Saturated and primitive smooth compactifications of ball quotients

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Let $X_i = (\mathbb{B}/\Gamma)'$, $1 \leq i \leq 2$ be smooth toroidal compactifications of quotients $\mathbb{B}/\Gamma_i$ of the complex 2-ball

$$\mathbb{B} = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 < 1\} = \mathrm{PSU}_{2,1}/\mathrm{PS}(U_2 \times U_1)$$

by lattices $\Gamma < PU(2,1)$, $D^{(i)} := X_i \setminus (\mathbb{B}/\Gamma_i)$ be the toroidal compactifying divisors and $\rho_i : X_i \to Y_i$ be compositions of blow downs with exceptional divisors $E(\rho_i)$ onto minimal surfaces Y_i. The present note establishes a bijective correspondence between the unramified coverings $f : X_2 \to X_1$ of degree d, which restrict to unramified coverings $f : D^{(2)} \to D^{(1)}$, $f : E(\rho_2) \to E(\rho_1)$ of degree d and the unramified coverings $\varphi : Y_2 \to Y_1$ of degree d of the corresponding minimal models, which restrict to unramified coverings $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})$, $\varphi : \rho_2(E(\rho_2) \to \rho_1(E(\rho_1))$ of degree d. The aforementioned covering relations among X_i define an artinian partial order $\succcurlyeq$ on the set $\mathcal{S}$ of the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$. The maximal elements with respect to $\succcurlyeq$ are called saturated and the minimal elements with respect to $\succcurlyeq$ are said to be primitive. Our considerations reduce the study of $X \in \mathcal{S}$ to the study of the primitive $X \in \mathcal{S}$. For an arbitrary totally ordered subset $\{X_\alpha\}_{\alpha \in A} \subset \mathcal{S}$, all the minimal models Y_α of X_α have one and a same universal cover and one and a same Kodaira dimension. We discuss the saturated and the primitive $X \in \mathcal{S}$ of non-positive Kodaira dimension. The covering relations among the smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ are studies in Uludag's [Uludag], Stover's [Stover], Di Cerbo and Stover's [DiCerboStover1] and other articles.

Here is a synopsis of the article. Let $\rho_1 : X_1 \to Y_1$ be a composition of blow downs of a smooth projective surface X_1 onto a smooth projective surface Y_1. The first section establishes a bijective correspondence between the unramified coverings $f : X_2 \to X_1$ of degree d and the unramified covering $\varphi : Y_2 \to Y_1$ of degree d through fibered product commutative diagrams (4) with appropriate compositions of blow downs $\rho_2 : X_2 \to Y_2$. In order to induce $\varphi : Y_2 \to Y_1$ by $f : X_2 \to X_1$, one observes that $\varphi\rho_2$ is the Stein factorization of the proper holomorphic map $\rho_1 f : X_2 \to Y_1$. If $D^{(i)} \subset X_i$ are (possibly reducible) divisors, which do not contain irreducible components of the exceptional divisors $E(\rho_i)$ of $\rho_i : X_i \to Y_i$, then f is shown to restrict to an unramified covering $f : D^{(2)} \to D^{(1)}$ of degree d if and only if φ restricts to an unramified covering $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)}$ of degree d. In particular, if $\rho_1 : X_1 = (\mathbb{B}/\Gamma_1)' \to Y_1$ is a composition of blow downs of a smooth

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toroidal compactification $X_1 = (\mathbb{B}/\Gamma_1)'$ onto a minimal surface Y_1 then the unramified coverings $f : X_2 = (\mathbb{B}/\Gamma_2)' \to (\mathbb{B}/\Gamma_1)' = X_1$ of degree d, which restrict to unramified coverings $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_2$ of degree d are in a bijective correspondence with the unramified coverings $\varphi : Y_2 \to Y_1$ by appropriate minimal models Y_2 of X_2. Under the aforementioned correspondence, f restricts to an unramified covering $f : D^{(2)} = X_2 \backslash (\mathbb{B}/\Gamma_2) \to X_1 \backslash (\mathbb{B}/\Gamma_1) = D^{(1)}$ of degree d of the corresponding compactifying divisors if and only if φ restricts to an unramified covering $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})$ of degree d. In such a way, the presence of a finite unramified cover $X_2 = (\mathbb{B}/\Gamma_2)'$ of $X_1 = (\mathbb{B}/\Gamma_1)'$ can be detected by the means of an arbitrary minimal model Y_1 of X_1 and its finite unramified covers Y_2.

Let X_2, Y_2 be smooth projective surfaces and $\rho_2 = \beta_1 \dots \beta_r : X_2 \to Y_2$ be a commposition of blow downs with exceptional divisors $E(\beta i) \subset \beta_{i+1} \dots \beta_r(X_2)$. The second section introduces compatibility conditions on the finite unramified coverings $f : X_2 \to f(X_2)$ or $\varphi : Y_2 \to \varphi(Y_2)$ with ρ_2 in such a way that the existence of $f : X_2 \to f(X_2)$ to be equivalent to the existence of $\varphi : Y_2 \to \varphi(Y_2)$. In particular, for a smooth toroidal compactification $X_2 = (\mathbb{B}/\Gamma_2)'$ with toroidal compactifying divisor $D^{(2)} := X_2 \setminus (\mathbb{B}/\Gamma_2)$ and a composition of blow downs $\rho_2 = \beta_1 \dots \beta_r : X_2 \to Y_2$ onto a minimal surface Y_2, there exists an unramified covering $f : X_2 \to f(X_2) =: X_1$, which is compatible with ρ_2 and restricts to an unramified covering $f : \mathbb{B}/\Gamma_2 \to f(\mathbb{B}/\Gamma_2)$ of degree d if and only if there is an unramified covering $\varphi : Y_2 \to \varphi(Y_2) =: Y_1$ of a minimal model Y_1 of X_1, which is compatible with ρ_2 and restricts to an unramified covering $\varphi : \rho_2(D^{(2)}) \to \varphi\rho_2(D^{(2)})$ of degree d. Moreover, $X_1 = (\mathbb{B}/\Gamma_1)'$ is a smooth toroidal compactification and if $\rho_1 : X_1 \to Y_1$ is a composition of blow downs onto Y_1 then $\varphi\rho_2(D^{(2)}) = \rho(D^{(1)})$ for the compactifying divisor $D^{(1)} := X_1 \setminus (\mathbb{B}/\Gamma_1)$ of $\mathbb{B}/\Gamma_1$. A smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ is primitive if there is no unramified covering $f : X \to f(X)$ of degree d, which restricts to an unramified covering $f : \mathbb{B}/\Gamma \to f(\mathbb{B}/\Gamma)$ of degree d and is compatible with some composition of blow downs $\rho : X \to Y$ onto a minimal surface Y. Due to the established duality between the finite unramified coverings $f : X \to f(X)$ and $\varphi : Y \to \varphi(Y)$ of one and a same degree, the primitiveness of $X = (\mathbb{B}/\Gamma)'$ can be detected by the properties of Y.

The last, third section studies the finite unramified Galois coverings $f : X = (\mathbb{B}/\Gamma)' \to f(X)$ of smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$, which admit a blow down $\beta : X \to Y$ of $n \in \mathbb{N}$ smooth irreducible rational (-1)-curves onto a minimal surface Y. Di Cerbo and Stover have shown in [DiCerboStover2] that the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ with abelian or bi-elliptic minimal model Y have the aforementioned property. We establish that for such $X = (\mathbb{B}/\Gamma)'$ the compatibility of the unramified coverings $\varphi : Y \to \varphi(Y)$ of degree d, restricting to unramified coverings $\varphi : \beta(D) \to \varphi\beta(D)$ of degree d with $\beta : X \to Y$ is automatic, as far as $\beta(E(\beta)) = \beta(D)^{\rm sing}$ coincides with the singular locus of $\beta(D)$. The relative automorphism group $\mathrm{Aut}(Y, \beta(D)) = \mathrm{Aut}(Y, \beta(D), \beta(D)^{\rm sing})$ admits an isomorphism $\Phi : \mathrm{Aut}(Y, \beta(D)) \to \mathrm{Aut}(X, D)$ onto the relative automorphism group $\mathrm{Aut}(X, D) = \mathrm{Aut}(X, D, E(\beta))$. Let $N(\pi_1(Y))$ be the normalizer of the fundamental group $\pi_1(Y)$ of Y in the biholomorphism group $\mathrm{Aut}(\widetilde{Y})$ of the universal cover $\widetilde{Y}$ of Y. It is well known that the biholomorphism group $\mathrm{Aut}(Y)$ of Y is the quotient $\mathrm{Aut}(Y) = N(\pi_1(Y))/\pi_1(Y)$. If an unramified covering $\varphi : Y \to \varphi(Y)$ of degree d restricts to an unramified covering $\varphi : \beta(D) \to \varphi\beta(D)$ of degree d then any $g_o \in N(\pi_1(Y)) \cap \pi_1(\varphi(Y))$ is shown to induce a biholomorphism $\overline{g_o} := g_o\pi_1(Y) : \beta(D) \to \beta(D)$ and, therefore, a factorization $f = f_o\zeta$ of the associated unramified covering $f : X = (\mathbb{B}/\Gamma)' \to X_0 =$

$(\mathbb{B}/\Gamma_0)'$ of $\varphi : Y \to \varphi(Y)$ through the unramified Galois covering $\zeta : X \to X/\langle \Phi(g_o)\rangle$ and an unramified covering $f_o : X/\langle \Phi(g_o)\rangle \to X_0 = (\mathbb{B}/\Gamma_0)'$. In particular, for a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ with abelian minimal model Y, we establish that any unramified covering $f : X = (\mathbb{B}/\Gamma)' \to X_0 = (\mathbb{B}/\Gamma_0)'$ of degree d, which restricts to an unramified covering $f : \mathbb{B}/\Gamma \to \mathbb{B}/\Gamma_0$ of degree d, factors through a Galois covering $X \to X/\langle g\rangle$, $g \in \mathrm{Aut}(X, D)$, which restricts to a Galois covering $\mathbb{B}/\Gamma \to (\mathbb{B}/\Gamma)/\langle g\rangle$. The third section discusses also the saturation and the primitiveness of the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ with Kodaira dimension $\kappa(X) = -\infty$, as well as the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ with $K3$ or Enriques minimal model.

1 Unramified pull back of a smooth compactification

Lemma 1. *Let M be a complex manifold and N be a complex analytic subvariety of M or an open subset of M.*

(i) If $f : M \to f(M)$ is an unramified covering of degree d then $f : N \to f(N)$ is an unramified covering of degree d exactly when $f : M \setminus N \to f(M) \setminus f(N)$ is an unramified covering of degree d.

(ii) Let us suppose that $f : M \to f(M)$ is a holomorphic map onto a complex manifold, $f(N) \cap f(M \setminus N) = \emptyset$ and $f : N \to f(N)$, $f : M \setminus N \to f(M \setminus N)$ are unramified coverings of degree d. Then $f : M \to f(M)$ is an unramified covering of degree d.

Proof. (i) Let $X := N$ or $X := M \setminus X$. Then $f : X \to f(X)$ is an unramified covering of degree $\deg(f|_X) = \deg(f|_M) = d$ exactly when $f^{-1}(f(X)) = X$. If so, then the intersection $f^{-1}(f(M \setminus X)) \cap X = \emptyset$ is empty, whereas $f^{-1}(f(M \setminus X)) = M \setminus X$, the union $f(M) = f(X) \coprod f(M \setminus X)$ is disjoint and $f : M \setminus X \to f(M \setminus X) = f(M) \setminus f(X)$ is an unramified covering of degree d.

