ABELIAN VARIETIES EXAMPLES

Examples 1

1. If X_3 is an abelian variety, describe the group of n-torsion points on X.

2. Show that a smooth plane cubic curve has genus 1. Show that the intersection of two quadrics in $\mathbb{P}^3$ also has genus 1 if it is smooth.

3. Verify that $\Pi \in M_{g\times 2g}(\mathbb{C})$ is the period matrix ofa complex torus if and only if $\begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix}$ is nonsingular.

4. Prove that every homomorphism $f\colon T = V/\Lambda \to T' = V'/\Lambda'$ is induced by a unique $\mathbb{C}$-linear map $F\colon V \to V'$ such that $F(\Lambda) \subset \Lambda'$.

5. Give an example of an ample line bundle $\mathcal{L}$ on an abelian variety such that neither $\mathcal{L}$ nor $\mathcal{L}^2$ is very ample.

6. Prove that isogeny is an equivalence relation. How do you know it isn't just isomorphism?

Examples 2

1. Verify that the Abel-Jacobi map is well-defined.

2. Every abelian variety has a non-vanishin global g-form, where g is the dimension. Why? What does this tell you about abelian varieties and divisors on them?

3. Suppose C is a smooth projective curve, E is an elliptic curve and there is a nonconstant map $C \to E$. What does this tell us about $\operatorname{Jac} C$?

4. Show that there is an embedding from C into $\operatorname{Jac} C$, depending only on the choice of a base point $O \in C$.

5. Show that if D is a dicisor on C and $\deg D \geq g = g(C)$, then D is linearly equivalent to an effective divisor.

Examples 3

1. What are the elements of finite order in $\mathrm{SL}(2,\mathbb{Z})$? What does this tell you about the values of k for which modular forms of weight k for $\mathrm{SL}(2,\mathbb{Z})$ might exist?

2. All abelian varieties have non-trivial automorphisms: why? Identify some abelian varieties with even more automorphisms. A level-n structure is a choice of basis of the group of n-torsion points. Do you expect an abelian variety with a level-n structure (say $n \geq 3$) to have automorphisms?

3. Is every pp abelian surface the Jacobian of a genus 2 curve?

4. Verify that two points Z, $Z' \in \mathbb{H}_g$ give isomorphic pp abelian varieties if and only if they are equivalent under $\mathrm{Sp}(2g,\mathbb{Z})$.

Examples 4

1. Show that if A is an abelian variety and $\hat{A}$ is its dual, then $A \cong \hat{A}$ if and only if A has a principal polarisation.

2. The Abel-Prym map π_C associated to an étale double cover $\pi\colon \tilde{C} \to C$ is given by

$$\tilde{C} \longrightarrow \operatorname{Jac} C \xrightarrow{\sim} \widehat{\operatorname{Jac} \tilde{C}} \longrightarrow \hat{P} \xrightarrow{\sim} P$$

where $\widehat{\operatorname{Jac}\tilde{C}} \to \hat{P}$ is dual to the inclusion $P \hookrightarrow \operatorname{Jac}\tilde{C}$. Show that the Abel-Prym map is injective.

3. Show that the differential of π_C is injective unless C is hyperelliptic, so that $\pi_C\colon C \to P$ is an embedding.

[See Lange-Birkenhake, p. 383, and Barth-Peters-Van de Ven for guidance.]