 1$ or $2$ and so $v^*$ belongs to $C_2 \oplus C_2 \leq D(L_{6,2p^2})$, which is of order 4. If $v^2 = \pm 2p^2$, then $\operatorname{div} v = 1$ or $2$ or $p$ or $2p$ or $2p^2$ and $v^*$ belongs to $C_2 \oplus C_{2p^2} \leq D(L_{6,2p^2})$, which is of order $4p^2$. And so, by the Eichler criterion, up to $\widetilde{\mathrm{O}}^+(L_{6,2p^2})$ equivalence, there are at most $8(p^2+1)$ such elements. $\square$

## 5.4 Extension of pluricanonical forms

A $\Gamma_{6,2p^2}$-invariant pluricanonical form on $\mathcal{D}_{L_{6,2p^2}}$ will extend to a smooth model of $\mathcal{F}_{L_{6,2p^2}}$ if it vanishes to sufficiently high order over the interior obstructions. By using the Reid-Tai criterion and a result of Tai, we show that the required order can be determined effectively by toric methods. We begin by establishing bounds on the order of the elliptic elements of $\mathrm{O}(L_{6,2p^2})$.

**Lemma 5.4.1.** *If $g \in \mathrm{O}(L_{6,2p^2})$ is of finite order $m$, then $m \leq 30$.*

*Proof.* The element $g \in \mathrm{O}(L_{6,2p^2})$ has a representation on $L_{6,2p^2} \otimes \mathbb{Q}$, which is of degree 6. By general theory on the representations of the cyclic group over $\mathbb{Q}$, if $h$ is of order

$d$, there is a unique faithful irreducible representation of degree $\phi(d)$. Therefore, if $q^r | o(g)$, then $q \leq 7$ and $q^r$ is one of $2, 2^2, 2^3, 3, 3^2, 5$. One checks that the only $d$ with such factors satisfying $\phi(d) \leq 6$ are

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 14 | 18 |
| $\phi(d)$ | 1 | 1 | 2 | 2 | 4 | 2 | 6 | 4 | 6 | 4 | 4 | 6 | 6 |

and so $m \leq 30$. □

**Theorem 5.4.2.** *[Tai82] If $G \leq \mathrm{GL}(n, \mathbb{C})$, $X = \mathbb{C}^n/G$ and $X_g = \mathbb{C}^n/\langle g \rangle$ for $g \in G$, then a $G$-invariant pluricanonical form $\eta$ on $\mathbb{C}^n$ extends to $\widetilde{X}$ if and only if $\eta$ extends to $\widetilde{X_g}$ for every $g \in G$.*

**Theorem 5.4.3.** *If $\Omega$ is a $\Gamma$-invariant pluricanonical form on $\mathcal{D}_{L_{6,2p^2}}$, then $\Omega$ defines a pluricanonical form on a smooth model of $\mathcal{F}_{L_{6,2p^2}}$ if $\Omega$ vanishes to suitably high order over the pre-image of the obstructions under the map*

$$\pi : \mathcal{D}_{L_{6,2p^2}} \to \mathcal{F}_{L_{6,2p^2}}.$$

*Moreover, the order of vanishing required can be determined effectively.*

*Proof.* Suppose that $[w] \in \mathcal{F}_{L_{6,2p^2}}$ is singular or lies in the branch divisor. We assume that $\mathcal{F}_{L_{6,2p^2}}$ is locally isomorphic to $\mathbb{C}^4/G$ in a neighbourhood of $[w]$. By Theorem 5.4.2, we only need to check that $\Omega$ extends to $\widetilde{\mathbb{C}^4/\langle g \rangle}$ for each $g \in G$. Because of Lemma 5.4.1, this is a finite problem.

A computer search (by using Theorem 3.8.2 and 3.8.5) of the possible representations of elliptic $g \in \mathrm{O}(L_{6,2p^2})$ on $\mathrm{Hom}(\mathbb{W}, \mathbb{W}^\perp/\mathbb{W})$ found that, at most, 34 possible non-canonical cyclic quotient singularities can arise. These are listed in Appendix B. Cyclic quotient singularities are toric singularities and can be resolved effectively by the usual method of subdivision as in [Ful93]. One can then compute the order of vanishing required by $\Omega$ to ensure extension. □

## 5.5 Automorphisms of deformation generalised Kummer manifolds and finite quotient singularities

In this section, we classify the possible local forms of the singularities in $\mathcal{F}_{L_{6,2p^2}}(\Gamma)$ and say a little about the automorphisms of deformation generalised Kummer manifolds. If $G \leq \mathrm{O}^+(L_{6,2p^2})$ is a finite group, then $G$ fixes a point in $[w] \in \mathcal{D}_{L_{6,2p^2}}$. Furthermore, by general symmetric space theory [Hel78], if

$$[w] \in \mathcal{D}_{L_{6,2p^2}} \cong \mathrm{SO}(2,4)/\,\mathrm{SO}(2) \times \mathrm{SO}(4),$$

then the isotropy subgroup $G_{[w]} \leq \mathrm{O}^+(2,4)$ of $[w]$ lies in the maximal compact subgroup $\mathrm{SO}(2) \times \mathrm{SO}(4)$ of $\mathrm{O}^+(2,4)$. Therefore, the isotropy subgroup $G$ of $[w]$ in $\mathrm{O}^+(L_{6,2p^2})$ is a finite subgroup of $\mathrm{SO}(2) \times \mathrm{SO}(4)$.

Due to the work of Zassenhaus, the finite subgroups of $\mathrm{SO}(n)$ can effectively (albeit expensively) be calculated and tables exist up to $n = 4$ (see, also, [CS03] for $\mathrm{SO}(4)$). However, we choose to exploit the 2:1 cover of $\mathrm{SO}(4)$ by $\mathrm{SU}(2) \times \mathrm{SU}(2)$ given by the exceptional isomorphism between $\mathrm{SO}^+(4)$ and $\mathrm{SU}(2) \times \mathrm{SU}(2)$ (see, for example, [Kna02]), as it results in a slightly simpler statement. Indeed, if one is only interested in computing a full list of possible singularities, it is sufficient to classify the representations of finite groups in $\mathrm{SO}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ and, for the sake of simplificity, this is the approach we take.

There is also a 2:1 cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$ and so one can classify the finite subgroups of $\mathrm{SO}(2) \times \mathrm{SO}(4)$ (as is explained in [Ste08]) in terms of the finite subgroups of $\mathrm{SO}(3)$, which were known to Plato. The finite subgroups of $\mathrm{SO}(3)$ are

1. The cyclic group $C_n = \langle a \mid a^n = e \rangle$
2. The dihedral group $D_{2n} = \langle a, i \mid a^n = i^2 = (ai)^2 = e \rangle$
3. The tetrahedral group $\mathbb{T} = \langle r, s, t \mid r^3 = s^2 = t^2 = rst \rangle$
4. The octahedral group $\mathbb{O} = \langle r, s, t \mid r^3 = s^2 = t^4 = rst \rangle$
5. The icosahedral group $\mathbb{I} = \langle r, s, t \mid r^3 = s^2 = t^5 = rst \rangle$

and the pre-images in SU(2) are known as the *binary polyhedral groups*. The case of the pre-image of the cyclic group $C_n$ is exceptional, and yields $C_{2n}$. The binary polyhedral groups are

1. The cyclic group $C_n = \langle a \mid a^n = e\rangle$
2. The binary dihedral group $BD_{2n} = \langle r, s, t \mid r^2 = s^2 = t^n = rst\rangle$
3. The binary tetrahedral group $B\mathbb{T} = \langle r, s, t \mid r^2 = s^3 = t^3 = rst\rangle$
4. The binary octahedral group $B\mathbb{O} = \langle r, s, t \mid r^2 = s^3 = t^4 = rst\rangle$
5. The binary icosahedral group $B\mathbb{I} = \langle r, s, t \mid r^2 = s^3 = t^5 = rst\rangle$

**Remark 5.5.1.** *If $X$ is a polarised deformation generalised Kummer manifold then, by using Lemma 5.4.1, one obtains a list (see Theorem 5.5.3) of the possible images of* $\mathrm{Aut}(X)$ *on $H^2(X, \mathbb{Z})$ up to abstract isomorphism (see also [BNWS11] [Ogu12]).*

As in Chapter 5.0.1, around a point $[w] \in \mathcal{F}_{L_{6,2p^2}}$, the space is locally isomorphic to the quotient of $\mathrm{Hom}(\mathbb{W}, \mathbb{W}^\perp/\mathbb{W}) =: V$ by the stabiliser $G \leq \Gamma_{6,2p^2}$ of $[w]$. The action of $G$ on $V$ corresponds to a 4-dimensional twisted representation of $G$. Therefore, in order to classify the local form of the singularities of $\mathcal{F}_{L_{6,2p^2}}(\Gamma)$, it is sufficient to classify the four dimensional complex representations of the finite subgroups of $\mathrm{SO}(2) \times \mathrm{SO}(4)$.

The character tables of the binary polyhedral groups, and their associated irreducible representations are given in Appendix A. The case of $C_n$ is well known, the case of $BD_{2n}$ can be found in [Ste08], and we used the computer algebra system GAP to compute for $B\mathbb{T}$, $B\mathbb{O}$ and $B\mathbb{I}$. The full set of representations can be determined by semisimplicity and Proposition 5.5.2.

**Proposition 5.5.2.** *[JL01] If $G$ and $H$ are finite groups with whose irreducible representations are given by $\rho_i$ and $\theta_j$, respectively, then the irreducible representations of $G \times H$ are given precisely by $\rho_i \otimes \theta_j$.*

We may summarise the above discussion as follows:

**Theorem 5.5.3.** *Around* $[w] \in \mathcal{F}_{L_{6,2p^2}}$*, the space* $\mathcal{F}_{L_{6,2p^2}}$ *is locally isomorphic to* $\mathbb{C}^4/G$ *where* $G \leq \mathrm{GL}(4, \mathbb{C})$ *and* $G \cong G_1 \times G_2 \times G_3$ *where* $G_1$ *is cyclic, and* $G_2$ *and* $G_3$ *are binary polyhedral groups. Every element in* $G$ *has order not exceeding 56 and the action of* $G$ *on* $\mathbb{C}^4$ *is given precisely by the degree 4 representations of* $G$*, which can be deduced from Appendix A.*

# Chapter 6

# Toroidal compactifications and singularities in the boundary

## 6.1 Toroidal compactifications

In this section, we describe the construction of a toroidal compactification of $\mathcal{F}_{L_{6,2p^2}}$ and study the singularities in the boundary. We begin by describing the Baily-Borel compactification, which is a canonical compactification that can be defined for any Hermitian symmetric space $H$, or an arithmetic quotient of $H$. Our notation will often view $H$ as the symmetric space $H = G/K$. However, all of the the spaces we consider will be of the form $H = \mathrm{SO}(2,n)/\mathrm{SO}(2) \times \mathrm{SO}(n)$ and can be concretely realised in terms of the quadric

$$\mathcal{D}_L = \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x,x) = 0, (x,\overline{x}) > 0\}$$

for a lattice $L$ of signature $(2,n)$.

