BIRATIONAL GEOMETRY EXAMPLES

Questions 1

1. Let $X_3 \subset \mathbb{P}^3$ be a nonsingular cubic, $L, M \subset X$ disjoint lines. Show that the map $\phi\colon X \dashrightarrow L \times M = \mathbb{P}^1 \times \mathbb{P}^1$ is a morphism and it contracts 5 lines to points.
2. Suppose $L = (x = y = 0)$, $M = (z = t = 0)$ and $L_5 = (y = t = 0)$ lie on a nonsingular $X_3 \subset \mathbb{P}^3$. Prove that the map

$$\alpha\colon X \dashrightarrow \mathbb{P}^2$$

defined by

$$(x, y, z, t) \mapsto (xt, yz, yt)$$

is a morphism and it contracts $L_1, \ldots, L_4, L_5', L_5''$ (notation as in lectures).
3. Let $X_3 \subset \mathbb{P}^3_k$ be a nonsingular cubic over a perfect field k. Let $G = \mathrm{Gal}(\bar{k}/k)$, acting on the 27 lines of X. The following are equivalent:
(a) $\rho = \mathrm{rk}(\mathrm{Pic}\, X) = 1$
(b) The sum of the lines in each G-orbit is $\sim$ a hyperplane section.
(c). No Galois orbit consists of disjoint lines.
4. Find all the lines on a Fermat hypersurface $ax^3 + by^3 + cz^3 + dt^3 = 0$. If $a \in \mathbb{Q}$ is not a cube, then $X = (x^3 + y^3 + z^3 - at^3$ has $\rho = 1$.

Questions 2

1. By explicit blow up, resolve the following (germs of) plane curve singularities at the origin of $\mathbb{C}^2$:

$$x^2 + y^k = 0$$
$$x^3 + y^5 = 0$$

2. The canonical class of X is the line bundle

$$K_X = \bigwedge^{\top} T_X = \Omega^n_X \qquad \text{(Where } n = \dim X\text{)}.$$

Show that $K_{\mathbb{P}^n} = \mathcal{O}(-n-1)$.
3. ("Adjunction formula") If $Y \subset X$ is a nonsingular divisor (i.e. $\dim Y = \dim X - 1$) then $K_Y = K_X + Y_{|Y}$ [Look at the exact sequence $0 \to T_Y \to T_{X|Y} \to NY \to 0$]
4. Let $P \in X = X_3 \subset \mathbb{P}^3$ be a point. Consider, as in the lectures, the projection from P, π_P:

$$\begin{array}{ccccc} & & Y & & \\ bl_p = \epsilon & \swarrow & & \searrow & \pi \\ X & & \overset{\pi_P}{\dashrightarrow} & & \mathbb{P}^2 \end{array}$$

Show that, if $P \notin$ line of X, then $\pi\colon Y \to \mathbb{P}^2$ is 2:1, branched along a quartic curve $B \subset \mathbb{P}^2$. Find the equation of B in terms of a suitable equation for X.
5. Consider a birational map $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by a linear system $\mathcal{M} \subset |\mathcal{O}(d)|$. Show that $\exists P \in \mathbb{P}^2$ with $\mathrm{mult}_P\, \mathcal{M} > d/3$. [Follow the proof of Segre's Theorem.]

Assume there are P_1, P_2, P_3 with $m_i = \operatorname{mult}_{P_i} \mathcal{M} > d/3$. Choose coordinates such that $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, $P_3 = (0,0,1)$. Let $\alpha\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the standard Cremona transformation

$$(x_0, x_1, x_2) \dashrightarrow \left(\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}\right).$$

Show that $\phi \circ \alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is given by a linear system $\mathcal{M}_1 \subset |\mathcal{O}(d_1)|$ with $d_1 < d$.
6. (*) Classify all $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ birational, given by $\mathcal{M} \subset |\mathcal{O}_{\mathbb{P}^2}(2)|$. Show that ϕ is a composite of linear maps and standard Cremona Transformations.

