

Any smooth toric threefold contains all curves

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It is very easy to see that every smooth projective curve can be embedded in \mathbb{P}^3 . Eisenbud and Harris, in conversation, asked whether the same is true if \mathbb{P}^3 is replaced by an arbitrary smooth rational projective 3-fold X and Eisenbud suggested starting with the case where X is toric. In this note we answer the toric question, affirmatively. For simplicity, we work over the field of complex numbers.

Theorem *If Y is any smooth projective curve and X is any smooth projective toric 3-fold, then Y can be embedded in X .*

In fact we prove slightly more (see Theorem 2.4): such embeddings are commonplace, being determined by general elements of suitable linear systems. The proof uses only elementary facts about toric varieties. It is not obvious whether one should expect the same answer for the non-toric case.

I thank David Eisenbud for asking me the question, and Dan Ryder for listening to me while I tried to answer it.

1 Maps to toric varieties

We need a good description of maps to a smooth projective toric 3-fold. In fact there are several descriptions available of maps to toric varieties, due to Cox [Co], Kajiwara [Ka] and others. The version that we use here appeared in [Sa, Section 2]: the proof, which is largely due to Tadao Oda, is very short, so we give it here. We refer to [Od] for general background on toric varieties.

Let Δ be a finite (but not necessarily complete) smooth fan for a free \mathbb{Z} -module N of rank r . Denote the corresponding toric variety by X , and write M for the dual lattice $\text{Hom}(N, \mathbb{Z})$, with pairing $\langle \cdot, \cdot \rangle: M \times N \rightarrow \mathbb{Z}$. The torus is then $T = \text{Spec}(\mathbb{C}[M])$, where $\mathbb{C}[M] = \bigoplus_{m \in M} \mathbb{C}e(m)$ is the semigroup ring of M over \mathbb{C} .

As usual, $\Delta(d)$ denotes the set of d -dimensional cones in Δ . For each $\rho \in \Delta(1)$, we denote by $V(\rho)$ the corresponding irreducible Weil divisor on X and by n_ρ the generator of the semigroup $N \cap \rho$.

Theorem 1.1 *Let Y be a normal algebraic variety over \mathbb{C} . A morphism $f: Y \rightarrow X$ such that $f(Y) \cap T \neq \emptyset$ corresponds to a collection of effective Weil divisors $D(\rho)$ on Y indexed by $\rho \in \Delta(1)$ and a group homomorphism $\varepsilon: M \rightarrow \mathcal{O}_Y(Y_0)^\times$ to the multiplicative group of invertible regular functions on $Y_0 = Y \setminus \bigcup_{\rho \in \Delta(1)} D(\rho)$, such that*

$$D(\rho_1) \cap D(\rho_2) \cap \cdots \cap D(\rho_s) = \emptyset \quad \text{if } \rho_1 + \rho_2 + \cdots + \rho_s \notin \Delta \quad (1)$$

and

$$\operatorname{div}(\varepsilon(m)) = \sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle D(\rho) \quad \text{for all } m \in M. \quad (2)$$

Proof. Suppose $f: Y \rightarrow X$ is a morphism with $f(Y) \cap T \neq \emptyset$. For each $\rho \in \Delta(1)$, we take $D(\rho)$ to be the pull-back Weil divisor $f^{-1}(V(\rho))$, which is well-defined since Y is normal, X is smooth and $f(Y) \not\subset V(\rho)$.

If $\rho_1 + \cdots + \rho_s \notin \Delta$, then $V(\rho_1) \cap \cdots \cap V(\rho_s) = \emptyset$ so $D(\rho_1) \cap \cdots \cap D(\rho_s) = \emptyset$. In this case $Y_0 = f^{-1}(T)$, which is nonempty by assumption, and $f|_{Y_0}$ induces

$$f|_{Y_0}^*: \mathbb{C}[M] \rightarrow \mathcal{O}_Y(Y_0)^\times, \quad (3)$$

The composite $\varepsilon := f|_{Y_0}^* \circ \mathbf{e}$ satisfies (2), since

$$\operatorname{div}(\mathbf{e}(m)) = \sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle V(\rho) \quad \text{for all } m \in M. \quad (4)$$