(ii) The union $f(M) = f(N) \coprod f(M \setminus N)$ is disjoint, so that $f^{-1}(f(M \setminus N)) = M \setminus N$, $f^{-1}(f(N)) = N$ and $f : M \to f(M)$ is an unramified covering of degree d.

□

Lemma 2. *Let $f : X \to X'$ be an unramified covering of degree d of smooth projective surfaces.*

(i) Suppose that $D = \coprod_{j=1}^{k} D_j$ is a divisor on X with disjoint smooth irreducible components D_j and f restricts to an unramified covering $f : D \to f(D)$ of degree d. Then $f(D) = \cup_{j=1}^{k} f(D_j)$ has smooth irreducible components $f(D_j)$, f restricts to unramified coverings $f : D_j \to f(D_j)$ for all $1 \leq j \leq k$ and $f(D_i) \cap f(D_j) = \emptyset$ for $f(D_i) \not\equiv f(D_j)$.

In particular, D_j are smooth elliptic curves of and only if $f(D_j)$ are smooth elliptic curves.

(ii) If C' is a smooth irreducible rational curve on X' then the complete preimage $f^{-1}(C') = \coprod_{i=1}^{d} C_i$ consists of d disjoint smooth irreducible rational curves C_i and f restricts to isomorphisms $f : C_i \to C'$ for all $1 \leq i \leq d$.

Proof. (i) The unramified covering $f : D \to f(D)$ is a local biholomorphism, so that $f(D)$ is a smooth divisor on X'. Thus, all the irreducible components $f(D_j)$ of $f(D)$ are smooth curves and $f(D_i) \cap f(D_j) \neq \emptyset$ requires $f(D_i) \equiv f(D_j)$. For any $1 \leq i \leq k$ let $J(i)$ be the set of those $1 \leq j \leq k$, for which $f(D_j) \equiv f(D_i)$. Then there exists a subset $I \subseteq \{1, \dots, k\}$ with $\coprod\limits_{i \in I} J(i) = \{1, \dots, k\}$ and $f(D) = \coprod\limits_{i \in I} f(D_i)$. By the very definition of $J(i)$, there holds the inclusion $\coprod\limits_{jijJ(i)} D_j \subseteq f^{-1}(f(D_i))$. Since f restricts to an unramified covering $f : D \to f(D)$ of degree d, any $p \in f^{-1}(f(D_i))$ belongs to D_s for some $1 \leq s \leq k$. Then $f(p) \in f(D_i)$ specified that $s \in J(i)$, whereas $f^{-1}(f(D_i)) \subseteq \coprod\limits_{j \in J(i)} D_j$ and $f^{-1}(f(D_i)) = \coprod\limits_{j \in J(i)} D_j$. Thus, for any $i \in I$ the morphism f restricts to an unramified covering $f : \coprod\limits_{j \in J(i)} D_j \to f(D_i)$ of degree d. By definition, any $f(p) \in f(D_i)$ with $p \in \coprod\limits_{j \in J(i)} D_j$ has a trivializing neighborhood U on $f(D_i)$, whose pull back $f^{-1}(U) = \coprod\limits_{q \in f^{-1}(p)} V_q$ is a disjoint union of neighborhoods V_q of $q \in f^{-1}(p)$ on $\coprod\limits_{j \in J(i)} D_j$ with biholomorphic restrictions $f : V_q \to U$. For a sufficiently small U one can assume that $V_q \subset D_j$ for $q \in D_j$. That is why f restricts to unramified coverings $f : D_j \to f(D_j) = f(D_i)$. In particular, D_j are smooth elliptic curves exactly when $f(D_j)$ are smooth elliptic curves.

(ii) Let $f^{-1}(C') = \sum\limits_{i=1}^{k} C_i$ be a union of k irreducible corves C_i, $d_i := \deg [f|_{C_i} : C_i \to C']$ and $\mathrm{Br}(f|_{C_i}) := \{q \in C' \mid |f^{-1}(q) \cap C_i| < d_i\}$ be the branch locus of $f|_{C_i}$ for $1 \leq i \leq k$. Any $\mathrm{Br}(f|_{C_i})$ is a finite set, as well as the intersection $\cup_{1 \leq i < j \leq k} C_i \cap C_j$ of different irreducible components, so that

$$\Sigma := \left[\cup_{i=1}^{k} \mathrm{Br}(f|_{C_i})\right] \cup [\cup_{1 \leq i < j \leq k} f(C_i \cap C_j)]$$

is a finite subset of C'. For any $q \in C' \setminus \Sigma$ one has $f^{-1}(q) = \coprod\limits_{i=1}^{k} f^{-1}(q) \cap C_i$, whereas

$$d = |f^{-1}(q)| = \sum_{i=1}^{k} |f^{-1}(q) \cap C_i| = \sum_{i=1}^{k} d_i.$$

If $q_j \in \mathrm{Br}(f|_{C_j})$ then $f^{-1}(q_j) = \cup_{i=1}^{k} f^{-1}(q_j) \cap C_i$ with $|f^{-1}(q_j) \cap C_j| < d_j$, so that

$$d = |f^{-1}(q_j)| \leq \sum_{i=1}^{k} |f^{-1}(q_j) \cap C_i| < \sum_{i=1}^{k} d_i = d.$$

This is an absurd, justifying $\mathrm{Br}(f|_{C_j}) = \emptyset$ for all $1 \leq j \leq k$. Similarly, for any $p \in C_i \cap C_j$ there holds

$$d = |f^{-1}(p)| < \sum_{i=1}^{k} |f^{-1}(p) \cap C_i| = \sum_{i=1}^{k} d_i = d.$$

The contradiction shows that the irreducible components C_i of $f^{-1}(C')$ are disjoint. The unramified coverings $f|_{C_i} : C_i \to C'$ of the smooth irreducible rational curve C' are of degree $d_i = 1$, due to $\pi_1(C') = \{1\}$. Therefore $d = \sum\limits_{i=1}^{k} d_i = k$ and $f^{-1}(C') = \coprod\limits_{i=1}^{d} C_i$ consists of d

disjoint smooth irreducible rational curves with biholomorphic restrictions $f|_{C_i} : C_i \to C'$ for all $1 \leq i \leq d$.

$\square$

A (-1)-curve L_i on a smooth projective surface Y is a smooth irreducible rational curve with self-intersection $L_i^2 = -1$. Throughout, we say that a smooth projective surface Y is minimal if it does not contain a (-1)-curve. This is slightly different from the contemporary viewpoint of the Minimal Model Program, which considers a smooth projective surface Y to be minimal if its canonical divisor K_Y is nef (i.e., $K_Y.C \geq 0$ for all effective curves $C \subset Y$). The numerical effectiveness of K_Y excludes the existence of (-1)-curves on Y. If Y is of Kodaira dimension $\kappa(Y) = -\infty$ then K_Y is not nef, regardless of the presence of (-1)-curves on Y. That is the reason for exploiting the older, out of date notion of minimality of a smooth projective surface, which requires the non-existence of (-1)-curves on Y. By a theorem of Castelnuovo (Theorem V.5.7 [Ha]), for any smooth irreducible projective surface X there is a birational morphism $\rho : X \to Y$ onto a minimal smooth projective surface Y, which is a composition of blow downs of (-1)-curves. If X is of Kodaira dimension $\kappa(X) \geq 0$ then the minimal model Y of X is unique (up to an isomorphism). This is no more true when X is birational to a rational or a ruled surface.

Lemma 3. *(i) Let* $\mathrm{Bl} : X_1 \to Y_1$ *be a blow down of a* (-1)*-curve* $L_1 \subset X_1$ *and* $\varphi : Y_2 \to Y_1$ *be an unramified covering of degree* d*. Then the fibered product commutative diagram*

$$\begin{array}{ccc} X_2 := X_1 \times_{Y_1} Y_2 & \xrightarrow{\beta} & Y_2 \\ \Big\downarrow f & & \Big\downarrow \varphi \\ X_1 & \xrightarrow{\mathrm{Bl}} & Y_1 \end{array} \qquad (1)$$

consists of an unramified covering $f : X_2 \to X_1$ *of degree* d *and the blow down* $\beta : X_2 \to Y_2$ *of the disjoint union* $f^{-1}(L_1) = \coprod_{j=1}^{d} L_{1,j}$ *of the* (-1)*-curves* $L_{1,j}$*.*

(ii) Let $\rho_1 : \mathrm{Bl}_1 \ldots \mathrm{Bl}_{r-1}\mathrm{Bl}_r : T_r := X_1 \to Y_1 =: T_0$ *be a composition of blow downs* $\mathrm{Bl}_i : T_i \to T_{i-1}$ *of* (-1)*-curves* $L_i \subset T_i$ *and* $\varphi : Y_2 \to Y_1$ *be an unramified covering of degree* d*. Then the fibered product commutative diagrams*

$$\begin{array}{ccc} S_i := T_i \times_{T_{i-1}} S_{i-1} & \xrightarrow{\beta_i} & S_{i-1} \\ \Big\downarrow \varphi_i & & \Big\downarrow \varphi_{i-1} \\ T_i & \xrightarrow{\mathrm{Bl}_i} & T_{i-1} \end{array} \qquad (2)$$

fit into a commutative diagram

$$\begin{array}{ccccccc} S_r & & \ldots S_i := T_i \times_{T_{i-1}} S_{i-1} & \xrightarrow{\beta_i} & S_{i-1} & & \ldots S_0 := Y_2 \\ \Big\downarrow f & & \Big\downarrow \varphi_i & & \Big\downarrow \varphi_{i-1} & & \Big\downarrow \varphi = \varphi_0 \\ T_r := X & & \ldots T_i & \xrightarrow{\mathrm{Bl}_i} & T_{i-1} & & \ldots T_0 := Y_1 \end{array} \qquad (3)$$

and induce a fibered product commutative diagram

$$\begin{array}{ccc} X_2 = X_1 \times_{Y_1} Y_2 & \xrightarrow{\rho_2} & Y_2 \\ {\scriptstyle f} \downarrow & & \downarrow {\scriptstyle \varphi} \\ X_1 & \xrightarrow{\rho_1} & Y_1 \end{array} \tag{4}$$

with an unramified covering $f : X_2 \to X_1$ *of degree* d *and a composition* $\rho_2 = \beta_1 \dots \beta_{r-1}\beta_r :$ $X_2 \to Y_2$ *of blow downs of* $\varphi_i^{-1}(L_i) = \coprod_{j=1}^{d} L_{i,j}$ *for all* $1 \leq i \leq r$.