A more extensive overview can be found in [BJ06] or [GHS13] (our treatment follows [GHS13]). In general, one defines the Baily-Borel compactification of $H$ by taking the closure of $H$ in the embedding $H \subset H^\vee$ given by the Harish-Chandra embedding. For the domain $\mathcal{D}_L$, this is simply the Zariski closure of $\mathcal{D}_L$ inside $\mathbb{P}(L \otimes \mathbb{C})$ (where $\mathbb{P}(L\otimes\mathbb{C})$ lies in side the compact dual $\mathcal{D}_L^\vee = \{x \in \mathbb{P}(L\otimes\mathbb{C}) \mid (x,x) = 0\}$). We shall refer to the Baily-Borel compactification of $\mathcal{D}_L$ by $\mathcal{D}_L^{BB}$. Given $\mathcal{D}_L^{BB}$, we define boundary components as follows:

**Definition 6.1.1.** *Let $x, y \in \mathcal{D}_L^{BB}$. We define an equivalence relation on $\mathcal{D}_L^{BB}$ by letting $x \sim y$ if and only if there exist finitely many holomorphic maps*

$$f_i : \Delta = \{z \in \mathbb{C} \mid |z| < 1\} \to \mathcal{D}_L^{BB}$$

*such that $x \in f_1(\Delta)$ and $y \in f_k(\Delta)$ and $f_i(\Delta) \cap f_{i+1}(\Delta) \neq \emptyset$ for $1 \geq 1 < k$. The equivalence classes are called the boundary components of $\mathcal{D}_L^{BB}$.*

The Baily-Borel compactification can be decomposed as

$$\mathcal{D}_L^{BB} = \mathcal{D}_L \bigsqcup_{P \in \mathcal{P}} F_P$$

where $\mathcal{P}$ is a set of *certain* parabolic subgroups of $G$ and $F_P$ is the symmetric space of $P$. In order to describe the Baily-Borel compactification of the arithmetic quotient $\mathcal{D}_L/\Gamma$, we need to restrict our attention to *rational boundary components.*

**Definition 6.1.2.** *We define the normaliser $N(F_P)$ and the centraliser $Z(F_P)$ of the boundary component $F_P$ inside $G$ by*

$$N(F_P) = \{g \in G \mid g(F_P) = F_P\}$$
$$Z(F_P) = \{g \in G \mid g|_{F_P} = \mathrm{id}\}.$$

**Definition 6.1.3.** *A boundary component $F$ of $\mathcal{D}_L^{BB}$ is called a rational boundary component if*

1. *The normalizer $N(F)$ of $F$ is a parabolic subgroup of $G$ and defined over $\mathbb{Q}$.*
2. *The centralizer $Z(F)$ contains a co-compact subgroup that is*
   (a) *normal in $N(F)$.*
   (b) *an algebraic subgroup over $\mathbb{Q}$.*

The group $\Gamma$ acts on the set of rational boundary components. Moreover, if $F_P$ is a rational boundary component, $\Gamma_P = N(F_P) \cap \Gamma$ is a discrete group and so $F_P/\Gamma_P$ is also an arithmetic quotient of a Hermitian symmetric space. One can then define the Baily-Borel compactification $(\mathcal{D}_L/\Gamma)^*$ of $\mathcal{D}_L/\Gamma$ by taking the quotient of

$$\mathcal{D}_L^* = \mathcal{D}_L \bigsqcup_{\substack{P \in \mathcal{P} \\ P \text{ rational}}} F_P$$

by $\Gamma$ [BB66].

**Theorem 6.1.4.** *[GHS13] [BB66] The Baily-Borel compactification $(\mathcal{D}_L/\Gamma)^*$ is an irreducible normal complex projective variety. It contains $\mathcal{D}_L/\Gamma$ as a Zariski-open subset and can be decomposed as*

$$(\mathcal{D}_L/\Gamma)^* = \mathcal{D}_L/\Gamma \sqcup \bigsqcup_{\substack{P \in \mathcal{P} \\ P \text{ rational}}} F_P/\Gamma_P$$

*where $\mathcal{P}$ runs over all the $\Gamma$-equivlence classes of parabolic subgroups determining rational boundary components.*

When $\Gamma \leq \mathrm{O}(L)$, the maximal parabolic subgroups of $\Gamma$ are precisely the stabilisers of totally isotropic subspaces in $L \otimes \mathbb{Q}$, and we can refine the above decomposition:

**Definition 6.1.5.** *Let $(\mathcal{D}_L/\Gamma)^*$ be the Baily-Borel compactification of $\mathcal{D}_L/\Gamma$ as in Theorem 6.1.4 where $L$ is a lattice of signature $(2, n)$. If $\Gamma_P \leq \Gamma$ is the stabiliser of a totally isotropic subspace of rank 1, we say that $F_P/\Gamma_P$ is a rank 1 boundary component; If $\Gamma_P \leq \Gamma$ is the stabiliser of a totally isotropic subspace of rank 2, we say that $F_P/\Gamma_P$ is a rank 2 boundary component. Collectively, the boundary components in $(\mathcal{D}_L/\Gamma)^*$ are called cusps.*

In the above situation, the rank 1 boundary components are points and the rank 2 boundary components are modular curves.

We can now begin to define toroidal compactifications of $\mathcal{F}_L(\Gamma)$. Toroidal compactifications exist for general arithmetic quotients of Hermitian symmetric domains, but we shall restrict our attention to the case $\mathcal{F}_L(\Gamma)$. Full details can be found in the monograph [AMRT10]. Toroidal compactifications are an especially appealing class of compactification to work with if one is interested in proving general type results because their singularities are, at worst, quotients of toric singularities, and these are usually easy to resolve.

The construction begins with a local construction at each cusp $F$, and ends by gluing the local constructions together. Ordinarily (for example, in the case of abelian surfaces [HKW93]), one has to check that certain compatibility conditions are satisfied for the gluing procedure to be well defined, but if $\Gamma \leq \mathrm{O}(2,n)$, these conditions are automatically satisfied. This turns out to be a major simplification.

For a boundary component $F$, we define the domain $\mathcal{D}_L(F)$ as

$$\mathcal{D}_L(F) = F \times V(F) \times U(F)_{\mathbb{C}} \tag{6.1}$$

where, if $W(F)$ is the unipotent radical of $N(F)$, $U(F)$ is the centre of $W(F)$ and $V(F) = W(F)/U(F)$ is a complex vector space. We have the natural maps

$$\begin{tikzcd}
\mathcal{D}_L(F) \arrow[dd, "\pi_F"'] \arrow[dr, "\pi'_F"] & \\
& \mathcal{D}_L(F)' \arrow[dl, "p_F"] \\
F &
\end{tikzcd}$$

where $\mathcal{D}_L(F)' = \mathcal{D}_L(F)/U(F)_{\mathbb{C}}$. The domain $\mathcal{D}_L$ can then be realised as a Siegel domain inside $\mathcal{D}_L(F)$ by the tube domain condition

$$\mathcal{D}_L = \{x \in \mathcal{D}_L(F) \mid \mathrm{Im}(\mathrm{pr}_U(x)) \in C(F)\}$$

for a cone $C(F) \subset U(F)$ where $\mathrm{pr}_U$ is the projection map from $\mathcal{D}(F)$ to $U(F)_{\mathbb{C}}$ in

Equation (6.1). If we define the map $\phi_F$ by

$$\phi_F : \mathcal{D}_L(F) \to U(F)$$

by $\phi_F : x \mapsto \mathrm{Im}(\mathrm{pr}_U(x))$, we obtain the diagram

$$\begin{tikzcd}
C(F) & \subset & U(F) & & \\
\mathcal{D}_L \arrow[u, "\phi_F"] \arrow[rrd] \arrow[dd, "\pi_F"'] & \subset & \mathcal{D}_L(F) \arrow[u, "\phi_F"'] \arrow[d, "\pi'_F"] & \subset & \mathcal{D}_L^\vee \\
 & & \mathcal{D}_L(F)' \arrow[lld, "p_F"] & & \\
F & & & &
\end{tikzcd}$$

Indeed, the spaces $\pi'_F : \mathcal{D}_L(F) \to \mathcal{D}_L(F)'$ and $p_F : \mathcal{D}_L(F)' \to F$ are principal homogeneous spaces for $U(F)_{\mathbb{C}}$ and $V(F)$, respectively. The group $N(F)_{\mathbb{Z}} = \Gamma \cap N(F)$ acts on $\mathcal{D}_L(F)$ and if we restrict to $U(F)_{\mathbb{Z}} := \Gamma \cap U(F)$, we obtain a principal fibre bundle

$$\mathcal{D}_L(F)/U(F)_{\mathbb{Z}} \to \mathcal{D}_L(F)' \tag{6.2}$$

whose fibre is $U(F)_{\mathbb{C}}/U(F)_{\mathbb{Z}}$, which is an algebraic torus $T(F)$. To obtain a partial compactification over the cusp $F$, one first obtains a fan by taking an $N(F)_{\mathbb{Z}}$-invariant decomposition of the cone $C(F)$ into rational polyhedral cones. The fan defines a toric variety $X_{\Sigma(F)} \supset T(F)$ and we can replace $T(F)$ in the bundle of equation (6.2) with $X_{\Sigma(F)}$ to obtain a new bundle over $\mathcal{D}_L(F)$ with fibre $X_{\Sigma(F)}$. One then takes the closure of $\mathcal{D}_L/U(F)_{\mathbb{Z}}$ in the new bundle and then the quotient by $N(F)_{\mathbb{Z}}$ to obtain a partial compactification for the cusp $F$. The final step involves glueing the partial compactifications together by identifying the copies of $\mathcal{D}_L/\Gamma$ contained in each one. In general, the decomposition of $C(F)$ is not arbitrary and compatibility conditions must be satisfied by fans that occur when one cusp lies in the closure of another cusp.

These are conditions are automatically satisfied for subgroups of $\mathrm{O}(2,n)$: two cusps $F_1$ and $F_2$ have intersecting closures if and only if their associated isotropic subspaces $E_1, E_2 \in L \otimes \mathbb{Q}$ satisfy $E_1 \subset E_2$. Therefore, we need only consider the intersection of a 1 dimensional boundary component with a 0 dimensional boundary component. For the orthogonal group $\mathrm{O}(2,n)$, $\dim U(F) = 1$ at a one dimensional cusps and so the decomposition of $C(F) \subset U(F)$ is trivial.

For the rest of this section, we shall work with an explicit description of a toroidal compactication of $\mathcal{F}_L(\Gamma)$ but, because of the following results, only the one dimensional boundary components will concern us.

