VECTOR BUNDLES EXAMPLES

Exercises 1

1.1.1 Show that the Zariski topology on a quasi-projective variety X is a topology, but is non-Hausdorff unless X is finite.

1.1.2 Let X be an irrducible variety and $f: X \to Y$ a morphism. Show that $\overline{f(X)}$ (the closure of $f(X)$ in the Zariski topology) is irreducible.

1.1.3 Show that $\mathbb{P}^1$ is irreducible. (Try $\mathbb{P}^n$ if you like.)

1.1.4 Let X be a quasi-projective variety. Show that the diagonal $\Delta \subset X \times X$, $\Delta = \{(x, x) \mid x \in X\}$, is Zariski-closed in $X \times X$, but that Δ is *not* closed in the product of the Zariski topologies on the two copies of X unless X is finite.

1.2.1 Let X be an irreducible variety. Show that $\mathbb{C}(X)$ is a field.

1.3.1 Consider the following curves in $\mathbb{P}^2$

$$\text{a.}\quad y^2z - x^3 = 0$$

$$\text{b.}\quad y^2z - x^3 - x^2z = 0$$

$$\text{c.}\quad y^2z - x^3 + xz^2 = 0$$

Show that (a) and (b) each has one singular point, while (c) is non-singular. Sketch the real affine part of each curve.

[Here by the *real affine* part we mean

$$\{(x, y) \in \mathbb{R}^2 \mid f(x, y, 1) = 0\}. \quad]$$

Exercises 2

2.1.1 Let

$$\begin{aligned} L &= \{(x, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid v \text{lies on the line in } \mathbb{C}^{n+1} \text{corresponding to } x\} \\ &= \{((x_0 : \ldots : x_n), (\lambda x_0, \ldots, \lambda x_n)) \mid x_i \in \mathbb{C}\text{not all zero}, \lambda \in \mathbb{C}\}. \end{aligned}$$

Show that the projection $p: L \to \mathbb{P}^n$

$$p(x, v) = x$$

makes L into a line bundle over $\mathbb{P}^n$ (usually denoted by $\mathcal{O}(-1)$).

[Hint. For $0 \le i \le n$, let U_i denote the Zariski open set in $\mathbb{P}^n$ defined by $x_i \ne 0$. Consider the restriction of L to U_i.]

2.1.2 Show that $\mathcal{O}(-1)$ is defined with respect to the covering $\{U_i\}$ of $\mathbb{P}^n$ by the transition functions $g_{ij} = \frac{x_i}{x_j}$.

2.1.3 Define the line bundle $\mathcal{O}(1)$ over $\mathbb{P}^n$ by $\mathcal{O}(1) = \mathcal{O}(-1)^*$. Then define $\mathcal{O}(a)$ for $a \in \mathbb{Z}$ as follows:

$$\mathcal{O}(a) = \begin{cases} \overbrace{\mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1)}^{a} & \text{for } a > 0 \\ \mathcal{O} & \text{for } a = 0 \\ \underbrace{\mathcal{O}(-1) \otimes \cdots \otimes \mathcal{O}(-1)}_{a} & \text{for } a < 0. \end{cases}$$

Show that, for any a, $b \in \mathbb{Z}$, $\mathcal{O}(a+b) = \mathcal{O}(a) \otimes \mathcal{O}(b)$. Show also that, with respect to the open covering $\{U_i\}$, $\mathcal{O}(a)$ is defined by the transition functions $g_{ij} = \left(\frac{x_i}{x_j}\right)^a$.

2.2.1 Show that, for $a \geq 0$, $\Gamma\big(\mathcal{O}(a)\big)$ can be identified with the space of homogeneous polynomials of degree a in $x_0, \ldots, x_n$. Show also that, for $a < 0$, $\Gamma\big(\mathcal{O}(a)\big) = 0$.

Exercises 3

3.1.3 Show that $K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$.

In problems 3.2.1–3.2.4, C is a non-singular curve and K is its canonical bundle. The genus of C is g and is defined by $g = h^1(\mathcal{O})$.

3.2.1 Prove the Riemann-Roch theorem for line bundles over C.

3.2.2 Using Riemann-Roch and Serre duality, show that $\deg K = 2g - 2$ and $h^0(K) = g$.

3.2.3 Show that every line bundle L of degree $> 2g-2$ over C has $h^0 = d + 1 - g$. Show further that L is very ample whenever $\deg L > 2g$.

3.2.4 Show that, on $\mathbb{P}^1$, at least one of $h^0\big(\mathcal{O}(a)\big)$ and $h^1\big(\mathcal{O}(a)\big)$ is 0. For what values of a is it true that $h^0\big(\mathcal{O}(a)\big) = h^1\big(\mathcal{O}(a)\big) = 0$?

3.3.1 For given a, b, find all vector bundles E on $\mathbb{P}^1$ for which there exists an exact sequence

$$0 \longrightarrow \mathcal{O}(a) \longrightarrow E \longrightarrow \mathcal{O}(b) \longrightarrow 0.$$

3.4.1 Let E be an indecomposable vector bundle on an elliptic curve C. Show that there exists a unique line bundle L of degree 0 and an exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow L \longrightarrow 0.$$

(This completes the classification of rank-2 bundles given in the lectures.)

[Hint. Any vector bundle F of degree 2 has $h^0(F) \geq 2$. Deduce that, if F is decomposable, then F possesses a subbundle isomorphic to $\mathcal{O}(x)$ for some $x \in C$.]

Exercises 4

4.1.1 Show that every line bundle over a non-singular curve C is stable.

4.1.2 Show that, if E is stable (semistable) and L is a line bundle, then $E \otimes L$ is stable (semistable).

4.1.3 Show that, if E is stable, then E is simple (i.e. $h^0(\operatorname{End} E) = 1$ or equivalently the only endomorphisms of E are the scalar multiples of the identity.)

4.1.4 Let E be a semistable bundle of rank n and degree d over C with $d > n(2g-1)$. Prove

a. E is generated by its sections (i.e., given and point v in the fibre E_x of E over the point $x \in C$, $\exists$ section s of E such that $s(x) = v$)

b. $h^1(E) = 0$.

4.1.5 Show that the only stable bunles on $\mathbb{P}^1$ are the line bundles.

4.1.6 Show that $\exists$ stable bundles of rank n and degree d over an elliptic curve C if and only if $(n, d) = 1$. Describe $M(n, d)$ in this case.

4.1.7 Suppose $g \geq 2$ and $d \in \mathbb{Z}$. Show that $\exists$ stable bundles of rank 2 and degree d over C.

[Hint. Consider extensions of the form

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0$$

where $\deg L_2 - \deg L_1 = 1$ or 2. In the first case, it is easy to show that *any* non-trivial extension is stable; in the second, one can show that $\exists$ extensions which are stable.]

4.1.8 Try generalising 4.1.7 to arbitrary n.

4.2.1 For an alternative proof of 4.1.8, try to prove that R_d is always non-empty if $g \geq 2$

Exercises 5

5.1.1 Let U be a non-empty Zariski-open subset of an irreducible variety X. Show that U is irreducible.

5.1.2 Let E be a vector bundle over a curve C (it is not necessary to assume C non-singular), and suppose that E is generated by its sections. Show that there exists an exact sequence

$$0 \longrightarrow \mathcal{O}^{n-1} \longrightarrow E \longrightarrow L \longrightarrow 0,$$

where $L = \det E$.

RESEARCH PROBLEM

For what values of d is $B(2, d, 4) \neq \emptyset$?