Proof. (i) By the very definition of a blow down $\mathrm{Bl} : X_1 \to Y_1$ of L_1 to $\mathrm{Bl}(L_1) = q_1 \in Y_1$, one has $X_1 \setminus L_1 = Y_1 \setminus \{q_1\}$. Then

$$X_2 := X_1 \times_{Y_1} Y_2 = [(X_1 \setminus L_1) \times_{Y_1} Y_2] \coprod [L_1 \times_{Y_1} Y_2]$$

decomposes into the disjoint union of

$$(X_1 \setminus L_1) \times_{Y_1} Y_2 = \{(x_1, y_2) \,|\, x_1 = \mathrm{Bl}(x_1) = \varphi(y_2)\} \simeq Y_2 \setminus \varphi^{-1}(q_1) \quad \text{and}$$

$$L_1 \times_{Y_1} Y_2 = \{(x_1, y_2) \,|\, q_1 = \mathrm{Bl}(x_1) = \varphi(y_2)\} = L_1 \times \varphi^{-1}(q_1).$$

If $\varphi^{-1}(q_1) = \{p_{1,j} \,|\, 1 \leq j \leq d\}$ then X_2 is the blow up of Y_2 at $\{p_{1,j} \,|\, 1 \leq j \leq d\}$. Due to $\mathrm{Bl}f = \varphi\beta$, the exceptional divisor of β is $\beta^{-1}(\{p_{1,j} \,|\, 1 \leq j \leq d\}) = \beta^{-1}\varphi^{-1}(q_1) = (\varphi\beta)^{-1}(q_1) = (\mathrm{Bl}f)^{-1}(q_1) = f^{-1}\mathrm{Bl}^{-1}(q_1) = f^{-1}(L_1) = \coprod_{j=1}^{d} L_{1,j}$. According to Corollary 17.7.3 (i) from Grothendieck's [Groth4], $f : X_2 \to X_1$ is an unramified covering, since $\varphi : Y_2 \to Y_1$ is an unramified covering.

(ii) By an increasing induction on $1 \leq i \leq r$, one applies (i) to the fibered product commutative diagrams (2) and justifies (ii).

□

Lemma 4. *(i) In the notations from Lemma 3 (i) and the fibered product commutative diagram (1), let* $D^{(2)}$ *be a (possibly reducible) divisor on* X_2*, which does not contain an irreducible component of the exceptional divisor of* β *and* $D^{(1)}$ *be a (possibly reducible) divisor on* X_1*, which does not contain the exceptional divisor* L_1 *of* Bl. *Then the restriction* $f :$ $D^{(2)} \to D^{(1)}$ *is an unramified covering of degree* $d = \deg[f : X_2 \to X_1]$ *if and only if* $\varphi : \beta(D^{(2)}) \to \mathrm{Bl}(D^{(1)})$ *is an unramified covering of degree* d.

(ii) In the notations from Lemma 3 (ii) and the fibered product commutative diagram (4), let $D^{(2)}$ *be a (possibly reducible) divisor on* X_2*, which does not contain an irreducible component of the exceptional divisor of* ρ_2 *and* $D^{(1)}$ *be a (possibly reducible) divisor on* X_1*, which does not contain an irreducible component of the exceptional divisor of* ρ_1*. Then the restriction* $f : D^{(2)} \to D^{(1)}$ *is an unramified covering of degree* d *if and only if the restriction* $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})$ *is an unramified covering of degree* d.

Proof. (i) If $f : D^{(2)} \to D^{(1)}$ is an unramified covering of degree d then $f^{-1}(D^{(1)} \cap L_1) = f^{-1}(D^{(1)}) \cap f^{-1}(L_1) = D^{(2)} \cap f^{-1}(L_1)$ and the restriction $f : D^{(1)} \cap f^{-1}(L_1) \to D^{(1)} \cap L_1$ is an unramified covering of degree d. After denoting $f^{-1}(L_1) = \coprod_{j=1}^{d} L_{1,j}$, $\beta(L_{1,j}) = p_{1,j}$ and $\mathrm{Bl}(L_1) = q_1$, one applies Lemma 1 (i), in order to conclude that

$$\varphi \equiv f : \beta(D^{(2)}) \setminus \{p_{1,j} \,|\, 1 \leq j \leq d\} \equiv D^{(2)} \setminus f^{-1}(L_1) \longrightarrow D^{(1)} \setminus L_1 \equiv \mathrm{Bl}(D^{(1)}) \setminus \{q_1\}$$

is an unramified covering of degree d. Now, φ restricts to $\varphi : \{p_{1,j} \,|\, 1 \leq j \leq d\} \to \{q_1\}$, so that

$$\varphi : \beta(D^{(2)}) = \beta(D^{(2)}) \setminus \{p_{1,j} \,|\, 1 \leq j \leq d\} \coprod \{p_{1,j} \,|\, 1 \leq j \leq d\} \longrightarrow$$
$$\longrightarrow \left[\mathrm{Bl}(D^{(1)}) \setminus \{q_1\}\right] \coprod \{q_1\} = \mathrm{Bl}(D^{(1)})$$

is an unramified covering of degree d by Lemma 1 (ii).

Conversely, assume that $\varphi : \beta(D^{(2)}) \to \mathrm{Bl}(D^{(1)})$ is an unramified covering of degree d. Choose a sufficiently small neighborhood V of $q_1 = \mathrm{Bl}(L_1)$ on Y_1, such that $\varphi^{-1}(V) = \coprod_{j=1}^{d} U_j$ is a dsijoint union of neighborhoods U_j of $p_{1,j}$, $1 \leq j \leq d$ on Y_2 with biholomorphic restrictions $\varphi : U_j \to V$ of φ. Bearing in mind that $\mathrm{Bl}_1 : X_1 \to Y_1$ is the blow up of Y_1 at q_1, one decomposes

$$\mathrm{Bl}(D^{(1)}) = \left[\mathrm{Bl}(D^{(1)}) \setminus V\right] \coprod \left[\mathrm{Bl}(D^{(1)}) \cap V\right] \quad \text{and}$$
$$D^{(1)} = \left[\mathrm{Bl}(D^{(1)}) \setminus V\right] \coprod \mathrm{Bl}^{-1}(\mathrm{Bl}(D^{(1)}) \cap V).$$

Similarly, $\beta : X_2 \to Y_2$ is the blow up of Y_2 at $\varphi^{-1}(q_1) = \{p_{1,j} \,|\, 1 \leq j \leq d\}$, so that there are decompositions

$$\beta(D^{(2)}) = \left[\beta(D^{(2)}) \setminus \varphi^{-1}(V)\right] \coprod \left[\beta(D^{(2)}) \cap \varphi^{-1}(V)\right] \quad \text{and}$$
$$D^{(2)} = \left[\beta(D^{(2)}) \setminus \varphi^{-1}(V)\right] \coprod \beta^{-1}(\beta(D^{(2)}) \cap \varphi^{-1}(V)).$$

According to $\varphi^{-1}(\mathrm{Bl}(D^{(1)}) \cap V) = \varphi^{-1}(\mathrm{Bl}(D^{(1)})) \cap \varphi^{-1}(V) = \beta(D^{(2)}) \cap \varphi^{-1}(V)$, the restriction $\varphi : \beta(D^{(2)}) \cap \varphi^{-1}(V) \to \mathrm{Bl}(D^{(1)}) \cap V$ is an unramified covering of degree d. Now, Lemma 1 (ii) applies to provide that

$$f \equiv \varphi : \beta(D^{(2)}) \setminus \varphi^{-1}(V) \longrightarrow \mathrm{Bl}(D^{(1)}) \setminus V$$

is an unramified covering of degree d. According to Lemma 1 (ii), it sufficed to show that

$$f : \beta^{-1}(\beta(D^{(2)}) \cap \varphi^{-1}(V)) \longrightarrow \mathrm{Bl}^{-1}(\mathrm{Bl}(D^{(1)}) \cap V)$$

is an unramified covering of degree d, in order to conclude that $f : D^{(2)} \to D^{(1)}$ is an unramified covering of degree d. To this end, note that

$$\varphi^{-1}(\mathrm{Bl}(D^{(1)}) \cap V) = \beta(D^{(2)}) \cap \varphi^{-1}(V) = \beta(D^{(2)}) \cap \left(\coprod_{j=1}^{d} U_j\right) = \coprod_{j=1}^{d} \left[\beta(D^{(2)}) \cap U_j\right],$$

so that

$$\varphi : \coprod_{j=1}^{d} \left[\beta(D^{(2)}) \cap U_j\right] \longrightarrow \mathrm{Bl}(D^{(1)}) \cap V$$

is an unramified covering of degree d. Therefore, the biholomorphisms $\varphi : U_j \to V$ restrict to biholomorphisms $\varphi : \beta(D^{(2)}) \cap U_j \to \mathrm{Bl}(D^{(1)}) \cap V$. According to $\varphi(p_{1,j}) = q_1$, there arise biholomorphisms

$$\varphi : (\beta(D^{(2)}) \cap U_j) \setminus \{p_{1,j}\} \longrightarrow (\mathrm{Bl}(D^{(1)}) \cap V) \setminus \{q_1\}.$$

By the very definition of a blow up at a point, these induce biholomorphisms

$$f : \left[(\beta(D^{(2)}) \cap U_j) \setminus \{p_{1,j}\}\right] \coprod L_{1,j} \longrightarrow \left[(\mathrm{Bl}(D^{(1)}) \cap V) \setminus \{q_1\}\right] \coprod L_1$$

for all $1 \leq j \leq d$. Bearing in mind that

$$\coprod_{j=1}^{d} \left\{ \left[(\beta(D^{(2)}) \cap U_j) \setminus \{p_{1,j}\}\right] \coprod L_{1,j} \right\} = \beta^{-1}(\beta(D^{(2)}) \cap \varphi^{-1}(V)),$$

one concludes that φ induces an unramified covering

$$f : \beta^{-1}(\beta(D^{(2)}) \cap \varphi^{-1}(V)) \longrightarrow \mathrm{Bl}^{-1}(\mathrm{Bl}(D^{(1)}) \cap V)$$

of degree d.

(ii) Along the commutative diagram (3), if $f : D^{(2)} \to D^{(1)}$ is an unramified covering of degree d then by a decreasing induction on $r \geq i \geq 1$ and making use of (i), one observes that $\varphi_i : \beta_{i+1} \dots \beta_r(D^{(2)}) \to \mathrm{Bl}_{i+1} \dots \mathrm{Bl}_r(D^{(1)})$ is an unramified covering of degree d, whereas $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})$ is an unramified covering of degree d. Conversely, suppose that $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})$ is an unramified covering of degree d. Then by an increasing induction on $1 \leq i \leq r$ and making use of (i), one concludes that

$$\varphi_i : \beta_{i+1} \dots \beta_r(D^{(2)}) \to \mathrm{Bl}_{i+1} \dots \mathrm{Bl}_r(D^{(1)})$$

is an unramified covering of degree d. As a result, $f : D^{(2)} \to D^{(1)}$ is an unramified covering of degree d.

□

Corollary 5. *Let $X_1 = (\mathbb{B}/\Gamma_1)$ be a smooth toroidal compactification, $\rho_1 : X_1 \to Y_1$ be a composition of blow downs onto a minimal surface Y_1, $\varphi : Y_2 \to Y_1$ be an unramified covering of degree d and (4) be the defining commutative diagram of the fibered product $X_2 = X_1 \times_{Y_1} Y_2$. Then:*

(i) there is a subgroup Γ_2 of Γ_1 of index $[\Gamma_1 : \Gamma_2] = d$, such that $X_2 = (\mathbb{B}/\Gamma_2)'$ is the toroidal compactification of $\mathbb{B}/\Gamma_2$;

(ii) $f : X_2 \to X_1$ restricts to unramified coverings $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_1$, respectively, $f : D^{(2)} := X_2 \setminus (\mathbb{B}/\Gamma_2) \to X_1 \setminus (\mathbb{B}/\Gamma_1) =: D^{(1)}$ of degree d;

(iii) the composition $\rho_2 : X_2 \to Y_2$ of blow downs maps onto a minimal surface Y_2;

(iv) φ restricts to an unramified covering $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})$ of degree d.