Conversely, suppose $\{D(\rho)\}_{\rho \in \Delta(1)}$ and ε satisfy (1) and (2). For $\sigma \in \Delta$, put $\hat{\sigma} = \{\rho \in \Delta(1) \mid \rho \not\leq \sigma\}$. Then the corresponding open piece U_σ of X satisfies

$$\begin{aligned} U_\sigma &= X \setminus \bigcup_{\rho \in \hat{\sigma}} V(\rho) \\ &= \bigcap_{\rho \in \hat{\sigma}} (X \setminus V(\rho)) \\ &\cong \operatorname{Spec}(\mathbb{C}[M \cap \sigma^\vee]). \end{aligned}$$

Put $Y_\sigma = f^{-1}(U_\sigma) = Y \setminus \bigcup_{\rho \in \hat{\sigma}} D(\rho)$. Then $Y = \bigcup_{\sigma \in \Delta} Y_\sigma$ since the U_σ cover X (or one can check this directly).

For each $\sigma \in \Delta$, $M \cap \sigma^\vee$ is the semigroup consisting of $m \in M$ such that $\mathbf{e}(m)$ is regular on U_σ . Thus $\varepsilon(M \cap \sigma^\vee)$ consists of regular functions on Y_σ , and defines a morphism $f_\sigma: Y_\sigma \rightarrow U_\sigma$. These morphisms glue together to give a morphism $f: Y \rightarrow X$. \square

2 Curves

Now we apply Theorem 1.1 to the case where Y is a smooth projective curve and X is projective of dimension 3.

Let L be an effective (hence ample) divisor on Y . Let $\Delta(1) = \{\rho_1, \dots, \rho_r\}$, so $r - 3 = \text{rk Pic } X > 0$. We write n_j and V_j (rather than n_{ρ_j} and $V(\rho_j)$) for the generator and the divisor corresponding to $\rho_j \in \Delta(1)$.

Let $\{m_1, m_2, m_3\}$ be a \mathbb{Z} -basis for M and put $a_{ij} = \langle m_i, n_j \rangle$ for $1 \leq i \leq 3$, $1 \leq j \leq r$.

The system of linear equations $\sum_{j=1}^r a_{ij}\xi_j = 0$ has rank at most 3, so we can find nontrivial rational or integer solutions. In the projective case we can do better.

Lemma 2.1 *If X is projective, then $\sum_{j=1}^r a_{ij}\xi_j = 0$ has integer solutions with $\xi_j > 0$ for all j .*

Proof. Let H be a very ample divisor on X . We have $\sum_j a_{ij}V_j = 0$ in $\text{Pic } X$, since it is the divisor of $\mathbf{e}(m_i)$. But H^2V_j is the degree of the surface V_j in the projective embedding of X under $|H|$ and is therefore positive, so it is enough to take $\xi_j = H^2V_j$. \square

On our curve Y we take the line bundles $\mathcal{D}_j = \mathcal{O}_Y(\xi_j H)$, with ξ_j as in Lemma 2.1. We may also assume that $\xi_j > 2g(Y) - 2$ for all j , so that any nonzero linear combination of the \mathcal{D}_j with nonnegative integer coefficients is very ample.

We want to specify a map $f: Y \rightarrow X$ by means of data as in Theorem 1.1. Thus we must give elements D_j of the linear system $|\mathcal{D}_j|$.

Lemma 2.2 *If the D_j are general in $|\mathcal{D}_j|$ then they are reduced divisors and their supports are pairwise disjoint. In particular they satisfy Condition 1 of Theorem 1.1.*

Proof. This follows from the very ampleness of the linear systems $|\mathcal{D}_j|$. \square

To specify a map $f: Y \rightarrow X$ we now need only choose ε according to Theorem 1.1. This amounts to choosing suitable trivialisations of each of the three bundles $\mathcal{O}_Y(\sum a_{ij}\mathcal{D}_j)$, i.e. non-vanishing sections of $\mathcal{O}_Y(\sum a_{ij}\mathcal{D}_j)$ with order $-a_{ij}$ along D_j . Such trivialisations are unique up to multiplication by non-zero scalars. This means that the map $f = f_{\mathbb{D}, \mathbf{t}}$ is determined by choices of $\mathbb{D} = (D_1, \dots, D_r \in |\mathcal{D}_1| \times \dots \times |\mathcal{D}_r|)$ together with a choice of an element $\mathbf{t} \in (\mathbb{C}^*)^3 = T \subset \text{Aut } X$. In other words, choosing the D_j determines f up to composition with an element of T acting as an automorphism of X .

We note that the action of T has no effect on the question of whether or not the map is an embedding, and accordingly we suppress \mathbf{t} in the notation.

