Moduli spaces of irreducible symplectic manifolds

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Abstract

We study the moduli spaces of polarised irreducible symplectic manifolds. By a comparison with locally symmetric varieties of orthogonal type of dimension 20, we show that the moduli space of polarised deformation $\mathrm{K3}^{[2]}$ manifolds with polarisation of degree $2d$ and split type is of general type if $d \geq 12$.

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0 Introduction

A simply-connected compact complex Kähler manifold is called an irreducible symplectic manifold if it has an everywhere nondegenerate 2-form. Irreducible symplectic manifolds are also known as irreducible hyperkähler manifolds, and for brevity are frequently referred to simply as symplectic manifolds, omitting the word "irreducible". They have been extensively studied by Beauville, Bogomolov, Debarre, Fujiki, Huybrechts, Markman, Namikawa and O'Grady among others. Irreducible symplectic manifolds have even complex dimension: in the surface case they are the K3 surfaces. However, relatively few examples are known. Backgound, and considerable detail, may be found in Huybrechts' lecture notes [Huy3].

The second cohomology $H^2(X,\mathbb{Z})$ of a symplectic manifold X carries a nondegenerate quadratic form q_X of signature $(3, b_2(X)-3)$, called the Beauville form, or Beauville-Bogomolov form. Usually the lattice $L = \big(H^2(X,\mathbb{Z}), q_X\big)$ is not unimodular, nor is it known to be necessarily even, although it is even in all known examples. A polarisation on X is a choice of ample line bundle on X, or equivalently the cohomology class h of an ample line bundle. The (Beauville) degree of the polarisation is defined to be $d = q_X(h)$: it is positive. There is a period map for symplectic manifolds: the global Torelli theorem, however, is known to fail in some cases (see [Deb], [Nam1]).

Our aim in this paper is to study the moduli of polarised symplectic manifolds by means of the period map. In Section 1 we describe this construction precisely, prove that the moduli spaces exist and show how they are related to locally symmetric varieties of orthogonal type: see Theorem 1.5. These varieties are associated with the orthogonal complement L_h of h in L. What lattice L_h is depends in general on the choice of h, not just on the degree as in the case of K3 surfaces.

In Section 2 we specialise to the case of deformation $\mathrm{K3}^{[n]}$ manifolds: that is, symplectic manifolds deformation equivalent to $\mathrm{Hilb}^n(S)$ for a K3 surface S. In this case $L = 3U \oplus 2E_8(-1) \oplus \langle -2(n-1) \rangle$. Here one has a better understanding of the map from the moduli space to the locally symmetric variety, thanks to the work of Markman [Mar3]. We show in Theorem 2.3 that in this case one may consider the quotient by the group $\widetilde{\mathrm{O}}(L, h)$ of automorphisms of L that fix h and act trivially on the discriminant group $L^\vee/L$.

To continue further we need to study the orthogonal groups that can arise. We do this in Section 3, where we mainly study the lattice $L_{2t} = 3U \oplus 2E_8(-1) \oplus \langle -2t \rangle$. This leads us to a description of the possible types of polarisation for deformation $\mathrm{K3}^{[n]}$ manifolds. There are two special types, having only one orbit of polarising vectors.

For the rest of the paper we are concerned with the case $n = 2$ and with with the simplest polarisation, namely the split type, where the lattice L_h is $L_{2,2d} = 2U \oplus 2E_8(-1) \oplus \langle -2 \rangle \oplus \langle -2d \rangle$. Our main theorem, Theorem 4.1, states that every component of the corresponding moduli space is of general type as long as $d \geq 12$. O'Grady [OG4] studied the case $d = 1$ and showed that the moduli space is unirational. There seem to be very few other previous results about dimension 20 moduli spaces of orthogonal type. Voisin [Vo1] proved that one of them is birational to the moduli space of cubic fourfolds, and thus unirational, but the type of the polarisation in that case is not split. In the split case there are only nine possibly unirational moduli spaces (for $d = 9$ and $d = 11$ the Kodaira dimension is non-negative): for polarised K3 surfaces there are still forty-three such possibilities.

The proof of Theorem 4.1 is similar in style to the corresponding result for K3 surfaces proved in [GHS1] (see also [Vo2]), but there are many differences. We use the low-weight cusp form trick, which guarantees that once the stable orthogonal group $\widetilde{\mathrm{O}}(L_{2,2d})$ has a cusp form with suitable vanishing of weight less than the dimension of the moduli space then the components are of general type.

We construct the cusp form by means of the quasi pull-back of the Borcherds form, as in [GHS1]. To do so one requires a vector in E_7 orthogonal to at least 2 and at most 14 roots, of length $2d$.

Here there is a significant technical difficulty. The proof that these vectors exist involves estimating the number of ways of representing certain integers by various root lattices of odd rank. In Theorem 5.1 we give a new, clear, formulation of Siegel's formula for this number in the odd rank case. It may be expressed either in terms of Zagier L-functions or in terms of the H. Cohen numbers. This analytic estimate shows that the vectors we want exist for $d \geq 20$, and we can improve this bound slightly by means of a computer search.

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1 Irreducible symplectic manifolds

In this section we collect the necessary results concerning symplectic manifolds and their moduli. The main aim is to relate moduli spaces of polarised symplectic manifolds to quotients of homogeneous domains by an arithmetic group.

We begin with the basic definitions and facts about irreducible symplectic manifolds.

Definition 1.1 *A complex manifold X is called an irreducible symplectic manifold if the following conditions are fulfilled:*

(i) *X is a compact Kähler manifold;*

(ii) *X is simply connected;*

(iii) *$H^0(X, \Omega^2_X) \cong \mathbb{C}\sigma$ where σ is an everywhere nondegenerate holomorphic 2-form.*

Irreducible symplectic manifolds are also known as irreducible hyperkähler manifolds, and very often simply as symplectic manifolds. The symplectic surfaces are the K3 surfaces. In higher dimension the known examples are the Hilbert schemes $\mathrm{Hilb}^n(S)$ of a K3 surface S and deformations of them; generalised Kummer varieties and their deformations; and two examples of dimensions 6 and 10, constructed by O'Grady using moduli spaces of sheaves on abelian surfaces and K3 surfaces respectively ([OG1], [OG2]).

It follows immediately from the definition that X must have even dimension $2n$ and that its canonical bundle ω_X is trivial. Moreover $h^{2,0}(X) = h^{0,2}(X) = 1$ and $h^{1,0}(X) = h^{0,1}(X) = 0$. By a result of Bogomolov [Bog], the deformation space of X is unobstructed. This result was generalised to Ricci-flat manifolds by Tian [Ti] and Todorov [Tod], and algebraic proofs were given by Kawamata [Kaw1] and Ran [Ran] (see also [Fuj]). Since

$$T_{[0]} \operatorname{Def}(X) \cong H^1(X, T_X) \cong H^1(X, \Omega^1_X)$$

the dimension of the deformation space is $b_2(X) - 2$.

The main discrete invariants for symplectic manifolds are the *Beauville form* (also known as the *Beauville-Bogomolov form*) and the *Fujiki constant* or *Fujiki invariant.* The Beauville form is an indivisible integral symmetric bilinear form on $H^2(X,\mathbb{Z})$ of signature $(3, b_2(X)-3)$. Its role in the theory of irreducible symplectic manifolds is similar to the role of the intersection form for K3 surfaces. To define it, let $\sigma \in H^{2,0}(X)$ be such that $\int_X (\sigma\overline{\sigma})^n = 1$ and define

$$q'_X(\alpha) = \frac{n}{2}\int_X \alpha^2(\sigma\overline{\sigma})^{n-1} + (1-n)\left(\int_X \alpha\sigma^{n-1}\overline{\sigma}^n\right)\left(\int_X \overline{\alpha}\sigma^n\overline{\sigma}^{n-1}\right).$$

After multiplication by a positive constant γ the quadratic form $q_X = \gamma q'_X$ defines an indivisible, integral, symmetric bilinear form $(\quad,\quad)_X$ on $H^2(X,\mathbb{Z})$: this is the Beauville form. Clearly $(\sigma,\sigma)_X = 0$ and $(\sigma,\overline{\sigma})_X > 0$. Let $v(\alpha) = \alpha^{2n}$ be given by the cup product. Then, by a result of Fujiki [Fuj, Theorem 4.7], there is a positive rational number c, the *Fujiki invariant* such that

$$v(\alpha) = cq_X(\alpha)^n$$

for all $\alpha \in H^2(X,\mathbb{Z})$.

In [OG3] O'Grady introduced the notion of *numerical equivalence* among symplectic manifolds. Two symplectic manifolds X and X' of dimension $2n$ are said to be numerically equivalent if there exists an isomorphism $f\colon H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(X',\mathbb{Z})$ of abelian groups with $\int_X \alpha^{2n} = \int_{X'} f(\alpha)^{2n}$ for all $\alpha \in H^2(X,\mathbb{Z})$. The equivalence class of X is called the *numerical type* of X, denoted by $\mathbf{N}$. Clearly, two symplectic manifolds are numerically equivalent if they have the same Beauville form and Fujiki invariant. O'Grady [OG3, Section 2.1] showed that the converse is also true unless $b_2(X) = b_2(X') = 6$ and n is even, in which case the numerical type determines c_X but *a priori* one only has $q_X = \pm q_{X'}$. (There are, however, no known examples of irreducible symplectic manifolds with $b_2 = 6$.)

We fix an abstract lattice L which is isomorphic to $H^2(X,\mathbb{Z})$ equipped with the Beauville form $(\quad,\quad)_X$ (the Beauville lattice) and consider its associated period domain

$$\Omega_L = \{[\mathbf{w}] \in \mathbb{P}(L\otimes\mathbb{C}) \mid (\mathbf{w},\mathbf{w}) = 0,\ (\mathbf{w},\overline{\mathbf{w}}) > 0\}$$

which, since the signature is $(3, b_2(X)-3)$, is connected. A *marking* of a symplectic manifold X is an isomorphism $\psi\colon H^2(X,\mathbb{Z}) \xrightarrow{\sim} L$ of lattices. We can associate to each marked symplectic manifold (X,ψ) its period point $[\psi(\sigma)] \in \Omega_L$. Now let $f\colon \mathcal{X} \to U$ be a representative of $\operatorname{Def}(X)$. This means that U is a polydisc, $X_0 := f^{-1}(0) \cong X$ and f is a proper submersive map whose Kodaira-Spencer map

$$T_{f,0}\colon T_{U,0} \longrightarrow H^1(X,T_X)$$

is an isomorphism. We can use the marking ψ to define an isomorphism $\psi_U : R^2f_*(\mathbb{Z}) \xrightarrow{\sim} L_U$ (we shall tacitly shrink U wherever necessary) and thus a period map

$$\begin{aligned}\varphi_U\colon U &\longrightarrow \Omega_L\\ t &\longmapsto [\psi_t(\sigma_{X_t})].\end{aligned}$$

The local Torelli theorem for symplectic manifolds, proved by Beauville [Be], says that ϕ_U is a local isomorphism (in the complex topology).

The surjectivity of the period map was proved by Huybrechts in [Huy1, Theorem 8.1]. To formulate his result we consider a fixed lattice L which appears as the Beauville lattice of some symplectic manifold. Let $\mathfrak{M}_L$ be the corresponding moduli space of marked symplectic manifolds, i.e. as a set $\mathfrak{M}_L = \{(X, \psi\colon H^2(X,\mathbb{Z}) \xrightarrow{\sim} L\}/\approx$ where the equivalence relation $\approx$ is induced by $\pm f^*$ with $f : X \to X'$ a biholomorphic map. The space $\mathfrak{M}_L$ admits a natural smooth complex structure which, however, is not Hausdorff. The period map $\varphi\colon \mathfrak{M}_L \to \Omega_L$ is a holomorphic map and Huybrechts has shown that every connected component of $\mathfrak{M}_L$ maps surjectively onto Ω_L. For a discussion of moduli of marked symplectic manifolds see the paper [Huy6] by Huybrechts.

The situation improves considerably when one considers moduli of polarised symplectic manifolds. A *polarisation* on a symplectic manifold X is the choice of an ample line bundle $\mathcal{L}$ on X. Since the irregularity of X is 0 this is the same as the choice of a class $h \in H^2(X,\mathbb{Z})$ representing an ample line bundle on X. Clearly $q_X(h) > 0$. Conversely, Huybrechts has shown ([Huy1, Theorem 3.11]: see also [Huy2, Theorem 2]) that a symplectic manifold X is projective if and only if there exists a class $h \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ with $q_X(h) > 0$. It should be noted, however, that neither line bundle associated to $\pm h$ need be ample. There is, however, a small deformation of the pair (X, h) with this property.

We now fix an abstract lattice L of rank $b_2 = b_2(X)$ such that $H^2(X,\mathbb{Z}) \cong L$ and let $h \in L$ be a primitive element with $h^2 > 0$. Then the lattice

$$L_h = h_L^{\perp} < L$$

has signature $(2, b_2 - 3)$. It defines a homogeneous domain, which in this case has two connected components

$$\Omega_{L_h} = \mathcal{D}(L_h) \cup \mathcal{D}'(L_h). \tag{1}$$

If (X, h) is a pair with $h \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ and $\psi\colon H^2(X,\mathbb{Z}) \to L$ is a marking then the period point $[\psi(\sigma)] \in \Omega_{L_h}$. Hence for every deformation $\mathcal{X} \to U$ of the pair (X, h) the period map defines a holomorphic map $\varphi_U\colon U \to \Omega_{L_h}$.

In this paper we are interested in the moduli spaces of polarised symplectic manifolds. We shall fix the dimension $2n$ and the numerical type $\mathbf{N}$ of the symplectic manifolds that we consider. We have already remarked that this determines the Beauville lattice and Fujiki invariant unless $b_2 = 6$ and n is even, in which case the quadratic form is only determined up to sign. We shall consider polarised symplectic manifolds (X, h) of fixed numerical type and given value $q_X(h) = d > 0$. The degree of the associated line bundle $\mathcal{L}$ is $\deg(\mathcal{L}) = h^{2n} = cd^n$ where c is the Fujiki invariant. Instead of working with the (geometric) degree of a polarisation we prefer to work with the number d, which we will call the *Beauville degree* of the polarisation. We first note the following variant of a result of Huybrechts [Huy4, Theorem 4.3], which is itself an application of the finiteness theorem of Kollár and Matsusaka [KM, Theorem 3].

Proposition 1.2 *For fixed numerical type there are only finitely many deformation types of polarised symplectic manifolds (X, h) of dimension $2n$ and given Beauville degree $d = q_X(h) > 0$.*

Proof. Since the numerical type determines the Fujiki invariant c our choices also fix the degree $h^{2n} = cq_X(h)^n > 0$. The result follows immediately from [Huy4, Corollary 26.17]. $\square$

Now we define the moduli spaces we are interested in. We first fix a possible Hilbert polynomial, say $P(m)$. Note that this is more than fixing the degree of the polarisation. By Matsusaka's big theorem we can find a constant m_0 such that for all polarised manifolds $(X, \mathcal{L})$ with Hilbert polynomial $P(m)$ the line bundles $\mathcal{L}^{\otimes m}$ are very ample for $m \geq m_0$ and have no higher cohomology. Then we have embeddings $\varphi_{|\mathcal{L}^{\otimes m_0}|} \colon X \to \mathbb{P}^{N-1}$ where $N = h^0(X, \mathcal{L}^{\otimes m_0}) = P(m_0)$. Such an embedding depends on the choice of a basis of $H^0(X, \mathcal{L}^{\otimes m_0})$. Let H be an irreducible component of the Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^{N-1})$ that contains at least one point $\eta \in H$ corresponding to a symplectic manifold X_η. We denote by H^0 the open part of H parametrising smooth varieties.