Proof. By Lemma 3 (ii), the fibered product diagram (4) consists of an unramified covering $f: X_2 \to X_1$ of degree d and a composition $\rho_2 : X_2 \to Y_2$ of blow downs. The surface Y_2 is minimal. Otherwise any (-1)-curve L'_i on Y_2 maps isomorphically onto a (-1)-curve $\varphi(L'_i) \subset Y_1$, according to Lemma 2 (ii). That contradicts the minimality of Y_1 and shows the minimality of Y_2.

The unramified covering $f : X_2 \to X_1 = (\mathbb{B}/\Gamma_1)'$ of degree d restricts to an unramified covering $f : f^{-1}(\mathbb{B}/\Gamma_1) \to \mathbb{B}/\Gamma_1$ of degree d. The smoothness of $\mathbb{B}/\Gamma_1$ excludes the existence of isolated branch points of the Γ_1-Galois covering $\zeta_1 : \mathbb{B} \to \mathbb{B}/\Gamma_1$. However, ζ_1 can ramify along divisors and $\mathbb{B}$ is not the usual universal cover of the complex manifold $\mathbb{B}/\Gamma_1$. Nevertheless, $\mathbb{B}$ is the orbifold universal cover of $\mathbb{B}/\Gamma_1$ and the orbifold universal covering map $\zeta_1 : \mathbb{B} \to \mathbb{B}/\Gamma_1$ factors through a (possibly ramified) covering $\zeta_2 : \mathbb{B} \to f^{-1}(\mathbb{B}/\Gamma_1)$ and the covering $f : f^{-1}(\mathbb{B}/\Gamma_1) \to \mathbb{B}/\Gamma_1$, i.e., $\zeta_1 = f\zeta_2$. Since $\pi_1^{\mathrm{orb}}(\mathbb{B}) = \{1\}$ is a normal subgroup of $\Gamma_2 := \pi_1^{\mathrm{orb}}(f^{-1}(\mathbb{B}/\Gamma))$, the covering ζ_2 is Galois and its Galois group Γ_2 is a subgroup of $\Gamma_1 = \pi_1^{\mathrm{orb}}(\mathbb{B}/\Gamma_1)$ of index $[\Gamma_1 : \Gamma_2] = d$. In particular, $f^{-1}(\mathbb{B}/\Gamma_1) = \mathbb{B}/\Gamma_2$. By Lemma 1 (i), f restricts to an unramified covering $f : D^{(2)} := X_2 \setminus (\mathbb{B}/\Gamma_2) \to X_1 \setminus (\mathbb{B}/\Gamma_1) =: D^{(1)}$ of degree d of the toroidal compactifying divisor $D^{(1)} = \coprod_{j=1}^{k} D_j^{(1)}$ of $\mathbb{B}/\Gamma_1$. Note that for any $1 \leq j \leq k$ the restriction $f : f^{-1}(D_j^{(1)}) \to D_j^{(1)}$ is an unramified covering of degree d, whereas a local biholomorphism. Therefore $f^{-1}(D_j^{(1)}) = \cup_{i=1}^{r_j} D_{j,i}^{(2)}$ is smooth and has disjoint smooth irreducible components $D_{j,i}^{(2)}$. As a result,

$$D^{(2)} = f^{-1}(D^{(1)}) = \coprod_{j=1}^{k} f^{-1}(D_j^{(1)} = \coprod_{j=1}^{k} \coprod_{i=1}^{r_j} D_{j,i}^{(2)}$$

has disjoint smooth irreducible components $D_{j,i}^{(2)}$. By assumption, $D_j^{(1)}$ are smooth elliptic curves, so that all $D_{j,i}^{(2)}$ are smooth elliptic curves by Lemma 2 (i). That is why, $X_2 = (\mathbb{B}/\Gamma_2)'$ is the toroidal compactification of $\mathbb{B}/\Gamma_2$. According to Lemma 4 (ii), $\varphi : Y_2 \to Y_1$ restricts to an unramified covering $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})$ of degree d.

□

Lemma 6. *(i) Let $f : X_2 \to X_1$ be an unramified covering of degree d of smooth projective surfaces and $\mathrm{Bl} : X_1 \to Y_1$ be a blow down of a (-1)-curve $L_1 \subset X_1$. Then the Stein factorization $\varphi\beta$ of $\mathrm{Bl}f$ consists of the blow down $\beta : X_2 \to Y_2$ of $f^{-1}(L_1) = \coprod_{j=1}^{d} L_{1,j}$ and an unramified covering $\varphi : Y_2 \to Y_1$ of degree d, so that $X_2 = X_1 \times_{Y_1} Y_2$ is the fibered product of X_1 and Y_2 over Y_1.*

(ii) Let $\rho_1 = \mathrm{Bl}_1 \ldots \mathrm{Bl}_r : T_r := X_1 \to Y_1 =: T_0$ be a composition of blow downs of (-1)-curves $L_i \subset T_i$ and $f : X_2 \to X_1$ be an unramified covering of degree d. Then the Stein factorization $\varphi\rho_2$ of $\rho_1 f : X_2 \to Y_1$ closes the fibered product commutative diagram (4) with the composition $\rho_2 = \beta_1 \ldots \beta_r : S_r := X_2 \to Y_2 := S_0$ of the blow downs $\beta_i : S_i \to S_{i-1}$ of $\varphi_i^{-1}(L_i) = \coprod_{j=1}^{d} L_{i,j}$ for all $1 \leq i \leq r$ and an unramified covering $\varphi : Y_2 \to Y_1$ of degree d.

Proof. (i) If $\mathrm{Bl}f = \varphi\beta : X_2 \to Y_1$ is the Stein factorization of $\mathrm{Bl}f$ and $q_1 := \mathrm{Bl}(L_1)$

then $(\mathrm{Bl}f)^{-1}(q_1) = f^{-1}\mathrm{Bl}^{-1}(q_1) = f^{-1}(L_1) = \coprod_{j=1}^{d} L_{1,j}$ has irreducible components $L_{1,j}$ by Lemma ??. For any $q \in Y_1 \setminus \{q_1\}$ one has $(\mathrm{Bl}f)^{-1}(q) = f^{-1}\mathrm{Bl}^{-1}(q) = f^{-1}(q)$ of cardinality $|f^{-1}(q)| = d$. Therefore, the surjective morphism $\beta : X_2 \to Y_2$ with connected fibres is the blow down of $L_{1,j}$, $\forall 1 \leq j \leq d$. According to Lemma 1 (i), the restriction $f : X_2 \setminus f^{-1}(L_1) \to X_1 \setminus L_1$ is an unramified covering of degree d, since $f : f^{-1}(L_1) \to L_1$ is an unramified covering of degree d. In such a way, there arises a commutative diagram

$$\begin{array}{ccc} X_2 \setminus f^{-1}(L_1) & \xrightarrow{\beta=\mathrm{Id}} & Y_2 \setminus \beta f^{-1}(L_1) \\ \Big\downarrow f & & \Big\downarrow \varphi \\ X_1 \setminus L_1 & \xrightarrow{\mathrm{Bl}=\mathrm{Id}} & Y_1 \setminus \{q_1\} \end{array}$$

and $\varphi : Y_2 \setminus \beta f^{-1}(L_1) \to Y_1 \setminus \{q_1\}$ is an unramified covering of degree d. If $p_{1,j} := \beta(L_{1,j})$ then $\beta^{-1}\varphi^{-1}(q_1) = (\varphi\beta)^{-1}(q_1) = (\mathrm{Bl}f)^{-1}(q_1) = \coprod_{j=1}^{d} L_{1,j}$ reveals that $\varphi^{-1}(q_1) = \{p_{1,j} \,|\, 1 \leq j \leq d\}$ consists of d points and $\varphi : Y_2 \to Y_1$ is an unramified covering of degree d. By Lemma 3 (i), the fibered product $X_2' := X_1 \times_{Y_1} Y_2$ is the blow up of Y_2 at $\varphi^{-1}(q_1) = \{p_{1,j} \,|\, 1 \leq j \leq d\}$, so that $X_2' = X_2$.

According to Grothendieck's Corollary 17.7.3 (i) from [Groth4], it suffices to show that $X_2' = X_2$, in order to conclude that $\varphi : Y_2 \to Y_1$ is an unramified covering of degree d. We have justified straightforwardly that $\varphi : Y_2 \to Y_1$ is an unramified covering of degree d, in order to use it towards the coincidence of X_2 with the fibered product $X_2' := X_1 \times_{Y_1} Y_2$.

(ii) is an immediate consequence of the fact that the composition of morphisms with connected fibres has connected fibres.

□

Corollary 7. *Let $f : X_2 \to X_1 = (\mathbb{B}/\Gamma_1)'$ be an unramified covering of degree d of a smooth toroidal compactification $X_1 = (\mathbb{B}/\Gamma_1)'$, $\rho_1 : X_1 \to Y_1$ be a composition of blow downs onto a minimal surface Y_1 and $D^{(1)} := X_1 \setminus (\mathbb{B}/\Gamma_1)$ be the toroidal compactifying divisor of $\mathbb{B}/\Gamma_1$. Then:*

(i) there exist a composition $\rho_2 : X_2 \to Y_2$ of blow downs onto a minimal surface Y_2 and an unramified covering $\varphi : Y_2 \to Y_1$ of degree d, which exhibits $X_2 = X_1 \times_{Y_1} Y_2$ as a fibered product of X_1 and Y_2 over Y_1;

(ii) there is a subgroup $\Gamma_2 < \Gamma_1$ of index $[\Gamma_1 : \Gamma_2] = d$, such that $X_2 = (\mathbb{B}/\Gamma_2)'$ is the toroidal compactification of $\mathbb{B}/\Gamma_2$ and f restricts to unramified coverings $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_1$, $f : D^{(2)} := X_2 \setminus (\mathbb{B}/\Gamma_2) \to X_1 \setminus (\mathbb{B}/\Gamma_2) =: D^{(1)}$ of degree d;

(iii) φ restricts to an unramified covering $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})$ of degree d.

Proof. (i) is an immediate consequence of Lemma 6 (ii) and the fact that any inramified cover Y_2 of a minimal surface Y_1 is minimal.

(ii) The unramified covering $f : X_2 \to X_1 = (\mathbb{B}/\Gamma_1)'$ of degree d restricts to an unramified covering $f : f^{-1}(\mathbb{B}/\Gamma_1) \to \mathbb{B}/\Gamma_1$ of degree d. As in the proof of Corollary 5, there is a subgroup $\Gamma_2 < \Gamma_1$ of index $[\Gamma_1 : \Gamma_2] = d$, such that $X_2 = (\mathbb{B}/\Gamma_2)'$ is the toroidal compactification of $\mathbb{B}/\Gamma_2$ and f restricts to unramified coverings $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_1$, $f : D^{(2)} := X_2 \setminus (\mathbb{B}/\Gamma_2) \to X_1 \setminus (\mathbb{B}/\Gamma_1) =: D^{(1)}$ of degree d.

(iii) is an immediate consequence of Lemma 4 (ii).

