Castelnuovo-Mumford Regularity over Scrolls and Splitting Criteria

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Abstract

We introduce and study a notion of Castelnuovo-Mumford regularity suitable for scrolls obtained as projectivisations of sums of line bundles on $\mathbb{P}^m$. We show that this is a natural generalisation of the well known regularity on projective and multiprojective spaces and we prove Horrocks-type splitting criteria for vector bundles.

Introduction

Castelnuovo-Mumford regularity was defined initially on projective space, but many generalizations exist to other varieties, mostly rational varieties. Typically, these all reduce to classical Castelnuovo-Mumford regularity on $\mathbb{P}^N$ but are otherwise defined independently, depending on the class of varieties to be studied. In particular, if some variety falls into more than one such class, the competing definitions may not agree. Over the years, extensions of this notion have been proposed to handle other ambient varieties beyond projective space, including Grassmannians of lines [1], quadrics [3],

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multiprojective spaces [2, 6, 10], n-dimensional smooth projective varieties with an n-block collection [6], and weighted projective spaces [13].
A very general definition, for all simplicial toric varieties (and more) was given by Maclagan and Smith in [12]. Because of its wide scope, that definition is not always the most suitable in particular cases, so it makes sense to consider the possibilities for toric varieties of a special type, even though the definition of [12] applies to them.
We work with scrolls: that is, projectivisations of sums of line bundles on $\mathbb{P}^m$. This is a natural and widespread generalisation of the classical notion of scroll (which is the case $m = 1$). If $X = \mathbb{P}\mathcal{V}$ is a scroll in this sense then $\rho(X) = 2$ and the Picard group has obvious generators $\mathbf{f}$, the pullback of $\mathcal{O}_{\mathbb{P}^m}(1)$, and $\mathbf{h}$, the relatively ample line bundle $\mathcal{O}_{\mathbb{P}\mathcal{V}}(1)$. In fact it was shown in [8] that these are exactly the smooth toric varieties of Picard rank 2.
These choices allow us to give a definition of Castelnuovo-Mumford regularity that has many of the properties of regularity proved by Mumford in [14] for projective space. In particular we show that also for our notion of regularity, a regular coherent sheaf is globally generated. We compare our version with the regularity of Maclagan and Smith (with suitable choices) in this case.
In the second part of the paper we use the new notion of regularity to prove splitting criteria for vector bundles analogous to those of Horrocks (see [11]) on projective spaces. We compare our splitting criteria with those obtained for the same varieties by Brown and Sayrafi [5] (recently extended to arbitrary smooth projective toric varieties in [15]).
Both the notion of regularity and the splitting criteria are simpler for rational normal scrolls, i.e. when $m = 1$.
Smooth toric varieties of low Picard rank were described more fully in [4]. Also in [4] there is a characterisation of polyscrolls (recursive definition: a point is a polyscroll and if X is a polyscroll and $\mathcal{V} = \bigoplus \mathcal{L}_i$ is a direct sum of line bundles on X then $X' = \mathbb{P}\mathcal{V}$ is a polyscroll). It is possible that our definitions and some of our results can be extended to polyscrolls.

1 Scrolls and Regularity

We fix a decomposible vector bundle $\mathcal{V} = \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^m}(a_i)$ of rank $n+1$ on $\mathbb{P}^m$: we assume throughout that $a_0 \leq a_1 \leq \ldots \leq a_n$. The associated projective space bundle $X := \mathbb{P}\mathcal{V}$ is by definition $\mathrm{Proj}(\mathrm{Sym}\,\mathcal{V})$, adopting the notational conventions of [9, Section II.7]. The associated line bundle $\mathcal{O}_X(1)$ is relatively ample over $\mathbb{P}^m$, and is ample on X if $a_0 > 0$. We put $c := \sum_{i=0}^{n} a_i$ and we let $\pi\colon \mathbb{P}(\mathcal{V}) \to \mathbb{P}^m$ be the projection. We denote by $\mathbf{h}$ and $\mathbf{f}$ the classes in $\mathrm{Pic}\,X$ of $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ and the pullback $\pi^*\mathcal{O}_{\mathbb{P}^m}(1)$, respectively.

Definition 1.1. *If $X = \mathbb{P}\mathcal{V}$ with $\mathcal{V}$ as above, we call the pair $(X, \mathcal{V})$ an* abstract scroll *or simply a* scroll*. If $a_0 \geq 0$ we say that $(X, \mathcal{V})$ is a* semipositive scroll*, or a* positive scroll *if $a_0 > 0$.*

Note that $\mathbf{h}$ depends on $\mathcal{V}$ rather than only on X. If $\mathcal{V}' = \mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^m}(w) = \bigoplus \mathcal{O}_{\mathbb{P}^m}(a_i + w)$ then $\mathbb{P}\mathcal{V}' \cong X$ but $c' = c + (n+1)w$ and $\mathbf{h}' = \mathbf{h} + w\mathbf{f}$. In particular, this, and the fact that $\mathbf{h}$ is ample if and only if $a_0 > 0$, allow us to recover $\mathcal{V}$ given X and $\mathbf{h}$.
If $(X, \mathcal{V})$ is a positive scroll then $\mathbf{h}$ is globally generated and the image $\phi_{|\mathbf{h}|}(X)$ is a geometric scroll, i.e. a subvariety of some $\mathbb{P}^N$ in which the fibres of π appear as linear projective subspaces.
For conciseness, if $\mathcal{F}$ is a sheaf on X we will often write $\mathcal{F}\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] := \mathcal{F}(a\mathbf{h} + b\mathbf{f})$. If $I \subseteq \{0, \ldots, n\}$ we write $|I|$ for the cardinality of I and we set $a_I = \sum_{i \in I} a_i$. With this notation, we have $\omega_X \cong \mathcal{O}_X\left[\begin{smallmatrix} -(n+1) \\ c-1-m \end{smallmatrix}\right]$.
The following two easy lemmas are useful for computation.

Lemma 1.2. *Let X be a scroll.*

(i) $H^i(X, \mathcal{O}_X\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]) \cong H^i(\mathbb{P}^m, \operatorname{Sym}^a \mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^m}(b))$ *if* $a \geq 0$*;*

(ii) $H^i(X, \mathcal{O}_X\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right])) = 0$ *if* $-n \leq a < 0$*;*

(iii) $H^i(X, \mathcal{O}_X\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]) \cong H^{n+m-i}(\mathbb{P}^m, \operatorname{Sym}^{-a-n-1} \mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^m}(c - b - 1 - m))$ *if* $a < -n$*.*

Proof. See [9, Exercise III.8.4]. □

Lemma 1.3. *Again let X be a scroll. Then, for $0 < i < n + m = \dim X$,*

(i) if $a \geq 0$, then $H^i(X, \mathcal{O}_X\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]) = 0$ for any $b \geq -m$.

(ii) if $a < -n$, then $H^i(X, \mathcal{O}_X\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]) = 0$ for any $b < c$.

Proof. Both parts follow from 1.2(i). □

Recall the dual of the relative Euler exact sequence of a scroll X:

$$0 \longrightarrow \Omega^1_{X|\mathbb{P}^m}(\mathbf{h}) \longrightarrow \mathcal{B} := \bigoplus_{i=0}^{n} \mathcal{O}_X(a_i\mathbf{f}) \longrightarrow \mathcal{O}_X(\mathbf{h}) \longrightarrow 0, \qquad (1)$$

and so we have $\omega_{X|\mathbb{P}^m} \cong \mathcal{O}_X\left[\begin{smallmatrix} -(n+1) \\ c \end{smallmatrix}\right]$. The long exact sequence of exterior powers associated to (1) is

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_X\left[\begin{smallmatrix} -n \\ c \end{smallmatrix}\right] \longrightarrow \wedge^n\mathcal{B}((-n+1)\mathbf{h}) \xrightarrow{d_{n-1}} \wedge^{n-1}\mathcal{B}((-n+2)\mathbf{h}) \xrightarrow{d_{n-2}} \\ &\cdots \xrightarrow{d_1} \mathcal{B} \longrightarrow \mathcal{O}_X(\mathbf{h}) \longrightarrow 0. \end{aligned} \qquad (2)$$

Now (2) splits into

$$0 \longrightarrow \Omega^i_{X|\mathbb{P}^m}(i\mathbf{h}) \longrightarrow \wedge^i \mathcal{B} \longrightarrow \Omega^{i-1}_{X|\mathbb{P}^m}(i\mathbf{h}) \longrightarrow 0 \tag{3}$$

for each $i = 1, \dots, n$, and we have $\mathrm{Im}(d_i \otimes \mathcal{O}_S((i-1)\mathbf{h})) \cong \Omega^i_{X|\mathbb{P}^1}(i\mathbf{h}) \subset \wedge^i \mathcal{B}$. We will often use the following exact sequences, obtained as pullbacks of Koszul sequences from $\mathbb{P}^m$:

$$0 \to \mathcal{O}_X(-m\mathbf{f}) \to \mathcal{O}_X^{e_m}(-(m-1)\mathbf{f}) \to \cdots \to \mathcal{O}_X^{e_1} \to \mathcal{O}_X(\mathbf{f}) \to 0, \tag{4}$$

with $e_j = \binom{m+1}{j}$, and

$$0 \to \mathcal{O}_X\left[\begin{smallmatrix} -n \\ c-m-1 \end{smallmatrix}\right] \to \mathcal{O}_X^{e_m}\left[\begin{smallmatrix} -n \\ c-m \end{smallmatrix}\right] \to \cdots \to \mathcal{O}_X^{e_1}\left[\begin{smallmatrix} -n \\ c-1 \end{smallmatrix}\right] \to$$
$$\to \bigoplus_{i=0}^{n} \mathcal{O}_X\left[\begin{smallmatrix} -(n-1) \\ c-a_i \end{smallmatrix}\right] \to \cdots \to \bigoplus_{|I|=r} \mathcal{O}_X\left[\begin{smallmatrix} -(n-r) \\ c-a_I \end{smallmatrix}\right] \to \cdots \to \mathcal{B} \to \mathcal{O}_X(\mathbf{h}) \to 0. \tag{5}$$

1.1 Main definition

Our main definition is the following notion of regularity on a scroll $(X, \mathcal{V})$. For p, $q \in \mathbb{Z}$ we set $\mathbf{p} = p\mathbf{h} + q\mathbf{f}$ and we write $\mathcal{F}(\mathbf{p}) = \mathcal{F}[\begin{smallmatrix} p \\ q \end{smallmatrix}]$.