Questions 3

1. Let X be a nonsingular surface, $C \subset S$ reduced and irreducible. Prove that, if $H^0(K_C) = (0)$, then $C \cong \mathbb{P}^1$. [Hint: consider the normalisation $\nu\colon \tilde{C} \to C$ and study the inclusion $\mathcal{O}_C \subset \nu_*\mathcal{O}_{\tilde{C}}$.]
Let $D \subset X$ be a reduced connected curve with $H^0(K_D) = (0)$. Prove that D is a "tree of $\mathbb{P}^1$s".
Let L be a line bundle on D, and assume $\deg(L_{|C}) \geq 0$ for every irreducible component C of D. Show that L is generated by global sections, $h^1(D, L) = 0$, and $h^0(D, L) = 1 + \deg D$.
2. Let $C \subset X$ be reduced, irreducible, $p_a(C) = 1$, and L a line bundle of degree d on C. Denote $R(C, L) = \bigoplus_{n \geq 0} H^0(C, nL)$. Prove that
If $d = 1$,

$$R = \frac{k[x, y, z}{(z^2 + y^3 + a_4 y + a_6)}$$

where $x \in R_1$, $y \in R_2$, $z \in R_3$ and $a_i \in k[x]$ has degree i
If $d = 2$,

$$R = \frac{k[x_1, x_2, y}{\big(y^2 + a_4(x_1, x_2)\big)}$$

where $x_i \in R_1$, $y \in R_2$
If $d \geq 3$ R is generated by R_1, in particular
If $d = 3$,

$$R = \frac{k[x_1, x_2, x_3]}{\big(a_3(x_1, x_2, x_3)\big)} \qquad \text{(Where } \deg a_3 = 3)$$

If $d = 4$,

$$R = \frac{k[x_1, x_2, x_3, x_4]}{\big(q(x_1, x_2, x_3, x_4), q'(x_1, x_2, x_3, x_4)\big)} \qquad \text{(Where } \deg q = \deg q' = 2).$$

3. Let D be a divisor on a variety X. Assume:
(i) $R\big(D, \mathcal{O}_D(D)\big)$ is generated by elements of degree $\leq r$
(ii) $H^1\big(X, \mathcal{O}(jD)\big) = (0)$ for all $j > 0$.
Prove that $R(X, D)$ is generated by elements of degree $\leq r$.
4, (*) Prove that a del Pezzo surface of degree $d \geq 3$ is a surface $X_d \subset \mathbb{P}^d$ of degree d. [You must show that a general member $D \in |-K|$ is reduced, then use Q2.3.]

Questions 4

1. Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map, given by a linear system $\mathcal{M} \subset |\mathcal{O}(d)|$. In the notation of the lectures, show that $m_1 + m_2 + m_3 > d$. [Hint: we know that $m_1 > d/3$. Show that $m_2, m_3 > \frac{d - m_1}{2}$.]

2. Show that the nonstandard quadratic maps

$$(x_0, x_1, x_2) \dashrightarrow (x_0x_2, x_1x_2, x_0^2)$$
$$(x_0, x_1, x_2) \dashrightarrow (x_0^2, x_0x_1, x_1^2 + x_0x_2)$$

are composites of linear and standard quadratic maps.

3. Show that a de Jonquières map:

$$y \dashrightarrow \frac{ay+b}{cy+d}$$

(a, b, c, $d \in k[x]$) can be written as a composite of linear and quadratic maps.

4. Show that $\operatorname{Pic} \mathbb{F}_a = \mathbb{Z}[A] \oplus \mathbb{Z}[B]$ and

$$\begin{cases} A^2 = 0, \ AB = 1 \\ B^2 = -a. \end{cases}$$

Show $K_{\mathbb{F}_a} = -(2+a)A - 2B$.

If $\mathcal{M} \subset |\mathcal{O}(qA + bB)|$ is a mobile linear system, then $q \geq ba$.

[You may want to establish, and use, the following description of $\mathbb{F}_a$:

$$\mathbb{F}_a = (\mathbb{C}^2_{t_1,t_2} \setminus 0) \times (\mathbb{C}^2_{x_1,x_2} \setminus 0)/\mathbb{C}^\times \times \mathbb{C}^\times$$

where $\mathbb{C}^\times \times \mathbb{C}^\times$ acts as

$$\begin{aligned}(t_1, t_2; x_1, x_2) &\longrightarrow (\lambda t_1, \lambda t_2, x_1, -a x_2) \\ &\longrightarrow (t_1, t_2, \mu x_1, \mu x_2) \qquad]\end{aligned}$$