## 6.2 Rank 1 boundary components

All of the results in this subsection can be found in [GHS07]. Suppose that $\dim E = 1$; then, as $V(F)$ is trivial,

$$\mathcal{D}_L(F) \cong F \times U(F)_{\mathbb{C}} = U(F)_{\mathbb{C}}.$$

We let $M(F) = U(F)_{\mathbb{Z}}$ and $T(F) = U(F)_{\mathbb{C}}/U(F)_{\mathbb{Z}}$. We obtain the partial compactification in the direction of $F$ by taking the closure of $\mathcal{D}_L(F)/U(F)_{\mathbb{Z}}$ in the bundle formed by replacing $T(F)$ in Equation (6.2) with a toric variety $X_\Sigma(F)$, and then taking the quotient by $G(F) = N(F)_{\mathbb{Z}}/U(F)_{\mathbb{Z}}$. However, in this case, the resulting bundle is $X_\Sigma(F)$. Indeed, while it is not immediate from the construction, one can choose $X_\Sigma(F)$ so that $X_\Sigma(F)$ is smooth and so that $G(F)$ acts on the closure of $\mathcal{D}_L(F)/U(F)_{\mathbb{Z}}$ in $X_\Sigma(F)$ (an explanation may be found in [FC90]) and so at the 0 dimensional boundary components, determining the singularities is reduced to a purely toric problem.

**Theorem 6.2.1.** *[GHS07] (Theorem 2.17) If $X_\Sigma \supset T$ is a smooth toric variety on which a finite group of torus automorphisms $G \leq \mathrm{Aut}(T)$ acts, then $X_\Sigma/G$ has canonical singularities*

Therefore, the singularities at in the zero dimensional boundary components can be ignored. One may still have to check if the branch divisor presents an obstruction but, because of the following theorem, this can also be ignored.

**Theorem 6.2.2.** *[GHS07] (Corollary 2.22) There are no divisors at the boundary over a zero dimensional cusp $F$ that are fixed by a non-trivial element of $G(F)$.*

We therefore need only to consider the one dimensional boundary components.

## 6.3 Rank 2 boundary components

We describe the compactification at the one dimensional boundary components explicitly, as in [Sca87], [Kon93] and [GHS07].

**Lemma 6.3.1.** *Let $E \leq L_{6,2p^2}$ be a primitive, totally isotropic subspace of rank 2 corresponding to the boundary component $F$. Then there exists a $\mathbb{Z}$-basis $\{v_1, \ldots, v_6\}$ of $L_{6,2p^2}$ such that $\{v_1, v_2\}$ is a basis for $E$ and $\{v_1, \ldots, v_4\}$ is a basis for $E^\perp$. The basis can be chosen so that the bilinear form $Q$ has Gram matrix*

$$Q = ((v_i, v_j)) = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C \\ {}^tA & {}^tC & D \end{pmatrix} \tag{6.3}$$

*where $B$ is the form on $E^\perp/E$ and*

$$A = \begin{pmatrix} 0 & a_1 \\ a_1 a_2 & 0 \end{pmatrix}.$$

*Here $a_1$ and $a_2$ are the elementary divisors of the group $D(L_{6,2p^2})/H_E^\perp$. Moreover,*

$$(a_1, a_1 a_2) \in \{(1,1), (1,2p), (1,6p)\}.$$

*Proof.* As $E$ and $E^\perp$ are primitive, the claim about the existence of a basis on which $Q$ assumes the form of Equation (6.3) is immediate. We next consider the matrix $A$. By considering the Smith normal form of $A$, we see that $A$ embeds $\langle v_5, v_6\rangle$ in the dual $\langle v_5^*, v_6^*\rangle$ and so the elementary divisors of $A$ correspond to the elementary divisors of the abelian group $\langle v_5^*, v_6^*\rangle/\langle v_5, v_6\rangle$. If $H_E = E^{\perp\perp}/E \leq D(L_{6,2p^2})$, then $H_E^{\perp} = \langle v_1^*, \ldots, v_4^*\rangle$ in $D(L_{6,2p^2})$ and so $\langle v_5^*, v_6^*\rangle/\langle v_5, v_6\rangle \cong D(L_{6,2p^2})/H_E^{\perp}$. We next determine $H_E$ and $H_E^{\perp}$. As $E$ is totally isotropic in $L_{6,2p^2}$, $H_E$ is totally isotropic in $D(L_{6,2p^2})$. If $D(L_{6,2p^2})$ is identified with $((-1/6) \oplus (-1/2p^2), C_6 \oplus C_{2p^2})$, then $(x, y) \in D(L_{6,2p^2})$ is isotropic if and only if

$$p^2x^2 + 3y^2 = 0 \mod 6p^2.$$

As $(3, p) = 1$, $p|y$ and so, $p^2x^2 + 3p^2y_1^2 = 0 \mod 6p^2$ and $x^2 + py_1^2 = 0 \mod 6$.

By considering squares modulo 6, we conclude that $x = 0$ or 3 and that $x$ and $y$ must have different parities. The isotropic elements in $D(L_{6,2p^2})$ are, therefore,

$$(x, y) \in \{(0, 2kp), (3, (2k + 1)p) \mid k \in \mathbb{Z}\}.$$

The primitive isotropic subspaces of rank 1 in $D(L_{6,2p^2})$ are generated by $x_1 = (0, 2p)$ and $x_2 = (3, p)$ and the single rank 2 totally isotropic subspace is generated by $\langle x_1, x_2\rangle$.

If $H_E = \langle x_1\rangle$,

$$H_E^{\perp} = \{(a, b) \in D(L_{6,2p^2}) \mid pa + 6b \equiv 0 \mod 6p\}$$

and so $p|b$, $6|a$ and $H_E^{\perp} = \langle(0, p)\rangle \cong C_{2p}$.

If $H_E = \langle x_2\rangle$,

$$H_E^{\perp} = \{(a, b) \in D(L_{6,2p^2}) \mid pa + b \equiv 0 \mod 2p\}$$

and so $p|b$, $2|(a + b)$ and $H_E^{\perp} = \langle(1, p), (2, 0)\rangle$. If $y_1 = (1, p)$ and $y_2 = (2, 0)$, we also have the relations

$$6py_1 = 0$$
$$3y_2 = 0$$

and so $p(2y_1 - y_2) = 0$. Moreover, because $p \equiv \pm 1$ modulo (6), $2py_1 = \pm y_2$ and so

$H_E^\perp = \langle y_1 \rangle = \langle (1,p) \rangle \cong C_3 \oplus C_{2p}$. If $H_E = \langle x_1, x_2 \rangle$ then $H_E^\perp = \langle y_1 \rangle = \langle (1,p) \rangle \cong C_3 \oplus C_{2p}$. We conclude that,

1. If $H_E = \{0\}$, then $H_E^\perp = D(L_{6,2p^2})$ and $D(L_{6,2p^2}/H_E^\perp \cong \{0\}$.

2. If $H_E = \langle x_1 \rangle$, then $H_E^\perp = \langle (0,p) \rangle \cong C_{2p}$ and $D(L_{6,2p^2})/H_E^\perp \cong C_6 \oplus C_p$.

3. If $H_E = \langle x_2 \rangle$, then $H_E^\perp = \langle (1,p) \rangle \cong C_3 \oplus C_{2p}$ and $D(L_{6,2p^2})/H_E^\perp \cong C_2 \oplus C_p$.

4. If $H_E = \langle x_1, x_2 \rangle$, then $H_E^\perp = \langle (1,p) \rangle \cong C_3 \oplus C_{2p}$ and $D(L_{6,2p^2})/H_E^\perp \cong C_2 \oplus C_p$.

The result follows. $\square$

It is likely that the following lemma was proved in [Bri83], but we prove it here as we were unable to locate a copy.

**Lemma 6.3.2.** *Let $L$ be a lattice of signature $(2,n)$ and let $E \subset L$ be a primitive totally isotropic subspace of rank 2. If $H_E := E^{\perp\perp}_{L^\vee}$, then the discriminant form of the lattice $E^\perp/E$ is given by*

$$D(E^\perp/E) \cong H_E^\perp/H_E \subset D(L)$$

*Proof.* Let $E \leq L$ be a primitive totally isotropic subspace of rank 2. As $E$ and $E^\perp$ are primitive in $L$, then as a $\mathbb{Z}$-module, $L \cong (E^\perp/E) \oplus E \oplus F$ for some $F \leq L$. As a $\mathbb{Z}-$module, $L^\vee = \mathrm{Hom}(L, \mathbb{Z})$ assumes the following form

$$L^\vee \cong (E^\perp/E)^\vee \oplus (E \oplus F)^\vee$$

Moreover, $E^{\perp\perp} \subset L^\vee$ is primitive in $(E \oplus F)^\vee$ and we can take a basis $\{e_1^*, f_1^*, e_2^*, f_2^*\}$ of $(E \oplus F)^\vee$ so that $E^{\perp\perp} = \langle e_1^*, e_2^* \rangle$ and such that the bilinear form on $(E \oplus F)^\vee \subset L^\vee$ is equal to $U \oplus U$. Because $(E^\perp/E)$ is non-degenerate, $(E^\perp/E)^\vee$ has a basis $\mathcal{B}$ in $(E^\perp/E) \otimes \mathbb{Q}$. With respect to the basis $\{e_1^*, f_1^*, e_2^*, f_2^*\} \cup \mathcal{B}$, the form on $L^\vee$ is $U \oplus U \oplus L_0$. Therefore,

$$D(L) = L^\vee/L \cong \frac{\langle e_1^*, f_1^*, e_2^*, f_2^* \rangle}{E \oplus F} \oplus D(E^\perp/E).$$

As $H_E = \langle e_1^*, e_2^* \rangle / E$, therefore $D(E^\perp/E) \cong H_E^\perp/H_E \subset D(L)$. $\square$

**Corollary 6.3.3.** *Only the case* $(a_1, a_1a_2) = (1, 1)$ *or* $(1, 2p)$ *occurs in Lemma 6.3.1.*

*Proof.* By Lemma 6.3.2, the negative definite lattice $B$ has discriminant form $D(B) = H_E^{\perp}/H_E \leq D(L_{6,2p^2})$ and so if $(a_1, a_1a_2) = (1, 6p)$, then $D(B) = ((1/2), C_2)$. By using tables in [CS99], we see that no such $B$ can exist. The other cases may exist, though. If $(a_1, a_1a_2) = (1, 2p)$, $D(B) = ((1/3), C_3)$ and $B = A_2(-1)$. If $(a_1, a_1a_2) = (1, 1)$, $D(B) = ((-1/6) \oplus (-1/2p^2), C_6 \oplus C_{2p^2})$ and $B$ may be equal to $\langle -6 \rangle \oplus \langle -2p^2 \rangle$. $\square$