Definition 1.4. *A coherent sheaf $\mathcal{F}$ on X is said to be* (p, q)-regular *if*

(a) $h^{n+j}(\mathcal{F}(\mathbf{p})\left[\begin{smallmatrix} -n \\ c-j-1 \end{smallmatrix}\right]) = 0$ *for* $0 \le j \le m$ *but* $(n, j) \ne (0, 0)$*, and*

(b) $h^{i+j}(\mathcal{F}(\mathbf{p})\left[\begin{smallmatrix} -i \\ i-j \end{smallmatrix}\right]) = 0$ *for* $0 \le j \le m$ *and* $0 \le i < n$ *but* $(i, j) \ne (0, 0)$*.*

We will say regular *to mean* $(0, 0)$*-regular. We define the* regularity *of* $\mathcal{F}$*, denoted* $\mathrm{Reg}(\mathcal{F})$*, to be the least integer* p *such that* $\mathcal{F}$ *is* $(p, 0)$*-regular. We set* $\mathrm{Reg}(\mathcal{F}) = -\infty$ *if there is no such integer.*

Example 1.5. Some special cases of this are familiar.

(i) If $m = 0$, then $X = \mathbb{P}^n$ and $\mathbf{f} = 0$. The conditions 1.4(a) and 1.4(b) reduce respectively to $h^n(\mathcal{F}(\mathbf{p})(-n\mathbf{h})) = 0$ and $h^i(\mathcal{F}(\mathbf{p})(-i\mathbf{h})) = 0$ for $1 \le i < n$, giving the usual notion of Castelnuovo-Mumford regularity on $\mathbb{P}^n$.

(ii) If $n = 0$ then $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^m}(c)) = \mathbb{P}^m$ and $\mathbf{h} = c\mathbf{f}$. Indeed, $\omega_X \cong \mathcal{O}_X\left[\begin{smallmatrix} -1 \\ c-1-m \end{smallmatrix}\right] = \mathcal{O}_X(-c\mathbf{f} + (c-1-m)\mathbf{f}) = \mathcal{O}_X((-1-m)\mathbf{f})$. Condition 1.4(b) is vacuous, and 1.4(a) reduces to $h^j(\mathcal{F}(\mathbf{p})((c-j-1)\mathbf{f})) = 0$ for $1 \le j \le m$. So for $c = 1$ this again gives the usual notion of Castelnuovo-Mumford regularity on $\mathbb{P}^m$, and for $c > 1$ it gives a notion of Castelnuovo-Mumford regularity over the Veronese varieties $(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(c))$.

(iii) If $m = 1$ and $n = 2$ then X is a rational normal scroll surface and the regularity of Definition 1.4 agrees with the definition given in [7]. For $n > 2$ see Remark 1.16 below.

(iv) If $c = n + 1$ then $X = \mathbb{P}^n \times \mathbb{P}^m$ and $\mathbf{h} = [\mathcal{O}(1,1)]$. The regularity of Definition 1.4 agrees with the definition given in [2], since if $0 \le j \le m$ and $0 \le i < n$ but $(i,j) \ne (0,0)$ then

$$h^{n+m-j}(\mathcal{F}(\mathbf{p})\left[\begin{smallmatrix} -n \\ c-1-m-j \end{smallmatrix}\right]) = h^{n+m-j}(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{F}(\mathbf{p})(-n, -m-j))$$

and

$$h^{i+j}(\mathcal{F}(\mathbf{p})\left[\begin{smallmatrix} -i \\ i-j \end{smallmatrix}\right]) = h^{i+j}(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{F}(\mathbf{p})(-i, -j)).$$

1.2 Positivity

Although the definitions and examples in Subsection 1.1 make sense for arbitrary choices of a_i, for applications it is usually necessary to have some information about the positivity of $\mathbf{h}$. Therefore we assume henceforth that X is a positive scroll, i.e. $a_0 > 0$, so that $\mathbf{h}$ is ample on X.

Lemma 1.6. *If $\mathcal{F}$ is a regular coherent sheaf on a positive scroll X, then*

$$h^{n+m}(\mathcal{F}\left[\begin{smallmatrix} a-n \\ c-1-m+b \end{smallmatrix}\right]) = 0 \text{ for any } a,\, b \ge 0.$$

Proof. From (4) we get $h^{n+m}(\mathcal{F}\left[\begin{smallmatrix} -n \\ c-1-m+t \end{smallmatrix}\right]) = 0$ for any $t \ge 0$. From (5) tensored by $\mathcal{F}\left[\begin{smallmatrix} -(n-1) \\ c-2 \end{smallmatrix}\right]$ we get $h^{n+1}\big(\mathcal{F}\left[\begin{smallmatrix} -(n-1) \\ c-2+t \end{smallmatrix}\right]\big) = 0$ and again by (4) we obtain $h^{n+1}\big(\mathcal{F}\left[\begin{smallmatrix} -(n-1) \\ c-2+t \end{smallmatrix}\right]\big) = 0$ for $t \ge 0$. In the same way $h^{n+1}(\mathcal{F}\left[\begin{smallmatrix} a-n \\ c-2+b \end{smallmatrix}\right]) = 0$ for any $a \ge 0$ and for any $b \ge 0$. □

Notice that, if F is a smooth divisor in $|\mathbf{f}|$ (which exists since $\mathcal{O}_X(\mathbf{f})$ is globally generated) then $F = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{m-1}}(a_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{m-1}}(a_n))$, so our notion of regularity is available on F too.

Lemma 1.7. *If $\mathcal{F}$ is a regular coherent sheaf on X, and F a smooth divisor in $|\mathbf{f}|$, then $\mathcal{F}_{|F}$ is a regular coherent sheaf on F.*

Proof. We may assume $n,\, m > 0$. For $0 \le j \le m-1$ we consider the exact cohomology sequence

$$H^{n+j}(\mathcal{F}\left[\begin{smallmatrix} -n \\ c-j-1 \end{smallmatrix}\right]) \to H^{n+j}(\mathcal{F}_{|F}\left[\begin{smallmatrix} -n \\ c-j-1 \end{smallmatrix}\right]) \to H^{n+j+1}(\mathcal{F}\left[\begin{smallmatrix} -n \\ c-j-2 \end{smallmatrix}\right]).$$

The first and the third terms vanish by condition 1.4(a), so the middle term vanishes. Similarly, for $1 \le i \le n-1$ and $0 \le j \le m-1$, we consider

$$H^{i+j}(\mathcal{F}\left[\begin{smallmatrix} -i \\ i-j \end{smallmatrix}\right]) \to H^{i+j}(\mathcal{F}_{|F}\left[\begin{smallmatrix} -i \\ i-j \end{smallmatrix}\right]) \to H^{i+j+1}(\mathcal{F}\left[\begin{smallmatrix} -i \\ i-j-1 \end{smallmatrix}\right])$$

and since the first and third terms vanish by condition 1.4(a), then the middle term vanishes. Hence $\mathcal{F}_{|F}$ is regular. □

Lemma 1.8. *If $\mathcal{F}$ is a regular coherent sheaf on X, then it is also $(0,q)$-regular for any $q \geq 0$.*

Proof. We proceed by induction on m. The case $m = 0$ is proved in [14]. Now we assume the result for $m-1$ and we prove it for m. For $0 \leq j \leq m-1$ and for any $t \geq 0$ we have the exact sequence

$$H^{n+j}(\mathcal{F}\left[\begin{smallmatrix} -n \\ c-j-2+t \end{smallmatrix}\right]) \to H^{n+j}(\mathcal{F}\left[\begin{smallmatrix} -n \\ c-j-1+t \end{smallmatrix}\right]) \to H^{n+j}(\mathcal{F}_{|F}\left[\begin{smallmatrix} -n \\ c-j-1+t \end{smallmatrix}\right]).$$