Lemma 1.3 *H^0 has the following properties:*

(i) *Every point in H^0 parametrises a symplectic manifold;*

(ii) *H^0 is smooth.*

Proof. The universal family $\mathcal{X}^0$ over H^0 is a flat family of projective manifolds and thus every X in $\mathcal{X}^0$ is a compact Kähler manifold, is simply connected and has trivial canonical bundle, since this is true for X_η. Moreover, since the second Betti number is constant in $\mathcal{X}^0$, we have $h^{2,0}(X) = 1$ for every X in $\mathcal{X}^0$, by semi-continuity. Thus (i) follows from Beauville's classification theorem [Be, §5, Théorème 2].

To prove (ii) we proceed along the lines of [Sz, Theorem 1.3]. Let $X \subset \mathbb{P}^{N-1}$ correspond to a point in H^0. It follows from the restriction of the

Euler sequence on $\mathbb{P}^{N-1}$ to X that

$$H^1(X, T_{\mathbb{P}^{N-1}}|_X) \cong H^2(X, \mathcal{O}_X) \cong \mathbb{C}.$$

The long exact sequence of the normal bundle sequence yields

$$\cdots \longrightarrow H^0(X, N_{X/\mathbb{P}^{N-1}}) \xrightarrow{\alpha} H^1(X, T_X) \longrightarrow H^1(X, T_{\mathbb{P}^{N-1}}|_X) \longrightarrow \cdots$$

The image of α is contained in the hyperplane $V_h = h^\perp \subset H^1(X, T_X) \cong H^1(X, \Omega_X^1)$, which corresponds to deformations of the pair (X, h) where h is the class of $\mathcal{L} = \mathcal{O}_X(1)$. Since $H^1(X, T_{\mathbb{P}^{N-1}})$ is 1-dimensional the image of α is equal to V_h and hence the Hilbert scheme is unobstructed and thus smooth. $\square$

Definition 1.4 *Let L be a lattice. We denote the discriminant group of L by $D(L) = L^\vee/L$. The stable orthogonal group $\widetilde{\mathrm{O}}(L)$ is defined by*

$$\widetilde{\mathrm{O}}(L) = \ker(\mathrm{O}(L) \to \mathrm{O}(D(L))). \tag{2}$$

For a primitive element $h \in L$ with $h^2 = d > 0$, we define the groups

$$\mathrm{O}(L, h) = \{g \in \mathrm{O}(L) \mid g(h) = h\} \tag{3}$$

and

$$\widetilde{\mathrm{O}}(L, h) = \{g \in \widetilde{\mathrm{O}}(L) \mid g(h) = h\}. \tag{4}$$

For any subgroup $\Gamma \subset \mathrm{O}(L)$ we define the projective group $\mathrm{P}\Gamma = \Gamma/(\pm 1)$. If L is indefinite we define Γ^+ to be the subgroup of Γ of elements with real spin norm 1.

There are two ways to choose the definition of the spin norm. We have chosen it in such a way that any -2-reflection has spin norm 1.

We can consider $\mathrm{O}(L, h)$ and $\widetilde{\mathrm{O}}(L, h)$ as subgroups of $\mathrm{O}(L_h)$, where L_h is, as usual, the lattice perpendicular to h in L. We shall discuss the relationships among these three groups in Section 3.

Note that in our case the lattice L has signature $(3, b_2 - 3)$ and hence L_h has signature $(2, b_2 - 3)$. Thus the lattice L_h determines a homogeneous domain $\Omega_h = \Omega_{L_h}$ of type IV on which the three groups act, and if $\Gamma < \mathrm{O}(L_h)$ then Γ^+ is the subgroup of Γ that preserves the component $\mathcal{D}$.

The following theorem is crucial for the rest of the paper as it allows us to compare moduli spaces of polarised symplectic manifolds to suitable quotients of type IV homogeneous domains by an arithmetic group.

Theorem 1.5 *There exists a quasi-projective coarse moduli space $\mathsf{M}_{2n,\mathbf{N},d}$ parametrising primitively polarised symplectic manifolds of dimension $2n$,* *numerical type* $\mathbf{N}$ *and Beauville degree* d*. We choose any one of the irreducible components of* $\mathsf{M}_{2n,N,d}$ *and denote it by* $\mathcal{M}_d$*. Such a choice determines a primitive vector* $h \in L$ *(or possibly* $h \in L(-1)$ *if* $b_2 = 6$ *and* n *is even) with* $q(h) = d$ *such that there is a map*

$$\varphi\colon \mathcal{M}_d \longrightarrow (\mathrm{O}(L,h) \setminus \Omega_h)^0 .$$

Here $(\mathrm{O}(L,h) \setminus \Omega_h)^0$ *is a connected component of* $\mathrm{O}(L,h) \setminus \Omega_h$*. The map* φ *is a morphism of quasi-projective varieties which is dominant and finite onto its image.*

Proof. We first note that by Proposition 1.2 there are only finitely many possible Hilbert polynomials for a given choice of the discrete data $2n$, $\mathbf{N}$ and d. Hence it is enough to know that quasi-projective moduli spaces of symplectic manifolds for fixed Hilbert polynomial exist. This is a consequence of Viehweg's general theory: see [Vi, Theorem 1.13] and the discussion there. Indeed, every component $\mathcal{M}_d$ of $\mathsf{M}_{2n,\mathbf{N},d}$ is a quotient of the form $\mathrm{SL}(N,\mathbb{C})\backslash H^0$ for some component H of a suitable Hilbert scheme (see the discussion of Lemma 1.3).

We now want to relate the components $\mathcal{M}_d$ to quotients of the form $\mathrm{O}(L,h)\backslash\Omega_h$. For this we want to construct a map $\tilde{\varphi}\colon H^0 \to \mathrm{O}(L,h)\backslash\Omega_h$ and then argue that it factors through the quotient by $\mathrm{SL}(N,\mathbb{C})$. We first observe that every component H^0 determines an $\mathrm{O}(L)$-orbit of primitive vectors $h \in L$ with $q(h) = d$. Indeed, choosing a local marking ψ_t near a given point in H^0 we obtain a vector $h_t = \psi_t\big(c_1(\mathcal{O}_{X_t}(1))\big)$ with $q(h_t) = d$, and any two local markings differ by an element of $\mathrm{O}(L)$. Since H^0 is connected and the number of $\mathrm{O}(L)$-orbits is finite this associates to each H^0 a unique $\mathrm{O}(L)$-orbit.

Let h be a representative of the orbit defined by H^0. We shall be interested only in *h-markings*, that is, markings ψ with $\psi\big(c_1(\mathcal{O}_X(1))\big) = h$. They exist locally on all of H^0, and an h-marking on an open set $U \subset H^0$ defines, via the period map, a holomorphic map $\varphi_U\colon U \to \Omega_h$. Two h-markings differ by an element $\mathrm{O}(L,h)$, so we obtain a holomorphic map $\tilde{\varphi}\colon H^0 \to \mathrm{O}(L,h)\backslash\Omega_h$.

If $M \in \mathrm{SL}(N,\mathbb{C})$ maps $(X,\mathcal{O}_X(1))$ to $(X',\mathcal{O}_{X'}(1))$, it induces an isomorphism

$$M^*\colon \big(H^2(X',\mathbb{Z}), c_1(\mathcal{O}_{X'}(1))\big) \xrightarrow{\sim} \big(H^2(X,\mathbb{Z}), c_1(\mathcal{O}_X(1))\big).$$

If $\psi\colon H^2(X,\mathbb{Z}) \to L$ and $\psi'\colon H^2(X',\mathbb{Z}) \to L$ are h-markings then there exists an element $g \in \mathrm{O}(L,h)$ with $\psi \circ M^* = g \circ \psi'$. This shows that the map $\tilde{\varphi}$ factors through the quotient by $\mathrm{SL}(N,\mathbb{C})$, giving the required map

$$\varphi\colon \mathcal{M}_d \longrightarrow (\mathrm{O}(L,h)\backslash\Omega_h)^0 .$$

Our next aim is to show that the map φ is a morphism of quasi-algebraic varieties. For this we use a theorem of Borel [Bl] which says the following:

if Y is a quasi-projective variety and $f\colon Y \to \Gamma\backslash\Omega$ a holomorphic map to an arithmetic quotient of a homogeneous domain, where Γ is torsion free, then f is a morphism of quasi-projective varieties. Here $\Gamma\backslash\Omega$ carries the natural structure as a quasi-projective variety, which comes from the Baily-Borel compactification. We cannot apply this theorem immediately, as $\mathrm{O}(L,h)$ will in general not be torsion free. However, we can avoid this difficulty by using level covers.

We shall proceed in close analogy to [Po, Proposition 2.17] and [Sz, Section 2]. As the arguments are very similar to the ones used in these papers we shall omit the details. Since every point in H^0 parametrises a symplectic manifold, and is thus in particular never ruled, it follows from [MM] that the action of $\mathrm{SL}(N,\mathbb{C})$ on H^0 is proper and that the stabiliser of any point is finite and reduced. Let $\mathrm{O}(L,h)(l)$ be the congruence subgroup of $\mathrm{O}(L,h)$ of level l, i.e. the intersection of $\mathrm{O}(L,h)$ with the full level-l subgroup $\mathrm{O}(L)(l)$ of $\mathrm{O}(L)$. We shall assume $l \geq 3$.

One can then construct a finite étale Galois cover $H^0(l) \to H^0$ whose fibres are in bijective correspondence with $\mathrm{O}(L,h)/\mathrm{O}(L,h)(l)$. The action of the group $\mathrm{SL}(N,\mathbb{C})$ on H^0 lifts to $H^0_k(l)$. By construction we obtain a commutative diagram (for details of this see [Po, 2.9]):

$$\begin{tikzcd}
\mathrm{SL}(N,\mathbb{C})\backslash H^0(l) \arrow[r, "\varphi(l)"] \arrow[d, "f"] & \mathrm{O}(L,h)(l)\backslash\Omega_h \arrow[d, "g"] \\
\mathcal{M}_d \arrow[r, "\varphi"] & \mathrm{O}(L,h)\backslash\Omega_h
\end{tikzcd}$$

where all varieties are quasi-projective and where the vertical maps are finite surjective morphisms given by the action of a finite group. By Borel's result the holomorphic map $\varphi(l)$ is a morphism and hence φ is also a morphism.

Finally we want to prove that φ is dominant and has finite fibres. Since it is a morphism of quasi-projective varieties it is enough to show that φ has no positive-dimensional fibres. As in [Sz, Lemma 2.7] one can construct a further finite étale covering $H^0(\rho) \to H^0(l)$ with the property that $\mathrm{SL}(N,\mathbb{C})$ acts freely on $H^0(\rho)$. Let $\mathcal{X}^0 \to H^0$ be the universal family and denote its pullback to $H^0(\rho)$ by $\mathcal{X}^0(\rho)$. The group $\mathrm{SL}(N,\mathbb{C})$ also acts on $\mathcal{X}^0(\rho)$. Locally $\mathcal{X}^0(\rho) \to H^0(\rho)$ represents the Kuranishi family of the polarised symplectic manifold, and by the infinitesimal Torelli theorem the period map is a local isomorphism near every point of $H^0(\rho)$. Hence the induced morphism $\mathrm{SL}(N,\mathbb{C})\backslash H^0(\rho) \to \mathrm{O}(L,h)\backslash\Omega_h$ has no positive-dimensional fibres, and since $H^0(\rho) \to H^0$ is finite the same also holds for φ. □

The techniques used here give finiteness results for irreducible symplectic manifolds similar to those proved by Szendrői (e.g. [Sz, Theorem 4.2]) for Calabi-Yau manifolds.

Corollary 1.6 *Let X be an irreducible symplectic manifold. Given a pos-*

itive integer k there exist only finitely many minimal models of X which possess a polarisation whose degree is bounded by k.

Proof. We first note that any minimal model Y of X is again smooth. The following argument for this is due to D. Huybrechts: given Y there is a birational map between X and Y which is an isomorphism in codimension 1. Choose a generic ample line bundle $\mathcal{L}$ on Y and consider the corresponding line bundle $\mathcal{M}$ on X. Then $(c_1(\mathcal{M}), D) > 0$ for every divisor D on X. By [Huy3, Proposition 27.4], there is a third birationally equivalent pair $(Z, \mathcal{N})$ where Z is smooth and $\mathcal{N}$ is ample. But this implies that the birational map between Y and Z is an isomorphism, by [Huy5, p. 501]. Alternatively we can use Kawamata's recent general result [Kaw2] that any two minimal models are connected by flops, and Namikawa's observation [Nam2] that flops of symplectic manifolds preserve smoothness.

Since the degree of the polarisation is bounded, the results of Kollár and Matsusaka [KM] imply that the minimal models in question have only finitely many possible Hilbert polynomials. Thus they belong to finitely many components of moduli spaces of polarised symplectic manifolds. Since all minimal models are smooth and birationally equivalent they are deformation equivalent by Huybrechts [Huy1, p. 65], and hence have the same numerical type. Moreover their Hodge structures are isomorphic. In other words all minimal models define the same period point and since every such period point corresponds to at most finitely many polarised symplectic manifolds in a given component of moduli the result follows. $\square$

Remark 1.7 *The map $\varphi\colon \mathcal{M}_d \to (\mathrm{O}(L,h)\backslash\Omega_h)^0$ will in general not be surjective as there are period points in Ω_h which parametrise pairs (X,h) where h is not ample.*

This phenomenon already occurs for K3 surfaces. Unlike in the K3 case it is, however, not clear which open part of the period domain belongs to ample divisors. There are some results about this in special cases, due to Hassett and Tschinkel [HT].

We shall use Theorem 1.5 in Section 4 to prove general type results for some moduli spaces of symplectic manifolds by proving that the quotients $\mathrm{O}(L,h)\backslash\Omega_h$ are of general type.

2 Deformation $\mathrm{K3}^{[n]}$ manifolds and monodromy

For the remainder of the paper we concentrate on a special case.

Definition 2.1 *A deformation* $\mathrm{K3}^{[n]}$ *manifold is a symplectic manifold that is deformation equivalent to* $\mathrm{Hilb}^n(S)$ *for some* K3 *surface* S.

(Compare the definition of numerical $\mathrm{K3}^{[2]}$ in [OG3].) If X is a deformation $\mathrm{K3}^{[n]}$ manifold then $H^2(X,\mathbb{Z}) \cong L_{2n-2}$ (as a lattice with the Beauville form), where for any $t \in \mathbb{N}$ we put

$$L_{2t} = 3U \oplus 2E_8(-1) \oplus \langle -2t \rangle. \tag{5}$$

For deformation $\mathrm{K3}^{[n]}$ manifolds, the numerical type is determined completely by the dimension $2n$, and the (Beauville) degree of a polarisation is always even. The Fujiki invariant is $\frac{(2n)!}{2^n n!}$.

We study deformation $\mathrm{K3}^{[n]}$ manifolds by using monodromy operators, whose theory was developed by Markman [Mar1], [Mar2], [Mar3]. We consider a flat family $\pi\colon \mathcal{X} \to B$ of compact complex manifolds with fibre X over the point $b \in B$. Associated to such a family we obtain a monodromy representation

$$\pi_1(B,b) \longrightarrow \mathrm{Aut}(H^*(X,\mathbb{Z})).$$

We define the *group of monodromy operators* to be the subgroup $\mathrm{Mon}(X)$ of $\mathrm{Aut}(H^*(X,\mathbb{Z}))$ generated by the image of all monodromy representations. If we restrict to the second cohomology, we obtain a representation $\pi_1(B,b) \to \mathrm{Aut}(H^2(X,\mathbb{Z}))$ and correspondingly a subgroup $\mathrm{Mon}^2(X) \subset \mathrm{Aut}(H^2(X,\mathbb{Z}))$. If X is a symplectic manifold then monodromy transformations preserve the Beauville form and we obtain a subgroup $\mathrm{Mon}^2(X) \subset \mathrm{O}(H^2(X,\mathbb{Z}))$.