**Lemma 6.3.4.** *There exists a basis* $\{v_1, \ldots v_6\}$ *for* $L_{6,2p^2} \otimes \mathbb{Q}$ *such that* $\{v_1, v_2\}$ *form a* $\mathbb{Z}-$*basis for* $E$ *and* $\{v_1, \ldots v_4\}$ *form a* $\mathbb{Z} -$ *basis for* $E^{\perp}$ *and*

$$Q = ((v_i, v_j)) = \begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ A & 0 & 0 \end{pmatrix}$$

*where* $A$ *and* $B$ *are as described previously in Lemma 6.3.1.*

*Proof.* This is essentially Lemma 2.24 of [GHS07]. Let $R = -B^{-1}C \in M_2(\mathbb{Z}[1/\det B])$ and let $R' \in M_2(\mathbb{Z}[1/\det B])$ satisfy

$$D - {}^tCB^{-1}C + {}^tR'A + {}^tAR' = 0$$

and define the base change matrix

$$N = \begin{pmatrix} I & 0 & R' \\ 0 & I & R \\ 0 & 0 & I \end{pmatrix}.$$

$\square$

**Lemma 6.3.5.** *The groups $N(F)$, $W(F)$ and $U(F)$ are given by*

$$N(F) = \left\{ \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \mid \begin{array}{c} {}^tUAZ = A,\ {}^tXBX = B,\ {}^tVAZ = 0,\ {}^tXBY +^t VAZ = 0 \\ {}^tYBY +^t ZAW +^t WAZ = 0,\ \det(U) > 0 \end{array} \right\}$$

$$W(F) = \left\{ \begin{pmatrix} I & V & W \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix} \mid BY +^t VA = 0,\ {}^tYBY + AW +^t WA = 0 \right\}$$

$$U(F) = \left\{ \begin{pmatrix} I & 0 & \begin{pmatrix} 0 & a_1a_2x \\ -x & 0 \end{pmatrix} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

*Proof.* Direct calculation (as [Kon93]). $\square$

We also need a description for $N(F)_{\mathbb{Z}}$. As mentioned in Proposition 2.27 of [GHS07], if $g \in N(F)$ is given on the above basis then $g \in N(F)_{\mathbb{Z}}$ if

$$N^{-1}gN = \begin{pmatrix} U & V & -VB^{-1}C + W + UR' - R'Z \\ 0 & X & Y - XB^{-1}C + B^{-1}CZ \\ 0 & 0 & Z \end{pmatrix} \in \mathrm{GL}(6, \mathbb{Z}).$$

We next identify $D_L(F)$ with $(z, w_1, w_2, \tau) \in \mathbb{C} \times \mathbb{C}^2 \times \mathbb{H}$ as a Siegel domain (as explained in [Kon93] or [GHS07]). The identification proceeds by choosing homogeneous coordinates $[t_1 : \ldots : t_6]$ on $\mathbb{P}(L \otimes \mathbb{C})$. The map $\mathcal{D}_L(F) \to \mathbb{P}(L \otimes \mathbb{C})$ is given by $t_6 := 1$, $t_1 \mapsto z \in \mathbb{C}$, $t_3 \mapsto w_1 \in \mathbb{C}$, $t_5 \mapsto \tau$ and $t_2 \mapsto \frac{-2\delta z\tau - (w_1,w_2)B^t(w_1,w_2)}{2\delta a_2}$.

**Proposition 6.3.6.** *Let*

$$g = \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \in N(F)$$

*where* $Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *The action of* $g$ *on* $\mathcal{D}_L(F)$ *is given by*

$$\begin{cases} z \mapsto \frac{z}{\det Z} + (c\tau + d)^{-1} \left( \frac{c}{2\delta \det Z} {}^t\underline{w} B \underline{w} + \underline{V_1} \underline{w} + W_{11}\tau + W_{12} \right) \\ \underline{w} \mapsto (c\tau + d)^{-1} \left( X\underline{w} + Y \left( \begin{smallmatrix} \tau \\ 1 \end{smallmatrix} \right) \right) \\ \tau \mapsto \frac{a\tau + b}{c\tau + d} \end{cases}$$

*Proof.* As in [GHS07]. □

## 6.4 Bounds on the boundary components

We wish to examine the non-canonical singularities in $X$. Because of Theorem 6.2.1 (as in [GHS07]) the compactification may be chosen so that all the singularities at the 0 dimensional cusps are canonical. Therefore, we need only to consider the compactification at the 1 dimensional cusps. The boundary components of $\mathcal{F}_L(\Gamma)$ correspond to precisely to $\Gamma$-orbits of totally isotropic subspaces in $L \otimes \mathbb{Q}$, with the zero dimensional cusps corresponding to the orbits of isotropic lines and the one dimensional cusps corresponding to the orbits of totally isotropic planes. We begin by using the approach of [Sca87] to determine the $\mathrm{O}(L_{6,2})$-orbits of totally isotropic planes in $L_{6,2} \otimes \mathbb{Q} = L_{6,2p^2} \otimes \mathbb{Q}$. This involves showing that given a totally isotropic subspace $E \leq L_{6,2}$, the bilinear form on $L_{6,2}$ can be put into a certain normal form.

**Lemma 6.4.1.** *If* $E \leq L_{6,2}$ *is primitive and totally isotropic of rank 2, then* $E^\perp/E \cong \langle -6 \rangle \oplus \langle -2 \rangle$ *or* $E^\perp/E \cong A_2(-1)$.

*Proof.* We consider the subspaces $H_E \leq D(L_{6,2})$. As $E$ is totally isotropic, $H_E \leq D(L_{6,2})$ is totally isotropic. As usual, identify $D(L_{6,2})$ with $C_6 \oplus C_2$. If $(a, b) \in D(L_{6,2})$ is isotropic, then $a^2/6 - b^2/2 = 0$ $(\mathbb{Q}/\mathbb{Z})$ which has solutions $(a, b) = (0, 0)$ or $(a, b) = (3, 1)$. If $H_E = \{(0, 0)\}$, then $H_E^\perp/H_E = D(L_{6,2})$ with form $((-1/6) \oplus (1/2), C_6 \oplus C_2)$. If $H_E = \langle (3, 1) \rangle$, then $H_E^\perp = \langle (1, 1) \rangle$ and $H_E^\perp/H_E \cong \langle (2, 0) \rangle$ with form $((1/3), C_3)$. By using tables in [CS99], we see that there are two negative definite even lattices of determinant 12: $\langle -6\rangle \oplus \langle -2\rangle$ and $\left(\begin{smallmatrix} -4 & -2 \\ -2 & -4 \end{smallmatrix}\right)$ but, as calculated previously, the discriminant form of the second lattice is inequivalent to $((1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3)$. Therefore, in the case $H_E = \langle (0,0)\rangle$ we have $E^{\perp}/E \cong \langle -6\rangle \oplus \langle -2\rangle$; similarly, by using tables in [CS99], in the case $H_E = \langle (3,1)\rangle$ we have $E^{\perp}/E \cong A_2(-1)$. $\square$

**Lemma 6.4.2.** *Let $E \leq L_{6,2} \otimes \mathbb{Q}$ be a totally isotropic subspace of rank 2. Then there exists a $\mathbb{Z}-$basis $\{v_1, \dots v_6\}$ of $L_{6,2}$ such that $\{v_1, v_2\}$ is a basis for $E$ and $\{v_1, \dots, v_4\}$ is a basis for $E^{\perp}$ and the inner product on $L_{6,2}$ becomes*

$$Q = ((v_i, v_j)) = \begin{pmatrix} 0 & 0 & P \\ 0 & B & C \\ P & {}^tC & Q \end{pmatrix}$$

*where*

1. *If $H_E = \langle (1,1)\rangle$, then $B = \langle -6\rangle \oplus \langle -2\rangle$ and $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $Q = C = 0$.*
2. *If $H_E = \langle (3,1)\rangle$, then $B = A_2(-1)$ and $P = \left(\begin{smallmatrix} 0 & 1 \\ 3 & 0 \end{smallmatrix}\right)$ and $Q = \left(\begin{smallmatrix} 2d & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for $d \in \{0,1,2\}$ and $C = \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)$ for $c \in \{0,1,2\}$.*

*Proof.* We start by taking a basis $\{v_1, \dots v_6\}$ of $L_{6,2}$ for which $\{v_1, v_2\}$ is a basis for $E$ and $\{v_1, \dots v_4\}$ is a basis for $E^{\perp}$. Suppose that on this basis

$$Q = ((v_i, v_j)) = \begin{pmatrix} 0 & 0 & A_0 \\ 0 & B_0 & C_0 \\ {}^tA_0 & {}^tC_0 & D_0 \end{pmatrix}.$$

By Lemma 6.4.1, $H_E = \langle (0,0)\rangle$ or $H_E = \langle (3,1)\rangle$. If $H_E = \langle (0,0)\rangle$ then, by the Elementary Divisor Theorem, there exist $U, Z \in \mathrm{GL}(2,\mathbb{Z})$ such that

$$UA_0Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Moreover, there exists $X \in \mathrm{GL}(2,\mathbb{Z})$ such that ${}^tXB_0X = B = \langle -6\rangle \oplus \langle -2\rangle$, and so the matrix $g_1 := \operatorname{diag}(U, X, Z) \in \mathrm{GL}(6, \mathbb{Z})$ transforms $Q$ to $Q'$ where

$$Q' =^t g_1 Q g_1 = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C_1 \\ {}^tA & {}^tC_1 & D_1 \end{pmatrix}.$$

Now consider

$$g_2 := \begin{pmatrix} I & -{}^tA^tC_1 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \in \mathrm{GL}(6, \mathbb{Z}).$$

The map $g_2$ transforms $Q'$ to $Q''$ where

$$Q'' = \begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ {}^tA & 0 & D_2 \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We next require that $D_2$ be put into the correct form. Consider

$$g_3 := \begin{pmatrix} I & 0 & W \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \in \mathrm{GL}(6, \mathbb{Z})$$

$g_3$ sends $D_2 \mapsto D_2 + {}^tWA + {}^tAW$. One checks that ${}^tWA + {}^tAW$ contains all matrices of the form

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

where $a, b, c \in \mathbb{Z}$. Therefore, there exists $W$ so that $g_3$ sends

$$D_2 \mapsto \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}$$

where $d_{11}$ and $d_{22}$ are taken modulo 2. However, as the form $Q$ is even, $d_{11}$ and $d_{22}$ are both even. Therefore, there exists $W$ so that $g_3$ sends $D_2$ to 0. The matrix $g_3 g_2 g_1 \in \mathrm{GL}(6, \mathbb{Z})$ gives the required base change.