If $q = 1$ then first term vanishes by conditions 1.4(a) and the third term vanishes by Lemma 1.7 and the inductive hypothesis, so the middle term vanishes. By recursion on t we get $H^{n+j}(\mathcal{F}\left[\begin{smallmatrix} -n \\ c-j-1+q \end{smallmatrix}\right]) = 0$. Moreover $H^{n+j}(\mathcal{F}\left[\begin{smallmatrix} -n \\ c-j-1+q \end{smallmatrix}\right]) = 0$ also for $j = m$ by Lemma 1.6. Thus $\mathcal{F}(q\mathbf{f})$ satisfies the conditions 1.4(a).
The proof that $\mathcal{F}(q\mathbf{f})$ satisfies the condition 1.4(b) is similar. For $1 \leq i \leq n$ and $0 \leq j \leq m$, we have

$$H^{i+j}(\mathcal{F}\left[\begin{smallmatrix} -i \\ i-j-1+t \end{smallmatrix}\right]) \to H^{i+j}(\mathcal{F}\left[\begin{smallmatrix} -i \\ i-j+t \end{smallmatrix}\right]) \to H^{i+j}(\mathcal{F}_{|F}\left[\begin{smallmatrix} -i \\ i-j+t \end{smallmatrix}\right],$$

for any t. If $t = 1$ then first term vanishes by condition 1.4(b). We want to show that the third term also vanishes. Since $\mathcal{F}_{|F}$ is regular by Lemma 1.7, and $\mathcal{F}_{|F}(\mathbf{f})$ is regular by the inductive hypothesis, we have $H^{i+j}(\mathcal{F}_{|F}\left[\begin{smallmatrix} -i \\ i-j+1 \end{smallmatrix}\right]) = 0$ for $1 \leq i \leq n-1$ and $0 \leq j \leq m-1$, and also for $(i,j) = (n,m)$ since $\dim(F) < n+m$. Moreover the regularity of $\mathcal{F}_{|F}(\mathbf{f})$ implies $H^{i+j}(\mathcal{F}_{|F}\left[\begin{smallmatrix} -i \\ i-j+1 \end{smallmatrix}\right]) = 0$ for $j = n$ and $0 \leq j \leq m-1$.
Thus the middle term vanishes in all relevant cases and again by recursion on t we get $H^{i+j}(\mathcal{F}\left[\begin{smallmatrix} -i \\ i-j+q \end{smallmatrix}\right]) = 0$ for $1 \leq i \leq n$ and $0 \leq j \leq m$. Hence $\mathcal{F}(q\mathbf{f})$ also satisfies condition 1.4(b), and hence $\mathcal{F}$ is $(0,q)$-regular. □

Notice that, if S is a smooth divisor in $|\mathbf{h} - a_0\mathbf{f}|$ (again, $\mathcal{O}_X(\mathbf{h} - a_0\mathbf{f})$ is globally generated) then $S = \mathbb{P}(\mathcal{O}_{\mathbb{P}^m}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^m}(a_n))$, so again the regularity of Definition 1.4 is available. We continue to use $\mathbf{h}$ and $\mathbf{f}$ for the generators of $\mathrm{Pic}(S)$: in X they are obtained as the intersection products $(\mathbf{h} - a_0\mathbf{f})\mathbf{h}$ and $(\mathbf{h} - a_0\mathbf{f})\mathbf{f}$.
When $n = 1$, we have $S = \mathbb{P}(\mathcal{O}_{\mathbb{P}^m}(a_1)) \cong \mathbb{P}^m$, and $(\mathbf{h} - a_0\mathbf{f})\mathbf{f} = \mathbf{f}$, the ample generator $\mathcal{O}_{\mathbb{P}^m}(1)$ of $\mathrm{Pic}\, S$. Then $(\mathbf{h} - a_0\mathbf{f})\mathbf{h} = c\mathbf{f} - a_0\mathbf{f} = a_1\mathbf{f}$, so on S we have $\mathbf{h} = a_1\mathbf{f}$.
When $m = 0$, we have $S = \mathbb{P}^{n-1}$, simply a hyperplane section in $\mathbb{P}^n$.

Lemma 1.9. *Suppose that $n > 0$ and $m > 0$. If $\mathcal{F}$ is a regular coherent sheaf on X and S is a smooth divisor in $|\mathbf{h} - a_0\mathbf{f}|$, then $\mathcal{F}_{|S}$ is regular on S.*

Proof. Consider the exact cohomology sequence, for $0 \leq j \leq m$,

$$H^{n-1+j}(\mathcal{F}\left[\begin{smallmatrix} -(n-1) \\ c-a_0-j-1 \end{smallmatrix}\right]) \to H^{n-1+j}(\mathcal{F}_{|S}\left[\begin{smallmatrix} -(n-1) \\ c-a_0-j-1 \end{smallmatrix}\right]) \to H^{n+j}(\mathcal{F}\left[\begin{smallmatrix} -n \\ c-j-1 \end{smallmatrix}\right]).$$

The third term vanishes by condition 1.4(a) and Lemma 1.6, and the first term vanishes by conditions 1.4(b) and Lemma 1.8 (notice that $c-a_0-j-1 \geq n-1-j$ because $c-a_0 = a_1+\cdots+a_n \geq n$), so the middle term vanishes, so conditions 1.4(a) hold for $\mathcal{F}_{|S}$.
For $1 \leq i < n-1$ and $1 \leq j \leq m$ but $(i,j) \neq (0,0)$, we consider the exact sequence

$$H^{i+j}(\mathcal{F}\left[\begin{smallmatrix}-i\\ i-j\end{smallmatrix}\right]) \to H^{i+j}(\mathcal{F}_{|S}\left[\begin{smallmatrix}-i\\ i-j\end{smallmatrix}\right]) \to H^{i+j+1}\big(\mathcal{F}\left[\begin{smallmatrix}-(i+1)\\ i-j+a_0\end{smallmatrix}\right]\big).$$

The first term vanishes by 1.4(b) and the third term vanishes by Lemma 1.8 (notice that $i-j+a_0 \geq i+1-j$), so the middle term vanishes, so conditions 1.4(b) hold for $\mathcal{F}_{|S}$. $\square$

Proposition 1.10. *If $\mathcal{F}$ is a regular coherent sheaf on X, then*

(i) $\mathcal{F}\left[\begin{smallmatrix}p\\ q\end{smallmatrix}\right]$ is regular for p, $q \geq 0$.

(ii) $H^0(\mathcal{F}(\mathbf{f}))$ is spanned by (hence, equal to) $H^0(\mathcal{F}) \otimes H^0(\mathcal{O}(\mathbf{f}))$, and $H^0(\mathcal{F}(\mathbf{h}))$ is spanned by $H^0(\mathcal{F}(a_0\mathbf{f})) \oplus \cdots \oplus H^0(\mathcal{F}(a_n\mathbf{f}))$.

Proof. To prove (i) it is enough to show that $\mathcal{F}\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right] = \mathcal{F}(\mathbf{h})$ is regular, since $\mathcal{F}\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right] = \mathcal{F}(\mathbf{f})$ is regular by Lemma 1.8.
We have $H^{n+m}\big(\mathcal{F}\left[\begin{smallmatrix}-(n-1)\\ c-1-j\end{smallmatrix}\right]\big) = 0$ by Lemma 1.6. For $0 \leq j \leq m-1$ we consider the exact sequence

$$H^{n+j}(\mathcal{F}\left[\begin{smallmatrix}-n\\ c+a_0-1-j\end{smallmatrix}\right]) \to H^{n+j}\big(\mathcal{F}\left[\begin{smallmatrix}-(n-1)\\ c-1-j\end{smallmatrix}\right]\big) \to H^{n-1+j+1}\big(\mathcal{F}_{|S}\left[\begin{smallmatrix}-(n-1)\\ c-1-j\end{smallmatrix}\right]\big).$$

The first term vanishes by hypothesis (and the fact that $c+a_0-1-j > c-i-j$) and the third term vanishes by Lemma 1.9 (and the fact that $c-1-j \geq c-a_0-1-j-1$). Thus the middle term vanishes, so 1.4(a) holds for $\mathcal{F}(\mathbf{h})$.
We verify the condition 1.4(b) for $\mathcal{F}(\mathbf{h})$ by induction on n. We need to show that if $0 \leq i \leq n-1$ and $0 \leq j \leq m$ but $(i,j) \neq (0,0)$ then $h^{i+j}\big(\mathcal{F}\left[\begin{smallmatrix}-(i-1)\\ i-j\end{smallmatrix}\right]\big) = 0$.
If $n = 1$, then $1 \leq j \leq m$ then $S = \mathbb{P}^m$ and we have the exact sequence

$$H^j(\mathcal{F}((a_0-j)\mathbf{f})) \to H^j(\mathcal{F}(\mathbf{h}-j\mathbf{f})) \to H^j(\mathcal{F}_{|S}(\mathbf{h}-j\mathbf{f}))$$

in which the first term vanishes by hypothesis (since $a_0-j > -j$) and the third term vanishes because $H^j(\mathcal{F}_{|S}(\mathbf{h}-j\mathbf{f})) = H^j(\mathbb{P}^m, \mathcal{F}_{|\mathbb{P}^m}(c-j))$, and $\mathcal{F}_{|\mathbb{P}^m}(t)$ is regular on $\mathbb{P}^m$ for any $t \geq 0$. Hence we have the required vanishing of the middle term for $n = 1$.
For the induction step, we may now assume the vanishing for $\mathcal{F}_{|S}(\mathbf{h})$ on S. Then, for $0 \leq i \leq n-1$ and $0 \leq j \leq m$ but $(i,j) \neq (0,0)$, we have

$$H^i(\mathcal{F}\left[\begin{smallmatrix}-i\\ i-j+a_0\end{smallmatrix}\right]) \to H^i(\mathcal{F}\left[\begin{smallmatrix}-(i-1)\\ i-j\end{smallmatrix}\right]) \to H^i(\mathcal{F}_{|S}\left[\begin{smallmatrix}-(i-1)\\ i-j\end{smallmatrix}\right])$$

in which the first term vanishes by 1.4(b) and Lemma 1.8, since $i-j+a_0 > i-j$, and the third term vanishes by the inductive hypothesis and Lemma 1.9. So the middle term vanishes, as required.
For the proof of (ii), we start with (4) and tensor by $\mathcal{F}$ to get

$$0 \to \mathcal{F}^{e_m}(-(m-1)\mathbf{f}) \to \cdots \to \mathcal{F}^{e_1} \to \mathcal{F}(\mathbf{f}) \to 0.$$