$\square$

Definition 8. *A smooth toroidal compactification $X_1 = (\mathbb{B}/\Gamma_1)'$ is saturated if there is no unramified covering $f : X_2 = (\mathbb{B}/\Gamma_2)' \to (\mathbb{B}/\Gamma_1)' = X_1$ of degree d, which restricts to an unramified covering $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_1$ of degree d.*

Bearing in mind that the fundamental group of a smooth projective variety is a birational invariant, one combines Corollary 5 with Corollary 7 and obtains the following

Corollary 9. *A smooth toroidal compactification $X_1 = (\mathbb{B}/\Gamma_1)'$ is saturated if and only if one and, therefore, any minimal model Y_1 of X_1 is simply connected.*

2 Unramified push forward of a smooth compactification

Let X_2 be a smooth projective surface, $\beta : X_2 \to Y_2$ be a blow down with exceptional divisor $E(\beta) = \coprod_{s=1}^{d} L_{1,s}$ and $f : X_2 \to X_1$ be an unramified covering of degree d, which restricts to an unramified covering $f : E(\beta) \to f(E(\beta))$ of degree d. According to Lemma 2 (ii), $L_1 := f(E(\beta))$ is a (-1)-curve on X_1. Then Lemma 6 (i) implies that there is a fibered product commutative diagram (1) with the blow down $\mathrm{Bl} : X_1 \to Y_1$ of L_1 and an unramified covering $\varphi : Y_2 \to Y_1$ of degree d, which shrinks $\beta(E(\beta)) = \{p_{1,j} := \beta(L_{1,j}) \,|\, 1 \leq j \leq d\}$ to a point $q_1 \in Y_1$. We say that φ is induced by f.

Suppose that $\rho_2 = \beta_1 \dots \beta_r : S_r := X_2 \to Y_2 =: S_0$ is a composition of blow downs

$$\beta_i : S_i := \beta_{i+1} \dots \beta_r(S_r) \longrightarrow S_{i-1} := \beta_i \dots \beta_r(S_r) \tag{5}$$

with exceptional divisors $E(\beta_i) = \coprod_{s=1}^{d} L_{i,s}$ for all $1 \leq i \leq r$. By a decreasing induction on $r \geq i \geq 1$, let us assume that there is a fibered product commutative diagram

$$\begin{array}{ccccccc}
S_r & \xrightarrow{\beta_r} & S_{r-1} & & \dots S_{i+1} & \xrightarrow{\beta_{i+1}} & S_i \\
\downarrow{\scriptstyle f=\varphi_r} & & \downarrow{\scriptstyle \varphi_{r-1}} & & \downarrow{\scriptstyle \varphi_{i+1}} & & \downarrow{\scriptstyle \varphi_i} \\
f(S_r) & \xrightarrow{\mathrm{Bl}_r} & \varphi_{r-1}(S_{r-1}) & & \dots \varphi_{i+1}(S_{i+1}) & \xrightarrow{\mathrm{Bl}_{i+1}} & \varphi_i(S_i)
\end{array}$$

with fibered product squares $\mathrm{Bl}_j \varphi_j = \varphi_{j-1}\beta_j$, such that φ_j restricts to an unramified covering $\varphi_j : E(\beta_j) \to L_j := \varphi_j(E(\beta_j))$ of degree d and φ_{j-1} shrinks the set $\beta_j(E(\beta_j)) = \{p_{j,s} := \beta_j(L_{j,s}) \,|\, 1 \leq s \leq d\}$ to a point $q_j \in \varphi_{j-1}(S_{j-1})$ for all $r \geq j \geq i+1$. If $\varphi_i : S_i \to \varphi_i(S_i)$ restricts to an unramified covering $\varphi_i : E(\beta_i) \to L_i := \varphi_i(E(\beta_i))$ of degree d then there is an unramified covering $\varphi_{i-1} : S_{i-1} \to \varphi_{i-1}(S_{i-1})$ of degree d, which shrinks $\beta_i(E_{\beta_i}) = \{p_{i,s} = \beta_i(L_{i,s}) \,|\, 1 \leq s \leq d\}$ to a point $q_i \in S_{i-1}$ and closes the fibered product commutative diagram $\varphi_{i-1}\beta_i = \mathrm{Bl}_i \varphi_i$. Thus, if an unramified covering $f : X_2 \to X_1$

of degree d induces unramified coverings $E(\beta_i) = \coprod_{s=1}^{d} L_{i,s} \to L_i$ of degree d for all $1 \leq i \leq r$ then there is an unramified covering $\varphi := \varphi_0 : Y_2 = S_0 \to \varphi_0(S_0) =: Y_1$ of degree d, which induces unramified coverings $\beta_i(E(\beta_i)) = \{p_{i,s} := \beta_i(L_{i,s}) \,|\, 1 \leq s \leq d\} \to \{q_i\} \subset \varphi_{i-1}(S_{i-1})$ of degree d for all $1 \leq i \leq r$.

Conversely, assume that Y_2 is a smooth projective surface, $\beta : X_2 \to Y_2$ is a blow down with exceptional divisor $E(\beta) = \coprod_{s=1}^{d} L_{1,s}$ and $\varphi : Y_2 \to Y_1$ is an unramified covering of degree d, which shrinks $\beta(E(\beta)) = \{p_{1,s} = \beta(L_{1,s}) \,|\, 1 \leq s \leq d\}$ to a point $q_1 \in Y_1$. According to Lemma 3 (i), there is a fibered product commutative diagram (1), where $\mathrm{Bl} : X_1 \to Y_1$ is the blow up of Y_1 at $q_1 \in Y_1$ and $f : X_2 \to X_1$ is an unramified covering of degree d, which restricts to an unramified covering $f : E(\beta) = \coprod_{s=1}^{d} L_{1,s} \to L_1 := \mathrm{Bl}^{-1}(q_1)$ of degree d. Let $\rho_2 = \beta_1 \ldots \beta_r : S_r := X_2 \to Y_2 =: S_0$ be a composition of blow downs (5) with exceptional divisors $E(\beta_i) = \coprod_{s=1}^{d} L_{i,s}$. By an increasing induction on $1 \leq i \leq r$, suppose that

$$\begin{array}{ccccccc} S_i & \xrightarrow{\beta_i} & S_{i-1} & \qquad & \ldots S_1 & \xrightarrow{\beta_1} & S_0 = Y_2 \\ \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_{i-1}} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi=\varphi_0} \\ \varphi_i(S_i) & \xrightarrow{\mathrm{Bl}_i} & \varphi_{i-1}(S_{i-1}) & & \ldots \varphi_1(S_1) & \xrightarrow{\mathrm{Bl}_1} & \varphi(Y_2) \end{array}$$

is a fibered product commutative diagram with fibered product squares $\varphi_{j-1}\beta_j = \mathrm{Bl}_j \varphi_j$, such that φ_{j-1} restricts to an unramified covering

$$\varphi_{j-1} : \beta_j(E(\beta_j)) = \{p_{j,s} := \beta_j(L_{j,s}) \,|\, 1 \leq s \leq d\} \longrightarrow \{q_j\} \subset \varphi_{j-1}(S_{j-1})$$

of degree d and φ_j restricts to an unramified covering

$$\varphi_j : E(\beta_j) = \coprod_{s=1}^{d} L_{j,s} \longrightarrow \varphi_j(E(\beta_j)) =: L_j$$

of degree d for all $1 \leq j \leq i$. If φ_i restricts to an unramified covering

$$\varphi_i : \beta_{i+1}(E(\beta_{i+1})) = \{p_{i+1,s} = \beta_{i+1}(L_{i+1,s}) \,|\, 1 \leq s \leq d\} \longrightarrow \{q_{i+1}\} \subset \varphi_i(S_i)$$

of degree d then there is an unramified covering

$$\varphi_{i+1} : S_{i+1} \longrightarrow \varphi_{i+1}(S_{i+1})$$

of degree d, which restricts to an unramified covering

$$\varphi_{i+1} : E(\beta_{i+1}) = \coprod_{s=1}^{d} L_{i+1,s} \longrightarrow L_{i+1} := \varphi_{i+1}(E(\beta_{i+1}))$$

of degree d and closes the fibered product commutative diagram $\varphi_i\beta_{i+1} = \mathrm{Bl}_{i+1}\varphi_{i+1}$ with the blow down $\mathrm{Bl}_{i+1} : \varphi_{i+1}(S_{i+1}) \to \varphi_i(S_i)$ of L_{i+1}. In such a way, if $\varphi : Y_2 \to Y_1$ is an unramified covering of degree d, which induces unramified coverings

$$\beta_i(E(\beta_i)) = \{p_{i,s} := \beta_i(L_{i,s}) \,|\, 1 \leq s \leq d\} \longrightarrow \{q_i\} \subset \varphi_{i-1}(S_{i-1})$$

of degree d for all $1 \leq i \leq r$ then $f := \varphi_r : X_2 \to f(X_2)$ is an unramified covering of degree d, which induces unramified coverings $E(\beta_i) = \coprod_{s=1}^{d} L_{i,s} \to L_i$ of degree d for all $1 \leq i \leq r$. The above considerations justify the following

Lemma-Definition 10. *Let X_2, Y_2 be smooth projective surfaces and*

$$\rho_2 = \beta_1 \dots \beta_r : S_r := X_2 \longrightarrow Y_2 =: S_0$$

be a composition of blow downs (5) with exceptional divisors $E(\beta_i)$ for all $1 \leq i \leq r$. Then the following are equivalent:

(i) there is an unramified covering $f : X_2 \to f(X_2)$ of degree d, which induces unramified coverings $E(\beta_i) = \coprod_{s=1}^{d} L_{i,s} \to L_i$ of degree d for all $1 \leq i \leq r$;

(ii) there is an unramified covering $\varphi : Y_2 \to \varphi(Y_2)$ of degree d, which induces unramified coverings $\beta_i(E(\beta_i)) = \{p_{i,s} = \beta_i(L_{i,s}) \,|\, 1 \leq s \leq d\} \to \{q_i\} \subset \varphi_{i-1}(S_{i-1})$ of degree d for all $1 \leq i \leq r$.

If there holds one and, therefore, any one of the aforementioned conditions then there is a fibered product commutative diagram (4), where

$$\rho_1 = \mathrm{Bl}_1 \dots \mathrm{Bl}_r : X_1 := \varphi(X_2) \to \varphi(Y_2) =: Y_1$$

is the composition of blow downs Bl_i of L_i for all $1 \leq i \leq r$ and we say that $f : X_2 \to f(X_2)$ and $\varphi : Y_2 \to \varphi(Y_2)$ are compatible with ρ.

Corollary 11. *Let $X_2 = (\mathbb{B}/\Gamma_2)'$ be a smooth toroidal compactification and $\rho_2 : X_2 \to Y_2$ be a composition of blow downs onto a minimal surface Y_2. If there is an unramified covering $f : X_2 = (\mathbb{B}/\Gamma_2)' \to f(X_2) =: X_1$ of degree d, which is compatible with ρ_2 and restricts to an unramified covering $f : \mathbb{B}/\Gamma_2 \to f(\mathbb{B}/\Gamma_2)$ of degree d then:*

(i) there is a fibered product commutative diagram (4) with an unramified covering $\varphi : Y_2 \to \varphi(Y_2) =: Y_1$ of degree d and a composition of blow downs $\rho_1 : X_1 \to Y_1$ onto a minimal surface Y_1;

(ii) there is a lattice Γ_1 of $\mathrm{Aut}(\mathbb{B}) = PU(2,1)$, containing Γ_2 as a subgroup of index $[\Gamma_1 : \Gamma_2] = d$ and such that $X_1 = (\mathbb{B}/\Gamma_1)'$ is the toroidal compactification of $\mathbb{B}/\Gamma_1$;

(iii) φ restricts to an unramified covering $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})$ of degree d, where $D^{(j)} := X_j \setminus (\mathbb{B}/\Gamma_j)$ are the compactifying divisors of $\mathbb{B}/\Gamma_j$, $1 \leq j \leq 2$.