If $H_E = \langle (3,1) \rangle$ then, by the Elementary Divisor Theorem, there exist $U, Z \in \mathrm{GL}(2, \mathbb{Z})$ such that

$$UA_0Z = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$$

Moreover, there exists $X \in \mathrm{GL}(2, \mathbb{Z})$ such that ${}^tXB_0X = B = A_2(-1)$, and so the matrix $g_4 := \mathrm{diag}(U, X, Z) \in \mathrm{GL}(2, \mathbb{Z})$ transforms $Q$ to $Q'$ where

$$Q' = ^t g_1 Q g_1 = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C_1 \\ {}^tA & {}^tC_1 & D_1 \end{pmatrix}.$$

and

$$A = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}.$$

Now consider

$$g_5 := \begin{pmatrix} I & P & 0 \\ 0 & I & Q \\ 0 & 0 & I \end{pmatrix} \in \mathrm{GL}(6, \mathbb{Z}).$$

for some $P, Q \in M_2(\mathbb{Z})$. We claim that $P$ and $Q$ can be chosen such that

$${}^tPA + BQ + C_1 = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$$

where $a$ is determined modulo 3. We have

$${}^tPA + BQ + C_1 = \begin{pmatrix} 3p_{21} - 2q_{11} - q_{21} + c_{11} & p_{11} - 2q_{12} + q_{22} + c_{12} \\ 3p_{22} - 2q_{21} - q_{11} + c_{21} & p_{12} - q_{12} - 2q_{22} + c_{22} \end{pmatrix}$$

The claim about the second column is immediate as $p_{11}$ and $p_{12}$ are both free. For the first column, we can work modulo 3 as $p_{21}$ and $p_{22}$ are free. As

$$\delta := 2q_{11} + q_{21} = -(2q_{21} + q_{11}) \mod 3$$

the first column can be mapped to ${}^t(0, c_{11} + c_{21})$ modulo 3 for an appropriate choice of $\delta$.

Therefore,

$$g_5 = \begin{pmatrix} I & P & 0 \\ 0 & I & Q \\ 0 & 0 & I \end{pmatrix}$$

with $P$ and $Q$ chosen as above transforms $Q'$ to

$$Q'' = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C_0 \\ {}^tA & {}^tC_0 & D_2 \end{pmatrix}$$

where $C_0$ is as in the statement of the theorem. We next require that $D_2$ be put into the correct form. Consider

$$g_6 := \begin{pmatrix} I & 0 & W \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \in \mathrm{GL}(6, \mathbb{Z})$$

for $W \in M_2(\mathbb{Z})$. The element $g_6$ sends

$$D_2 \mapsto D_2 + {}^tWA + {}^tAW.$$

One checks that the set $\{{}^tWA + {}^tAW \mid W \in M_2(\mathbb{Z})\}$ contains all matrices of the form

$$\begin{pmatrix} 6a & b \\ b & 2c \end{pmatrix}$$

where $a, b, c \in \mathbb{Z}$. Therefore, there exists $W$ so that $g_3$ sends

$$D_2 \mapsto \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}$$

where $d_{11}$ is taken modulo 6 and $d_{22}$ is taken modulo 2. As the form $Q$ is even, $d_{11}$ and $d_{22}$ are both even and therefore there exists $W$ so that $g_3$ sends $d_{11}$ to one of 0, 2 or 4 and the rest to 0. Therefore $g_6 g_5 g_3 \in \mathrm{GL}(6, \mathbb{Z})$ gives the required base change. $\square$

**Theorem 6.4.3.** *The modular variety $\mathcal{F}_\Gamma$ has at most $320(p^5 + p^2)$ rank 2 boundary components.*

*Proof.* If $E_1, E_2 \leq L_{6,2}$ are primitive totally isotropic subspaces of rank 2 with the same normal form, then by Lemma 6.4.2, there exist bases $\{v_1, \ldots, v_6\}$ and $\{w_1, \ldots, w_6\}$ of $L_{6,2}$ such that $\{v_1, v_2\}$, $\{w_1, w_2\}$ are bases for $E_1$ and $E_2$ respectively and

$$((v_i, v_i)) = ((w_i, w_j)) = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C \\ A & {}^tC & D \end{pmatrix}.$$

Accordingly, one can define $g \in \mathrm{O}(L_{6,2})$ by $g : v_i \mapsto w_i$ such that $g(E_1) = E_2$ and so there are at most 20 totally isotropic rank 2 subspaces of $L_{6,2}$ up to $\mathrm{O}^+(L_{6,2})$ equivalance. By Theorem 4.0.11,

$$|\,\mathrm{O}^+(L_{6,2}) : \mathrm{O}^+(L_6, h^s_{2p^2})| = 16(p^5 + p^2)$$

and so, up to $\mathrm{O}^+(L_6, h^s_{2p^2})$ equivalence, there are at most $320(p^5+p^2)$ rank 2 boundary components. $\square$

## 6.5 A reduction procedure and singularities in a boundary component

We next show that the set of fixed points can be reduced by application of special elements in $N(F)_{\mathbb{Z}}$. This enables us to produce an upper bound for the number of components of the singular locus. For a given boundary component $F$, we define $N = a_1 a_2 \det B$. Without loss of generality, we can assume that the basis chosen in Lemma 6.3.4 is such that the lattice given by $B$ has a basis given by the fundamental polyhedron.

**Lemma 6.5.1.** *Let $E$ be a rank 2 totally isotropic subspace corresponding to the boundary component $F$. Let $A = \operatorname{diag}(a_1, a_1 a_2)$, as in Lemma 6.3.4. Then the principal congruence subgroup of level $N$, $\Gamma(N)$, embeds in $N(F)$. The embedding is given by sending $Z \in \Gamma(N)$ to*

$$g_Z = \begin{pmatrix} Z' & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Z \end{pmatrix} \in N(F)_{\mathbb{Z}}$$

*where, if*

$$Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{we write} \qquad Z' = \begin{pmatrix} d & -ca_2 \\ -b/a_2 & a \end{pmatrix}.$$

*Proof.* Let

$$g = \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \in N(F).$$

If $g \in N(F)_{\mathbb{Z}}$ then, by Lemma 6.3.5, the following are integral matrices

$$U = X = Z \tag{6.4}$$

$$-VB^{-1}C + W + UR' - RZ \tag{6.5}$$

$$Y - XB^{-1}C + B^{-1}CZ. \tag{6.6}$$

Let

$$Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$$

and let $X = I$ and $V = W = Y = 0$. By Lemma 6.3.4, we can suppose that $R', B^{-1} \in M_2(\mathbb{Z}[1/\det B])$. By Lemma 6.3.5, ${}^tUAZ = A$ and so

$$U = \begin{pmatrix} d & -ca_2 \\ -b/a_2 & a \end{pmatrix}.$$

As $Z \in \Gamma(N)$, it follows that $U \in M_2(\mathbb{Z})$. Because of Equations (6.5) and (6.6), we obtain the following integral matrices:

$$UR' - R'Z \tag{6.7}$$

$$-B^{-1}C + B^{-1}CZ. \tag{6.8}$$

If

$$R' = \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

then

$$UR' - R'Z = \begin{pmatrix} -a_2cy - aw + dw - cx & -a_2cz - bw \\ -cz - bw/a_2 & -by + az - dz - bx/a_2 \end{pmatrix} \in M_2(\mathbb{Z}).$$

As $Z \in \Gamma(N)$, then $a \equiv d \equiv 1$ modulo $N$ and $b \equiv c \equiv 0$ modulo $N$ and so Equation (6.5) is satisfied. Furthermore, $Z \equiv I$ modulo $N$ and so $C - CZ \equiv 0$ modulo $N$. As $\det B | N$,

$$-B^{-1}C + B^{-1}CZ = B^{-1}(C - CZ) \in M_2(\mathbb{Z})$$

and so $\Gamma(N) \leq N(F)_{\mathbb{Z}}$. $\square$

**Lemma 6.5.2.** *Let $E$ be a rank 2 totally isotropic subspace corresponding to the boundary component $F$. Let $A = \operatorname{diag}(a_1, a_1a_2)$, as in Lemma 6.3.4. The group $W(F)_{\mathbb{Z}}$ contains all elements of the form*

$$g_Y = \begin{pmatrix} I & * & * \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix}$$

*where $Y \in M_2(N\mathbb{Z})$.*

*Proof.* If

$$g_Y = \begin{pmatrix} I & V & W \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix} \in W(F)_{\mathbb{Z}}$$

then by Lemma 6.3.5,

$$BY + {}^tVA = 0 \tag{6.9}$$

$${}^tYBY + AW + {}^tWA = 0. \tag{6.10}$$

Furthermore, by Lemma 6.3.4,

$$N^{-1}gN = \begin{pmatrix} I & V & W - VB^{-1}C \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix}$$

subject to the conditions that

$$W - VB^{-1}C \tag{6.11}$$

$$V = Y = 0 \tag{6.12}$$

are both integral. We look for solutions satisfying Equation (6.12). Equation (6.9) has a solution in $V$ if $Y \in M_2(a_1a_2\mathbb{Z})$ and Equation (6.11) is satisfied if $V \in M_2(\det B\mathbb{Z})$. Because of Equation (6.9), we can ensure that both are satisfied if $Y \in M_2(N\mathbb{Z})$.

If

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

then Equation (6.10) becomes

$$\begin{aligned} -{}^tYBY &= AW +^t WA \\ &= \begin{pmatrix} 2a_1w_{11} & a_1w_{12} + a_1a_2w_{12} \\ a_1a_2w_{21} + a_1w_{12} & 2a_1a_2w_{22} \end{pmatrix} \end{aligned}$$

and, by considering Equation (6.9), has a solution in $W$ if $Y \in M_2(2a_1a_2\mathbb{Z})$. All such conditions are clearly satisfied if $Y \in M_2(N\mathbb{Z})$. □

**Theorem 6.5.3.** *If $(a_1, a_1a_2) = (1, 1)$ the singular locus of a boundary component contains of the order of $p^6$ points and $p^5$ lines. The number of surfaces in the boundary component does not depend on $p$. If $(a_1, a_1a_2) = (1, 2p)$ the singular locus of a boundary component contains of the order of $p^{14}$ points, $p^{12}$ lines, and $p^9$ surfaces.*

*Proof.* By Proposition 6.3.6, $g$ acts on $(z, \underline{w}, \tau)$ by

$$\begin{aligned} z &\mapsto \frac{z}{\det Z} + (c\tau + d)^{-1}\left(\frac{c}{2\delta\det Z}{}^t\underline{w}B\underline{w} + \underline{V_1}\underline{w} + W_{11}\tau + W_{12}\right) \\ \underline{w} &\mapsto (c\tau + d)^{-1}\left(X\underline{w} + Y\left(\begin{smallmatrix}\tau \\ 1\end{smallmatrix}\right)\right) \\ \tau &\mapsto \frac{a\tau + b}{c\tau + d}. \end{aligned}$$

In particular (as noted in [GHS07]), $\tau$ is $\mathrm{SL}(2, \mathbb{Z})$ equivalent to $i$ or a cube root of unity $\omega$. Indeed, $\tau \in \mathrm{SL}(2, \mathbb{Z})i$ if $Z$ is of order 4 and $\tau \in \mathrm{SL}(2, \mathbb{Z})\xi_3$ if $Z$ is of order 3 or 6. Moreover, if