Since $H^1(\mathcal{F}(-\mathbf{f})) = \cdots = H^{m-1}(\mathcal{F}(-(m-1)\mathbf{f})) = 0$ by 1.4(b) with $i=0$ and $j = 1, \ldots, m-1$, we obtain

$$H^0(\mathcal{F}) \otimes H^0(\mathcal{O}_X(\mathbf{f})) \to H^0(\mathcal{F}(\mathbf{f})) \to 0,$$

which gives the first part of (ii) immediately. If instead we take (5) and tensor by $\mathcal{F}$ we get:

$$0 \to \mathcal{F}\left[\begin{smallmatrix} -n \\ c-m-1 \end{smallmatrix}\right] \to \mathcal{F}^{e_m}\left[\begin{smallmatrix} -n \\ c-m \end{smallmatrix}\right] \to \cdots \to \mathcal{F}^{e_m}\left[\begin{smallmatrix} -n \\ c-1 \end{smallmatrix}\right] \to \bigoplus_{i=0}^{n} \mathcal{F}\left[\begin{smallmatrix} -(n-1) \\ c-a_i \end{smallmatrix}\right] \to$$
$$\cdots \to \bigoplus_{i=0}^{n} \mathcal{F}(a_i\mathbf{f}) \to \mathcal{F}(\mathbf{h}) \to 0.$$

From 1.4(a) we have $H^{n+m}(\mathcal{F}\left[\begin{smallmatrix} -n \\ c-m-1 \end{smallmatrix}\right]) = \cdots = H^n(\mathcal{F}\left[\begin{smallmatrix} -n \\ c-1 \end{smallmatrix}\right]) = 0$.
From 1.4(b), with $j=0$ and $i>0$, and using Lemma 1.8, we obtain $H^i(\mathcal{F}\left[\begin{smallmatrix} -i \\ a_I \end{smallmatrix}\right]) = 0$ for any $I \subset \{0, \ldots, n\}$ with $|I| = i+1$, because then $a_I > i = i-j$. From this we get immediately

$$H^0(\mathcal{F}(a_0\mathbf{f})) \oplus \cdots \oplus H^0(\mathcal{F}(a_n\mathbf{f})) \to H^0(\mathcal{F}(\mathbf{h})) \to 0$$

as required. □

Corollary 1.11. *If $\mathcal{F}$ is a regular coherent sheaf on X then it is globally generated.*

Proof. Proposition 1.10 gives surjections $H^0(\mathcal{F})^r \to \bigoplus_{k=0}^n H^0(\mathcal{F}(a_k\mathbf{f}))$, for $r = h^0(\mathcal{O}(\mathbf{f}))$, and $\bigoplus_{k=0}^n H^0(\mathcal{F}(a_k\mathbf{f})) \to H^0(\mathcal{F}(\mathbf{h}))$. Thus we have a surjection $H^0(\mathcal{F})^r \to H^0(\mathcal{F}(\mathbf{h}))$. Take $l \gg 0$ such that $\mathcal{F}(l\mathbf{h})$ is globally generated. For a suitable positive integer s the diagram

$$\begin{array}{ccc} H^0(\mathcal{F})^s \otimes \mathcal{O}_X & \longrightarrow & H^0(\mathcal{F}(l\mathbf{h})) \otimes \mathcal{O}_X \\ \downarrow & & \downarrow \\ H^0(\mathcal{F})^r & \longrightarrow & \mathcal{F}(l\mathbf{h}) \end{array}$$

commutes and the left, top and right arrows are surjections. Therefore the bottom arrow is also a surjection, so $\mathcal{F}$ is generated by its sections. □

Corollary 1.12. *If $n > 0$, then $\mathcal{O}_X$, $\mathcal{O}_X(\mathbf{f})$ and $\mathcal{O}_X(\mathbf{h}-\mathbf{f})$ are all regular but not $(-1,0)$-regular, so* $\mathrm{Reg}(\mathcal{O}_X) = \mathrm{Reg}(\mathcal{O}_X(\mathbf{f})) = \mathrm{Reg}(\mathcal{O}_X(\mathbf{h}-\mathbf{f})) = 0$.

Proof. For $\mathcal{O}_X$ and $\mathcal{O}_X(\mathbf{f})$ the conditions 1.4(a), and 1.4(b) for $i \neq 0$, follow immediately from Lemma 1.2(ii). For $i = 0$ we require for 1.4(b) that $H^j(\mathcal{O}_X(-j\mathbf{f})) = H^j(\mathcal{O}((-j+1)\mathbf{f}) = 0$ for $j \leq m$, and both of these are cases of Lemma 1.3.
So $\mathcal{O}_X$ and $\mathcal{O}_X(\mathbf{f})$ are regular. On the other hand, they are not $(-1,0)$-regular because $\mathcal{O}_X(-\mathbf{h})$ and $\mathcal{O}_X(-\mathbf{h}+\mathbf{f})$ are not regular, since

$$h^{n+m}(\mathcal{O}_X(-\mathbf{h})\left[\begin{smallmatrix}-n\\ c-1-m\end{smallmatrix}\right]) = h^n(\mathcal{O}_X(-\mathbf{h}+\mathbf{f})\left[\begin{smallmatrix}-n\\ c-1\end{smallmatrix}\right]) = 1.$$

In the case of $\mathcal{O}_X(\mathbf{h}-\mathbf{f})$, for 1.4(a) we require that $h^{n+j}(\mathcal{O}_X\left[\begin{smallmatrix}-n+1\\ c-j-2\end{smallmatrix}\right]) = 0$ if $0 \leq j \leq m$. This holds by 1.2(ii) if $n \neq 1$ and by Lemma 1.3 if $n = 1$, since $c - j - 2 \geq -m$.
For 1.4(b) we require that $h^{i+j}(\mathcal{O}_X\left[\begin{smallmatrix}-i+1\\ i-j-1\end{smallmatrix}\right]) = 0$, for $0 \leq j \leq m$ and $0 \leq i < n$ but $(i,j) \neq (0,0)$. This holds by Lemma 1.2(ii) when $i > 1$, and by Lemma 1.3 when $i = 1$ since $i - j - 1 \geq -m$. For $i = 0$ it holds because, by Lemma 1.2(i), we have $H^j(\mathcal{O}_X(\mathbf{h}-(j+1)\mathbf{f}) = H^j(\mathbb{P}^m, \mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^m}(-j-1)) = 0$.
So $\mathcal{O}_X(\mathbf{h}-\mathbf{f})$ is regular. On the other hand, it is not $(-1,0)$-regular because $\mathcal{O}_X(-\mathbf{f})$ is not regular, as since $h^m(\mathcal{O}_X(-\mathbf{f}) \otimes \mathcal{O}_X(-m\mathbf{f})) = 1$. □

1.3 Comparison with multigraded regularity

The regularity defined in Subsection 1.1 is related to the multigraded regularity of Maclagan and Smith [12] as it applies in this special case.
Multigraded regularity, as defined in [12, Definition 6.2], makes sense on any smooth projective variety Y (and far more generally). It depends, as ours does, on a chosen $\mathbf{p} \in \operatorname{Pic} Y$, but also on a choice of a finite subset $\mathbf{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_\ell\} \subset \operatorname{Pic} Y$. If we further assume that the divisors $\mathbf{c}_i$ are nef, then [12, Corollary 6.6] gives the following very simplified definition for multigraded regularity.

Definition 1.13. *Suppose that $\mathcal{F}$ is a sheaf on a smooth projective toric variety Y and $\mathbf{p} \in \operatorname{Pic} Y$, and $\mathbf{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_\ell\}$ is a finite set of nef divisor classes on Y. Then we say that $\mathcal{F}$ is* multigraded $\mathbf{p}$-regular *(with respect to $\mathbf{C}$) if $H^i(\mathcal{F}(\mathbf{p} - \sum \lambda_r \mathbf{c}_r)) = 0$ for all $i > 0$ whenever $\lambda_i \in \mathbb{N}$ and $\sum \lambda_r = i$.*

Comparison of [12, Corollary 6.6] with [12, Definition 6.2], and the discussion there, gives the following.

