Blow-ups with log canonical singularities

Gregory Sankaran, Bath

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1 After the COW

In February 2018, in a London pub after a meeting of the COW seminar, Caucher Birkar asked:

Question 1.1 *Denote by $\mathbb{A}^4_{\mathbf{n}}$ the weighted blowup of $\mathbb{A}^4$ at $0 \in \mathbb{A}^4$ with coprime weights $\mathbf{n} = (n_1, n_2, n_3, n_4) \in \mathbb{N}^4$. If $\mathbb{A}^4_{\mathbf{n}}$ has terminal singularities, is the smallest of the weights bounded?*

In [6], Paco Santos (Santander) and I prove the following stronger statement, which was stated as a conjecture by Birkar in subsequent conversation.

Theorem 1.2 *In each fixed dimension d and for each $\varepsilon \in (0, 1]$ there is an integer $\ell_{\varepsilon,d} \in \mathbb{N}$ such that if $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ is primitive and the weighted blowup $\mathbb{A}^d_{\mathbf{n}}$ has only ε-log canonical (ε-lc) singularities then $n_{\min} := \min\{n_1, \ldots, n_d\} \leq \ell_{\varepsilon,d}$.*

Moreover we give a precise answer to Question 1.1.

Theorem 1.3 *If the weighted blowup $\mathbb{A}^4_{\mathbf{n}}$ has terminal singularities then $n_{\min} \leq 32$. Moreover, with finitely many exceptions $n_{\min} \leq 6$.*

In the Sarkisov programme for 3-folds we need to understand divisorial contractions $X \to F$ with F a Fano 3-fold. In general it seems easier to understand divisorial contractions from a given X than to a given F, but Kawakita [3] showed that a divisorial Mori contraction with centre a smooth point is a weighted blow-up, with weights $(1, a, b)$.

The second part of this is the answer to Question 1.1 in dimension 3, so Theorem 1.2 extends that to arbitrary dimension and to ε-lc. However, the first part does not generalise in a simple way because some Mori contractions are not weighted blow-ups at all.

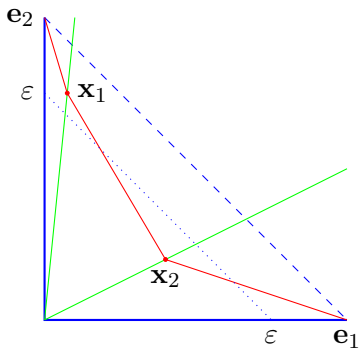
2 Hollow and empty

Our approach is toric, and uses some results about lattice polytopes.

Fix a subgroup Γ of the additive group $\mathbb{R}^d$. We say that a polytope Π is *hollow* (with respect to Γ) if $\Pi \cap \Gamma \subset \partial\Pi$, and *empty* if $\Pi \cap \Gamma$ consists of vertices of Π. If $\Gamma = \mathbb{Z}^d$ then a rational polyhedral cone σ with primitive generators $\mathbf{w}_i \in \mathbb{Z}^d$ determines an affine toric variety X_σ and a polytope $\Delta(\sigma) = \mathrm{Conv}(0, \mathbf{w}_1, \ldots)$, which is a simplex if and only if X_σ is $\mathbb{Q}$-factorial. For $0 \le \varepsilon \le 1$ we have (roughly):

Lemma 2.1 *X_σ is ε-log terminal if and only if $\varepsilon\Delta(\sigma)$ is empty, and ε-lc and only if $\varepsilon\Delta(\sigma)$ is hollow.*

In the picture, the lattice point $\mathbf{x}_1$ violates terminal and $\mathbf{x}_2$ violates ε-lc.



The weighted blowup $\mathbb{A}^d_{\mathbf{n}}$ is the toric variety associated with the fan in $\mathbb{R}^d$ that consists of all the faces of the cones $\sigma^j_{\mathbf{n}} = \mathbb{R}_{\ge 0}\mathbf{n} + \sum_{i\ne j} \mathbb{R}_{\ge 0}\mathbf{e}_i$. These are simplicial, so $\mathbb{A}^d_{\mathbf{n}}$ always has $\mathbb{Q}$-factorial singularities.