Let $\mathrm{Ref}(X)$ be the subgroup of $\mathrm{O}(H^2(X,\mathbb{Z}))$ generated by -2-reflections and by the negatives of $+2$-reflections. By the choice of spin norm made in Definition 1.4, this is a subgroup of $\mathrm{O}^+(H^2(X,\mathbb{Z}))$.

Theorem 2.2 (Markman [Mar3, Theorem 1.2].) *If X is a deformation $\mathrm{K3}^{[n]}$ manifold then*

$$\mathrm{Mon}^2(X) = \mathrm{Ref}(X).$$

Using a marking $\psi\colon H^2(X,\mathbb{Z}) \xrightarrow{\sim} L_{2n-2}$ we can think of $\mathrm{Mon}^2(X)$ as a subgroup of $\mathrm{O}^+(L_{2n-2})$. Since $\mathrm{Mon}^2(X)$ is a normal subgroup we obtain a well-defined subgroup $\mathrm{Mon}^2(L_{2n-2}) = \mathrm{Ref}(L_{2n-2}) = \mathrm{Ref}(X) \subset \mathrm{O}^+(L_{2n-2})$.

It follows from a result of Kneser [Kn, Satz 4] or from [Mar2, Lemma 4.10] that the groups satisfy

$$\mathrm{Ref}(L_{2n-2}) = \widehat{\mathrm{O}}^+(L_{2n-2}) = \{g \in \mathrm{O}^+(L_{2n-2}) \mid g|_{D(L_{2n-2})} = \pm\,\mathrm{id}\} \tag{6}$$

where $D(L_{2n-2})$ is the discriminant group of the lattice L_{2n-2} Note that the assumptions of Kneser's theorem are fulfilled since L_{2n-2} contains three copies of U.

Unlike in the case of K3 surfaces, for fixed degree $2d$ there is not a unique $\mathrm{O}^+(L_{2n-2})$-orbit of primitive vectors h with $h^2 = 2d$. We shall address this question in Section 3. Hence the moduli space of deformation $\mathrm{K3}^{[n]}$ manifolds with a primitive polarisation of degree $2d$ will in general have more than one component.

Theorem 2.3 *Let $\mathcal{M}_{2d}^{[n]}$ be an irreducible component of the moduli space of deformation* $\mathrm{K3}^{[n]}$ *manifolds with a primitive polarisation of degree $2d$. Then the map φ from Theorem 1.5, above, factors through the finite cover $\widetilde{\mathrm{O}}^+(L_{2n-2},h)\backslash\mathcal{D}_h \to \mathrm{O}^+(L_{2n-2},h)\backslash\mathcal{D}_h$: that is, there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{M}_{2d}^{[n]} \arrow[r, "\tilde{\varphi}"] \arrow[dr, "\varphi"] & \widetilde{\mathrm{O}}^+(L_{2n-2},h)\backslash\mathcal{D}_h \arrow[d] \\
 & \mathrm{O}^+(L_{2n-2},h)\backslash\mathcal{D}_h
\end{tikzcd}$$

Proof. Recall from the proof of Theorem 1.5 that $\mathcal{M}_{2d}^{[n]} = \mathrm{SL}(N,\mathbb{C})\backslash H^0$ for some suitable open part of a component H of the Hilbert scheme. We choose a base point in H^0 and denote the corresponding symplectic variety by X_0. Choose an h-marking $\psi_0\colon \big(H^2(X_0,\mathbb{Z}), c_1(\mathcal{O}_{X_0(1)})\big) \to (L_{2n-2},h)$. Now let Y be a variety corresponding to another point in H and choose a path σ_Y from X_0 to Y. Transporting the marking ψ_0 along this path we obtain an h-marking $\psi_{\sigma_Y}: \big(H^2(Y,\mathbb{Z}), c_1(\mathcal{O}_Y(1))\big) \to (L_{2n-2},h)$. Clearly this marking will depend on the path σ_Y. Let τ_Y be another path from X_0 to Y and ψ_{τ_Y} the corresponding marking. Then $\tau_Y \circ \sigma_Y^{-1}$ is a closed path based at Y and induces an automorphism $f^* = \psi_{\sigma_Y}^{-1} \circ \psi_\tau \in \mathrm{Mon}^2(Y)$. Let $f' = \psi_{\sigma_Y} \circ f^* \circ \psi_{\sigma_Y}^{-1} \in \mathrm{Mon}^2(L_{2n-2})$. Then $f' \circ \psi_{\sigma_Y} = \psi_{\sigma_Y} \circ f^* = \psi_{\tau_Y}$. This shows that we have a morphism

$$\varphi'\colon H^0 \longrightarrow (\mathrm{Mon}^2(L_{2n-2}) \cap \mathrm{O}^+(L_{2n-2},h))\backslash\mathcal{D}_h = \widehat{\mathrm{O}}^+(L_{2n-2},h)\backslash\mathcal{D}_h$$

where the last equality follows from Theorem 2.2 and equation (6).

We next claim that φ' factors through $\mathcal{M}_{2d}^{[n]}$. For this let $g \in \mathrm{SL}(N,\mathbb{C})$ be an element which maps Y to Z. Let σ_Y and σ_Z be paths from X_0 to Y and Z respectively, with corresponding markings ψ_{σ_Y} and ψ_{σ_Z}. We now consider the path $\sigma_Z \circ \sigma_Y^{-1}$ from Y to Z. Using the element g to identify Y and Z makes this a closed path. We can now argue as above and conclude that $\psi_{\sigma_Y} \circ g^* \circ {\psi_{\sigma_Z}}^{-1} \in \mathrm{Mon}^2(L_{2n-2}) \cap \mathrm{O}^+(L_{2n-2},h) = \widehat{\mathrm{O}}^+(L_{2n-2},h)$. (Strictly speaking we need a complex family to argue that this element is in $\mathrm{Mon}^2(L_{2n-2})$, but this can easily be achieved by a complex thickening of the closed path.)

We can put $\widetilde{\mathrm{O}}^+(L_{2n-2},h)$ in the formulation of the theorem because $\mathrm{P}\widetilde{\mathrm{O}}^+(L_{2n-2},h) \cong \mathrm{P}\widehat{\mathrm{O}}^+(L_{2n-2},h)$, and the groups act on the symmetric space through their projectivisations. $\square$

Remark 2.4 *The lifting of the map φ to $\tilde{\varphi}$ is not unique. Two markings ψ_0 and ψ_1 define the same lifting if and only if ${\psi_0 \circ \psi_1}^{-1}$ is trivial in $\mathrm{PO}^+(L_{2n-2},h)/\,\mathrm{P}\widetilde{\mathrm{O}}^+(L_{2n-2},h)$, so the different liftings are classified by* *the quotient* $\mathrm{PO}^+(L_{2n-2},h)/\widetilde{\mathrm{PO}}^+(L_{2n-2},h)$. *We shall compute the index of* $\widetilde{\mathrm{O}}^+(L_{2n-2},h)$ *in* $\mathrm{O}^+(L_{2n-2},h)$ *below (Proposition 3.11), in almost all cases.*

Theorem 2.3 should also be compared to Markman's consideration of the non-polarised case in [Mar3, Section 4.2].

Remark 2.5 *As in [Mar3] we can conclude from Theorem 2.3 that the global Torelli theorem for polarised deformation* $\mathrm{K3}^{[n]}$ *manifolds fails whenever* $[\mathrm{PO}^+(L_{2n-2},h):\widetilde{\mathrm{PO}}^+(L_{2n-2},h)]>1$. *This can occur: see Proposition 3.11, below.*

With Remark 2.5 in mind we pose the following question.

Question 2.6 *Is it true that for every* $\mathrm{O}^+(L_{2n-2})$*-orbit of some primitive vector* h *with* $h^2=2d>0$ *the part of the moduli space* $\mathsf{M}_{2n,\mathbf{N},2d}$ *corresponding to polarisations in the orbit of* h *is irreducible and that the map* $\tilde{\varphi}$ *has degree* 1*?*

A positive answer to both parts of Question 2.6 could be viewed as the correct version of the global Torelli theorem for deformation $\mathrm{K3}^{[n]}$ manifolds.

Remark 2.7 *For every class of symplectic manifolds, Theorem 1.5 remains true if we consider the monodromy group instead of the orthogonal group.*

3 Orthogonal groups

Let L be an even lattice. By lattice (or sublattice) we always mean a non-degenerate lattice (or sublattice). If $g\in\mathrm{O}(L)$ we denote by $\bar{g}$ its image in $\mathrm{O}(D(L))$.

Let S be a primitive sublattice of L: we are mainly interested in the case $S=L_h$ for some $h\in L$ with $h^2\neq 0$, but we want to consider this more general situation. Analogously to Definition 1.4 we define the groups

$$\mathrm{O}(L,S)=\{g\in\mathrm{O}(L)\mid g|_S\in\widetilde{\mathrm{O}}(S)\}\quad\text{and}\quad\widetilde{\mathrm{O}}(L,S)=\mathrm{O}(L,S)\cap\widetilde{\mathrm{O}}(L).$$

Note that $\mathrm{O}(L,\mathbb{Z}h)=\mathrm{O}(L,h)$ if $h^2\neq\pm 2$.

Let $S^\perp$ be the orthogonal complement of S in L. We have

$$S^\perp\oplus S<L<L^\vee<(S^\perp)^\vee\oplus S^\vee.$$

The overlattice L is defined by the finite subgroup

$$H=L/(S^\perp\oplus S)<(S^\perp)^\vee/S^\perp\oplus S^\vee/S=D(S^\perp)\oplus D(S)$$

which is an isotropic subgroup of $D(S^\perp)\oplus D(S)$. Following [Nik] we consider the projections

$$p_S\colon H\to D(S),\qquad p_{S^\perp}\colon H\to D(S^\perp).$$

Using the definitions and the fact that the lattices S and $S^\perp$ are primitive in L one can show (see [Nik, Prop. 1.5.1]) that these projections are injective and moreover that if $d_S \in p_S(H)$ then there is a unique $d_{S^\perp} \in p_{S^\perp}(H)$ such that $d_S + d_{S^\perp} \in H$.

Lemma 3.1 *$\alpha \in \mathrm{O}(S^\perp)$ can be extended to $\mathrm{O}(L)$ if and only if*

$$\bar\alpha(p_{S^\perp}(H)) = p_{S^\perp}(H)$$

and there exists $\beta \in \mathrm{O}(S)$ such that $p_S^{-1} \circ \bar\beta \circ p_S = p_{S^\perp}^{-1} \circ \bar\alpha \circ p_{S^\perp}$.

This is a reformulation of [Nik, Corollary 1.5.2]. The following is a particular case.

Lemma 3.2 *Let S be a primitive sublattice of an even lattice L.*

(i) $g \in \mathrm{O}(L,S)$ *if and only if* $g(S) = S$, $\bar g|_{D(S)} = \mathrm{id}$ *and* $\bar g|_{p_{S^\perp}(H)} = \mathrm{id}$.

(ii) $\alpha \in \mathrm{O}(S^\perp)$ *can be extended to* $\mathrm{O}(L,S)$ *if and only if* $\bar\alpha|_{p_{S^\perp}(H)} = \mathrm{id}$.

(iii) If $p_{S^\perp}(H) = D(S^\perp)$ *then* $\mathrm{O}(L,S)|_{S^\perp} \cong \widetilde{\mathrm{O}}(S^\perp)$.

(iv) Assume that the projection $\mathrm{O}(S^\perp) \to \mathrm{O}(D(S^\perp))$ *is surjective. Then*

$$\mathrm{O}(L,S)|_{S^\perp}/\widetilde{\mathrm{O}}(S^\perp) \cong \{\bar\gamma \in \mathrm{O}(D(S^\perp)) \mid \bar\gamma|_{p_{S^\perp}(H)} = \mathrm{id}\}.$$

Remark 3.3 *Let $g \in \mathrm{O}(L,S)$. Then $\bar g|_{p_{S^\perp}(H)} = \mathrm{id}$ is equivalent to $\bar g|_H = \mathrm{id}$ or to $\bar g|_{H^\vee} = \mathrm{id}$, where $H^\vee = ((S^\perp)^\vee \oplus S^\vee)/L^\vee$. The condition $\bar g|_{H^\vee} = \mathrm{id}$ is equivalent to the following: for any $v \in (S^\perp)^\vee$ we have $g(v) - v \in (S^\perp)^\vee \cap L^\vee$. But $(S^\perp)^\vee \cap L^\vee$ might be larger than $S^\perp$. This shows in terms of the dual lattices that $\bar g|_{p_{S^\perp}(H)} = \mathrm{id}$ is weaker than $g|_{S^\perp} \in \widetilde{\mathrm{O}}(S^\perp)$.*

Corollary 3.4 *If $|H| = |\det S^\perp|$ then $\mathrm{O}(L,S)|_{S^\perp} \cong \widetilde{\mathrm{O}}(S^\perp)$.*

Proof. This follows from the injectivity of $p_{S^\perp}$ on H and from Lemma 3.2(iii). For example, the condition of the corollary is true if L is an even unimodular lattice and S is any primitive sublattice of L. □

If $l \in L$ its *divisor* $\mathrm{div}(l)$ is the positive generator of the ideal $(l, L) \subset \mathbb{Z}$. Therefore $l^* = l/\mathrm{div}(l)$ is a primitive element of the dual lattice $L^\vee$ and $\mathrm{div}(l)$ is a divisor of $\det(L)$. We recall the the following classical criterion of Eichler (see [E, §10]).

Lemma 3.5 *Let L be a lattice containing two orthogonal isotropic planes. Then the $\widetilde{\mathrm{O}}(L)$-orbit of a primitive vector $l \in L$ is determined by two invariants: by its length $l^2 = (l,l)$ and its image $l^* + L$ in the discriminant group $D(L)$.*

We consider the special lattice $L_{2t} = 3U \oplus 2E_8(-1) \oplus \langle -2t \rangle$ defined in Equation (5) above. We shall need this for the application in Section 4. It has signature $(3, 20)$. We denote a generator of the 1-dimensional sublattice $\langle -2t \rangle$ by l_t, so $l_t^2 = -2t$. In what follows we study the groups $\operatorname{O}(L_{2t}, h_d)$ and $\widetilde{\operatorname{O}}(L_{2t}, h_d)$ where h_d is a primitive vector of length $2d$. In the next proposition we study the $\widetilde{\operatorname{O}}(L_{2t})$-orbits of the polarisation vectors h_d. We note that $\operatorname{div}(h_d)$ is a common divisor of $2d$ and $2t = -\det(L_{2t})$.

Proposition 3.6 *Let $h_d \in L_{2t}$ be primitive of length $2d > 0$ and $\operatorname{div}(h_d) = f$. We put*

$$g = (\tfrac{2t}{f}, \tfrac{2d}{f}), \qquad w = (g, f), \qquad g = wg_1, \qquad f = wf_1.$$

Then

$$2t = fgt_1 = w^2 f_1 g_1 t_1 \qquad \text{and} \qquad 2d = fgd_1 = w^2 f_1 g_1 d_1$$

where $(t_1, d_1) = (f_1, g_1) = 1$.