Lemma 1.14. *If $\mathcal{F}$ is multigraded $\mathbf{p}$-regular under the conditions of 1.13, then we also have $H^i(\mathcal{F}(\mathbf{p} + \mathbf{m} - \sum \lambda_r \mathbf{c}_r)) = 0$ for any $\mathbf{m} = \sum \mu_r \mathbf{c}_r$ with $\mu_r \in \mathbb{N}$ (i.e. $\mathbf{m} \in \mathbb{N}\mathbf{C}$ in the notation of [12]).*

To compare this with our definition, we shall take Y to be a semipositive scroll X, and $\mathbf{C} = \{\mathbf{h}, \mathbf{f}\}$. Note that $\mathbf{f}$ is nef by construction but $\mathbf{h}$ is nef because of the semipositivity. The condition in Definition 1.13 may be rewritten as

$$H^{i+j}(\mathcal{F}(\mathbf{p})\left[\begin{smallmatrix}-i\\ -j\end{smallmatrix}\right]) = 0 \quad \text{for all } i,\, j \in \mathbb{N} \text{ except } i = j = 0. \tag{6}$$

Proposition 1.15. *Let $(X, \mathcal{V})$ be a semipositive scroll and suppose that $\mathcal{F}$ is a multigraded* $\mathbf{p}$*-regular coherent sheaf on X, where $\mathbf{p} = p\mathbf{h} + q\mathbf{f}$. Then $\mathcal{F}$ is (p,q)-regular in the sense of Definition 1.4.*

Proof. It is sufficient to use the condition (6). Then applying Lemma 1.14 with $\mathbf{m} = i\mathbf{f}$ with $i < n$ and $j \leq m$ gives Definition 1.4(b), and the same with $i = n$ and $\mathbf{m} = (c-1)\mathbf{f}$ gives Definition 1.4(a). □

On the other hand, if X is the Hirzebruch surface $\mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ then $\mathcal{O}_{\mathbb{F}_2}$ is not multigraded $(0,0)$-regular: see [12, Example 1.2] and [12, Example 6.5]. However, $\mathcal{O}_{\mathbb{F}_2}$ is regular according to Definition 1.4, by Corollary 1.12.
The main difference between the two definitions concerns the top cohomology. Definition 1.13 requires the vanishing conditions

$$H^{m+n}(\mathcal{F}(\mathbf{p})\left[\begin{smallmatrix} -r \\ -m-n+r \end{smallmatrix}\right]) = 0 \text{ for } 0 \leq r \leq m+n,$$

which in this context reduce to the last one, $H^{m+n}(\mathcal{F}(\mathbf{p})\left[\begin{smallmatrix} -m-n \\ 0 \end{smallmatrix}\right]) = 0$. By contrast, Definition 1.4 requires only $H^{m+n}(\mathcal{F}(\mathbf{p})\left[\begin{smallmatrix} -n \\ c-m-1 \end{smallmatrix}\right]) = 0$, which is a much weaker condition (but makes sense only for scrolls).
In Section 2 we will exploit this to prove splitting criteria on scrolls, analogous to those of Horrocks [11] on $\mathbb{P}^n$.

1.4 Rational normal scrolls

We end this section with a discussion of the case $m = 1$: these are the rational normal scrolls $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n))$.

Remark 1.16. If X is a rational normal scroll and $\mathcal{F}$ is a coherent sheaf on X, then $\mathcal{F}$ is (p,q)-regular if and only if

(a) $h^{n+1}(\mathcal{F}\left[\begin{smallmatrix} -n \\ c-2 \end{smallmatrix}\right]) = h^n(\mathcal{F}(\mathbf{p})\left[\begin{smallmatrix} -n \\ c-1 \end{smallmatrix}\right]) = 0$,

(b) $h^{i+1}(\mathcal{F}(\mathbf{p})\left[\begin{smallmatrix} -i \\ i-1 \end{smallmatrix}\right]) = 0$ for $0 \leq i < n$,

(c) $h^i(\mathcal{F}(\mathbf{p})\left[\begin{smallmatrix} -i \\ i \end{smallmatrix}\right]) = 0$ for $0 < i < n$,

where, as before, $\mathcal{F}(\mathbf{p}) = \mathcal{F}\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$.

In this case a smooth hyperplane section $H \in |\mathbf{h}|$ of X is still a rational normal scroll of the same degree: i.e. $H = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a'_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a'_{n-1}))$ with $a'_0 + \cdots + a'_{n-1} = c$. We continue to use $\mathbf{h}$ and $\mathbf{f}$ for the generators of $\operatorname{Pic}(H)$ (in X they are obtained as the intersection products $\mathbf{h}^2$ and $\mathbf{hf}$).

Lemma 1.17. *If $\mathcal{F}$ is a regular coherent sheaf on the rational normal scroll X and H is a smooth divisor in $|\mathbf{h}|$, then $\mathcal{F}_{|H}$ is $(0,1)$-regular on H.*

Proof. In the exact sequence

$$H^n\left(\mathcal{F}\left[{-(n-1) \atop c-1}\right]\right) \to H^n\left(\mathcal{F}_{|H}\left[{-(n-1) \atop c-1}\right]\right) \to H^{n+1}\left(\mathcal{F}\left[{-n \atop c-1}\right]\right)$$

the third term vanishes by 1.16(a) and Lemma 1.7 and the first term vanishes by condition 1.16(b) and Lemma 1.8, since $c-1 \geq n-2$. So the middle term vanishes, so $H^n(S, \mathcal{F}_{|H}\left[{-(n-1) \atop c-1}\right]$. Similarly, in

$$H^{n-1}\left(\mathcal{F}\left[{-(n-1) \atop c}\right]\right) \to H^{n-1}\left(\mathcal{F}_{|H}\left[{-(n-1) \atop c}\right]\right) \to H^n\left(\mathcal{F}\left[{-n \atop c}\right]\right)$$

the third and first terms vanish by 1.16(a) and Lemma 1.8, respectively by 1.16(c) and Lemma 1.8, since $c \geq n-1$. So the middle term vanishes, so $H^{n-1}(H, \mathcal{F}_{|H}\left[{-(n-1) \atop c}\right]) = 0$, and we have verified 1.16(a) for $\mathcal{F}_{|H}(\mathbf{f})$.
For $0 \leq i \leq n-2$, in

$$H^{i+1}\left(\mathcal{F}\left[{-i \atop i}\right]\right) \to H^{i+1}\left(\mathcal{F}_{|H}\left[{-i \atop i}\right]\right) \to H^{i+2}\left(\mathcal{F}\left[{-(i+1) \atop i}\right]\right)$$

the first and the third terms vanish by 1.16(b) and Lemma 1.8. So the middle term vanishes, giving 1.16(b) for $\mathcal{F}_{|H}(\mathbf{f})$.
Finally, for $1 \leq i \leq n-2$, in

$$H^{i}\left(\mathcal{F}\left[{-i \atop i+1}\right]\right) \to H^{i}\left(\mathcal{F}_{|H}\left[{-i \atop i+1}\right]\right) \to H^{i+1}\left(\mathcal{F}\left[{-(i+1) \atop i+1}\right]\right)$$

the first and third terms vanish by 1.16(c) and Lemma 1.8. So the middle term vanishes, giving 1.16(c) for $\mathcal{F}_{|H}(\mathbf{f})$. □

2 Splitting criteria for vector bundles

In this section we assume $m, n > 0$. We apply the results of Section 1 in order to prove splitting criteria for vector bundles.

Theorem 2.1. *Suppose that $(X, \mathcal{V})$ is a positive scroll with $m, n > 0$, and let $\mathcal{E}$ be a rank r vector bundle on X. Then the following conditions are equivalent:*

(i) for any integer t we have the vanishing

(a) $h^{n+j}\left(\mathcal{E}\left[{t \atop c-j-1}\right]\right) = 0$ for $0 \leq j < m$ and

(b) $h^{i+j}\left(\mathcal{E}\left[{t \atop i-j}\right]\right) = 0$ for $0 \leq j \leq m$ and $0 \leq i < n$ but $(i,j) \neq (0,0)$;

(ii) there are r integers $t_1, \ldots, t_r$ such that $\mathcal{E} \cong \bigoplus_{i=1}^{r} \mathcal{O}_X(t_i\mathbf{h})$.

Proof. First suppose that $\mathcal{E}$ satisfies (i), and let t be an integer such that $\mathcal{E}(t\mathbf{h})$ is regular but $\mathcal{E}((t-1)\mathbf{h})$ is not. Comparing the definition of regularity (Definition 1.4 with $p = q = 0$) with (i) we see that $\mathcal{E}((t-1)\mathbf{h})$ is not

regular if and only if $H^{n+m}(\mathcal{E}\left[\begin{smallmatrix} t-n-1 \\ c-m-1 \end{smallmatrix}\right]) \neq 0$. In that case, by Serre duality $H^0(\mathcal{E}^\vee(-t\mathbf{h})) \neq 0$. But this, together with the fact that $\mathcal{E}(t\mathbf{h})$ is globally generated by Corollary 1.11, implies that $\mathcal{O}_X$ is a direct summand of $\mathcal{E}(t\mathbf{h})$. By induction on r, it follows that $\mathcal{E}$ satisfies (ii).
Conversely, if $\mathcal{E}$ satisfies (ii) then it satisfies (i) because $\mathcal{O}_X$ satisfies all the conditions in (i), by Lemma 1.3. □

In the case $X \cong \mathbb{P}^n \times \mathbb{P}^m$, i.e. $c = n+1$, Theorem 2.1 reduces to [2, Theorem 1.3]. Indeed,

$$h^{n+j}(\mathcal{E}\left[\begin{smallmatrix} t \\ c-j-1 \end{smallmatrix}\right]) = h^{n+j}(\mathcal{E}(t, t+n-j)) = h^{n+j}(\mathcal{E}(t-n, t-j))$$

for $0 \le j < m$, and

$$h^{i+j}(\mathcal{E}\left[\begin{smallmatrix} t \\ i-j \end{smallmatrix}\right]) = h^{i+j}(\mathcal{E}(t, t+i-j)) = h^{i+j}(\mathcal{E}(t-i, t-j))$$

for $0 \le j \le m$ and $0 \le i < n$ but $(i,j) \neq (0,0)$.