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

then

$$\tau = \frac{\alpha\theta + \beta}{\gamma\theta + \delta}$$

where $\theta \in \{i, \xi_3\}$ and so

$$\tau = \frac{(\alpha\gamma + \delta\beta) + (\alpha\delta + \beta\gamma)\operatorname{Re}\theta + (\alpha\delta - \beta\gamma)\operatorname{Im}\theta i}{\gamma^2 + \delta^2 + 2\gamma\delta\operatorname{Re}\theta}.$$

and we define $J$ by

$$J = 2(\gamma^2 + \delta^2 + 2\gamma\delta\operatorname{Re}\theta)$$

and $K_1$ and $K_2$ by

$$\tau = \frac{K_1}{J} + \frac{K_2}{J}\upsilon,$$

where $\upsilon \in \{i, \omega\}$. At $\underline{w}$,

$$\underline{w} = (c\tau + d)^{-1}\left(X\underline{w} + Y\left(\begin{smallmatrix}\tau\\1\end{smallmatrix}\right)\right). \tag{6.13}$$

For $Z$ defined by $g$, we define $\xi = (c\tau + d)^{-1}$ and $T$ by

$$T = I - \xi X.$$

As observed in [GHS07] Proposition 2.28, $\xi$ is a sixth or a fourth root of unity. This follows because $G_4(i) \neq 0$ and $G_6(\xi_3) \neq 0$ where $G_k$ is the weight-$k$ Eisenstein series (see [DS05]). In particular, $\xi$ is a sixth root of unity if $Z$ is of order 3 or 6 and a fourth root of unity if $Z$ is of order 4.

If $\det T \neq 0$, then

$$\underline{w} \in T^{-1}Y\left(\begin{smallmatrix}\tau\\1\end{smallmatrix}\right)$$

and so, by noting that $Y \in M_2(\mathbb{Z}[1/\det B])$, we have that $\underline{w} \in L \times L$ where

$$L = \frac{\langle 1, \tau\rangle}{\det T \det B}$$

(and where $\langle 1, \tau\rangle$ denotes the lattice in $\mathbb{C}$ generated by 1 and $\tau$). We can assume that the basis $\{v_1, \ldots, v_6\}$ is given so that $\{v_3, v_4\}$ defines the fundamental polyhedron of the lattice $B$. We can therefore assume that $X$ is one of the standard automorphisms of $B$ given in the introduction. The value of $\det B$ in each case is given in Table 6.1.

| $\xi$ | $i$ | $-1$ | $-i$ | $1$ | $\xi_6$ | $\omega$ | $\omega^2$ | $\xi_6^5$ |
|---|---|---|---|---|---|---|---|---|
| $\phi_1(x)^2$ | $2i$ | $4$ | $-2i$ | $0$ | $\xi_6-1$ | $3\xi_6$ | $-3\xi_6+3$ | $-\xi_6$ |
| $\phi_1(x)\phi_2(x)$ | $2$ | $0$ | $2$ | $0$ | $\xi_6+1$ | $\xi_3^2$ | $\xi_6+1$ | $\xi_3^2$ |
| $\phi_2(x)^2$ | $-2i$ | $0$ | $2i$ | $4$ | $-3\xi_6+3$ | $-\xi_6$ | $\xi_6-1$ | $3\xi_6$ |
| $\phi_3(x)$ | $-i$ | $1$ | $i$ | $3$ | $-2\xi_6+2$ | $0$ | $0$ | $2\xi_6$ |
| $\phi_4(x)$ | $0$ | $2$ | $0$ | $2$ | $-\xi_6+1$ | $\xi_6$ | $-\xi_6+1$ | $\xi_6$ |
| $\phi_6(x)$ | $i$ | $3$ | $-i$ | $1$ | $0$ | $2\xi_6$ | $-2\xi_6+2$ | $0$ |

**Table 6.1**

We next consider $L$ for each value of $\det B$. By direct calculation we find that,

If $Z$ is order 4, $\qquad L \leq \dfrac{\langle 1, i\rangle}{JK\det B} \qquad K = 1,2,3,4$

if $Z$ is order 3 or 6, $\qquad L \leq \dfrac{\langle 1, \sqrt{3}i\rangle}{2KJ\det B} \qquad K = 1,2,3,4,6.$

We next bound the number of components of the singular locus in each boundary component by using the elements defined in Lemma 6.5.1 and Lemma 6.5.2.

By Lemma 6.5.1, $\Gamma(N) \leq N(F)$. It is well known (see [DS05]) that

$$|\operatorname{SL}(2,\mathbb{Z}) : \Gamma(N)| = N^3 \prod_{p|N}\left(1-\frac{1}{p^2}\right)$$

and as

$$|\operatorname{O}^+(L_{6,2p^2}) : \widetilde{\operatorname{O}}^+(L_{6,2p^2})| = 16,$$

there are at most

$$16N^3 \prod_{p|N}\left(1-\frac{1}{p^2}\right) =: K_N$$

equivalence classes of $\tau$ modulo $N(F) \cap \operatorname{O}(L_6, h^s_{2p^2})$. If $Z$ is of order 4, then

$$w_j = \frac{x_{1j}}{JK\det B} + \frac{x_{2j}i}{JK\det B} \in \frac{\langle 1, i\rangle}{JK\det B}$$

and

$$g_Y : w_j \mapsto \frac{x_{1j} + KK_1\det BY_{j1} + Y_{j2}JK\det B}{JK\det B} + \frac{(x_{2j} + K\det BK_2Y_{j1})i}{JK\det B}$$

and as $Y \in M_2(N\mathbb{Z})$ can be chosen arbitrarily, $x_{1j}$ can be reduced modulo $NJK \det B$ and $x_{2j}$ can be reduced modulo $NKK_2 \det B$.

If $Z$ is of order 3 or 6, then

$$w_j = \frac{x_{1j}}{2JK \det B} + \frac{x_{2j}\sqrt{3}i}{2JK \det B} \in \frac{\langle 1, \sqrt{3}i \rangle}{2JK \det B}$$

and

$$g_Y : w_j \mapsto \frac{x_{1j} + 2KK_1 \det BY_{j1} + 2Y_{j2}JK \det B}{JK \det B} + \frac{(x_{2j} + 2K \det BK_2 Y_{j1})\sqrt{3}i}{2JK \det B}$$

and as $Y \in M_2(N\mathbb{Z})$ can be chosen arbitrarily, $x_{1j}$ can be reduced modulo $2NJK \det B$ and $x_{2j}$ can be reduced modulo $2NKK_2 \det B$. We consider the cases where $\det T = 0$ separately. They occur when

$(\chi_X(x), \xi) \in \{\ (\phi_1\phi_2, -1), (\phi_2^2, -1), (\phi_4, -i), (\phi_2^2, 1), (\phi_1\phi_2, 1), (\phi_6, \xi_6), (\phi_3, \xi_3), (\phi_3, \xi_3^2), (\phi_6, \xi_6^5)\}$. In each case we solve Equation (6.13) directly, and reduce as above. We find that:

1. If $(\chi_X(x), \xi) = (\phi_1\phi_2, -1)$, $w_2 \in \mathbb{C}$ is free and $w_1 \in \frac{\langle 1,i \rangle}{2J \det B}$ and so $w_1$ can be reduced to one of $2NJ \det B$ points.

2. If $(\chi_X(x), \xi) = (\phi_2^2, -1)$, $w_1, w_2 \in \mathbb{C}$ are free.

3. If $(\chi_X(x), \xi) = (\phi_4, -i)$, $w_2 \in \mathbb{C}$ is free and $w_1 = iw_2 + x_1$ for $x_1 \in w_1 \in \frac{\langle 1,i \rangle}{J \det B}$ and so $w_1$ can be reduced to one of $NJ \det B$ lines.

4. If $(\chi_X(x), \xi) = (\phi_2^2, 1)$, $w_1, w_2 \in \frac{\langle 1,i \rangle}{J \det B}$ or $w_1, w_2 \in \frac{\langle 1,\sqrt{3}i \rangle}{2J \det B}$ and so each of $w_1, w_2$ can be reduced to one of $2NJ \det B$ points.

5. If $(\chi_X(x), \xi) = (\phi_1\phi_2, 1)$, $w_1 \in \mathbb{C}$ is free and $w_2 \in \frac{\langle 1,i \rangle}{2J \det B}$ or $w_2 \in \frac{\langle 1,\sqrt{3}i \rangle}{4J \det B}$ and so $w_1$ can be reduced to one of $4NJ \det B$ points.

6. If $(\chi_X(x), \xi) = (\phi_6, \xi_6)$, $w_2 \in \mathbb{C}$ is free, $w_1 = -\xi_6 + x_1$ for $x_1 \in \frac{\langle 1,\sqrt{3}i \rangle}{2J \det B}$ and so $w_1$ can be reduced to one of $2NJ \det B$ points.

7. If $(\chi_X(x), \xi) = (\phi_3, \xi_3)$, $w_2 \in \mathbb{C}$ is free and $w_1 = \xi_3 + x_1$ for $x_1 \in \frac{\langle 1, \sqrt{3}i \rangle}{2J \det B}$ and so $w_1$ can be reduced to one of $2NJ \det B$ points.

8. If $(\chi_X(x), \xi) = (\phi_3, \xi_3^2)$, $w_2 \in \mathbb{C}$ is free and $w_1 = \xi_3 + x_1$ for $x_1 \in \frac{\langle 1, \sqrt{3}i \rangle}{2J \det B}$ and so $w_1$ can be reduced to one of $2NJ \det B$ points.

9. If $(\chi_X(x), \xi) = (\phi_6, \xi_6^5)\}$, $w_2 \in \mathbb{C}$ and $w_1 = x_1 + \xi_6 w_2$ for $x_1 \in \frac{\langle 1, \sqrt{3}i \rangle}{2J \det B}$ and so $w_1$ can be reduced to one of $2NJ \det B$ points.