Proof. (i) is an immediate consequence of Lemma 10.

Towards (ii), let us note that the composition $f\zeta_2 : \mathbb{B} \to f(\mathbb{B}/\Gamma_2)$ of the orbifold universal covering $\zeta_2 : \mathbb{B} \to \mathbb{B}/\Gamma_2$ with the unramified covering $f : \mathbb{B}/\Gamma_2 \to f(\mathbb{B}/\Gamma_2)$ is Galois, since $\pi_1^{\mathrm{orb}}(\mathbb{B}) = \{1\}$ is a normal subgroup of $\Gamma_1 := \pi_1^{\mathrm{orb}}(f(\mathbb{B}/\Gamma_2))$. Moreover, $\pi_1^{\mathrm{orb}}(\mathbb{B}/\Gamma_2) = \Gamma_2$ is a subgroup of Γ_1 of index $[\Gamma_1 : \Gamma_2] = d$ and $f(\mathbb{B}/\Gamma_2) = \mathbb{B}/\Gamma_1$. By Lemma 1 (i), $f : X_2 \to X_1$

restricts to an unramified covering $f : D^{(2)} = X_2 \setminus (\mathbb{B}/\Gamma_2) \to D^{(1)} := X_1 \setminus (\mathbb{B}/\Gamma_1)$ of degree d. The toroidal compactifying divisor $D^{(2)}$ of $\mathbb{B}/\Gamma_2$ has disjoint smooth elliptic irreducible components, so that Lemma 2 (i) applies to provide that $D^{(1)}$ consists of disjoint smooth elliptic irreducible components and $X_1 = (\mathbb{B}/\Gamma_1)'$ is the toroidal compactification of $\mathbb{B}/\Gamma_1$. According to Lemma 4 (ii), that suffices for $\varphi : Y_2 \to Y_1$ to restrict to an unramified covering $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})$.

□

Corollary 12. *Let $X_2 = (\mathbb{B}/\Gamma_2)'$ be a smooth toroidal compactification, $D^{(2)} := X_2 \setminus (\mathbb{B}/\Gamma_2)$ be the compactifying divisor of $\mathbb{B}/\Gamma_2$ and $\rho_2 : X_2 \to Y_2$ be a composition of blow downs onto a minimal surface Y_2. If $\varphi : Y_2 \to \varphi(Y_2)$ is an unramified covering of degree d, which is compatible with ρ_2 and restricts to an unramified covering $\varphi : \rho_2(D^{(2)}) \to \varphi\rho_2(D^{(2)})$ of degree d then:*

(i) there is a fibered product commutative diagram (4) with an unramified covering $f : X_2 \to f(X_2) =: X_1$ of degree d and a composition of blow downs $\rho_1 : X_1 \to Y_1$ onto a minimal surface Y_1;

(ii) there is a lattice Γ_1 of $\mathrm{Aut}(\mathbb{B}) = PU(2,1)$, containing Γ_2 as a subgroup of index $[\Gamma_1 : \Gamma_2] = d$ and such that $X_1 = (\mathbb{B}/\Gamma_1)'$ is the toroidal compactification of $\mathbb{B}/\Gamma_1$;

(iii) f restricts to an unramified covering $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_1$ of degree d.

Proof. Lemma 10 justifies (i). According to Lemma 4 (ii), f restricts to an unramified covering $f : D^{(2)} \to f(D^{(2)})$ of degree d. Then Lemma 1 (i) applies to provide that $f : X_2 \setminus D^{(2)} = \mathbb{B}/\Gamma_2 \to X_1 \setminus f(D^{(2)})$ is an unramified covering of degree d. The proof of Corollary 11 (ii) has established that this is sufficient for the existence of a lattice Γ_1 of $\mathrm{Aut}(\mathbb{B}) = PU(2,1)$, containing Γ_2 as a subgroup of index $[\Gamma_1 : \Gamma_2] = d$ and such that $X_1 \setminus f(D^{(2)}) = \mathbb{B}/\Gamma_1$. That justifies (iii). By assumption, $D^{(2)}$ consists of smooth elliptic irreducible components. Therefore $f(D^{(2)}$ has smooth elliptic irreducible components and $X_1 = (\mathbb{B}/\Gamma_1) \coprod f(D^{(2)})$ is the toroidal compactification of $\mathbb{B}/\Gamma_1$.

□

Definition 13. *Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification. If there is no unramified covering $f : X \to f(X)$ of degree d, which restricts to an unramified covering $f : \mathbb{B}/\Gamma \to f(\mathbb{B}/\Gamma)$ of degree d and is compatible with some composition of blow downs $\rho : X \to Y$ onto a minimal surface Y, we say that $X = (\mathbb{B}/\Gamma)'$ is primitive.*

The Euler characteristic of a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ is a natural number $e(X) = e(\mathbb{B}/\Gamma)$. That is why, there exists a primitive smooth toroidal compactification $X_0 = \overline{\mathbb{B}/\Gamma_0}$ and a finite sequence

$$X_n := X \xrightarrow{f_n} X_{n-1} \qquad \ldots X_i \xrightarrow{f_i} X_{i-1} \ldots \qquad X_1 \xrightarrow{f_1} X_0$$

of unramified coverings $f_i : X_i = (\mathbb{B}/\Gamma_i)' \to (\mathbb{B}/\Gamma_{i-1})' = X_{i-1}$ of degree d_i of smooth toroidal compactifications $X_j = (\mathbb{B}/\Gamma_j)'$, which restrict to unramified coverings $f_i : \mathbb{B}/\Gamma_i \to \mathbb{B}/\Gamma_{i-1}$ of degree d_i and are compatible with some compositions of blow downs $\rho_i : X_i \to Y_i$ onto minimal surfaces Y_i. Combining Corollary 11 with Corollary 12, one obtains the following

Corollary 14. *Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification with toroidal compactifying divisor $D := X \setminus (\mathbb{B}/\Gamma)$. Then X is primitive if and only if no one minimal model Y of X with a composition of blow downs $\rho : X \to Y$ admits an unramified covering $\varphi : Y \to \varphi(Y)$ of degree $d > 1$, which restricts to an unramified covering $\varphi : \rho(D) \to \varphi\rho(D)$ of degree d and is compatible with ρ.*

Let us suppose that a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ with toroidal compactifying divisor $D := X \setminus (\mathbb{B}/\Gamma)$ admits a blow down $\beta : X \to Y$ of $n \in \mathbb{N}$ smooth irreducible rational (-1)-curves onto a minimal surface Y and there is an unramified covering $\varphi : Y \to \varphi(Y)$ of degree d, which restricts to unramified coverings $\varphi : \beta(D) \to \varphi\beta(D)$ and $\varphi : \beta(E(\beta)) \to \varphi\beta(E(\beta))$ of degree d. Then the Euler number of the smooth surface $\varphi(Y)$ is $e(\varphi(Y)) = \frac{e(Y)}{d} \in \mathbb{Z}$ and the cardinality of $\varphi\beta(E(\beta))$ if $|\varphi\beta(E(\beta))| = \frac{|\beta(E(\beta))|}{d} = \frac{n}{d} \in \mathbb{N}$, so that $d \in \mathbb{N}$ divides $e(Y)$ and $n = |\beta(E(\beta))|$. As a result, d divides the greatest common divisor $\mathrm{GCD}(|\beta(E(\beta))|, e(Y))$.

Note that the compatibility of an unramified covering $\varphi : Y \to \varphi(Y)$ with $\beta : X \to Y$ reduces to $\varphi^{-1}(\varphi\beta(E(\beta)) = \beta(E(\beta))$ and is detected on Y. When $\rho = \beta_1 \dots \beta_r : X \to Y$ is a composition of $r \geq 2$ blow downs, the compatibility of an unramified covering $\varphi : Y \to \varphi(Y)$ of degree d with ρ cannot be traced out on the minimal model Y of X alone. Namely, if $S_0 := Y$, $T_0 := \varphi(Y)$ then in the notations from the commutative diagram (3), the unramified covering $\varphi_1 : S_1 \to T_1$ of degree d may restrict to an unramified covering $\varphi_1 : \beta_2(E(\beta_2)) \to \varphi_1\beta_2(E(\beta_2))$ of degree d, but $\varphi_0 := \varphi$ is not supposed to restrict to an unramified covering $\varphi : \beta_1\beta_2(E(\beta_2)) \to \varphi\beta_1\beta_2(E(\beta_2))$ of degree d. More precisely, if an irreducible component $L_{1,j}$ of $E(\beta_1)$ intersects $\beta_2(E(\beta_2))$ in at least two points then $|\beta_1\beta_2(E(\beta_2))| < d$ and $\varphi : \beta_1\beta_2(E(\beta_2)) \to \varphi\beta_1\beta_2(E(\beta_2))$ is of degree $< d$.

3 Saturated and primitive smooth compactifications of non-positive Kodaira dimension

Definition 15. *Let $X = (\mathbb{B}/\Gamma)'$ and $X_0 = (\mathbb{B}/\Gamma_0)'$ be smooth toroidal compactification. We say that X dominates X_0 and write $X \succeq X_0$ or $X_0 \preceq X$ if there exist a finite sequence of ball lattices*

$$\Gamma_n := \Gamma < \Gamma_{n-1} < \ldots < \Gamma_i < \Gamma_{i-1} < \ldots < \Gamma_1 < \Gamma_0,$$

with smooth toroidal compactifications $X_i = (\mathbb{B}/\Gamma_i)'$ of the corresponding ball quotients $\mathbb{B}/\Gamma_i$ and a finite sequence of unramified coverings

$$X_n := X \xrightarrow{f_n} X_{n-1} \qquad \ldots X_i \xrightarrow{f_i} X_{i-1} \ldots \qquad X_1 \xrightarrow{f_1} X_0$$

of degree $\deg[f_i : X_i \to X_{i-1}] = [\Gamma_{i-1} : \Gamma_i] = d_i \in \mathbb{N}$, which restrict to unramified coverings $f_i : \mathbb{B}/\Gamma_i \to \mathbb{B}/\Gamma_{i-1}$ of degree d_i and are compatible with some compositions $\rho_i = \beta_{i,1} \dots \beta_{i,r_i} : X_i \to Y_i$ of blow downs $\beta_{i,j}$ onto minimal surfaces Y_i.

It is clear that a smooth toroidal compactification $X = \overline{\mathbb{B}/\Gamma}$ is saturated if and only if it is maximal with respect to the partial order $\succeq$. Similarly, X is primitive exactly when it is minimal with respect to $\succeq$. Note that the partial order $\succeq$ on the set $\mathcal{S}$ of the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ is artinian, i.e., any subset $\mathcal{S}_o \subseteq \mathcal{S}$ has a minimal element

$X_o = (\mathbb{B}/\Gamma_o)' \in \mathcal{S}_o$. The minimal $X \in \mathcal{S}$ are exactly the primitive ones, but the minimal $X_o \in \mathcal{S}_o$ are not necessarily primitive, since such X_o is not supposed to be a minimal element of $\mathcal{S}$.