(i) If g_1 is **even**, *then such an h_d exists if and only if $(d_1, f_1) = (f_1, t_1) = 1$ and $-d_1/t_1$ is a quadratic residue modulo f_1. Moreover the number of $\widetilde{\operatorname{O}}(L_{2t})$-orbits of h_d with fixed f (if at least one h_d exists) is equal to*

$$w_+(f_1)\phi(w_-(f_1)) \cdot 2^{\rho(f_1)},$$

where $w = w_+(f_1)w_-(f_1)$ and $w_+(f_1)$ is the product of all powers of primes dividing (w, f_1), $\rho(n)$ is the number of prime factors of n and $\phi(n)$ is the Euler function.

(ii) If g_1 is **odd**, *and f_1 is* **even** *or f_1 and d_1 are* **both odd**, *then such an h_d exists if and only if $(d_1, f_1) = (t_1, 2f_1) = 1$ and $-d_1/t_1$ is a quadratic residue modulo $2f_1$. The number of $\widetilde{\operatorname{O}}(L_{2t})$-orbits of such h_d is equal to*

$$w_+(f_1)\phi(w_-(f_1)) \cdot 2^{\rho(f_1/2)} \qquad \text{if } f_1 \equiv 0 \mod 2$$

and to

$$w_+(f_1)\phi(w_-(f_1)) \cdot 2^{\rho(f_1)} \qquad \text{if } f_1 \equiv d_1 \equiv 1 \mod 2.$$

(iii) If g_1 and f_1 are **both odd** *and d_1 is* **even**, *then such an h_d exists if and only if $(d_1, f_1) = (t_1, 2f_1) = 1$, $-d_1/(4t_1)$ is a quadratic residue modulo f_1 and w is odd. The number of $\widetilde{\operatorname{O}}(L_{2t})$-orbits of such h_d is equal to*

$$w_+(f_1)\phi(w_-(f_1)) \cdot 2^{\rho(f_1)}.$$

(iv) For c a suitable integer, determined mod f and satisfying $(c,f)=1$, and $b=(d+c^2t)/f^2$, we have

$$(h_d)^{\perp}_{L_{2t}} \cong 2U \oplus 2E_8(-1) \oplus B \quad \textit{with} \quad B = \begin{pmatrix} -2b & c\frac{2t}{f} \\ c\frac{2t}{f} & -2t \end{pmatrix}.$$

The form B is a negative definite binary quadratic form of determinant $4dt/f^2$. The greatest common divisor of the elements of B is equal to $g_1(\frac{2b}{g_1}, w)$.

Proof. A primitive vector h_d with $(h_d, L_{2t}) = f\mathbb{Z}$ acn be written

$$h_d = fv + cl_t$$

where $v \in 3U \oplus 2E_8(-1)$. The coefficient c is coprime to f because h_d is primitive. According to Eichler's criterion (Lemma 3.5) the $\widetilde{\mathrm{O}}(L_{2t})$-orbit of h_d is uniquely determined by $h_d^* \equiv \frac{c}{f} l_t \mod L_{2t}$. Therefore it is determined by c mod f because the discriminant group of L_{2t} is cyclic.

We put $v^2 = 2b$. Then $2d = 2bf^2 - 2c^2t$, or

$$2f_1 b = g_1(d_1 + c^2 t_1) \tag{7}$$

with $(f_1, g_1) = (t_1, d_1) = (c, f_1) = 1$. If a c coprime to f and satisfying Equation (7) exists, then because $(f_1, t_1) | g_1 d_1$ we must have $(f_1, t_1) = 1$, and similarly we must have $(f_1, d_1) = 1$.

(i) First we consider the case when g_1 is **even**. Equation (7) is equivalent to the congruence

$$t_1 c_1^2 \equiv -d_1 \mod f_1, \qquad c_1 \equiv c \mod f_1. \tag{8}$$

If g_1 is even, then f_1 is odd. Since $(t_1, f_1) = (d_1, f_1) = 1$ then $-d_1/t_1$ mod f_1 is invertible modulo f_1. If $-d_1/t_1$ is not a quadratic residue modulo f_1 then the congruence (8) has no solutions, so we assume that $-d_1/t_1$ is a quadratic residue modulo f_1. Because f_1 is odd, the number of solutions c_1 of (8) taken modulo f_1 is equal to

$$\#\{x \mod f_1 \mid x^2 \equiv 1 \mod f_1\} = 2^{\rho(f_1)}.$$

Let us calculate the number of solutions c modulo f where $f = wf_1$. Any solution c_1 is coprime to f_1. Let us put

$$c = c_1 + (x + yw_-(f_1))f_1$$

where x is taken mod $w_-(f_1)$ and y is taken mod $w_+(f_1)$.

We note that $(f_1, w_-(f_1)) = 1$. We have $(c, wf_1) = 1$ if and only if $(c_1 + xf_1, w_-(f_1)) = 1$. For any fixed c_1 the numbers $c_1 + xf_1$ form the full residue system modulo $w_-(f_1)$ if x runs modulo $w_-(f_1)$. Therefore for any fixed c_1 there are exactly $w_+(f_1)\phi(w_-(f_1))$ solutions c modulo $f = f_1w$ such that $c \equiv c_1 \mod f_1$ and $(c, f) = 1$. This finishes the proof of (i).

(ii) Consider the case when g_1 is **odd**. In this case t_1 is also odd, since $(g_1, 2f_1) = 1$ so $d_1 + c^2t_1 \equiv 0 \mod 2f_1$: so if t_1 is even then d_1 is also even.

If f_1 is **even**, then because f_1^2 is divisible by $2f_1$ Equation (7) is equivalent to the congruence

$$t_1c_1^2 \equiv -d_1 \mod 2f_1, \qquad c_1 \equiv c \mod f_1. \tag{9}$$

Since $(f_1, d_1) = 1$ we know that d_1 is odd. Therefore

$$\#\{c_1 \mod f_1 \mid c_1^2 \equiv -d_1/t_1 \mod 2f_1\} = 2^{\rho(f_1/2)}$$

if $-d_1/t_1$ is a quadratic residue mod $2f_1$. The rest is similar to the case (i).

If f_1 and d_1 are **both odd**, then Equation (7) is equivalent to the congruence

$$t_1c_1^2 \equiv -d_1 \mod 2f_1, \qquad c_1 \equiv c \mod 2f_1. \tag{10}$$

The residue $-d_1/t_1 \mod 2f_1$ is invertible and it is always 1 modulo 2. Therefore the number of solutions modulo $2f_1$ is equal to $2^{\rho(f_1)}$ and they are all different modulo f_1. We put

$$c = c_1 + (x + yw_-(2f_1)2^{\delta_2(w)-1})2f_1$$

taking x mod $w_-(2f_1)$ and y mod $w_-(2f_1)2^{\delta_2(w)-1}$, where $2^{\delta_2(w)}$ is the 2-factor of w for w even and is 2 if w is odd (so the factor $2^{\delta_2(w)-1}$ only appears if w is even). For even or odd w we obtain the same formula for the number of solutions c. This proves (ii).

(iii) Consider the case when g_1 and f_1 are **both odd**, and d_1 is **even**. Equation (7) is equivalent to the congruence (10) and c_1 is always even, i.e., $c_1 = 2c_2$ and

$$c_2^2 \equiv -(d_1/2)/(2t_1) \mod f_1$$

and c_2 is considered modulo f_1. In particular w must be odd since otherwise $2|(c, w)$. For odd w we choose

$$c = 2c_2 + (x + w_-(f_1)y)2f_1$$

with x taken mod $w_-(f_1)$ and y taken $w_-(f_1)$, and this completes the proof of (iii).

(iv) Fix a representative of the $\widetilde{\mathrm{O}}(L_{2t})$-orbit of h_d of the form

$$h_d = fe_1 + fbe_2 + cl_t \in U \oplus \langle -2t\rangle \tag{11}$$

where e_1 and e_2 form a usual basis of the hyperbolic plane U ($e_1^2 = e_2^2 = 0$ and $e_1 \cdot e_2 = 1$). The orthogonal complement of h_d in $U \oplus \langle -2t \rangle$ is a lattice L_B of rank 2

$$L_B = (h_d)^{\perp}_{U+\langle -2t\rangle} = \langle e_1 - be_2,\ c\tfrac{2t}{f} e_2 + l_t \rangle. \tag{12}$$

with the quadratic form B as in (iv). Both vectors are orthogonal to h_d: they form a basis because using them one can reduce to zero the coordinates at e_1 and at l_t. In the notation above we obtain

$$B = g_1 \begin{pmatrix} -2b/g_1 & cwt_1 \\ cwt_1 & -w^2 f_1 t_1 \end{pmatrix}.$$

We have $(cwt_1, w^2 f_1 t_1) = wt_1(c, wf_1) = wt_1$. The greatest common divisor of the elements of B is equal to $g_1(\frac{2b}{g_1}, w)$ because $2b/g_1$ and t_1 are coprime. □

Corollary 3.7 *Let us assume that* $w = 1$. *If there exists a primitive vector* $h_d \in L_{2t}$ *such that* $h_d^2 = 2d$ *and* $\operatorname{div}(h_d) = f$, *then all such vectors belong to the same* $\mathrm{O}(L_{2t})$*-orbit.*

Proof. The natural projection $\mathrm{O}(L_{2t}) \to \mathrm{O}(D(L_{2t}))$ is surjective (see [Nik]). Furthermore (see [GH])

$$\mathrm{O}(D(L_{2t})) \cong \{\, x \mod 2t \mid x^2 \equiv 1 \mod 4t \,\}.$$

Therefore for $w = 1$ all solutions $c \mod f$ of the congruences (8) and (9) are equivalent modulo the action of this abelian 2-group. □

Example 3.8 *Let* $f = 1$. *From the Proposition 3.6 it follows that for any* t *and* d *there is only one* $\widetilde{\mathrm{O}}(L_{2t})$*-orbit of primitive vectors* h_d *with* $\operatorname{div}(h_d) = 1$. *Moreover* $c = 0$ *and so*

$$(h_d)^{\perp}_{L_{2t}} \cong L_{2t,2d} = 2U \oplus 2E_8(-1) \oplus \langle -2t \rangle \oplus \langle -2d \rangle. \tag{13}$$

A polarisation determined by an h_d with $\operatorname{div}(h_d) = 1$ is called *split.* We note that this is the only case when the matrix B is diagonal because c can be zero and coprime to f only if $f = 1$. This case will be the main subject of much of the rest of this paper.

Example 3.9 *Let* $f = 2$. *In this case* c *is odd, so we may take* $c = 1$. *A constant* b *and a vector* h_d *exist if and only if* $d + t \equiv 0 \mod 4$. *Moreover the* $\widetilde{\mathrm{O}}(L_{2t})$*-orbit of* h_d *is unique because* $D(L_{2t})$ *is cyclic and thus contains only one element of order* 2.

If $f > 2$, then Proposition 3.6 shows that the number of orbits is zero or strictly greater than one. Thus the cases $f = 1$ and $f = 2$ are the most natural polarisation types.

Example 3.10 *Let d and t be coprime. Examples 3.8 and 3.9 give us the full classification of possible $h_d \in L_{2t}$ (in particular if $t = 1$ or $d = 1$), since if $(t, d) = 1$ then $f = \operatorname{div}(h_d) = 1$ or 2.*

In the next proposition we show that if $w = 1$, then the groups $\widetilde{\mathrm{O}}(L_{2t}, h_d)$ and $\mathrm{O}(L_{2t}, h_d)$ have rather clear structure.

Proposition 3.11 *Let $h_d \in L_{2t}$ be a primitive vector such that $h_d^2 = 2d$ and $\operatorname{div}(h_d) = f$. Assume that $w = 1$, i.e. f and $(\frac{2t}{f}, \frac{2d}{f})$ are coprime. Then*

(i) $\widetilde{\mathrm{O}}(L_{2t}, h_d) \cong \widetilde{\mathrm{O}}((h_d)^{\perp}_{L_{2t}})$.

(ii) The factor group $\mathrm{O}(L_{2t}, h_d)/\widetilde{\mathrm{O}}(L_{2t}, h_d)$ is an abelian 2-group, which is of order $2^{\rho(t/f)}$ if f is odd. If f is even the order is equal to $2^{\rho(2t/f)+\delta}$, where

$$\delta = \begin{cases} 0 & \text{if } (2t/f) \equiv 1 \mod 2 \text{ or } (2t/f) \equiv 4 \mod 8, \\ -1 & \text{if } (2t/f) \equiv 2 \mod 4, \\ 1 & \text{if } (2t/f) \equiv 0 \mod 8. \end{cases}$$

Proof. We may take h_d in the form (11). We can fix a basis k_1, k_2 of $L_B^{\vee}$ given by

$$\begin{aligned} k_1 &= \frac{f}{2d}h_d - e_2 &&= \frac{f}{2d}\big(fe_1 + bfe_2 + cl_t\big) - e_2, \\ k_2 &= \frac{c}{2d}h_d + \frac{1}{2t}l_t &&= \frac{f}{2d}\big(ce_1 + cbe_2 + \frac{bf}{t}l_t\big). \end{aligned}$$

Up to sign, this is dual to the basis fixed in (12). We put

$$k_3 = fk_2 - ck_1 = ce_2 + \frac{f}{2t}l_t.$$

If $v \in L^{\vee}$ we shall denote by $\bar{v}$ the corresponding element in the discriminant group $D(L) = L^{\vee}/L$. We note that the orders of $\bar{k}_1$ and of $\bar{k}_3$ in $D(L_B) = D((h_d)^{\perp}_{L_{2t}})$ (see (12)) are equal to $\frac{2d}{f}$ and $\frac{2t}{f}$ respectively. Moreover $\bar{k}_1 \cdot \bar{k}_3 = 0$. Let us calculate the order of the intersection of the subgroups generated by $\bar{k}_1$ and $\bar{k}_3$ in $D(L_B)$. If $n\bar{k}_1 \in \langle \bar{k}_3 \rangle$ then $n = g_1 d_1 n_1$ because $m\bar{k}_3$ does not contain the e_1-component. Therefore $n\bar{k}_1 \equiv (\frac{n_1 c}{w}l_t + xe_2)$ mod L_B where $x \in \mathbb{Z}$ (see (12)), and $|\langle \bar{k}_1 \rangle \cap \langle \bar{k}_3 \rangle| = w$. It follows that $\bar{k}_1$ and $\bar{k}_3$ form a basis of $D(L_B)$ if $w = 1$.

As in the beginning of the section we consider the following series of lattices

$$\langle h_d \rangle \oplus h_d^{\perp} < L_{2t} < L_{2t}^{\vee} < \langle h_d^{\vee} \rangle \oplus (h_d^{\perp})^{\vee},$$

where $h_d^\vee = \frac{1}{2d}h_d$ and $h_d^\perp = (h_d)^\perp_{L_{2t}} \cong 2U \oplus 2E_8(-1) \oplus L_B$. The subgroup $H = L_{2t}/(\langle h_d\rangle \oplus h_d^\perp)) < D(h_d) \oplus D(L_B)$ has order $\frac{2d}{f}$. It is generated by the element $\bar{k}_1 - f\bar{h}_d/2d$. Therefore the projection

$$p(H) = p_{h_d^\perp}(H) = \langle \bar{k}_1\rangle$$

is the subgroup generated by $\bar{k}_1$. It follows if $w = 1$ the discriminant group is

$$D(h_d^\perp) = \langle \bar{k}_1\rangle \oplus \langle \bar{k}_3\rangle = p(H) \oplus \langle \bar{k}_3\rangle.$$

According to Lemma 3.2

$$\mathrm{O}(L_{2t}, h_d) \cong \{\gamma \in \mathrm{O}(h_d^\perp) \mid \bar{\gamma}|_{p(H)} = \mathrm{id}\}.$$

Let us consider an element $\gamma \in \mathrm{O}(h_d^\perp)$ satisfying $\bar{\gamma}|_{p(H)} = \mathrm{id}$ as an element of $\mathrm{O}(L_{2t}, h_d)$ (i.e., we put $\gamma(h_d) = h_d$). According to the decomposition above $\gamma \in \widetilde{\mathrm{O}}(L_{2t}, h_d)$ if and only if $\bar{\gamma}(\bar{k}_3) = \bar{k}_3$. Therefore $\widetilde{\mathrm{O}}(L_{2t}, h_d) \cong \widetilde{\mathrm{O}}(h_d^\perp)$.