Theorem 2.2. *Let $\mathcal{E}$ be a vector bundle on X. Then $\mathcal{E}$ is a direct sum of line bundles $\mathcal{O}_X$, $\mathcal{O}_X(\mathbf{f})$ and $\mathcal{O}_X(\mathbf{h}-\mathbf{f})$ with some twist $t\mathbf{h}$ if and only if the following conditions hold for any integer t.*

(a) $h^{n+j}(\mathcal{E}\left[\begin{smallmatrix} t \\ c-j-1 \end{smallmatrix}\right]) = 0$ *for* $1 \le j < m$,

(b) $h^{i+j}(\mathcal{E}\left[\begin{smallmatrix} t \\ i-j \end{smallmatrix}\right]) = 0$ *for* $0 \le j \le m$ *and* $0 \le i < n$ *but* $(i,j) \neq (0,0), (0,m)$,

(c) $h^{j+1}(\mathcal{E}^\vee\left[\begin{smallmatrix} t \\ -j \end{smallmatrix}\right]) = 0$ *for* $0 \le j < m$,

(d) $h^{|I|}(\mathcal{E}\left[\begin{smallmatrix} t \\ a_I-1 \end{smallmatrix}\right]) = h^{|I|}(\mathcal{E}^\vee\left[\begin{smallmatrix} t \\ a_I-1 \end{smallmatrix}\right]) = 0$ *if* $1 \le |I| \le n$.

Proof. As in Theorem 2.1 it is easy to check that $\mathcal{O}_X$, $\mathcal{O}_X(\mathbf{f})$ and $\mathcal{O}_X(\mathbf{h}-\mathbf{f})$ satisfy all the required vanishing.
Assume that (a)–(d) hold, and consider the integer t such that $\mathcal{E}(t\mathbf{h})$ is regular but $\mathcal{E}((t-1)\mathbf{h})$ is not. Up to a twist we may assume $t=0$.
Comparing (a) and (b) with the definition of regularity, we see that $\mathcal{E}(-\mathbf{h})$ is not regular if and only if one of the following conditions is satisfied:

(i) $h^{n+m}\big(\mathcal{E}\left[\begin{smallmatrix} -(n+1) \\ c-m-1 \end{smallmatrix}\right]\big) \neq 0$,

(ii) $h^{n}\big(\mathcal{E}\left[\begin{smallmatrix} -(n+1) \\ c-1 \end{smallmatrix}\right]\big) \neq 0$,

(iii) $h^{m}(\mathcal{E}\left[\begin{smallmatrix} -1 \\ -m \end{smallmatrix}\right]) \neq 0$.

We consider each of these possibilities in turn.
If (i) holds, then $\mathcal{O}_X$ is a direct summand, as in the proof of Theorem 2.1.
If (ii) holds, then we consider (2) tensored by $\mathcal{E}(-\mathbf{h}-\mathbf{f})$, which gives

$$0 \to \mathcal{E}\left[\begin{smallmatrix} -(n+1) \\ c-1 \end{smallmatrix}\right] \to \bigoplus_{i=0}^{n} \mathcal{E}\left[\begin{smallmatrix} -n \\ c-a_i-1 \end{smallmatrix}\right] \to \cdots \to \bigoplus_{i=0}^{n} \mathcal{E}\left[\begin{smallmatrix} -1 \\ a_i-1 \end{smallmatrix}\right] \to \mathcal{E}(-\mathbf{f}) \to 0.$$

Since by (d)

$$H^n\Big(\bigoplus_{i=0}^{n}\mathcal{E}\left[\begin{smallmatrix}-n\\ c-a_i-1\end{smallmatrix}\right]\Big)=\cdots=H^1\Big(\bigoplus_{i=0}^{n}\mathcal{E}\left[\begin{smallmatrix}-1\\ a_i-1\end{smallmatrix}\right]\Big)=0,$$

we have a surjection $H^0(\mathcal{E}(-\mathbf{f}))\to H^n\big(\mathcal{E}\left[\begin{smallmatrix}-(n+1)\\ c-1\end{smallmatrix}\right]\big)$. So $H^0(\mathcal{E}(-\mathbf{f}))\neq 0$, and there exists a nonzero map $f\colon\mathcal{E}\to\mathcal{O}_X(\mathbf{f})$.
On the other hand $H^n\big(\mathcal{E}\left[\begin{smallmatrix}-(n+1)\\ c-1\end{smallmatrix}\right]\big)\cong H^m(\mathcal{E}^\vee(-m\mathbf{f}))$. Then the exact sequence (4) tensored by $\mathcal{E}^\vee$ reads

$$0\to\mathcal{E}^\vee(-m\mathbf{f})\to\mathcal{E}^\vee(-(m-1)\mathbf{f})^{e_m}\to\cdots\to(\mathcal{E}^\vee)^{e_1}\to\mathcal{E}^\vee(\mathbf{f})\to 0.$$

But by (c)

$$H^1(\mathcal{E}^\vee)=\cdots=H^m(\mathcal{E}^\vee(-(m-1)\mathbf{f}))=0,$$

so we have a surjective map $H^0(\mathcal{E}^\vee(\mathbf{f}))\to H^n\big(\mathcal{E}\left[\begin{smallmatrix}-(n+1)\\ c-1\end{smallmatrix}\right]\big)$. Therefore $H^0(\mathcal{E}^\vee(\mathbf{f}))\neq 0$ and there exists a nonzero map $g\colon\mathcal{O}_X(\mathbf{f})\to\mathcal{E}$.
Let us consider the following commutative diagram:

$$\begin{array}{ccc}
H^n\big(\mathcal{E}\left[\begin{smallmatrix}-(n+1)\\ c-1\end{smallmatrix}\right]\big)\otimes H^m(\mathcal{E}^\vee(-m\mathbf{f})) & \xrightarrow{\ \sigma\ } & H^{n+m}\big(\mathcal{O}_X\left[\begin{smallmatrix}-(n+1)\\ c-1-m\end{smallmatrix}\right]\big)\\
\downarrow & & \downarrow\\
H^0(\mathcal{E}(-\mathbf{f}))\otimes H^1(\mathcal{E}^\vee(-\mathbf{f})) & \longrightarrow & H^1(\mathcal{O}_X(-\mathbf{f})\otimes\mathcal{O}_X(-\mathbf{f}))\\
\downarrow & & \downarrow\\
H^0(\mathcal{E}(-\mathbf{f}))\otimes H^0(\mathcal{E}^\vee(\mathbf{f})) & \xrightarrow{\ \tau\ } & H^0(\mathcal{O}_X(-\mathbf{f})\otimes\mathcal{O}_X(\mathbf{f}))\\
\| & & \|\\
\mathrm{Hom}(\mathcal{E},\mathcal{O}_X(\mathbf{f}))\otimes\mathrm{Hom}(\mathcal{O}_X(\mathbf{f}),\mathcal{E}) & & \mathrm{Hom}(\mathcal{O}_X(\mathbf{f}),\mathcal{O}_X(\mathbf{f})).
\end{array}$$

The spaces in the right column are all 1-dimensional. The map σ comes from Serre duality and it is not zero, the right vertical map are isomorphisms and the left vertical maps are surjective, so the map τ is also not zero. This means that the the map $f\circ g\colon\mathcal{O}_X(\mathbf{f})\to\mathcal{O}_X(\mathbf{f})$ is non-zero and hence it is an isomorphism. This isomorphism shows that $\mathcal{O}_X(\mathbf{f})$ is a direct summand of $\mathcal{E}$.
If (iii) holds then the exact sequence (4) tensored by $\mathcal{E}(-\mathbf{h})$ reads

$$0\to\mathcal{E}(-\mathbf{h}-m\mathbf{f})\to\mathcal{E}(-\mathbf{h}-(m-1)\mathbf{f})^{e_m}\to\cdots\to\mathcal{E}(-\mathbf{h})^{e_1}\to\mathcal{E}(-\mathbf{h}+\mathbf{f})\to 0.$$