The present section discusses the saturated and the primitive smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ of Kodaira dimension $\kappa(X) \leq 0$.

Proposition 16. *If $X = (\mathbb{B}/\Gamma)'$ is a smooth toroidal compactification of Kodaira dimension $\kappa(X) = -\infty$ then X is a rational surface or X has a ruled minimal model $\pi : Y \to E$ with an elliptic base E.*

Any smooth rational $X = (\mathbb{B}/\Gamma)'$ is both saturated and primitive.

There is no smooth saturated $X = (\mathbb{B}/\Gamma)'$, whose minimal model is a ruled surface $\pi : Y \to E$ with an elliptic base E.

Proof. (i) Let $\rho : X = (\mathbb{B}/\Gamma)' \to Y$ be a composition of blow downs onto a minimal surface Y of $\kappa(Y) = -\infty$, Then $Y = \mathbb{P}^2(\mathbb{C})$ is the complex projective plane or $\pi : Y \to E$ is a ruled surface with a base E of genus $g \in \mathbb{Z}^{\geq 0}$. The toroidal compactifying divisor $D := X \setminus (\mathbb{B}/\Gamma) = \coprod_{j=1}^{k} D_j$ has disjoint smooth irreducible elliptic components D_j. If $g \geq 2$ then the morphisms $\pi\rho : D_j \to E$ map to points $p_j := \pi\rho(D_j) \in E$, so that $\rho(D_j) \subseteq \pi^{-1}(p_j)$ for all $1 \leq j \leq k$. The exceptional divisor L of $\rho : X \to Y$ has finite image $\rho(L) = \{q_1, \ldots, q_m\}$ on Y and $\rho(L) \subseteq \coprod_{i=1}^{m} \pi^{-1}(\pi(q_i))$. Therefore

$$Y' := Y \setminus \left[\coprod_{i=1}^{m} \pi^{-1}(\pi(q_i)) \right] \subseteq Y \setminus \rho(L) \equiv X \setminus L$$

and ρ acts identically on Y'. Moreover,

$$Y'' := Y' \setminus \left[\coprod_{j=1}^{k} \pi^{-1}(p_j) \right] = Y \setminus \left[\left(\coprod_{i=1}^{m} \pi^{-1}(\pi(q_i)) \right) \coprod \left(\coprod_{j=1}^{k} \pi^{-1}(p_j) \right) \right] \subseteq \mathbb{B}/\Gamma.$$

However, Y'' contains (infinitely many) fibres $\pi^{-1}(e) \simeq \mathbb{P}^1(\mathbb{C})$, $e \in E$ of $\pi : Y \to E$ and that contradicts the Kobayashi hyperbolicity of $\mathbb{B}/\Gamma$. In such a way, we have shown that any minimal model Y of a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ of $\kappa(X) = -\infty$ is birational to $\mathbb{P}^2(\mathbb{C})$ or to a minimal ruled surface $\pi : Y \to E$ with an elliptic base E.

Any rational $X = (\mathbb{B}/\Gamma)'$ is simply connected and does not admit finite unramified coverings $X_1 \to X$ of degree $d > 1$. That is why X is saturated. Let us suppose that $f : X = (\mathbb{B}/\Gamma)' \to X_0 = (\mathbb{B}/\Gamma_0)'$ is an unramified covering of degree $d > 1$, which is compatible with some composition of blow downs $\rho : X \to Y$ onto a minimal rational surface Y and restricts to an unramified covering $f : \mathbb{B}/\Gamma \to \mathbb{B}/\Gamma_0$ of degree d. The Kodaira dimension is preserved under finite unramified coverings, so that $\kappa(X_0) = \kappa(X) = -\infty$. The surface X_0 is not simply connected, whereas non-rational. Therefore, there is a composition $\rho_0 : X_0 \to Y_0$ of blow downs onto a ruled surface $\pi_0 : Y_0 \to E_0$ with base E_0 of genus $g_0 \in \mathbb{N}$. The surjective morphism $\rho_0 f : X = (\mathbb{B}/\Gamma)' \to Y_0$ induces an embedding $(\rho_0 f)^* : H^{0,1}(Y_0) \to H^{0,1}(X)$. On one hand, the irregularity of Y_0 is $h^{0,1}(Y_0) := \dim_{\mathbb{C}} H^{0,1}(Y_0) = g_o \in \mathbb{N}$. On the other hand, the rational surface X has vanishing irregularity $h^{0,1}(X) = 0$. That contradicts the presence of a finite unramified covering $f : X \to X_0$ of degree $d > 1$ and shows that any smooth rational toroidal compactification $X = (\mathbb{B}/\Gamma)'$ is primite.

Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification, whose minimal model Y is a ruled surface $\pi : Y \to E$ with an elliptic base E. Since Y is birational to $\mathbb{P}^1(\mathbb{C}) \times E$ and the fundamental group is a birational invariant, one has $\pi_1(X) \simeq \pi_1(Y) \simeq \pi_1(E) \simeq (\mathbb{Z}^2, +)$. In particular, Y is not simply connected. According to Corollary 9, X cannot be saturated.

□

According to the Enriques-Kodaira classification, there are four types of minimal smooth projective surfaces Y of Kodaira dimension $\kappa(Y) = 0$. These are the abelian and the bi-elliptic surfaces with universal cover $\mathbb{C}^2$, as well as the $K3$ and the Enriques surfaces with $K3$ universal cover. If $\varphi : Y_2 \to Y_1$ is a finite unramified covering of smooth projective surfaces then the Kodaira dimension $\kappa(Y_1) = \kappa(Y_2)$ and the universal covers $\widetilde{Y_1} = \widetilde{Y_2}$ coincide. Let Y_2 be a smooth projective surface with a fixed point free involution $g_o : Y_2 \to Y_2$ and $\beta : X_2 \to Y_2$ be the blow up of Y_2 at a $\langle g_o \rangle$-orbit $\{p_{1,1}, p_{1,2} = g_o(p_{1,1})\} \subset Y_2$. Then by the very definition of a blow up, g_o iniduces a fixed point free involution $g_1 : X_2 \to X_2$, which leaves invariant the exceptional divisor $E(\beta) = L_{1,1} \coprod L_{1,2}$, $L_{1,i} := \beta^{-1}(p_{1,i})$ of β and there is a fibered product commutative diagram (4) with a $\langle g_o \rangle$-Galois covering $\varphi : Y_2 \to Y_1$, a $\langle g_1 \rangle$-Galois covering $f : X_2 \to X_1$ and the blow up $\mathrm{Bl} : X_1 \to Y_1$ of Y_1 at $\{q_1\} = \varphi(\{p_{1,1}, p_{1,2}\})$. Now, suppose that $\rho_2 = \beta_1 \ldots \beta_r : S_r := X_2 \to Y_2 =: S_0$ is a composition of blow downs with exceptional divisors $E(\beta_i) = L_{i,1} \coprod L_{i,2}$ and $g_o : S_0 \to S_0$ is a fixed point free involution. By an increasing induction on $1 \leq i \leq r$, if $g_{i-1} : S_{i-1} \to S_{i-1}$ is a fixed point free involution, which leaves invariant $\beta_i(E(\beta_i)) = \{p_{i,1}, p_{i,2}\}$ then there is a fixed point free involution $g_i : S_i \to S_i$, which leaves invariant $E(\beta_i) = L_{i,1} \coprod L_{i,2}$. In such a way, if a fixed point free involution $g_0 : S_0 \to S_0$ induces isomorphisms $L_{i,1} \to L_{i,2}$ for all $1 \leq i \leq r$ then there is a fixed point free involution $g_r : S_r \to S_r$ and a fibered product commutative diagram (4) with a $\langle g_o \rangle$-Galois covering $\varphi : Y_2 \to Y_1$, a $\langle g_r \rangle$-Galois covering $f : X_2 \to X_1$ and the composition $\rho_1 = \mathrm{Bl}_1 \ldots \mathrm{Bl}_r : X_1 \to Y_1$ of the blow downs of $E(\beta_i)/\langle g_i \rangle = L_i \simeq \mathbb{P}^1(\mathbb{C})$. If $g_o : S_0 \to S_0$ induces isomorphisms $L_{i,1} \to L_{i,2}$ of the irreducible components of $E(\beta_i) = L_{i,1} \coprod L_{i,2}$ for all $1 \leq i \leq r$, we say that g_o is compatible with $\rho_2 = \beta_1 \ldots \beta_r$.

Proposition 17. *Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification, $D := X \setminus (\mathbb{B}/\Gamma)$ be the toroidal compactifying divisor of $\mathbb{B}/\Gamma$ and $\rho = \beta_1 \ldots \beta_r : X \to Y$ be a composition of blow downs onto a $K3$ surface Y. Then:*

(i) X is a saturated compactification;

(ii) X is non-primitive exactly when there is a fixed point free involution $g_o : Y \to Y$, which is compatible with ρ and leaves invariant $\rho(D)$;

(iii) if X is non-primitive then there is a fibered product commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\rho} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X_0 & \xrightarrow{\rho_0} & Y_0 \end{array}$$

with a primitive smooth toroidal compactification $X_0 = (\mathbb{B}/\Gamma_0)'$, a composition of blow downs

$\rho_0 : X_0 \to Y_0$ *onto a minimal Enriques surface* Y_0 *and unramified double covers* $f : X \to X_0$, $\varphi : Y \to Y_0$.

Proof. (i) is an immediate consequence of $\pi_1(Y) = \{1\}$, according to Corollary 9.

(ii) and (iii) follow from Corollary 14 and the fact that a minimal projective surface Y_0 admits an unramified covering $\varphi : Y \to Y_0$ by a $K3$ surface Y if and only if Y_0 is the quotient of Y by a fixed point free involution $g_o : Y \to Y$. Such $Y_0 = Y/\langle g_o \rangle$ are called minimal Enriques surfaces and do not admit unramified coverings $\varphi_0 : Y_0 \to \varphi_0(Y_0)$ of degree > 1.

□

Proposition 18. *Let* $X = (\mathbb{B}/\Gamma)'$ *be a smooth toroidal compactification and* $\rho : \beta_1 \ldots \beta_r : X \to Y$ *be a composition of blow downs onto a minimal Enriques surface* Y. *Then:*

(i) X *is a primitive compactification;*

(ii) X *is not saturated;*

(iii) there is an unramified double cover $f : X_1 = \overline{\mathbb{B}/\Gamma_1} \to \overline{\mathbb{B}/\Gamma} = X$ *by a saturated smooth toroidal compactification* $X_1 = (\mathbb{B}/\Gamma_1)'$ *with* $K3$ *minimal model* Y_1.

Proof. (i) is due to the lack of an unramified covering $\varphi : Y \to \varphi(Y)$ of degree $d > 1$.

(ii) follows from $\pi_1(Y) = (\mathbb{Z}_2, +) \neq \{1\}$.

(iii) is an immediate consequence of the Enriques-Kodaira classification of the smooth projective surfaces.

□

Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification with abelian or bi-elliptic minimal model Y. According to Theorem 1.3 from Di Cerbo and Stover's article [DiCerboStover2], X can be obtained from Y by blow up $\beta : X \to Y$ of $n \in \mathbb{N}$ points $p_1, \ldots, p_n \in Y$.