After reduction by suitable $g_Y g_Z \in N(F) \cap \mathrm{O}^+(L_6, h^s_{2p^2})$, we conclude that the singular locus of each boundary component consists of at most $96K_N N^2 J K^2 K_2 \det B + 14K_N NJ \det B$ points; $K_N NJ \det B$ lines; and $K_N$ surfaces. We have at once that $|J| \leq 3N^3$ and $K \leq 6$. By Corollary 6.3.3,

$$\det B = \begin{cases} 12p^2 & \text{if } (a_1, a_1 a_2) = (1, 1) \\ 3 & \text{if } (a_1, a_1 a_2) = (1, 2p) \end{cases}$$

and one checks that

$$K_N = \begin{cases} 24 & \text{if } (a_1, a_1 a_2) = (1, 1) \\ 9216p^7(p^2 - 1) & \text{if } (a_1, a_1 a_2) = (1, 2p) \end{cases}$$

and so

$$96K_N N^2 J K^2 K_2 \det B + 14K_N NJ \det B = \begin{cases} o(p^6) & \text{if } (a_1, a_1 a_2) = (1, 1) \\ o(p^{14}) & \text{if } (a_1, a_1 a_2) = (1, 2p) \end{cases}$$

and

$$K_N NJ \det B = \begin{cases} o(p^5) & \text{if } (a_1, a_1 a_2) = (1, 1) \\ o(p^{12}) & \text{if } (a_1, a_1 a_2) = (1, 2p). \end{cases}$$

In each case, a sharp bound can be given. □

We end by remarking that as in [Kon93] and [GHS07], the action of

$$g = \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \in N(F)$$

on the tangent space is given by

$$\begin{pmatrix} \exp_{a_2}(t) & 0 & 0 \\ * & (c\tau + d)^{-1} X & 0 \\ * & * & (c\tau + d)^{-2} \end{pmatrix}.$$

Here,

$$t = (c\tau + d)^{-1} \left( \frac{c}{2\delta \det Z} {}^t\underline{w} B \underline{w} + c^t \underline{w} B \underline{w}/2a_1 + \underline{V_1 w} + W_{11}\tau + W_{12} \right)$$

and is, of course, equal to 0 at each boundary component. One can establish criteria for the extension of pluricanonical forms as in Chapter 4. We find that the only non-canonical singularities we must check are $\frac{1}{3}(3, 3, 1, 1)$ and $\frac{1}{6}(6, 2, 1, 1)$.

Appendix A

# Character tables

## A.1 The cyclic group $C_n$

Let $C_n = \langle a \mid a^n = e \rangle$ and let $\xi = e^{2\pi i/n}$.

| $\chi$ | $e$ | $a$ | $a^2$ | $\dots$ | $a^{n-2}$ | $a^{n-1}$ |
|---|---|---|---|---|---|---|
| $\chi_0$ | 1 | 1 | 1 | $\dots$ | 1 | 1 |
| $\chi_1$ | 1 | $\xi$ | $\xi^2$ | $\dots$ | $\xi^{n-2}$ | $\xi^{n-1}$ |
| $\chi_1$ | 1 | $\xi^2$ | $\xi^4$ | $\dots$ | $\xi^{2(n-2)}$ | $\xi^{2(n-1)}$ |
| $\vdots$ | | | | $\vdots$ | | $\vdots$ |
| $\chi_1$ | 1 | $\xi^{n-1}$ | $\xi^{2(n-1)}$ | $\dots$ | $\xi^{(n-2)(n-1)}$ | $\xi^{(n-1)(n-1)}$ |

**Table A.1:** *Characters of $C_n$*

The character $\rho_i$ corresponding to the character $\chi_i$ is given by

$$\rho_i : a \mapsto (\xi^i).$$

## A.2 The binary dihedral group $BD_{2n}$

Let $BD_{2n} = \langle a, b \mid a^n = b^2 = (ba)^2 \rangle$ and let $\xi = e^{2\pi i/n}$.

| $\chi$ | $e$ | $b^2$ | $a^k$ for $k = 1, \ldots, n-1$ | $b$ | $ba$ |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | $-1$ | $-1$ |
| $\chi_3$ | 1 | $-1$ | $(-1)^k$ | $i$ | $-i$ |
| $\chi_4$ | 1 | $-1$ | $(-1)^k$ | $-i$ | $i$ |
| $\chi'_1$ | 2 | $-2$ | $\xi^k + \xi^{-k}$ | 0 | 0 |
| $\chi'_2$ | 2 | $-2$ | $\xi^{2k} + \xi^{-2k}$ | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\chi'_{n-1}$ | 2 | $(-2)^{n-1}$ | $\xi^{(n-1)k} + \xi^{-(n-1)k}$ | 0 | 0 |

**Table A.2:** *Characters of $BD_{2n}$, $n$ even*

| $\chi$ | $e$ | $b^2$ | $a^k$ for $k = 1, \ldots, n-1$ | $b$ | $ba$ |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | $-1$ | $-1$ |
| $\chi_3$ | 1 | $-1$ | $(-1)^k$ | 1 | $-1$ |
| $\chi_4$ | 1 | $-1$ | $(-1)^k$ | $-1$ | 1 |
| $\chi'_1$ | 2 | $-2$ | $\xi^k + \xi^{-k}$ | 0 | 0 |
| $\chi'_2$ | 2 | $-2$ | $\xi^{2k} + \xi^{-2k}$ | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\chi'_{n-1}$ | 2 | $(-2)^{n-1}$ | $\xi^{(n-1)k} + \xi^{-(n-1)k}$ | 0 | 0 |

**Table A.3:** *Characters of $BD_{2n}$, $n$ odd*

If $n$ is even, the representations $\rho_i$ and $\rho'_i$ corresponding to the characters $\chi_i$ and $\chi'_i$ are given by

$$\begin{aligned}
&\rho_1 : a \mapsto (1) && \rho_1 : b \mapsto (1) \\
&\rho_2 : a \mapsto (1) && \rho_1 : b \mapsto (-1) \\
&\rho_3 : a \mapsto (-1) && \rho_1 : b \mapsto (i) \\
&\rho_4 : a \mapsto (-1) && \rho_1 : b \mapsto (-i) \\
&\rho'_i : a \mapsto \begin{pmatrix} \xi^i & 0 \\ 0 & \xi^{-i} \end{pmatrix} && \rho_1 : b \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

If $n$ is odd, the representations $\rho_i$ and $\rho'_i$ corresponding to the characters $\chi_i$ and $\chi'_i$

are given by

$$\begin{aligned}
&\rho_1 : a \mapsto (1) && \rho_1 : b \mapsto (1)\\
&\rho_2 : a \mapsto (1) && \rho_1 : b \mapsto (-1)\\
&\rho_3 : a \mapsto (-1) && \rho_1 : b \mapsto (1)\\
&\rho_4 : a \mapsto (-1) && \rho_1 : b \mapsto (-1)\\
&\rho_i' : a \mapsto \begin{pmatrix} \xi^i & 0 \\ 0 & \xi^{-i} \end{pmatrix} && \rho_1 : b \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

## A.3 The binary tetrahedral group $B\mathbb{T}$

Let $B\mathbb{T} = \langle a, b \mid a^3 = b^3 = (ab)^2 \rangle$ and let $\xi = e^{2\pi i/3}$.

| $\chi$ | $e$ | $a^3b$ | $a^3ba$ | $a^3$ | $aba$ | $b$ | $a$ |
|---|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | $\xi^2$ | 1 | 1 | $\xi$ | $\xi^2$ | $\xi$ |
| $\chi_3$ | 1 | $\xi$ | 1 | 1 | $\xi^2$ | $\xi$ | $\xi^2$ |
| $\chi_4$ | 2 | $-1$ | 0 | $-2$ | $-1$ | 1 | 1 |
| $\chi_5$ | 2 | $-\xi$ | 0 | $-2$ | $-\xi^2$ | $\xi$ | $\xi^2$ |
| $\chi_6$ | 2 | $-\xi^2$ | 0 | $-2$ | $-\xi$ | $\xi^2$ | $\xi$ |
| $\chi_7$ | 3 | 0 | $-1$ | 3 | 0 | 0 | 0 |

**Table A.4:** *Characters of $B\mathbb{T}$*

The representations $\rho_i$ corresponding to the characters $\chi_i$ are given by

$$
\begin{aligned}
&\rho_1 : a \mapsto (1) && b \mapsto (1) \\
&\rho_2 : a \mapsto (\xi) && b \mapsto (\xi^2) \\
&\rho_3 : a \mapsto (\xi^2) && b \mapsto (\xi) \\
&\rho_4 : a \mapsto \begin{pmatrix} -\xi & -\xi^2 \\ 0 & -\xi \end{pmatrix} && b \mapsto \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \\
&\rho_5 : a \mapsto \begin{pmatrix} -\xi & -\xi^2 \\ 0 & -1 \end{pmatrix} && b \mapsto \begin{pmatrix} 0 & -1 \\ -\xi & -\xi^2 \end{pmatrix} \\
&\rho_6 : a \mapsto \begin{pmatrix} \xi & \xi^2 \\ -1 & 0 \end{pmatrix} && b \mapsto \begin{pmatrix} \xi^2 & -\xi \\ 1 & 0 \end{pmatrix} \\
&\rho_7 : a \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} && b \mapsto \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

## A.4 The binary octahedral group $B\mathbb{O}$

Let $B\mathbb{O} = \langle a, b \mid a^3 = b^4 = (ab)^2 \rangle$. The character table of $B\mathbb{T}$ is given in Table A.5.

| $\chi$ | $e$ | $a^5ba^2$ | $ab^2$ | $(a^2b)^2$ | $a^3$ | $b$ | $a^3b^2a$ | $a^2b$ |
|---|---|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | $-1$ | 1 | 1 | 1 | $-1$ | 1 | $-1$ |
| $\chi_3$ | 2 | 0 | $-1$ | 2 | 2 | 0 | $-1$ | 0 |
| $\chi_4$ | 2 | 0 | $-1$ | 0 | $-2$ | $\sqrt{2}$ | 1 | $-\sqrt{2}$ |
| $\chi_5$ | 2 | 0 | $-1$ | 0 | $-2$ | $-\sqrt{2}$ | 1 | $\sqrt{2}$ |
| $\chi_6$ | 3 | 1 | 0 | $-1$ | 3 | $-1$ | 0 | $-1$ |
| $\chi_7$ | 3 | $-1$ | 0 | $-1$ | 3 | 1 | 0 | 1 |
| $\chi_8$ | 4 | 0 | 1 | 0 | $-4$ | 0 | $-1$ | 0 |

**Table A.5:** *Characters of* $B\mathbb{O}$

Let $\xi_8 = e^{\pi i/4}$ and $\xi_3 = e^{2\pi i/3}$. The representations $\rho_i$ corresponding to the characters $\chi_i$ are given by

$$\rho_1 : a \mapsto (1) \qquad b \mapsto (1)$$

$$\rho_2 : a \mapsto (1) \qquad b \mapsto (-1)$$

$$\rho_3 : a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\rho_4 : a \mapsto \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} \xi_8^3 + \xi_8^2 - \xi_8 & \xi_8 - \xi_8^3 \\ \xi_8^2 & -\xi_8^2 \end{pmatrix}$$

$$\rho_5 : a \mapsto \frac{1}{3}\begin{pmatrix} 2 - \xi_8 - \xi_8^3 & 2\xi_8^3 - \xi_8^2 - 2\xi_8 \\ \xi_8 - \xi_8^2 - \xi_8^3 & 1 + \xi_8 + \xi_8^3 \end{pmatrix} \qquad b \mapsto \frac{1}{3}\begin{pmatrix} 2 + \xi_8 - 2\xi_8^3 & -1 - \xi_8 - \xi_8^3 \\ 2 - \xi_8 - \xi_8^3 & \xi_8 - \xi_8^2 - \xi_8^3 \end{pmatrix}$$

$$\rho_6 : a \mapsto \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\rho_7 : a \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\rho_8 : a \mapsto \begin{pmatrix} -\xi_3^2 & 0 & 0 & 0 \\ 0 & 0 & \xi_3^2 & 0 \\ 0 & -\xi_3^2 & \xi_3^2 & 0 \\ -\xi_3 & 0 & 0 & -1 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} -\xi_3 & \xi_3 & -\xi_3 & 0 \\ \xi_3 & 0 & 0 & 0 \\ -\xi_3^2 & 0 & 0 & -\xi_3 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