We note that the natural projection $\mathrm{O}(h_d^\perp) \to \mathrm{O}(D(L_B))$ is surjective (see [Nik]). Therefore according to Lemma 3.2

$$\mathrm{O}(L_{2t}, h_d)/\widetilde{\mathrm{O}}(L_{2t}, h_d) \cong \{\gamma \in \mathrm{O}(h_d^\perp) \mid \bar{\gamma}|_{p(H)} = \mathrm{id}\}/\widetilde{\mathrm{O}}(h_d^\perp) \cong \mathrm{O}(\langle \bar{k}_3\rangle),$$

where $\langle \bar{k}_3\rangle = \{n\bar{k}_3 \mid n \mod \frac{2t}{f}\}$ and $\bar{k}_3^2 \equiv -\frac{f^2}{2t} \mod 2$. Therefore

$$\begin{aligned}\mathrm{O}(\langle \bar{k}_3\rangle) &\cong \{x \mod \tfrac{2t}{f} \mid x^2\bar{k}_3^2 \equiv \bar{k}_3^2 \mod 2\}\\ &= \{x \mod \tfrac{2t}{f} \mid x^2 f \equiv f \mod 2\tfrac{2t}{f}\}\end{aligned}$$

We supposed that $w = 1$. Therefore $f = f_1$ and $g = g_1$ are coprime. We have $\frac{2t}{f} = g_1t_1$ with $(f_1, t_1) = 1$ (see Proposition 3.6). It follows that the group $\mathrm{O}(\langle \bar{k}_3\rangle)$ is isomorphic to the group

$$\{x \mod \tfrac{2t}{f} \mid x^2 \equiv 1 \mod 2^{\varepsilon(f)}\tfrac{2t}{f}\},$$

where $\varepsilon(f) = 1$ if f is odd (in this case $\frac{2t}{f}$ is even) and $\varepsilon(f) = 0$ is f is even. The last group is well-known (compare with [GH]). □

Corollary 3.12 *We have that $\mathrm{O}(L_{2t}, h_d) \cong \widetilde{\mathrm{O}}(L_{2t}, h_d)$ in the following three cases: f is odd and $f = t$; or $f = 2t$; or $f = t$ and $2d/f$ is odd.*

Proof. If $f = t$ or $f = 2t$, then $g = (\frac{2t}{f}, \frac{2d}{f}) = 1$ or 2. If $g = 1$, then $w = 1$. If $g = 2$ then $w = (f, g) = 1$ for odd f and for even f such that $(2d)/f$ is odd. In all these case the index $[\mathrm{O}(L_{2t}, h_d) : \widetilde{\mathrm{O}}(L_{2t}, h_d)] = 1$ according Proposition 3.11. □

Remark 3.13 *The condition $w = 1$, i.e., that f and $(\frac{2t}{f}, \frac{2d}{f})$ are coprime, is valid for any f if $(2t, 2d)$ is square free. In particular this condition is true for any vector h_d if $2t$ is square free.*

Remark 3.14 *The finite group* $\mathrm{O}(D((h_d)^{\perp}_{L_{2t}}))$ *is cyclic for any* h_d *with* $\operatorname{div}(h_d) = f$ *if* $g_1 = (\frac{2t}{f}, \frac{2d}{f}) = 1$. *If* $g_1 > 1$ *the discriminant group is not cyclic, but it is the orthogonal sum of two cyclic groups if* $w = 1$. *Proposition 3.11 shows that we can consider this rather general case as a regular one.*

Since the classification of polarisation types in this section depends only on the discriminant group it immediately gives an identical classification for polarisations of deformations of generalised Kummer varieties.

4 Modular forms and root systems

For the rest of the paper we restrict to a special class of symplectic 4-folds. We consider the case of deformation $\mathrm{K3}^{[2]}$ manifolds with polarisation of degree $2d$ of split type, as in Example 3.8 above. We denote an irreducible component of the corresponding moduli space by $\mathcal{M}_{2d}^{[2],\mathrm{split}}$. (We do not know whether there is only one irreducible component: see Question 2.6.) According to Theorem 2.3 we have a dominant map

$$\mathcal{M}_{2d}^{[2],\mathrm{split}} \longrightarrow \widetilde{\mathrm{O}}(L_2, h_d)\backslash\Omega_{h_d}.$$

In this case, where $t = 1$, we have

$$\widetilde{\mathrm{O}}(L_{2,2d}) = \widetilde{\mathrm{O}}(L_2, h_d) = \mathrm{O}(L_2, h_d)$$

by Proposition 3.11(i) and Corollary 3.12, where $L_{2,2d}$ is as defined in Equation (13). In particular the vertical map in Theorem 2.3 is of degree 1, and an affirmative answer to Question 2.6 would imply that global Torelli holds for deformation $\mathrm{K3}^{[2]}$ manifolds with polarisation of split type.

It is more convenient to express this quotient in terms of the symmetric domain $\mathcal{D}(L_{2,2d})$ defined in Equation (1) above. Recall that $\widetilde{\mathrm{O}}^+(L_{2,2d})$ is the index 2 subgroup of $\widetilde{\mathrm{O}}(L_{2,2d})$ that preserves $\mathcal{D}(L_{2,2d})$. Then

$$\widetilde{\mathrm{O}}(L_2, h_d)\backslash\Omega_{h_d} = \widetilde{\mathrm{O}}^+(L_{2,2d})\backslash\mathcal{D}(L_{2,2d}).$$

In the rest of this paper we study the Kodaira dimension of the locally symmetric variety $\widetilde{\mathrm{O}}^+(L_{2,2d})\backslash\mathcal{D}(L_{2,2d})$ and of the moduli space $\mathcal{M}_{2d}^{[2],\mathrm{split}}$.

Theorem 4.1 *The variety* $\mathcal{M}_{2d}^{[2],split}$ *is of general type if* $d \geq 12$. *Moreover its Kodaira dimension is non-negative if* $d = 9$ *and* $d = 11$.

Let L be an even integral lattice of signature $(2, n)$ with $n \geq 3$. A modular form of weight k and character $\chi\colon \Gamma \to \mathbb{C}^*$ for a subgroup $\Gamma < \mathrm{O}^+(L)$ of finite index is a holomorphic function $F\colon \mathcal{D}_L^\bullet \to \mathbb{C}$ on the affine cone $\mathcal{D}_L^\bullet$ over $\mathcal{D}_L$ such that

$$F(tZ) = t^{-k}F(Z) \quad \forall\, t \in \mathbb{C}^* \qquad \text{and} \qquad F(gZ) = \chi(g)F(Z) \quad \forall\, g \in \Gamma.$$

A modular form is a cusp form if it vanishes at every cusp. For applications, we require the order of vanishing to be at least 1 (both here and in [GHS1], although it is not stated explicitly there). In general this is a slightly stronger requirement because the order of vanishing might be a rational number less than 1. However, it is easy to check that for trivial character and character det, which are the cases used here and in [GHS1], the vanishing order at any cusp is an integer.

We denote the linear spaces of modular and cusp forms of weight k and character χ for Γ by $M_k(\Gamma, \chi)$ and $S_k(\Gamma, \chi)$ respectively.

The next theorem follows from the results obtained in [GHS1].

Theorem 4.2 *Suppose there exists a non-zero cusp form F_a of some weight $a < 20$ and character* det *with respect to the modular group $\widetilde{\mathrm{O}}^+(L_{2,2d})$. Then the modular variety $\mathcal{M}_{2d}^{[2],split}$ is of general type.*

If there exists a non-zero cusp form F_{20} of weight 20 and character det *then $\mathcal{M}_{2d}^{[2],split}$ has non-negative Kodaira dimension.*

Proof. $\mathcal{M}_{2d}^{[2],\mathrm{split}}$ is a quasi-projective variety of dimension 20. It has a toroidal compactification having only canonical singularities, by [GHS1, Theorem 2]. By [GHS1, Theorem 1.1], the variety $\mathcal{M}_{2d}^{[2],\mathrm{split}}$ is of general type if there exists a non-zero cusp form $F_a \in S_a(\widetilde{\mathrm{O}}^+(L_{2,2d}))$ of weight $a < 20$ that vanishes along the ramification divisor of the projection

$$\pi\colon \mathcal{D}(L_{2,2d}) \longrightarrow \widetilde{\mathrm{O}}^+(L_{2,2d})\backslash\mathcal{D}(L_{2,2d}).$$

We note that according to [GHS1, Corollary 2.13] the ramification divisor is determined by the elements $\sigma \in \widetilde{\mathrm{O}}(L_{2,2d})$ such that σ or $-\sigma$ is a reflection with respect to a vector $r \in L_{2,2d}$. We classified those reflections using the results of [GHS1, §3].

Let $F_a \in S_a(\widetilde{\mathrm{O}}^+(L_{2,2d}), \det)$ be of weight $a < 20$ and suppose that $\sigma \in \widetilde{\mathrm{O}}(L_{2,2d})$ defines a component of the ramification divisor. Then

$$F_a(\pm\sigma(Z)) = \det(\pm\sigma) \cdot F_a(Z) = -F_a(Z)$$

because $\det(-\sigma) = (-1)^{20}\det(\sigma) = -1$. Therefore the cusp form F_a with character det automatically vanishes on the ramification divisor.

If $a = 20$ and F_{20} vanishes along the ramification divisor of π then F_{20} determines a section of the canonical bundle by a well-known result of Freitag [Fr, Hilfssatz 2.1, Kap. III]. □

One can estimate the obstructions to continuing of the pluricanonical forms across the ramification divisor using the exact formula for Mumford–Hirzebruch volume of the corresponding orthogonal groups (see [GHS2]). But this approach only gives good results for locally symmetric varieties of orthogonal type if the dimension is quite large: at least 33 in the cases considered in [GHS3].

If the dimension of the modular variety is smaller than 26 we can use the quasi pull-back (see Equation (14) below) of the Borcherds modular form Φ_{12} to construct cusp forms of small weight. The Borcherds form is a modular form of weight 12 for $\mathrm{O}^+(II_{2,26})$, where $II_{2,26}$ is the unimodular lattice $2U \oplus 3E_8(-1)$.

$\Phi_{12}(Z) = 0$ if and only if there exists $r \in II_{2,26}$ with $r^2 = -2$ such that $(r, Z) = 0$. Moreover, the multiplicity of the rational quadratic divisor in the divisor of zeros of Φ_{12} is 1 (see [Bo]). This form generates very important functions on the moduli spaces of polarised K3 surfaces (see [BKPS], [Ko] and [GHS1]). In the context of the moduli space of symplectic manifolds we can use the following specialisation of the quasi pull-back.

The Weyl group of E_8 acts transitively on the roots of E_8. If v is a root of $E_8(-1)$ then $v^\perp_{E_8(-1)} \cong E_7(-1)$. Let $l \in E_7(-1)$ satisfy $l^2 = -2d$. The choice of v and l determines an embedding of $L_{2,2d}$ into $II_{2,26}$. The embedding of the lattice also gives us an embedding of the domain $\mathcal{D}(L_{2,2d}) \subset \mathbb{P}(L_{2,2d} \otimes \mathbb{C})$ into $\mathcal{D}(II_{2,26}) \subset \mathbb{P}(II_{2,26} \otimes \mathbb{C})$.

We put $R_l = \{r \in E_7(-1) \mid r^2 = -2,\ (r, l) = 0\}$, and $N_l = \# R_l$. (It is clear that N_l is even.) We note that R_l is the set of roots orthogonal to the sublattice $\langle v \rangle \oplus \langle l \rangle$ in $E_8(-1)$. Then the quasi pull-back of Φ_{12} is given by the following formula (see [BKPS]):

$$F_l = \left. \frac{\Phi_{12}(Z)}{\prod_{\{\pm r\} \in R_l} (Z, r)} \right|_{\mathcal{D}_{L_{2,2d}}} \in M_{12 + \frac{N_l}{2}}(\widetilde{\mathrm{O}}^+(L_{2,2d}), \det). \tag{14}$$

It is a non-trivial modular form of weight $12 + \frac{N_l}{2}$. By [GHS1, Theorem 6.2] it is a cusp form if N_l is non empty. In [GHS1] we proved this for $l \in E_8(-1)$. But in the proof we used only the fact that any isotropic subgroup of the discriminant form of the lattice $2U \oplus 2E_8(-1) \oplus \langle -2d \rangle$ is cyclic (see [GHS1, Theorem 4.2]). The same is true for $L_{2,2d}$ because its discriminant group is cyclic (see §2). The weight of F_l is smaller than 20 if $N_l < 16$.

The problem therefore is to determine the d for which such a vector exists. Sufficient conditions are given in Theorem 4.5 below. We apply the method used in the proof of [GHS1, Theorem 7.1]. We first need some properties of the lattice E_7.

Lemma 4.3 *The Weyl group $W(E_7)$ acts transitively on the sets of sublattices of E_7 of types $A_1 \oplus A_1$ or A_2.*

Proof. $W(E_7)$ acts transitively on the roots. Moreover $(A_1)^{\perp}_{E_7} \cong D_6$ and $W(D_6)$ acts transitively on its roots. This proves the $A_1 \oplus A_1$ case.

Let $A_2^{(1)}$, $A_2^{(2)}$ be two different copies of A_2 in E_7. Without loss of generality we can assume that they have a common root a, i.e. $A_2^{(1)} = A_2(a,c) = \mathbb{Z}a + \mathbb{Z}c$ and $A_2^{(2)} = A_2(a,d)$, where $a \cdot c = a \cdot d = -1$. Any A_2-lattice contains six roots

$$R(A_2(a,c)) = \{ \pm a,\ \pm c,\ \pm(a+c) \}$$

and it is generated by any pair of linearly independent roots. If $c \cdot d = -1$ then $(a+d)\cdot c = -2$, $c = -(a+d)$ and $A_2(a,c) = A_2(a,d)$. Therefore $c \cdot d = 0$ or 1. (We recall that for any two non-collinear roots u and v one has $|u \cdot v| \le 1$.)

If $c \cdot d = 1$ then $(c-d)^2 = 2$ and $(c-d)\cdot a = 0$. The reflection $\sigma_{c-d}$with respect to the root $(c-d)$ transforms $A_2^{(1)}$ into $A_2^{(2)}$:

$$\sigma_{(c-d)}(c) = c - (c\cdot(c-d))(c-d) = d, \qquad \sigma_{(c-d)}(a) = a.$$

If $c \cdot d = 0$ then $A_2(a,c) + A_2(a,d)$ is a root lattice of type A_3 with 12 roots:

$$R(A_3(a,c,d)) = \pm\{a,\ c,\ d,\ a+c,\ a+d,\ a+c+d\}.$$

The roots $\pm(a+c+d)$ are the only roots in $A_3(a,c,d)$ orthogonal to a. We have $\sigma_{a+c+d}(c) = -(a+d)$. To find a reflection σ such that $\sigma(a) = a$ and $\sigma(-(a+d)) = d$ we have to go outside of $A_3 = A_2(a,c) + A_2(a,d)$. We recall that E_7 contains 126 roots (see [Bou]):

$$\begin{aligned} R(E_7) =& \{ \pm(e_7 - e_8) \} \cup \{ \pm e_i \pm e_j \mid 1 \le i < j \le 6 \} \\ & \cup \{ \pm\tfrac{1}{2}(e_7 - e_8) + \tfrac{1}{2}\sum_{i=1}^{6}(-1)^{\nu(i)} e_i \mid \sum_{i=1}^{6} \nu(i) \text{ is even} \} \end{aligned}$$

where e_i form the usual Euclidian basis in $\mathbb{Z}^8$. Without loss of generality we can assume that $a = e_7 - e_8$. Since $a \cdot d = -1$, we see that

$$d = -\tfrac{1}{2}(e_7 - e_8) + \tfrac{1}{2}\sum_{i=1}^{6}(-1)^{\nu(i)} e_i.$$

We put $r_j = (-1)^{\nu(j)} e_j + (-1)^{\nu(j+1)} e_{j+1}$ for $j = 1$, 3 and 5. We obtain

$$\sigma_{r_5} \circ \sigma_{r_3} \circ \sigma_{r_1}(d) = d - r_1 - r_3 - r_5 = -(a+d).$$

□

Corollary 4.4 *We have*

(i) The Weyl group $W(E_8)$ acts transitively on all its sublattices of types $3A_1$ and $A_1 \oplus A_2$ in E_8.