3 Bounding the weights

First consider the terminal case. $\mathbb{A}^d_{\mathbf{n}}$ has terminal singularities if and only if the simplices $\Delta(\sigma^j_{\mathbf{n}})$ are all empty. But that happens if and only if their union is empty, and their union is the convex hull $\mathrm{Conv}(0, \mathbf{e}_1, \ldots, \mathbf{e}_d, \mathbf{n})$. Split that the other way: into the part above the top face Δ_1 of the standard simplex Δ, which is is $\Delta_{\mathbf{n}} := \mathrm{Conv}(\mathbf{e}_1, \ldots, \mathbf{e}_d, \mathbf{n})$, and the part below, which is Δ. Since Δ is empty, $\mathbb{A}^d_{\mathbf{n}}$ has terminal singularities if and only if $\Delta_{\mathbf{n}}$ is empty.

To answer Question 1.1 we move the origin to $\mathbf{n}$ and look at the affine toric 4-fold that arises from $\Delta_{\mathbf{n}}$. The singularity is a non-Gorenstein cyclic quotient singularity. A partial list of terminal singularities of this type was given experimentally in [5]. They correspond to empty lattice simplices, but are grouped together in fives because X_σ depends on the choice of a vertex.

A member of the list in [5] consists of a quintuple $\mathbf{q} = (q_1, q_2, q_3, q_4, q_5)$ such that $\sum q_i = 0$, and an index p (temporarily prime): the action of $\mathbb{Z}/p$

with weights any four of the q_i on $\mathbb{A}^4$ gives a terminal quotient singularity. The q_i/p are the barycentric coordinates of a generator of $\mathbb{Z}^4$ modulo the lattice generated by the vertices, normalised so as to add to zero.

The index p is not very important here. Most $\mathbf{q}$ that occur work if p is big enough. [5] lists $\mathbf{q} = (1, -1, a, b, -a - b)$ and $\mathbf{q} = (1, -2, a, -2a, a + 1)$ (for almost any a and b) and 29 others such as $(9, 1, -2, -3, -5)$, that work for any p, and some special ("sporadic") cases that work for particular p.

The list of 29 is known to be complete [7], and A. Borisov [1] showed that there are only finitely many sporadic cases. This is almost enough to answer Question 1.1, but we can do better. Borisov's result was a corollary of the following result of J. Lawrence [4], which is the key to everything else.

Theorem 3.1 *Fix $d \in \mathbb{N}$ and an open subset $U \subset \mathbb{R}^d$, and let $\mathbb{G}$ be a closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. Then there are only finitely many maximal subgroups $G < \mathbb{G}$ such that $\mathbb{Z}^d \subset G$ and $G \cap U = \varnothing$.*

Iglesias-Valiño and Santos [2] used Theorem 3.1, facts from lower dimensions and a lot of computing power to classify all hollow lattice simplices in dimension 4 and thus all terminal 4-fold quotient singularities of any index. This subsumes [5, 7, 1], and allows us to prove Theorem 1.3. The worst case has index 245 and weights $\mathbf{n} = (32, 41, 71, 102)$; we can achieve $n_{\min} = 6$ with any index $V \geq 37$ and coprime to 30, by taking $\mathbf{n} = (V - 30, 6, 10, 15)$.

For the ε-lc case and arbitrary d, by the same argument as before we get

Lemma 3.2 *For $\varepsilon \in (0, 1]$, $\mathbb{A}^d_{\mathbf{n}}$ is ε-lc if and only if $\varepsilon\Delta_{\mathbf{n}}$ is hollow.*

We change coordinates affine-linearly by moving $\mathbf{n}$ to the origin and fixing all the $\mathbf{e}_i$. This moves 0 to $\mathbf{p} := \mathbf{n}/(-1 + \sum n_i) \in \mathbb{Q}^d$: we put $\Lambda_{\mathbf{p}} = \mathbb{Z}^d + \mathbb{Z}\mathbf{p}$ and $\Delta_{\mathbf{p},\varepsilon} = \mathbf{p} + \varepsilon(\Delta - \mathbf{p})$, that is, Δ shrunk towards $\mathbf{p}$ by a factor ε. In these terms we have

Lemma 3.3 *$\mathbb{A}^d_{\mathbf{n}}$ is ε-lc if and only if $\Delta_{\mathbf{p},\varepsilon}$ is hollow with respect to $\Lambda_{\mathbf{p}}$.*

From here a fairly simple application of Theorem 3.1 is enough to give Theorem 1.2 with $\varepsilon = 0$. The proof of the full result is more complicated but uses essentially the same ideas. One shows that a bound exists locally for $\mathbf{p}$ near each point of the compact set Δ_1.

References

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