Also, by (b),

$$h^1(\mathcal{E}(-\mathbf{h}))=\cdots=h^m(\mathcal{E}(-\mathbf{h}-(m-1)\mathbf{f}))=0,$$

so $h^0(\mathcal{E}(-\mathbf{h}+\mathbf{f})) \neq 0$. By Serre duality, $h^m(\mathcal{E}(-\mathbf{h}-m\mathbf{f})) = h^n(\mathcal{E}^\vee\left[\begin{smallmatrix} -n \\ c-1 \end{smallmatrix}\right])$. From the exact sequence

$$0 \to \mathcal{E}^\vee\left[\begin{smallmatrix} -n \\ c-1 \end{smallmatrix}\right] \to \bigoplus_{i=0}^{n} \mathcal{E}^\vee\left[\begin{smallmatrix} -(n-1) \\ c-a_i-1 \end{smallmatrix}\right] \to \cdots \to \bigoplus_{i=0}^{n} \mathcal{E}^\vee((a_i-1)\mathbf{f}) \to \mathcal{E}^\vee(\mathbf{h}-\mathbf{f}) \to 0$$

we get also $h^0(\mathcal{E}^\vee(\mathbf{h}-\mathbf{f})) \neq 0$, since by (d),

$$\begin{aligned} h^n(\bigoplus_{i=0}^{n} \mathcal{E}^\vee\left[\begin{smallmatrix} -(n-1) \\ c-a_i-1 \end{smallmatrix}\right]) &= h^1(\bigoplus_{i=0}^{n} \mathcal{E}\left[\begin{smallmatrix} -2 \\ a_i-m \end{smallmatrix}\right]) = \cdots = h^1(\bigoplus_{i=0}^{n} \mathcal{E}^\vee((a_i-1)\mathbf{f})) \\ &= h^n(\bigoplus_{i=0}^{n} \mathcal{E}\left[\begin{smallmatrix} -(n+1) \\ c-a_i-m \end{smallmatrix}\right]) = 0. \end{aligned}$$

This shows, by the same argument as before, that $\mathcal{O}_X(\mathbf{h}-\mathbf{f})$ is a direct summand of $\mathcal{E}$. □

If $c = n+1$ then $X = \mathbb{P}^n \times \mathbb{P}^m$ and Theorem 2.2 is an improvement of [2, Theorem 1.4]. To see this, first note that in this case (c) is implied by (a), because for $0 \leq j \leq m-1$ we have $h^{j+1}(\mathcal{E}^\vee\left[\begin{smallmatrix} t \\ j \end{smallmatrix}\right]) = h^{m+n-j-1}(\mathcal{E}\left[\begin{smallmatrix} -t-n \\ -t+j-m \end{smallmatrix}\right])$. Moreover, because $a_0 = \cdots = a_n = 1$, if $|I| = i$ we get

$$h^i(\mathcal{E}\left[\begin{smallmatrix} t \\ a_I-1 \end{smallmatrix}\right]) = h^i(\mathcal{E}\left[\begin{smallmatrix} t \\ t-1+i \end{smallmatrix}\right]) = h^i(\mathcal{E}\left[\begin{smallmatrix} t+1-i \\ t \end{smallmatrix}\right])$$

and

$$h^i(\mathcal{E}^\vee\left[\begin{smallmatrix} t \\ a_I-1 \end{smallmatrix}\right]) = h^i(\mathcal{E}^\vee\left[\begin{smallmatrix} t+1-i \\ t \end{smallmatrix}\right]) = h^{m+n-i}(\mathcal{E}\left[\begin{smallmatrix} -t-2-n+i \\ -t-m-1 \end{smallmatrix}\right]).$$

In what follows we will use the following truncations of (2) for $i = 1, \ldots, n$:

$$\begin{aligned} 0 \to \mathcal{O}_X\left[\begin{smallmatrix} -(n+1) \\ c \end{smallmatrix}\right] &\to \bigoplus_{|I|=n} \mathcal{O}_X\left[\begin{smallmatrix} -n \\ a_I \end{smallmatrix}\right] \to \cdots \\ &\to \bigoplus_{|I|=i+1} \mathcal{O}_X\left[\begin{smallmatrix} -(i+1) \\ a_I \end{smallmatrix}\right] \to \Omega^i_{X|\mathbb{P}^m} \to 0 \end{aligned} \tag{7}$$

and

$$0 \to \Omega^i_{X|\mathbb{P}^m} \to \bigoplus_{|I|=i} \mathcal{O}_X\left[\begin{smallmatrix} -i \\ a_I \end{smallmatrix}\right] \to \cdots \to \bigoplus_{|I|=1} \mathcal{O}_X\left[\begin{smallmatrix} -1 \\ a_I \end{smallmatrix}\right] \to \mathcal{O}_X \to 0. \tag{8}$$

Dualising (8) and tensoring with $\mathcal{O}_X\left[\begin{smallmatrix} -(n+1) \\ c \end{smallmatrix}\right]$ gives

$$\begin{aligned} 0 \to \mathcal{O}_X\left[\begin{smallmatrix} -(n+1) \\ c \end{smallmatrix}\right] &\to \bigoplus_{|I|=1} \mathcal{O}_X\left[\begin{smallmatrix} -n \\ c-a_I \end{smallmatrix}\right] \to \cdots \\ &\to \bigoplus_{|I|=i} \mathcal{O}_X\left[\begin{smallmatrix} -(n+1-i) \\ c-a_I \end{smallmatrix}\right] \to (\Omega^i_{X|\mathbb{P}^m})^\vee\left[\begin{smallmatrix} -(n+1) \\ c \end{smallmatrix}\right] \to 0, \end{aligned}$$

so, using the obvious fact that $c - a_I = a_{I'}$ where $I' = \{0, \ldots, n\} \smallsetminus I$, we get

$$(\Omega^i_{X|\mathbb{P}^m})^\vee\left[\begin{smallmatrix} -(n+1) \\ c \end{smallmatrix}\right] \cong \Omega^{n-i}_{X|\mathbb{P}^m}.$$

Theorem 2.3. *Let $\mathcal{E}$ be an indecomposible vector bundle on X such that* $\mathrm{Reg}(\mathcal{E}) = 0$*. Suppose that the following vanishing occurs.*

(a) $h^{n+j}\big(\mathcal{E}\left[\begin{smallmatrix} -(n+1) \\ c-j-1 \end{smallmatrix}\right]\big) = 0$ *for* $1 \leq j < m$*,*

(b) $h^{i+j}\big(\mathcal{E}\left[\begin{smallmatrix} -(i+1) \\ i-j \end{smallmatrix}\right]\big) = h^{i+j}\big(\mathcal{E}\left[\begin{smallmatrix} -(i+1) \\ i-j+1 \end{smallmatrix}\right]\big) = 0$ *for* $1 \leq j \leq m$ *and* $0 \leq i < n$*,*

(c) $h^{j+1}(\mathcal{E}^\vee(-j\mathbf{f})) = h^{j+1}(\mathcal{E}(-\mathbf{h} - j\mathbf{f})) = 0$ *for* $0 \leq j < m$*,*

(d) $h^{|I|}\big(\mathcal{E}\left[\begin{smallmatrix} -|I| \\ a_I-1 \end{smallmatrix}\right]\big) = h^{|I|}\big(\mathcal{E}^\vee\left[\begin{smallmatrix} -|I|+1 \\ a_I-1 \end{smallmatrix}\right]\big) = 0$ *if* $1 \leq |I| \leq n$*,*

(e) $h^k\big(\bigoplus_{|I|=1-k+i} \mathcal{E}\left[\begin{smallmatrix} -k \\ k+1-a_I \end{smallmatrix}\right]\big) = 0$ *for* $0 < k \leq i < n$ *and*

$h^k\big(\bigoplus_{|I|=k+1} \mathcal{E}^\vee\left[\begin{smallmatrix} -(k-1) \\ a_I-i-1 \end{smallmatrix}\right]\big) = 0$ *for* $0 < k \leq n-i < n$*.*

Then $\mathcal{E} \cong \mathcal{O}_X$*, or* $\mathcal{E} \cong \mathcal{O}_X(\mathbf{f})$*, or* $\mathcal{E} \cong \mathcal{O}_X(\mathbf{h} - \mathbf{f})$*, or* $\mathcal{E} \cong \Omega^i_{X|\mathbb{P}^m}\left[\begin{smallmatrix} i+1 \\ -(i+1) \end{smallmatrix}\right]$ *with* $1 < i < n$*.*

Proof. Comparing the definition of regularity with (a) and the first part of (b) we see that $\mathcal{E}(-\mathbf{h})$ is not regular if and only if one of the following conditions is satisfied:

(i) $h^{n+m}\big(\mathcal{E}\left[\begin{smallmatrix} -(n+1) \\ c-m-1 \end{smallmatrix}\right]\big) \neq 0$,

(ii) $h^{n}\big(\mathcal{E}\left[\begin{smallmatrix} -(n+1) \\ c-1 \end{smallmatrix}\right]\big) \neq 0$,

(iii) $h^{m}(\mathcal{E}\left[\begin{smallmatrix} -1 \\ -m \end{smallmatrix}\right]) \neq 0$,

(iv) $h^{i+m}\big(\mathcal{E}\left[\begin{smallmatrix} -(i+1) \\ i-m \end{smallmatrix}\right]\big) \neq 0$ for some i with $0 < i < n$.