(ii) The class number of the lattices A_5 and $A_1 \oplus D_4$ is equal to one.

(iii) The sublattices $4A_1$ in E_8 form two orbits with respect to $W(E_8)$.

Proof. (i) follows from the fact that $(A_1)^{\perp}_{E_8} \cong E_7$. To prove (ii) we note that $A_1 \oplus D_4$ is a maximal lattice because its discriminant group does not contain any isotropic vectors. (The square of any element in $D_4^{\vee}/D_4$ is equal to $1/2$ modulo 2.) Furthermore

$$(A_1)^{\perp}_{E_7} \cong D_6, \qquad (A_1)^{\perp}_{D_6} \cong A_1 \oplus D_4, \qquad (A_2)^{\perp}_{E_7} \cong A_5. \tag{15}$$

To see these one has to use the extended Dynkin diagram of the corresponding root lattice and to take into account the maximality of D_6, A_5 and $A_1 \oplus D_4$. The discriminant quadratic form is the invariant of the genus of an even quadratic lattice. Therefore if M is a lattice in the genus of $A_1 \oplus D_4$ then we can consider $M \oplus 2A_1$ as a sublattice of E_7. All such M are isomorphic according to the lemma. The same argument works for A_5.

To prove (iii) we remark that $(3A_1)^{\perp}_{E_8} \cong A_1 \oplus D_4$. The last lattice contains two orbits of roots. □

Theorem 4.5 *There exists a vector l in E_7 of length $2d$ orthogonal to at least 2 and at most 14 roots if*

$$30N_{A_1\oplus D_4}(2d) + 16N_{A_5}(2d) < 5N_{D_6}(2d) \tag{16}$$

or to at least 2 and at most 16 roots if

$$30N_{A_1\oplus D_4}(2d) + 16N_{A_5}(2d) < 6N_{D_6}(2d), \tag{17}$$

where $N_L(2d)$ denotes the number of representations of $2d$ by the lattice L.

Proof. Suppose that any vector $l \in E_7$ of length $2d$ is orthogonal to at least 16 roots if it is orthogonal to any. Let us fix a root a in E_7 orthogonal to l. Therefore $l \in a^{\perp}_{E_7} = D_6^{(a)}$. The others roots orthogonal to l are some roots in $D_6^{(a)}$ (60 roots) or roots in $R(E_7) \setminus R(D_6^{(a)})$ (66 roots). The last 66 roots form a bouquet $Q_a(16A_2)$ of 16 copies $A_2(a)$ of A_2 centred in $\pm a$. If l is orthogonal to any root from $A_2(a)$ different from a, then l is orthogonal to the whole lattice $A_2(a)$ and $l \in (A_2)^{\perp}_{E_7} \cong A_5$. If l is orthogonal to a root in $D_6^{(a)}$ then $l \in (2A_1)^{\perp}_{E_7} \cong A_1 \oplus D_4$. Therefore we have

$$l \in \bigcup_{i=1}^{30} (A_1 \oplus D_4)^{(i)} \cup \bigcup_{j=1}^{16} A_5^{(j)}. \tag{18}$$

Denote by $n(l)$ the number of components in (18) containing the vector l. We have calculated this vector exactly $n(l)$ times in the sum

$$30N_{A_1\oplus D_4}(2d)+16N_{A_5}(2d).$$

We need to estimate $n(l)$. We shall consider several cases.

1. Let $l\cdot c\neq 0$ for any $c\in Q_a(16A_2)\setminus\{\pm a\}$. Then l is orthogonal to at least 7 copies of A_1 in $D_6^{(a)}$ and $n(l)\geq 7$.

Now we suppose that there exist $c\in Q_a(16A_2)\setminus\{\pm a\}$ such that $l\cdot c=0$. Then l is orthogonal to $A_2(a,c)$ which is one of the 16 subsystems of the bouquet Q_a.

2. If l is orthogonal to only one copy of A_2 in Q_a, $A_2^{(i)}$ (6 roots) then l is orthogonal to at least 5 copies of A_1 in $D_6^{(a)}$. Thus $n(l)\geq 6$.

3. If l is orthogonal to exactly two copies of A_2 in Q_a, $A_2^{(i)}$ and $A_2^{(j)}$, then l is orthogonal to $A_3=A_2^{(i)}+A_2^{(j)}$ having 12 roots. Thus l is orthogonal to another $2A_1$ in $D_6^{(a)}$. But A_3 contains one more copy A_1 from $D_6^{(a)}$ orthogonal to a (see the proof of Lemma 4.3). Therefore $n(l)\geq 5$.

4. If l is orthogonal to three or more $A_2^{(i)}$ then their sum contains at least three $A_1<D_6^{(a)}$ and $n(l)\geq 6$.

Therefore we have proved that if any $l\in E_7$ with $l^2=2d$ is orthogonal to at least 16 roots then $n(l)\geq 5$ and

$$30N_{A_1\oplus D_4}(2d)+16N_{A_5}(2d)\geq 5N_{D_6}(2d).$$

If we replace 16 roots by 18 roots in the last condition then we obtain the second inequality of Theorem 4.5

$$30N_{A_1\oplus D_4}(2d)+16N_{A_5}(2d)\geq 6N_{D_6}(2d).$$

$\square$

5 Representations by quadratic forms of odd rank

To estimate the values of d for which the inequality of Theorem 4.5 is true we need exact formulae for the numbers $N_{A_1\oplus D_4}(2d)$ and $N_{A_5}(2d)$.

Let A be a symmetric even integral positive definite $m\times m$ matrix of determinant $\det A=|A|$, and

$$S(X)=\frac{1}{2}A[X]=\frac{1}{2}{}^tXAX$$

be the corresponding quadratic form taking integral values on $\mathbb{Z}^m$. The genus $\operatorname{gen} S$ of S contains a finite number of classes S_i. The integral orthogonal group $\mathrm{O}(S_i)$ is finite of order $|\,\mathrm{O}(S_i)|$. One defines the *mass* of the genus by

$$\mathrm{mass}(S) = \sum_{S_i \in \mathrm{gen}\, S} |\,\mathrm{O}(S_i)|^{-1}$$

and the *weight* w_i of the class S_i in the genus of S by

$$w_i = |\,\mathrm{O}(S_i)|^{-1}/\,\mathrm{mass}(S).$$

Siegel's main theorem on the quadratic forms (see [Si]) tells us that the number of representations of t by the genus of S, defined by

$$r(t, \mathrm{gen}\, S) = \sum_{S_i \in \mathrm{gen}\, S} w(S_i) r(t, S_i)$$

where $r(t, S_i)$ is the number of the representation of t by the quadratic form S_i, can be written in terms of the local densities $\alpha_p(t, S)$

$$r(t, \mathrm{gen}\, S) = \varepsilon_m \prod_{p \leq \infty} \alpha_p(t, S),$$

where $\varepsilon_m = 1$ for all $m \geq 2$ except $\varepsilon_2 = 1/2$. The local densities (or the local measures of the representations of t by S) are defined as follows

$$\alpha_p(t, S) = \lim_{a \to \infty} p^{-a(m-1)} \#\{\, X \in (\mathbb{Z}/p^a\mathbb{Z})^m \mid S(X) \equiv t \mod p^a \,\}$$

if p is a finite prime and

$$\alpha_\infty(t, S) = \lim_{V \to t} \frac{\mathrm{vol}\, S^{-1}(V)}{\mathrm{vol}\, V} = (2\pi)^{\frac{m}{2}} \Gamma(\tfrac{m}{2})^{-1} t^{\frac{m}{2}-1} |A|^{-\frac{1}{2}},$$

where $|A| = \det(A)$, V is a real neighbourhood of t and vol is the usual Euclidian volume in $\mathbb{R}$ or $\mathbb{R}^m$ (see [Si, Hilfssatz 26 and (71)]).

If $m \geq 5$ then the quantity $r(t, \mathrm{gen}\, S)$ coincides up to the factor $a_\infty(t, S)$ with the singular series, which gives us a good asymptotic estimate of the number of representations $r(t, S)$.

If the genus of S contains only one class then the Siegel formula gives us an exact formula for the number $r(t, S)$ of representations of t by S. In his first paper [Si] on the analytic theory of quadratic forms Siegel found exact formulae for the local densities if the prime p is not a divisor of the determinant of A. If the rank $m \geq 4$ is even we have the following formula (see [Iw, (11.74)])

$$r(t, \mathrm{gen}\, S) = a_\infty(t, S) L(\frac{m}{2}, \chi_{4D})^{-1} \Big(\sum_{a | t} \chi_{4D}(a) a^{1-\frac{m}{2}}\Big) \cdot \prod_{p | 2D} \alpha_p(t, S) \quad (19)$$

where $D = (-1)^{m/2}|A|$ is the discriminant of A and $\chi_{4D}(a) = \big(\frac{4D}{a}\big)$ is the quadratic character.

Usually the exact computation of the local densities for odd rank m is said to be more complicated: see for example the introduction to [Sh]. Here we give a well-organised formula for $r(t, \operatorname{gen} S)$ for odd rank m. For this purpose we use the Zagier L-function $L(s, \Delta)$ and the H. Cohen numbers $H(n, \Delta)$ (see [C] and [Za]). In these terms, surprisingly, the exact formula for odd rank is simpler that the formula (19) for even rank.

If $\Delta \equiv 0, 1 \mod 4$ then $\Delta = Df^2$, where D is the discriminant of the quadratic field $\mathbb{Q}(\sqrt{\Delta})$. By definition (see [Za, (7) and Proposition 3]) one has

$$L(s, \Delta) = \frac{\zeta(2s)}{\zeta(s)} \sum_{n=1}^{\infty} b_n(\Delta) n^{-s}, \tag{20}$$

where $b_n(\Delta) = \#\{x \mod 2n \mid x^2 \equiv \Delta \mod 4n\}$, and

$$L(s, \Delta) = L(s, \chi_D) \sum_{a|f} \mu(a) \left(\frac{D}{a}\right) a^{-s} \sigma_{1-2s}\big(\tfrac{f}{a}\big),$$

where $\sigma_s(t) = \sum_{d|t} d^s$ and $\mu(a)$ is the Möbius function. The main advantage of $L(s, \Delta)$ is the fact that it satisfies a simple functional equation.

The function

$$L^*(s, \Delta) = \begin{cases} \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \Delta^{\frac{s}{2}} L(s, \Delta) & \text{if } \Delta > 0 \\ \pi^{-\frac{s}{2}} \Gamma(\frac{s+1}{2}) |\Delta|^{\frac{s}{2}} L(s, \Delta) & \text{if } \Delta < 0 \end{cases}$$

has a meromorphic continuation to the whole complex plane and satisfies the functional equation

$$L^*(s, \Delta) = L^*(1 - s, \Delta)$$

(see [Za, Proposition 3]). Moreover $L(s, \Delta)$ is entire except for a simple pole (of residue $\frac{1}{2}$ if $\Delta = 0$ and 1 otherwise) if Δ is a square. This function is very useful for calculation of Fourier coefficients of various Eisenstein series (see [C], [Za] and [GS-P]).

To formulate our reorganisation of the Siegel formula for odd rank m we introduce some notation. We write

$$t = t_A t_1 t_2^2,$$

where t_1 is square free, $(t_1 t_2^2, |A|) = 1$ and t_A divides some power of $|A|$. We put

$$D = \operatorname{disc} \mathbb{Q}\left(\sqrt{(-1)^{\frac{m-1}{2}} 2t|A|}\right).$$

We note that $D > 0$ if $m \equiv 1 \mod 4$ and $D < 0$ if $m \equiv 3 \mod 4$. The determinant of A is always even if m is odd and A is even integral.

Theorem 5.1 *Let $m = 2m_1 + 1$ and $S(X) = \frac{1}{2}A[X]$. Then we have*

$$r(t, \operatorname{gen} S) =$$
$$(2\pi)^{\frac{m}{2}}\Gamma\big(\frac{m}{2}\big)^{-1} t^{\frac{m}{2}-1}|A|^{-\frac{1}{2}} L\big(\frac{m-1}{2},\, Dt_2^2\big)\zeta(m-1)^{-1}$$
$$\cdot \prod_{p\mid |A|} \frac{1-\chi_D(p)p^{\frac{1-m}{2}}}{1-p^{1-m}}\alpha_p(t,S) \qquad (21)$$

and

$$r(t, \operatorname{gen} S) =$$
$$\left(\frac{t_A}{|D_A|}\right)^{m_1-\frac{1}{2}} 2^{2m_1-\frac{1}{2}}|A|^{-\frac{1}{2}}\left|\frac{2m_1}{B_{2m_1}}\right|(-1)^{[m_1/2]}H(m_1, Dt_2^2)$$
$$\cdot \prod_{p\mid |A|} \frac{1-\chi_D(p)p^{\frac{1-m}{2}}}{1-p^{1-m}}\alpha_p(t,S) \qquad (22)$$

where D_A is the $|A|$-part of the discriminant D (i.e. D_A divides some power of $|A|$) and $H(m_1, Dt_2^2) = L(1-m_1, Dt_2^2)$ are the H. Cohen numbers.

We should like to note that the variant of the Siegel formula given in Theorem 5.1 is different from the formula given in [Sh]. Shimura used the L-function with a primitive character. We modify the local factors $\alpha_p(S)$ for the prime divisors of $|A|$ and use the function $L(s,\Delta)$ with a non-fundamental discriminant, i.e., we put some other non-regular p-factors inside the L-function. As a result our formulae (see Examples 5.2–5.4 below) are shorter.

Proof. From the definition of the local densities we see that

$$\alpha_p(t,S) = \alpha_p(2t,A) \quad \text{if } p \neq 2, \qquad \alpha_2(t,S) = 2\alpha_2(2t,A). \qquad (23)$$

We assume that p is not a divisor of $|A|$. Let $l_p = \operatorname{ord}_p(t)$ and $t = p^{l_p}t_{\bar{p}}$. According to [Si, Hilfssatz 16] the density $\alpha_p(2t,A)$ is given by

$$\alpha_p(2t,A) = (1-p^{1-m})\big(1+p^{2-m}+\cdots+p^{(2-m)\frac{l_p-1}{2}}\big)$$

for $l_p \equiv 1 \bmod 2$ and

$$\alpha_p(2t,A) = (1-p^{1-m})\left(1+p^{2-m}+\cdots+p^{(2-m)(\frac{l_p}{2}-1)}+\frac{p^{(2-m)\frac{l_p}{2}}}{1-\varepsilon_{A,t}(p)p^{\frac{1-m}{2}}}\right)$$

for $l_p \equiv 0 \bmod 2$, where $\varepsilon_{A,t}(p) = \left(\dfrac{(-1)^{\frac{m-1}{2}}|A|2t_{\bar{p}}}{p}\right)$. If $l_p = 0$ we take only the last summand in the second bracket (see [Si, Hilfssatz 12]). The numbers Dt_2^2 and $2t|A|$ differ by a square f^2 such that f divides some power of $|A|$. Therefore if p does not divide $|A|$ and l_p is even then

$$\varepsilon_{A,t}(p) = \chi_D(p) = \left(\tfrac{D}{p}\right) \neq 0.$$

If l_p is odd then $p|D$ and $\chi_D(p) = 0$.