Later we shall see that $f_{\mathbb{D}}$ will turn out to be an embedding for all sufficiently general $\mathbb{D} \in |\mathcal{D}_1| \times \cdots \times |\mathcal{D}_r|$. The next lemma shows that in order to determine whether the general $f_{\mathbb{D}}$ is an immersion, it is enough to check it over an affine piece of X .

Lemma 2.3 *Suppose that, for every $\tau \in \Delta(3)$, there is a nonempty open subset $A_\tau \subset \prod_j |\mathcal{D}_j|$ such that*

$$f_{\mathbb{D}}: Y_\tau = f_{\mathbb{D}}^{-1}(U_\tau) \rightarrow U_\tau$$

is a closed immersion if $\mathbb{D} \in A_\tau$. Then $f_{\mathbb{D}}: Y \rightarrow X$ is a closed immersion for general $\mathbb{D} \in \prod_j |\mathcal{D}_j|$.

Proof. It is enough to take $\mathbb{D} \in \bigcap_{\tau \in \Delta(3)} A_\tau$. □

Theorem 2.4 *If X is a projective smooth toric 3-fold, Y is a smooth projective curve and \mathcal{D}_j as above, the map $f_{\mathbb{D}, \mathbf{t}}: Y \rightarrow X$ is an embedding for almost all $\mathbb{D} \in \prod_j |\mathcal{D}_j|$.*

Proof. In view of Lemma 2.3 it remains to check that the set A_τ for which $f_{\mathbb{D}}$ is an embedding above U_τ is indeed nonempty.

After renumbering, we have $\tau = \rho_1 + \rho_2 + \rho_3$ and we consider the semi-group $M \cap \tau^\vee$. It is generated by $l_1, l_2, l_3 \in M$ with the property that $\langle l_i, n_i \rangle > 0$ and $\langle l_i, n_k \rangle = 0$ if $1 \leq k \leq 3$ and $k \neq i$. The function $p_i = \varepsilon_{\mathbb{D}, \mathbf{t}}(l_i) = f_{\mathbb{D}, \mathbf{t}}|_{Y_\tau} \circ \mathbf{e}(l_i) f_{\mathbb{D}, \mathbf{t}}$ is the i th coordinate function: it takes the value 0 on D_i and is nonzero on D_k for $1 \leq k \leq 3$, $k \neq i$.

We first pick D_j for $j > 3$ once and for all, only requiring them to be general in the sense of Lemma 2.2. Now choose D_3 so that D_3 is also reduced and disjoint from the other D_j chosen so far. This is enough to determine p_3 up to the torus action, since $\text{div}(p_3) = \langle l_3, n_3 \rangle D_3 + \sum_{j>3} \langle l_3, n_j \rangle D_j$ is independent of D_1 and D_2 . Similarly a choice of D_1 or of D_2 determines p_1 or p_2 up to the torus action, independently of the choice.

After making such a choice of D_3 , we claim that for general $D_2 \in |\mathcal{D}_2|$ the map $(p_2, p_3): Y_\tau \rightarrow \mathbb{A}^2$ is generically injective. We shall check this by exhibiting a choice of D_2 which makes this map injective near D_3 . Observe that for any pair $P, Q \in D_3$ (so $p_3(P) = p_3(Q) = 0$) we can find $D_2 \in |\mathcal{D}_2|$ such that $P \in D_2$ but $Q \notin D_2$ (although such a choice of D_2 will not be general in the sense of Lemma 2.2), because \mathcal{D}_2 is sufficiently ample. For this choice of D_2 , we have $0 = p_2(P) \neq p_2(Q)$, so $p_2(P) \neq p_2(Q)$ for general D_2 and hence for general D_2 the values of p_2 on the points of D_3 are all

different from one another. In particular (p_2, p_3) corresponding to a general D_2 is injective at any point of D_3 and is therefore injective generically.

By exactly the same argument, a general choice of $D_1 \in |\mathcal{D}_1|$ separates points not separated by the other choices. If P' and Q' are (possibly infinitely near) points such that $p_2(P') = p_2(Q')$ and $p_3(P') = p_3(Q')$, then $p_1(P') \neq p_1(Q')$ if $P' \in D_1$ and $Q' \notin D_1$. Such D_1 exist if \mathcal{D}_1 is sufficiently ample. So for general D_1 we also have $p_1(P') \neq p_1(Q')$, as required. \square

References

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