Conditions (i)–(iii) were considered in Theorem 2.2 (notice that the twists we used in conditions (c), (d) in the proof of Theorem 2.2 are exactly those of conditions (c), (d) and (e) with $i = k = 1$).
So we consider case (iv). Suppose that $0 < i < n$ and $h^{i+m}\big(\mathcal{E}\left[\begin{smallmatrix} -(i+1) \\ i-m \end{smallmatrix}\right]\big) \neq 0$. Let us consider the following exact sequence, obtained from the dual of (8) tensored by $\mathcal{E}\left[\begin{smallmatrix} -(i+1) \\ a+1 \end{smallmatrix}\right]$ and (4) tensored by $\mathcal{E}\left[\begin{smallmatrix} -(i+1) \\ i \end{smallmatrix}\right]$:

$$0 \to \mathcal{E}\left[\begin{smallmatrix} -(i+1) \\ i-m \end{smallmatrix}\right] \to \mathcal{E}\left[\begin{smallmatrix} -(i+1) \\ i+1-m \end{smallmatrix}\right]^{e_m} \to \cdots \to \mathcal{E}\left[\begin{smallmatrix} -(i+1) \\ i \end{smallmatrix}\right]^{e_1} \to$$
$$\to \bigoplus_{|I|=1} \mathcal{E}\left[\begin{smallmatrix} -i \\ i+1-a_I \end{smallmatrix}\right] \to \cdots \to \bigoplus_{|I|=i} \mathcal{E}\left[\begin{smallmatrix} -1 \\ i+1-a_I \end{smallmatrix}\right] \to [\Omega^i_{X|\mathbb{P}^m}]^\vee \otimes \mathcal{E}\left[\begin{smallmatrix} -(i+1) \\ i+1 \end{smallmatrix}\right] \to 0.$$

The second part of (b) gives

$$h^{i+m}\big(\mathcal{E}\Big[\begin{smallmatrix}-(i+1)\\ i+1-m\end{smallmatrix}\Big]\big) = \cdots = h^{i+1}\big(\mathcal{E}\Big[\begin{smallmatrix}-(i+1)\\ i\end{smallmatrix}\Big]\big) = 0$$

and (e) gives

$$h^i\big(\bigoplus_{|I|=1} \mathcal{E}\big[\begin{smallmatrix}-i\\ i+1-a_I\end{smallmatrix}\big]\big) = \cdots = h^1\big(\bigoplus_{|I|=i} \mathcal{E}\big[\begin{smallmatrix}-1\\ i+1-a_I\end{smallmatrix}\big]\big),$$

so we get $h^0\big(\mathcal{E} \otimes \Big(\Omega^i_{X|\mathbb{P}^m}\Big[\begin{smallmatrix}i+1\\ -(i+1)\end{smallmatrix}\Big]\Big)^\vee\big) \neq 0$.
By Serre duality, $h^{i+m}\big(\mathcal{E}\Big[\begin{smallmatrix}-(i+1)\\ i-m\end{smallmatrix}\Big]\big) = h^{n-i}\big(\mathcal{E}^\vee\Big[\begin{smallmatrix}-(n-i)\\ c-i-1\end{smallmatrix}\Big]\big)$. Let us consider the exact sequence obtained from (7) tensored by $\mathcal{E}^\vee\Big[\begin{smallmatrix}1+i\\ -(i+1)\end{smallmatrix}\Big]$:

$$0 \to \mathcal{E}^\vee\Big[\begin{smallmatrix}-(n-i)\\ c-i-1\end{smallmatrix}\Big] \to \bigoplus_{|I|=n} \mathcal{E}^\vee\Big[\begin{smallmatrix}-(n-i-1)\\ a_I-i-1\end{smallmatrix}\Big] \to \dots$$
$$\to \bigoplus_{|I|=i+1} \mathcal{E}^\vee((a_I - i - 1)\mathbf{f})) \to \mathcal{E}^\vee \otimes \Omega^i_{X|\mathbb{P}^m}\Big[\begin{smallmatrix}i+1\\ -(i+1)\end{smallmatrix}\Big]) \to 0.$$

By (e),

$$h^{n-i}\big(\bigoplus_{|I|=n} \mathcal{E}^\vee\Big[\begin{smallmatrix}-(n-i-1)\\ a_I-i-1\end{smallmatrix}\Big]\big) = \cdots = h^1\big(\bigoplus_{|I|=i+1} \mathcal{E}^\vee((a_I - i + 1)\mathbf{f}))\big) = 0,$$

so we also get $h^0\big(\mathcal{E}^\vee \otimes \Omega^i_{X|\mathbb{P}^m}\Big[\begin{smallmatrix}-(i+1)\\ i+1\end{smallmatrix}\Big]\big) \neq 0$. From this we conclude that $\mathcal{E} \cong \Omega^i_{X|\mathbb{P}^m}\Big[\begin{smallmatrix}-(i+1)\\ i+1\end{smallmatrix}\Big]$. □

If $c = n + 1$, so $X = \mathbb{P}^n \times \mathbb{P}^m$, then Theorem 2.3 is similar to [2, Theorem 3.5].

Remark 2.4. A splitting criterion for vector bundles on scrolls is given in [5, Theorem 1.5], proved by a spectral sequence argument. That result and ours do not seem to be directly comparable. Theorem 2.1 and Theorem 2.2 use less information about the cohomology: in particular, only some vanishing, whereas the result in [5] assumes a priori agreement between the dimensions of all the cohomology spaces of $\mathcal{E}$ and a split bundle. On the other hand, our results apply to a more restricted class of bundles.

In the case of rational normal scrolls $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n))$, namely when $m = 1$, the above splitting criteria (Theorems 2.1–2.3) become much simpler (Corollaries 2.5–2.7).

Corollary 2.5. *Let $\mathcal{E}$ be a rank r vector bundle on a rational normal scroll X. Then the following conditions are equivalent:*

(i) for any integer t we have the vanishing

(a) $h^n(\mathcal{E}\left[\begin{smallmatrix} t \\ c-1 \end{smallmatrix}\right]) = 0$,

(b) $h^{i+j}(\mathcal{E}\left[\begin{smallmatrix} t \\ i-j \end{smallmatrix}\right]) = 0$ *for* $j = 0, 1$ *and* $0 \le i < n$ *but* $(i,j) \neq (0,0)$;

(ii) there are r *integers* $t_1, \ldots, t_r$ *such that* $\mathcal{E} \cong \bigoplus_{i=1}^{r} \mathcal{O}_X(t_i\mathbf{h})$.

Corollary 2.6. *Let* $\mathcal{E}$ *be a vector bundle on a rational normal scroll* X. *Then* E *is a direct sum of line bundles* $\mathcal{O}_X$, $\mathcal{O}_X(\mathbf{f})$ *and* $\mathcal{O}_X(\mathbf{h}-\mathbf{f})$ *with some twist* $t\mathbf{h}$ *if and only if, for any integer* t,

(b) $h^{i+j}(\mathcal{E}\left[\begin{smallmatrix} t \\ i-j \end{smallmatrix}\right]) = 0$ *for* $0 \le i < n$ *and* $j = 0, 1$ *but* $(i,j) \neq (0,0), (0,1)$,

(d) $h^{|I|}(\mathcal{E}\left[\begin{smallmatrix} t \\ a_I-1 \end{smallmatrix}\right]) = h^{|I|}(\mathcal{E}^\vee\left[\begin{smallmatrix} t \\ a_I-1 \end{smallmatrix}\right]) = 0$ *if* $1 \le |I| \le n$.

Corollary 2.7. *Let* $\mathcal{E}$ *be an indecomposible vector bundle on a rational normal scroll* X *with* $\mathrm{Reg}(\mathcal{E}) = 0$. *Suppose that the following vanishing occurs.*

(b) $h^{i+1}\big(\mathcal{E}\left[\begin{smallmatrix} -(i+1) \\ i-1 \end{smallmatrix}\right]\big) = h^{i+1}\big(\mathcal{E}\left[\begin{smallmatrix} -(i+1) \\ i \end{smallmatrix}\right]\big) = 0$ *for* $0 \le i < n$,

(d) $h^{|I|}\big(\mathcal{E}\left[\begin{smallmatrix} -|I| \\ a_I-1 \end{smallmatrix}\right]\big) = h^{|I|}\big(\mathcal{E}^\vee\left[\begin{smallmatrix} -|I|+1 \\ a_I-1 \end{smallmatrix}\right]\big) = 0$ *if* $1 \le |I| \le n$,

(e) $h^k\big(\bigoplus_{|I|=1-k+i} \mathcal{E}\left[\begin{smallmatrix} -k \\ k+1-a_I \end{smallmatrix}\right]\big) = 0$ *for* $0 < k \le i < n$ *and*

$h^k\big(\bigoplus_{|I|=k+1} \mathcal{E}^\vee\left[\begin{smallmatrix} -(k-1) \\ a_I-i-1 \end{smallmatrix}\right]\big) = 0$ *for* $0 < k \le n-i < n$.

Then $\mathcal{E} \cong \mathcal{O}_X$, *or* $\mathcal{E} \cong \mathcal{O}_X(\mathbf{f})$, *or* $\mathcal{E} \cong \mathcal{O}_X(\mathbf{h}-\mathbf{f})$, *or* $\mathcal{E} \cong \Omega^i_{X|\mathbb{P}^m}\left[\begin{smallmatrix} i+1 \\ -(i+1) \end{smallmatrix}\right]$ *with* $1 < i < n$.

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