Let us reorganise the p-factors in the Siegel formulae for the local densities. We put

$$\alpha_p(t,S) = \left(\frac{1-p^{1-m}}{1-\chi_D(p)p^{\frac{1-m}{2}}}\right) p^{(2-m)\lfloor\frac{l_p}{2}\rfloor} \left(1+\sum_{1\le j\le l_p/2} p^{(m-2)j}\right)\cdot\left(1-\chi_D(p)p^{\frac{1-m}{2}})\right). \quad (24)$$

This formula is valid for both even and odd l_p. If $l_p = 1$ (i.e. if p divides only t_1, not t_2) then $\alpha_p(t,S) = 1 - p^{1-m}$.

Taking the product over all divisors of t_2 we obtain the factor

$$t_2^{2-m}\sum_{d|t_2} d^{m-2}\prod_{p|d}(1-\chi_D(p)p^{\frac{1-m}{2}}) = \sum_{a|t_2}\mu(a)\chi_D(a)a^{\frac{1-m}{2}}\sigma_{2-m}\left(\tfrac{t_2}{a}\right).$$

Using the functional equation we can express $L(m_1, Dt_2^2)$ in terms of $L(1-m_1, Dt_2^2) = H(m_1, Dt_2^2)$. Together with the formula for the Bernoulli numbers

$$(-1)^{m_1+1}\frac{B_{2m_1}}{2m_1} = \pi^{-\frac{1}{2}-2m_1}\Gamma(m_1)\Gamma(m_1+\tfrac{1}{2})\zeta(2m_1)$$

it gives us the second formula (22). We note that $(-1)^{[m_1/2]}H(m_1, Dt_2^2)$ are positive rational numbers with bounded denominators. The denominators are 120 for $m_1 = 2$, 252 for $m_1 = 3$, 240 for $m_1 = 4$, etc. (see [C]). □

The exact formulae for the local densities $\alpha_p(t,S)$, $S(X) = \frac{1}{2}A[X]$, for all prime divisors of the determinant of A including $p = 2$ were calculated in many papers. See for example Malyshev [Mal], who used a classical method of Gauss sums, and Yang [Ya], who calculated the local densities in terms of local Whittaker integrals. In the examples below we use the formulae of [Ya].

Example 5.2 *The sum of five squares.*

Let $S_5(X) = x_1^2+\cdots+x_5^2$. In this example we are finishing the calculation of Siegel (see [Si, §10]) who found $r(t, S_5)$ for odd t. According to Theorem 5.1 we have

$$r(t,S_5) = t^{3/2}\frac{120}{\pi^2}L(2, Dt_2^2)\frac{1-\chi_D(2)2^{-2}}{1-2^{-4}}\alpha_2(t,S_5),$$

where $t = 2^a t_1 t_2^2$ with $a = 2b$ or $2b+1$ as in Theorem 5.1, $D = \operatorname{disc}\mathbb{Q}(\sqrt{t})$. The formula for $\alpha_2(t,S)$ (see [Ya, pp. 323–324]) is rather too long to give here. After some tedious transformations we obtain that

$$\alpha_2(t,S_5) = 1 - \sum_{k=1}^{b} 2^{-3k+1} + (-1)^D 2^{-3b-2} - \chi_D(2)2^{-3b-3},$$

where $\chi_D(2) = 0$ if $D \equiv 0 \mod 4$ and $\chi_D(2) = 1$ or -1 if $D \equiv 1 \mod 8$ or $D \equiv 5 \mod 8$ respectively. In terms of the Cohen numbers we have the numerical formula

$$r(t,S_5) = \begin{cases} -40H(2,Dt_2^2)\cdot \dfrac{2^{3b+2}+3}{7} & \text{if } D \equiv 0 \mod 4,\\ \dfrac{-120H(2,Dt_2^2)}{4+\chi_D(2)}\cdot\left(\dfrac{5\cdot 2^{3b+3}+2}{7} - \chi_D(2)\right) & \text{if } D \equiv 1 \mod 4.\end{cases}$$

We note that Dt_2^2 is equal to t up to a power of 2.

Let L be an even integral quadratic lattice and $q_L(x)$ and $b_L(x,y)$ be the corresponding finite quadratic and bilinear forms on the discriminant group $D(L) = L^\vee/L$. For q_L we have the local decomposition

$$q_L = \bigoplus_p (q_L)_p = \bigoplus_p q_{L\otimes\mathbb{Z}_p},$$

where $q_{L\otimes\mathbb{Z}_p}$ is the finite quadratic form with values in $\mathbb{Q}_p/\mathbb{Z}_p$ $(p \neq 2)$ or in $\mathbb{Q}_2/2\mathbb{Z}_2$ and $(q_L)_p$ is the discriminant form on the p-component of the finite abelian group $L^\vee/L$ with values in $\mathbb{Q}^{(p)}/\mathbb{Z} \cong \mathbb{Q}_p/\mathbb{Z}_p$ or in $\mathbb{Q}^{(2)}/2\mathbb{Z}$ ($\mathbb{Q}^{(p)}$ is the ring of fractions whose denominator is a power of p). A similar decomposition is valid for b_L. We recall that any quadratic form over the p-adic integers $\mathbb{Z}_p$ $(p \neq 2)$ is equivalent to a diagonal form. For $p = 2$ it can be represented as a sum of forms of types $2^n ux^2$ $(u \in \mathbb{Z}_2^*/(\mathbb{Z}_2^*)^2)$, $2^n(2x_1x_2)$ and $2^n(x_1^2 + 2x_1x_2 + x_2^2)$.

Example 5.3 *The root lattice* $A_1 \oplus D_4$.

The quadratic form $S_{1,4}$ of this lattice is similar to the sum of five squares. More exactly

$$S_{1,4} = x_1^2 + \frac{1}{2}(x_2^2 + x_3^2 + x_4^2 + x_5^2), \quad \text{where } x_2 + x_3 + x_4 + x_5 \text{ is even.}$$

The determinant of $A_1 \oplus D_4$ is equal to 8. The discriminant form of D_4 is equal to the discriminant form of $V(2) = 2(2x_1^2 + 2x_1x_2 + 2x_2^2)$. Using this we obtain that over $\mathbb{Z}_2$

$$\frac{1}{2}(A_1 \oplus D_4)\otimes\mathbb{Z}_2 \cong \langle 2\rangle \oplus \frac{1}{2}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} \oplus \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$

We use the notation of the previous example for $t = 2^a t_1 t_2^2$, $a = 2b$ or $2b+1$. Again using [Ya] we obtain

$$\alpha_2(t, S_{1,4}) = 1 - \sum_{k=1}^{b} 2^{-3k} + (-1)^D 2^{-3(b+1)} - \chi_D(2) 2^{-3b-4}. \qquad (25)$$

The second formula (22) of Theorem 5.1 is

$$r(t, S_{1,4}) = \begin{cases} -8H(2, Dt_2^2) \cdot 2^{3b+3} \alpha_2(t, S_{1,4}), & \text{if } D \equiv 0 \mod 4, \\ \dfrac{-120H(2, Dt_2^2)}{4 + \chi_D(2)} \cdot 2^{3b+4} \alpha_2(t, S_{1,4}), & \text{if } D \equiv 1 \mod 4. \end{cases}$$

The first formula (21) of Theorem 5.1 gives us an expression which we shall use later in our estimations of $N_{A_1 \oplus D_4}(2d)$:

$$r(t, S_{1,4}) = t^{3/2} 16 \frac{\zeta(2)}{\zeta(4)} L(2,\, Dt_2^2) \frac{1 - \chi_D(2) 2^{-2}}{1 - 2^{-4}} \alpha_2(t, S_{1,4}). \qquad (26)$$

See (20) in order to understand the form of the factors.

Example 5.4 *The root lattice A_5.*

Let $D = \operatorname{disc} \mathbb{Q}(\sqrt{3t})$ and $t = 2^a 3^c t_1 t_2^2$. According to Theorem 5.1

$$r(t, \tfrac{1}{2}A_5) = t^{3/2} \frac{32}{\sqrt{3}} \frac{\zeta(2) L(2, Dt_2^2)}{\zeta(4)} \prod_{p=2,3} \frac{1 - \chi_D(p) p^{-2}}{1 - p^{-4}} \alpha_p(t, \tfrac{1}{2}A_5).$$

The discriminant form of the lattice A_5 is the cyclic group of order 6 generated by the element $\bar{v}$ such that $\bar{v} \cdot \bar{v} \equiv \frac{5}{6} \mod 2\mathbb{Z}$. For the local part of the discriminant group we have

$$\begin{aligned} D(A_5)_3 &= \langle 2\bar{v} \rangle, \quad (2\bar{v})^2 \equiv \tfrac{1}{3} \mod \mathbb{Z}_3, \\ D(A_5)_2 &= \langle 3\bar{v} \rangle, \quad (3\bar{v})^2 \equiv \tfrac{3}{2} \mod 2\mathbb{Z}_2. \end{aligned}$$

It follows that

$$A_5 \otimes \mathbb{Z}_3 \cong x_1^2 + x_2^2 + x_3^2 + 2x_4^2 + 3x_5^2$$

and

$$A_5 \otimes \mathbb{Z}_2 \cong 2x_1x_2 + 2x_3x_4 + 6x_5^2 \cong 2U \oplus \langle 6 \rangle.$$

Put $t' = 2^a t_1 t_2^2$, so that $t = 3^c t'$ and $(3, t') = 1$. The formula for α_3 (see [Ya, p. 317]) after some transformations can be written as follows

$$\alpha_3(3^c t', \tfrac{1}{2}A_5) = 1 - \sum_{k=1}^{3\lfloor \frac{c}{2} \rfloor + 2} \left(\frac{k}{3}\right) 3^{-k} + \left(\frac{t'}{3}\right) 3^{-\frac{3c+3}{2}}, \qquad (27)$$

where $\left(\frac{k}{3}\right)$ is the Legendre symbol and we add the last term only if c is odd. For $p = 2$, we put $a = 2b$ or $2b + 1$ and obtain

$$\alpha_2(t, \tfrac{1}{2}A_5) = 1 + \sum_{k=1}^{b} 2^{-3k-1} - (-1)^D 2^{-3b-4} + \chi_D(2) 2^{-3b-5}. \tag{28}$$

In terms of the Cohen numbers we have

$$r(2t, A_5) = \left(\frac{t_{A_5}}{D_{A_5}}\right)^{3/2} \frac{1}{\sqrt{3}} 2^5 \cdot 30(-H(2, Dt_2^2)) \prod_{p=2,3} \frac{1 - \chi_D(p)p^{-2}}{1 - p^{-4}} \alpha_p(t, \tfrac{1}{2}A_5),$$

where $t_{A_5} = 2^a 3^c$ and D_{A_5} are the products of the powers of 2 and 3 in t and D.

Proposition 5.5 *The inequality*

$$30N_{A_1 \oplus D_4}(2m) + 16N_{A_5}(2m) < 5N_{D_6}(2m)$$

is true for any $m \geq 20$ and for $m = 17$. The inequality

$$30N_{A_1 \oplus D_4}(2m) + 16N_{A_5}(2m) < 6N_{D_6}(2m).$$

is true if $m \geq 12$.

Proof. First we estimate $N_{D_6}(2m)$ from below. By definition

$$D_6 = \{(x_i) \in \mathbb{Z}^6 \mid x_1 + \cdots + x_6 \in 2\mathbb{Z}\}.$$

Therefore the number $N_{D_6}(2m)$ is equal to the number of representation of $2m$ by six squares. It is classically known (see [Iw, p. 187]) and it can be easily proved using Eisenstein series or the Siegel main formula that

$$N_{D_6}(2m) = 64\tilde{\sigma}_2(m, \chi_4) - 4\sigma_2(m, \chi_4)$$

where $\chi_4(m) = \left(\frac{-4}{m}\right)$ is the unique non-trivial Dirichlet character modulo 4 and for any Dirichlet character χ we put

$$\sigma_k(m, \chi) = \sum_{d \mid m} \chi(d) d^k, \qquad \tilde{\sigma}_k(m, \chi) = \sum_{d \mid m} \chi\left(\frac{m}{d}\right) d^k.$$

Let $a_p = \operatorname{ord}_p(m)$. For any quadratic character χ modulo Δ we have

$$\begin{aligned}\tilde{\sigma}_k(m, \chi) &= m^k \sum_{p \mid m} \frac{1 - (\chi(p)p^{-k})^{(a_p+1)}}{1 - \chi(p)p^{-k}} \\ &\geq m^k \prod_{p \mid m,\ (p, \Delta)=1} (1 - p^{-k}).\end{aligned}$$

This is because

$$\tilde{\sigma}_k(m,\chi)=\sum_{d|m}\chi(d)\big(\frac{m}{d}\big)^k=m^k\sum_{d|m}\chi(d)d^{-k}$$

and

$$\frac{1-(\chi(p)p^{-k})^{(a_p+1)}}{1-\chi(p)p^{-k}}\geq\frac{1-p^{-k(a_p+1)}}{1+p^{-k}}\geq\frac{1-p^{-2k}}{1+p^{-k}}=1-p^{-k}.$$

If $(m,\Delta)=1$ then $\tilde{\sigma}_k(m,\chi)=\chi(m)\sigma_k(m,\chi)$ since χ is a real character. Moreover for any prime divisor p of the module Δ of χ

$$\tilde{\sigma}_k(p^a m_1,\chi)=p^{ak}\tilde{\sigma}_k(m_1,\chi),\qquad \sigma_k(p^a m_1,\chi)=\sigma_k(m_1,\chi).$$

Therefore

$$N_{D_6}(2m)\geq 60\tilde{\sigma}_2(m,\chi_4)>60\zeta(2)^{-1}(1-2^{-2})^{-1}m^2=\frac{480}{\pi^2}\,m^2. \tag{29}$$

Next we have to estimate from above the Dirichlet series

$$\sum_{n\geq 1}\frac{b_n(\Delta)}{n^s}=\frac{\zeta(s)L(s,\Delta)}{\zeta(2s)}$$

(see (20)) for $s=2$. If $(n,\Delta)=1$ then

$$b_n(\Delta)\leq b_n(1)=2^{\rho(n)},$$

where $\rho(n)$ is the number of prime divisors of n, with equality if and only if $\big(\frac{\Delta}{p}\big)=1$ for any odd prime divisor of n and $\big(\frac{\Delta}{8}\big)=1$ if n is even. If in n there is at least one non-residue modulo Δ then $b_n(\Delta)=0$. Therefore if $(p,\Delta)=1$ (it might be that $p=2$) then the local p-factor of the Dirichlet series is equal to

$$1+2\sum_{m\geq 1}p^{-ms}=\frac{p^s+1}{p^s-1}. \tag{30}$$

Let us assume that $\Delta=p^{2k}\Delta'$ (it might be that $p=2$) with $(p,\Delta')=1$. Considering the congruence class of $b_{p^m}(p^{2k}\Delta')$ for all powers of p we see that the local p-factor of the Dirichlet series equals

$$1+\sum_{m=1}^{2k}\frac{p^{\lfloor\frac{m}{2}\rfloor}}{p^{ms}}+2p^k\sum_{m\geq 2k+1}p^{-ms}. \tag{31}$$

If $\Delta=p^{2k+1}\Delta'$ then the local factor is smaller: the last term in 31, $2p^k\sum_{m\geq 2k+1}p^{-ms}$, is replaced by one summand, $p^{k-(2k+1)s}$. A direct calculation shows that for $s=2$ the regular factor (30) is larger than the non-regular factor (31) for any prime $p\geq 2$. Therefore

$$\frac{\zeta(2)L(2,\Delta)}{\zeta(4)}\leq\prod_p\frac{p^2+1}{p^2-1}=\frac{\zeta(2)^2}{\zeta(4)}=\frac{5}{2}.$$

The next step is an estimation from above of the 2-factor in $N_{A_1\oplus D_4}(2m) = r(m, S_{1,4})$ and the 2- and 3-factors in $N_{A_5}(2m) = r(m, \frac{1}{2}A_5)$. Elementary calculation using (25) gives us

$$\frac{1-\chi_D(2)2^{-2}}{1-2^{-4}}\,\alpha_2(m, S_{1,4}) \le \frac{5}{4}, \tag{32}$$

with equality if $D \equiv 5 \mod 8$ and m is odd.