## A.5 The binary icosahedral group $B\mathbb{I}$

Let $B\mathbb{I} = \langle a, b \mid a^3 = b^5 = (ab)^2 \rangle$ The character table of $B\mathbb{I}$ is given in Table A.6.

| $\chi$ | $e$ | $a(ba^2b)^2$ | $b^2a^2$ | $a$ | $a(a^2b^2)^2$ | $a^3b^2$ | $ab^3a$ | $b$ | $a^3$ |
|---|---|---|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 2 | $\frac{-1+\sqrt{5}}{2}$ | $\frac{-1-\sqrt{5}}{2}$ | 1 | 0 | $\frac{1-\sqrt{5}}{2}$ | $-1$ | $\frac{1+\sqrt{5}}{2}$ | $-2$ |
| $\chi_3$ | 2 | $\frac{-1-\sqrt{5}}{2}$ | $\frac{-1+\sqrt{5}}{2}$ | 1 | 0 | $\frac{1+\sqrt{5}}{2}$ | $-1$ | $\frac{1-\sqrt{5}}{2}$ | $-2$ |
| $\chi_4$ | 3 | $\frac{1+\sqrt{5}}{2}$ | $\frac{1-\sqrt{5}}{2}$ | 0 | $-1$ | $\frac{1+\sqrt{5}}{2}$ | 0 | $\frac{1-\sqrt{5}}{2}$ | 3 |
| $\chi_5$ | 3 | $\frac{1-\sqrt{5}}{2}$ | $\frac{1+\sqrt{5}}{2}$ | 0 | $-1$ | $\frac{1-\sqrt{5}}{2}$ | 0 | $\frac{1+\sqrt{5}}{2}$ | 3 |
| $\chi_6$ | 4 | $-1$ | $-1$ | $1\frac{1-\sqrt{5}}{2}$ | 0 | $-1$ | 1 | $-1$ | 4 |
| $\chi_7$ | 4 | $-1$ | $-1$ | $-1$ | 0 | 1 | 1 | 1 | $-4$ |
| $\chi_8$ | 5 | 0 | 0 | $-1$ | 1 | 0 | $-1$ | 0 | 5 |
| $\chi_9$ | 6 | 1 | 1 | 0 | 0 | $-1$ | 0 | $-1$ | $-6$ |

**Table A.6:** *Characters of $B\mathbb{I}$*

Let $\xi_5 = e^{2\pi i/5}$. The representations $\rho_i$ corresponding to the characters $\chi_i$ are given by

$$\rho_1 : a \mapsto \begin{pmatrix} 1 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} 1 \end{pmatrix}$$

$$\rho_2 : a \mapsto \begin{pmatrix} -\xi_5^3 & -\xi_5^3 \\ -\xi_5-\xi_5^4 & -\xi_5-\xi_5^2-\xi_5^4 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} -\xi_5^3 & -\xi_5-\xi_5^3-\xi_5^4 \\ 0 & -\xi_5^2 \end{pmatrix}$$

$$\rho_3 : a \mapsto \begin{pmatrix} -\xi_5-\xi_5^2-\xi_5^3 & -\xi_5^2-\xi_5^3-\xi_5^4 \\ \xi_5 & -\xi_5^4 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} -\xi_5 & 0 \\ -\xi_5^2-\xi_5^3 & -\xi_5^4 \end{pmatrix}$$

$$\rho_4 : a \mapsto \begin{pmatrix} \xi_5^2+\xi_5^3 & \xi_5^2+\xi_5^3 & 1 \\ 0 & 0 & 1 \\ \xi_5^2+\xi_5^3 & -1 & -\xi_5^2-\xi_5^3 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} 1 & 0 & 0 \\ \xi_5^2+\xi_5^3 & \xi_5^2+\xi_5^3 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\rho_5 : a \mapsto \begin{pmatrix} 0 & -1 & 0 \\ -\xi_5^2-\xi_5^3 & \xi_5^2+\xi_5^3 & -1 \\ \xi_5^2+\xi_5^3 & 1 & -\xi_5^2-\xi_5^3 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} -\xi_5^2-\xi_5^3 & \xi_5^2+\xi_5^3 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\rho_6 : a \mapsto \begin{pmatrix} 0 & 0 & 0 & \xi_5^3 \\ \xi_5-\xi_5^3 & 1 & -\xi_5^2-\xi_5^3-\xi_5^4 & \xi_5+\xi_5^2+\xi_5^3 \\ \xi_5^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} \xi_5 & 0 & 0 & 0 \\ -\xi_5^2+\xi_5^4 & \xi_5^2+\xi_5^3+\xi_5^4 & \xi_5^3+\xi_5^4 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \xi_5^4 & 0 \end{pmatrix}$$

$$\rho_8 : a \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_7 : a \mapsto \begin{pmatrix} 0 & 0 & -\xi_5 & 0 \\ \xi_5-\xi_5^2+\xi_5^3 & -1 & \xi_5+\xi_5^3 & \xi_5+\xi_5^2+\xi_5^4 \\ 0 & 0 & 0 & 1 \\ \xi_5^4 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho_7 : b \mapsto \begin{pmatrix} -3\xi_5-2\xi_5^2-\xi_5^3-4\xi_5^4 & \xi_5-\xi_5^2+\xi_5^3-\xi_5^4 & -\xi_5-2\xi_5^2-2\xi_5^4 & 3\xi_5+2\xi_5^3+2\xi_5^4 \\ -\xi_5^2+\xi_5^3-\xi_5^4 & \xi_5 & -\xi_5^2-\xi_5^4 & \xi_5+\xi_5^4 \\ 2\xi_5+2\xi_5^2+3\xi_5^4 & -\xi_5+\xi_5^2-\xi_5^3+\xi_5^4 & -\xi_5+\xi_5^2-\xi_5^3+\xi_5^4 & -4\xi_5-\xi_5^2-2\xi_5^3-3\xi_5^4 \\ -\xi_5-2\xi_5^2-2\xi_5^4 & \xi_5+\xi_5^3 & \xi_5-\xi_5^2+\xi_5^3-\xi_5^4 & 2\xi_5+\xi_5^3+2\xi_5^4 \end{pmatrix}$$

$$\rho_9 : a \mapsto \begin{pmatrix} \xi_5 & \xi_5^2+\xi_5^3+\xi_5^4 & -\xi_5-\xi_5^2-\xi_5^3 & 0 & -\xi_5-\xi_5^2 & -\xi_5^4 \\ -\xi_5 & -\xi_5-\xi_5^2-\xi_5^4 & -\xi_5^4 & -1 & \xi_5 & \xi_5^4 \\ -\xi_5^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\xi_5^4 & 0 & 0 & 0 & 0 \\ -\xi_5^2 & \xi_5 & \xi_5^2+2\xi_5^3+\xi_5^4 & -\xi_5 & \xi_5+2\xi_5^2+\xi_5^3+\xi_5^4 & -\xi_5-\xi_5^2-2\xi_5^3-\xi_5^4 \\ -\xi_5-\xi_5^2 & -\xi_5^2-\xi_5^3-\xi_5^4 & \xi_5+2\xi_5^2+2\xi_5^3 & -1 & \xi_5+\xi_5^2-\xi_5^4 & -\xi_5-\xi_5^2-\xi_5^3 \end{pmatrix}$$

$$\rho_9 : b \mapsto \begin{pmatrix} -1 & -\xi_5-\xi_5^2-\xi_5^3 & -\xi_5^3-\xi_5^4 & 0 & -\xi_5^2-\xi_5^3-\xi_5^4 & \xi_5^3 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -\xi_5 & 0 & 0 & 0 \\ \xi_5 & -1 & -\xi_5-\xi_5^2 & 0 & \xi_5^2+\xi_5^3+\xi_5^4 & -\xi_5^3-\xi_5^4 \\ \xi_5^3 & -\xi_5^2 & \xi_5+\xi_5^2-\xi_5^4 & \xi_5^2 & \xi_5-\xi_5^3 & -\xi_5+\xi_5^4 \\ -1 & \xi_5^2+\xi_5^4 & -\xi_5^3 & -\xi_5^4 & 1 & \xi_5^3 \end{pmatrix}.$$

Appendix B

# Non-canonical singularities

1. $\frac{1}{6}(6,6,2,3)$
2. $\frac{1}{6}(6,6,1,1)$
3. $\frac{1}{6}(6,1,1,2)$
4. $\frac{1}{6}(6,1,2,4)$
5. $\frac{1}{6}(6,1,1,3)$
6. $\frac{1}{6}(6,6,1,2)$
7. $\frac{1}{6}(1,1,1,1)$
8. $\frac{1}{6}(1,1,1,2)$
9. $\frac{1}{10}(10,10,6,7)$
10. $\frac{1}{10}(10,10,1,7)$
11. $\frac{1}{10}(10,10,2,5)$
12. $\frac{1}{10}(10,10,1,6)$
13. $\frac{1}{10}(10,10,2,3)$
14. $\frac{1}{10}(10,10,1,2)$
15. $\frac{1}{10}(10,10,2,6)$
16. $\frac{1}{10}(10,1,2,5)$
17. $\frac{1}{10}(1,1,2,3)$
18. $\frac{1}{12}(12,1,2,3)$
19. $\frac{1}{12}(12,1,2,10)$
20. $\frac{1}{12}(12,1,3,4)$
21. $\frac{1}{12}(12,2,3,4)$
22. $\frac{1}{12}(1,2,3,4)$
23. $\frac{1}{12}(12,12,3,4)$
24. $\frac{1}{12}(12,12,2,3)$
25. $\frac{1}{12}(12,12,1,2)$
26. $\frac{1}{12}(12,12,1,4)$
27. $\frac{1}{12}(12,12,1,3)$
28. $\frac{1}{12}(12,1,2,4)$

29. $\frac{1}{20}(2,3,4,5)$

30. $\frac{1}{30}(30,3,4,6)$

31. $\frac{1}{30}(30,3,5,6)$

32. $\frac{1}{30}(30,4,5,6)$

33. $\frac{1}{30}(30,3,4,5)$

34. $\frac{1}{30}(3,4,5,6)$

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