For the local 3-factor in $\frac{1}{2}A_5$ we obtain (using (27))

$$\frac{1-\chi_D(3)3^{-2}}{1-3^{-4}}\,\alpha_3(m, \tfrac{1}{2}A_5) \le \frac{11}{12} \tag{33}$$

with equality if $m = 3m'$, where $m' \equiv 2 \mod 3$.

For the local 2-factor in $\frac{1}{2}A_5$ we obtain

$$\frac{1-\chi_D(2)2^{-2}}{1-2^{-4}}\,\alpha_2(m, \frac{1}{2}A_5) \le \frac{10}{7}. \tag{34}$$

In this case we must analyse the case when $m = 2^a m'$ and $b = \lfloor \frac{a}{2} \rfloor$ goes to infinity (see (28)). If $D \equiv 0 \mod 4$ or $\equiv 5 \mod 8$ then the local density tends to its supremum as b tends to infinity. This value is equal to $\frac{15}{14}$. Therefore the left-hand side of (34) is smaller than $\frac{10}{7}$. If $D \equiv 1 \mod 8$ then α_2 takes its maximal values $\frac{35}{32}$ for $b = 0$. In this case the left-hand side of (34) is equal to $\frac{7}{8}$.

Now we can combine all our estimates. We have

$$N_{A_1\oplus D_4}(2m) \le 50m^{3/2}, \qquad N_{A_5}(2m) \le \frac{2200}{21\sqrt{3}}m^{3/2}.$$

Using (29) we obtain that the inequalities (16) and (17) of Theorem 4.5 and Proposition 5.5 are valid for $m > 102$ and for $m > 71$ respectively. For many m smaller or equal to 102 we can write a better estimate for the number of representations. But we can use the exact formulae for the theta series of D_6, $A_1 \oplus D_4$ and A_5 in therms of Jacobi theta series in order to check the inequality for $m \le 102$.

The theta series of the lattice A_n is given by the formula (see [CS, Ch. 4, (56)])

$$\theta_{A_5}(\tau) = \frac{\sum_{k=0}^{5} \vartheta_3(\tau, \frac{k}{6})^6}{6\vartheta_3(6\tau)},$$

where

$$\vartheta_3(\tau, z) = \sum_{n\in\mathbb{Z}} \exp(\pi i(n^2\tau + 2nz)),$$

and $\vartheta_3(\tau) = \vartheta_3(\tau, 0)$. For the lattice D_n one has (see [CS, Ch. 4, (87), (10)]) that

$$\theta_{D_n}(\tau) = \frac{1}{2}(\vartheta_3(\tau)^n + \vartheta_3(\tau+1)^n).$$

d	p	λ_d
9	8	-1,2,3,1,2,1,3
11	8	3,3,0,-1,-2,-1,0
12	7	2,1,2,-2,0,0,1
13	7	2,3,-1,1,0,0,-1
14	6	2,0,3,0,2,1,1
15	7	1,-2,0,2,4,2,0
16	6	1,0,-1,3,0,0,-2
18	5	3,2,3,2,0,0,-2
19	6	2,3,2,-3,-4,-2,1

Table 1: Short vectors in E_7 orthogonal to few roots

Using these formulae we can compute (using PARI) the first 102 Fourier coefficients of the function

$$5\theta_{D_6} - 30\theta_{A_1\oplus D_4} - 16\theta_{A_5}.$$

We find that these coefficients are negative exactly for $d < 20$ and $d \neq 17$. Hence the first inequality of the proposition holds as stated. Repeating the same calculation with 6 instead of 5 we obtain the second inequality. □

We have now proved a slightly weakened version of Theorem 4.1. To obtain the full result we need the following observation.

Proposition 5.6 *For* $d = 12$, 13, 14, 15, 16, 18 *and* 19 *there exist vectors* $l_d \in E_7$ *that satisfy* $l_d^2 = 2d$ *that are orthogonal to at least* 2 *and at most* 14 *roots. For* $d = 9$ *and* $d = 11$ *there exist vectors of length* $l^2 = 2d$ *that are orthogonal to exactly* 16 *roots.*

Proof. These were found by a computer search. We give one example in each case. We express the vectors in terms of the simple roots v_i, $1 \leq i \leq 7$ which are given in terms of the standard basis $e_1, \ldots, e_8$ of $\mathbb{Q}^8$ by

$$\begin{aligned} v_i &= e_{i+2} - e_{i+1} \quad \text{for } 1 \leq i \leq 6, \\ v_7 &= \tfrac{1}{2}(e_1 + e_2 + e_3 + e_4) - \tfrac{1}{2}(e_5 + e_6 + e_7 + e_8) \end{aligned}$$

(see [Bou]). The examples are shown in Table 1: the vector $l_d = \sum \lambda_{d,i} v_i$ with $\lambda_d = (\lambda_{d,1}, \ldots, \lambda_{d,7}) \in \mathbb{Z}^7$ is orthogonal to exactly $2p_d$ roots of $E_7 = \sum_{i=1}^{7} \mathbb{Z} v_i \subset \mathbb{Q}^8$. There are other vectors with the required properties (for instance, we found one with $d = 19$ and $p = 7$), but none for smaller d. □

References

[AMRT] A. Ash, D. Mumford, M. Rapoport, Y. Tai, *Smooth compactification of locally symmetric varieties.* Lie Groups: History, Frontiers and Applications, Vol. IV. Math. Sci. Press, Brookline, Mass., 1975.

[BB] W.L. Baily Jr., A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains.* Ann. of Math. (2) **84** (1966), 442–528.

[Be] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle.* J. Differential Geometry **18** (1983), 755–782.

[Bo] R.E. Borcherds, *Automorphic forms on* $\mathrm{O}_{s+2,2}(\mathbb{R})$ *and infinite products.* Invent. Math. **120** (1995), 161–213.

[BKPS] R.E. Borcherds, L. Katzarkov, T. Pantev, N.I. Shepherd-Barron, *Families of* K3 *surfaces.* J. Algebraic Geom. **7** (1998), 183–193.

[Bl] A. Borel, *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem.* J. Differential Geometry **6** (1972), 543–560.

[Bog] F. Bogomolov, *Hamiltonian Kählerian manifolds.* (Russian) Dokl. Akad. Nauk SSSR **243** (1978), 1101–1104.

[Bou] N. Bourbaki, *Groupes et Algèbres de Lie, Chapitres 4, 5 et 6.* Hermann, 1968.

[C] H. Cohen, *Sums involving the values at negative integers of L functions of quadratic characters.* Math. Ann. **217** (1975), 271–285.

[CS] J.H. Conway, N.J.A. Sloane, *Sphere packings, lattices and groups.* Grundlehren der mathematischen Wissenschaften **290**. Springer-Verlag, New York, 1988.

[Deb] O. Debarre, *Un contre-exemple au théorème de Torelli pour les variétés symplectiques irréductibles.* C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), 681–684.

[E] M. Eichler, *Quadratische Formen und orthogonale Gruppen.* Grundlehren der mathematischen Wissenschaften **63**. Springer-Verlag, Berlin-New York, 1974.

[Fr] E. Freitag, *Siegelsche Modulfunktionen.* Grundlehren der mathematischen Wissenschaften **254**. Springer-Verlag, Berlin–Göttingen–Heidelberg, 1983.

[Fuj] A. Fujiki, *On the de Rham cohomology group of a compact Kähler symplectic manifold*. In *Algebraic geometry, Sendai, 1985*. Adv. Stud. Pure Math. **10**, 105–165. North-Holland, Amsterdam 1987.

[GH] V. Gritsenko, K. Hulek, *Minimal Siegel modular threefolds.* Math. Proc. Cambridge Philos. Soc. **123** (1998), 461–485.

[GHS1] V. Gritsenko, K. Hulek, G.K. Sankaran, *The Kodaira dimension of the moduli of K3 surfaces.* Invent. Math. **169** (2007), 519-567.

[GHS2] V. Gritsenko, K. Hulek, G.K. Sankaran, *The Hirzebruch-Mumford volume for the orthogonal group and applications.* Documenta Math. **12** (2007), 215–241.

[GHS3] V. Gritsenko, K. Hulek, G.K. Sankaran, *Hirzebruch-Mumford proportionality and locally symmetric varieties of orthogonal type.* `math.AG/0609744`, 19 pp.

[GS-P] V. Gritsenko, R. Schulze-Pillot, *Eisenstein series on four-dimensional hyperbolic space.* Acta Arith. **67** (1994), 241–268.

[HT] B. Hassett, Y. Tschinkel, *Moving and ample cones of holomorphic symplectic fourfolds.* `arXiv:0710.0390 [math.AG]`, 16pp.

[Huy1] D. Huybrechts, *Compact hyper-Kähler manifolds: basic results.* Invent. Math. **135** (1999), 63–113.

[Huy2] D. Huybrechts, *Erratum: Compact hyper-Kähler manifolds: basic results.* Invent. Math. **152** (2003), 209–212.

[Huy3] D. Huybrechts, *Compact hyperkähler manifolds.* In *Calabi-Yau manifolds and related geometries (Nordfjordeid, 2001)*, 161–225, Universitext, Springer, Berlin, 2003.

[Huy4] D. Huybrechts, *Finiteness results for compact hyperkähler manifolds.* J. Reine Angew. Math. **558** (2003), 15–22.

[Huy5] D. Huybrechts, *The Kähler cone of a compact hyperkähler manifold.* Math. Ann. **326** (2003), 499–513.

[Huy6] D. Huybrechts, *Moduli spaces of hyperkähler manifolds and mirror symmetry.* Intersection theory and moduli, 185–247, ICTP Lect. Notes **XIX**, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[Iw] H. Iwaniec, *Topics in classical automorphic forms.* Graduate Studies in Math. **17**, Amer. Math. Soc., 1997.

[Kaw1] Y. Kawamata, *Unobstructed deformations. A remark on a paper of Z. Ran.* J. Algebraic Geom. **1** (1992), 183–190.

[Kaw2] Y. Kawamata, *Flops connect minimal models.* `arXiv:0704.1013 [math.AG]`, 5 pp.

[KM] J. Kollár, T. Matsusaka, *Riemann-Roch type inequalities.* Am. J. Math. **105** (1983), 229–252.

[Kn] M. Kneser, *Erzeugung ganzzahliger orthogonaler Gruppen durch Spiegelungen.* Math. Ann. **255** (1981), 453–462.

[Ko] S. Kondo, *On the Kodaira dimension of the moduli space of* K3 *surfaces. II.* Compositio Math. **116** (1999), 111–117.

[Mal] A. V. Malyshev, *On the representation of integers by positive quadratic forms.* (Russian) Trudy Mat. Inst. Steklov **65** (1962) 212 pp. (MR0142511)

[Mar1] E. Markman, *On the monodromy of moduli spaces of sheaves on* K3 *surfaces.* `math.AG/0305042`, 69 pp.

[Mar2] E. Markman, *On the monodromy of moduli spaces of sheaves on* K3 *surfaces II.* `math.AG/0305043`, 76 pp.

[Mar3] E. Markman, *Integral constraints on the monodromy group of the hyperkahler resolution of a symmetric product of a* K3 *surface.* `math.AG/0601304`, 54 pp.

[MM] T. Matsusaka, D. Mumford, *Two fundamental theorems on deformations of polarized varieties.* Am. J. Math. **86** (1964), 668–684.

[Nam1] Y. Namikawa, *Counter-example to global Torelli problem for irreducible symplectic manifolds.* Math. Ann. **324** (2002), 841–845.

[Nam2] Y. Namikawa, *On deformations of* $\mathbb{Q}$*-factorial symplectic varieties.* J. Reine Angew. Math. **599** (2006), 97–110.

[Nik] V.V. Nikulin, *Integral symmetric bilinear forms and some of their applications.* Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), 111–177. English translation in Math. USSR, Izvestiya **14** (1980), 103–167.

[OG1] K. O'Grady, *Desingularized moduli spaces of sheaves on a* K3. J. Reine Angew. Math. **512** (1999), 49–117.

[OG2] K. O'Grady, *A new six-dimensional irreducible symplectic variety.* J. Algebraic Geom. **12** (2003), 435–505.

[OG3] K. O'Grady, *Irreducible symplectic* 4*-folds numerically equivalent to* $(\mathrm{K3})^{[2]}$. `math.AG/0504434`, 44 pp.

[OG4] K. O'Grady, *Irreducible symplectic* 4*-folds and Eisenbud-Popescu-Walter sextics.* Duke Math. J. **134** (2006), 99–137.

[Po] H. Popp, *On moduli of algebraic varieties. I* Invent. Math. **22** (1973), 1–40.

[Ran] Z. Ran, *Deformations of manifolds with torsion or negative canonical bundle.* J. Algebraic Geom. **1** (1992), 279–291.

[Sh] G. Shimura, *The representation of integers as sums of squares.* Amer. J. Math. **124** (2002), 1059–1081.

[Si] C.L. Siegel, *Über die analytische Theorie der quadratischen Formen.* Ann. of Math. **36** (1935), 527–606.

[Sz] B. Szendrői, *Some finiteness results for Calabi-Yau threefolds.* J. London Math. Soc. (2) **60** (1999), 689–699.

[Ti] G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric.* In *Mathematical aspects of string theory* (S.-T. Yau, Ed.). Advanced Series in Mathematical Physics **1**, 629–646. World Scientific, Singapore 1987.

[Tod] A.N. Todorov, *The Weil-Petersson geometry of the moduli space of* $\mathrm{SU}(n \geq 3)$ *(Calabi-Yau) manifolds I.* Comm. Math. Phys. **126** (1989), 325–346.

[Vi] E. Viehweg, *Quasi-projective moduli for polarized manifolds.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **30**. Springer-Verlag, Berlin, 1995.

[Vo1] C. Voisin, *Théorème de Torelli pour les cubiques de* $\mathbb{P}^5$. Invent. Math. **86** (1986), 577–601.

[Vo2] C. Voisin, *Géométrie des espaces de modules de courbes et de surfaces* K3 *[d'après Gritsenko-Hulek-Sankaran, Farkas-Popa, Mukai, Verra...].* Séminaire BOURBAKI 59ème année, 2006–2007, n 981.

[Ya] T. Yang, *An explicit formula for local densities of quadratic forms.* J. Number Theory **72** (1998), 309–356.

[Za] D. Zagier, *Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields.* Modular functions of one variable, VI (Bonn, 1976), 105–169. Lecture Notes in Math., **627**: Springer, Berlin, 1977.

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