Proposition 19. *Let* $X = (\mathbb{B}/\Gamma)'$ *be a smooth toroidal compactification with a blow down* $\beta : X \to Y$ *onto a minimal surface* Y *with exceptional divisor* $E(\beta) = \coprod_{i=1}^{n} L_i$ *and* $D := X \setminus (\mathbb{B}/\Gamma)$ *be the toroidal compactifying divisor of* $\mathbb{B}/\Gamma$. *Then:*

(i) β *transforms* $E(\beta)$ *onto the singular locus* $\beta(E(\beta)) = \beta(D)^{\text{sing}}$ *of* $\beta(D) \subset Y$*;*

(ii) X *is non-primitive if and only if there is an unramified covering* $\varphi : Y \to \varphi(Y)$ *of degree* $d > 1$, *which restricts to an unramified covering* $\varphi : \beta(D) \to \varphi\beta(D)$ *of degree* d*;*

(iii) the relative automorphism group $\mathrm{Aut}(Y, \beta(D)) = \mathrm{Aut}(Y, \beta(D), \beta(D)^{\text{sing}})$ *admits an isomorphism*

$$\Phi : \mathrm{Aut}(Y, \beta(D)) \longrightarrow \mathrm{Aut}(X, D)$$

with the relative automorphism group $\mathrm{Aut}(X, D) = \mathrm{Aut}(X, D, E(\beta))$*;*

(iv) $g_o \in \mathrm{Aut}(Y, \beta(D)$ *is fixed point free if and only if it corresponds to a fixed point free* $g = \Phi(g_o) \in \mathrm{Aut}(X, D)$.

Proof. (i) If $D = \coprod_{j=1}^{k} D_j$ has irreducible components D_j then the singular locus of $\beta(D)$ is

$$\beta(D)^{\text{sing}} = \left[\cup_{j=1}^{k} \beta(D_j)^{\text{sing}} \right] \cup \left[\cup_{1 \leq i < j \leq k} \beta(D_i) \cap \beta(D_j) \right].$$

Since D_j are smooth irreducible elliptic curves, $\beta(D)^{\text{sing}} \subseteq \beta(E(\beta))$. Conversely, any (-1)-curve L_i on $X = (\mathbb{B}/\Gamma)'$ intersects $D = \coprod_{j=1}^{k} D_j$ in at least three points, due to the Kobayashi hyperbolicity of $\mathbb{B}/\Gamma$. In fact, $|L_i \cap F| \geq 4$, according to Theorem 1.1 (2) from Di Cerbo and Stover's article [DiCerboStover2]. Therefore, the multiplicity of $\beta(L_i) = p_i$ with respect to $\beta(D)$ is ≥ 4 and $p_i \in \beta(D)^{\text{sing}}$. That justifies $\beta(E(\beta)) \subseteq \beta(D)^{\text{sing}}$ and $\beta(E(\beta)) = \beta(D)^{\text{sing}}$.

(ii) By Corollary 14 and (i), $X = (\mathbb{B}/\Gamma)'$ is non-primitive if and only if there is an unramified covering $\varphi : Y \to \varphi(Y)$ of degree $d > 1$, which restricts to unramified coverings $\varphi : \beta(D) \to \varphi\beta(D)$ and $\varphi : \beta(D)^{\text{sing}} \to \beta(D)^{\text{sing}}$ of degree d. Let us observe that any unramified covering $\varphi : \beta(D) \to \varphi\beta(D)$ of degree d restricts to an unramified covering $\varphi : \beta(D)^{\text{sing}} \to \beta(D)^{\text{sing}}$ of degree d, as far as the local biholomorphism $\varphi : \beta(D) \to \varphi\beta(D)$ preserves the multiplicities of the points with respect to $\beta(D)$ and $\beta(D)^{\text{sing}}$ consists of the points of $\beta(D)$ of multiplicity ≥ 2.

(iii) If a holomorphic automorphism $g_o : Y \to Y$ restricts to a holomorphic automorphism $g_o : \beta(D) \to \beta(D)$ then g_o preserves the multiplicities of the points with respect to $\beta(D)$ and $\beta(D)^{\text{sing}}$ is $\langle g_o \rangle$-invariant. That justifies $\text{Aut}(Y, \beta(D)) \leq \text{Aut}(Y, \beta(D), \beta(D)^{\text{sing}})$ and $\text{Aut}(Y, \beta(D)) = \text{Aut}(Y, \beta(D), \beta(D)^{\text{sing}})$.

In order to show the existence of a group isomorphism

$$\Phi : \text{Aut}(Y, \beta(D), \beta(D)^{\text{sing}}) \longrightarrow \text{Aut}(X, D, E(\beta)),$$

let us pick up $g_o \in \text{Aut}(Y, \beta(D), \beta(D)^{\text{sing}})$. Then $X \setminus E(\beta) = Y \setminus \beta(E(\beta)) = Y \setminus \beta(D)^{\text{sing}}$ is acted by $\Phi(g_o)|_{X \setminus E(\beta)} := g_o|_{Y \setminus \beta(D)^{\text{sing}}}$. By the very definition of a blow up at a point, the bijection $g_o : \beta(D)^{\text{sing}} \to \beta(D)^{\text{sing}}$ with $g_o(\beta(L_{1,i})) = \beta(L_{1,j})$ induces isomorphisms $\Phi(g_o) : L_{1,i} \to L_{1,j}$ and provides an element $\Phi(g_o) \in \text{Aut}(X, E(\beta))$. After observing that $\Phi(g_o)(D \setminus E(\beta)) = g_o(\beta(D) \setminus \beta(D)^{\text{sing}}) = \beta(D) \setminus \beta(D)^{\text{sing}} = D \setminus E(\beta)$, one concludes that $\Phi(g_o)$ transforms the Zariski closure D of $D \setminus E(\beta)$ onto itself and $\Phi(g_o) \in \text{Aut}(D)$.

The correspondence Φ is a group homomorphism since g_o and $\Phi(g_o)$ coincide on Zariski open subsets of Y, respectively, X. Towards the bijectiveness of Φ, let $g \in \text{Aut}(X, D, E(\beta))$ and note that $Y \setminus \beta(D)^{\text{sing}} = X \setminus E(\beta)$. That allows to define $\phi^{-1}(g)|_{Y \setminus \beta(D)^{\text{sing}}} := g|_{X \setminus E(\beta)}$. The isomorphism $g : E(\beta) \to E(\beta)$ of the exceptional divisor $E(\beta)$ of β induces a permutation $\Phi^{-1}(g) : \beta(D)^{\text{sing}} \to \beta(D)^{\text{sing}}$ of the finite set $\beta(D)^{\text{sing}}$ and provides an automorphism $\Phi^{-1}(g) \in \text{Aut}(Y, \beta(D)^{\text{sing}})$. Bearing in mind that $\Phi^{-1}(g)(\beta(D) \setminus \beta(D)^{\text{sing}}) = g(D \setminus E(\beta)) = D \setminus E(\beta) = \beta(D) \setminus \beta(D)^{\text{sing}}$, one concludes that $\Phi^{-1}(g) \in \text{Aut}(\beta(D))$ is an automorphism of the Zariski closure $\beta(D)$ of $\beta(D) \setminus \beta(D)^{\text{sing}} = \beta(D)^{\text{smooth}}$.

Note that any automorphism $g \in \text{Aut}(X, D)$ acts on the set of the smooth irreducible rational curves on X. Moreover, g preserves the self-intersection number of such a curve and $\langle g \rangle$ acts on the set $E(\beta) = \coprod_{i=1}^{n} L_i$ of the (-1)-curves on X. Thus, $g \in \text{Aut}(X, D, E(\beta))$ and $\text{Aut}(X, D) \subseteq \text{Aut}(X, D, E(\beta))$, whereas $\text{Aut}(X, D, E(\beta)) = \text{Aut}(X, D)$.

(iv) If $g \in \text{Aut}(X, D)$ has no fixed points on X then $g_o := \Phi^{-1}(g) \in \text{Aut}(Y, \beta(D))$ restricts to $g_o|_{Y \setminus \beta(E(\beta))} = g|_{X \setminus E(\beta)}$ without fixed points. The assumption $g_o(p_i) = p_i = \text{Bl}(L_i)$ for some $1 \leq i \leq n$ implies that g restricts to an automorphism $g : L_i \to L_i$. Any biholomorphism $g \in \text{Aut}(L_i) = \text{Aut}(\mathbb{P}^1(\mathbb{C})) = PGL(2, \mathbb{C})$ of the projective line $L_i = \mathbb{P}^1(\mathbb{C})$ is a fractional linear transformation and has two fixed points, counted with their multiplicities. That contradicts the lack of fixed points of g on X and implies that the associated automorphism $g_o = \Phi^{-1}(g) \in \mathrm{Aut}(Y, \beta(D))$ has no fixed points on Y.

Conversely, if $g_o \in \mathrm{Aut}(Y, \beta(D))$ has no fixed points on Y and $g := \Phi(g_o)$ then the restriction $g|_{X \setminus E(\beta)} = g_o|_{Y \setminus \beta(\beta)}$ has no fixed points. If $g(x) = x$ for some $x \in E(\beta) = \coprod_{i=1}^{n} L_i$ then $x \in L_i$ for some $1 \leq i \leq n$ and $g(L_i) = L_i$. As a result, g_o fixes $p_i = \beta(L_i) \in Y$, which is an absurd. In such a way, any fixed point free $g_o \in \mathrm{Aut}(Y, \beta(D))$ corresponds to a fixed point free $g = \Phi(g_o) \in \mathrm{Aut}(X, D)$.

□

Proposition 20. *Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification with toroidal compactifying divisor $D := X \setminus (\mathbb{B}/\Gamma)$ and a blow down $\beta : X \to Y$ of $n \in \mathbb{N}$ smooth irreducible rational (-1)-curves. Then $\mathrm{Aut}(X, D)$ is a finite group.*

Proof. By Proposition 19 (iii), $\mathrm{Aut}(X, D) = \mathrm{Aut}(X, D, E(\beta))$. Any $g \in \mathrm{Aut}(X, D)$ acts on $D = \coprod_{j=1}^{k} D_j$ and induces a permutation of the smooth elliptic irreducible components D_1, ..., D_k of D. In such a way, there arises a representation

$$\Sigma_1 : \mathrm{Aut}(X, D) \longrightarrow \mathrm{Sym}(D_1, \ldots, D_k) = \mathrm{Sym}(k).$$

The image of Σ_1 in the finite group $\mathrm{Sym}(k)$ is a finite group, so that it suffices to show the finiteness of $\ker(\Sigma_1)$, in order to conclude that $\mathrm{Aut}(X, D)$ is a finite group. Similarly, $\mathrm{Aut}(X, D) = \mathrm{Aut}(X, D, E(\beta))$ acts on the exceptional divisor $E(\beta) = \coprod_{i=1}^{n} L_i$ of $\beta : X \to Y$ and defines a representation

$$\Sigma_2 : \mathrm{Aut}(X, D) \longrightarrow \mathrm{Sym}(L_1, \ldots, L_n) = \mathrm{Sym}(n).$$

Since $\Sigma_2(\ker(\Sigma_1))$ is a finite group, it suffices to show that $G := \ker(\Sigma_2) \cap \ker(\Sigma_1)$ is a finite group. For any $1 \leq i \leq n$, $1 \leq j \leq k$ and $g \in G$, the finite set $L_i \cap D_j$ is transformed into itself, according to $g(L_i \cap D_j) \subseteq g(L_i) \cap g(D_j) = L_i \cap D_j$. Therefore, there is a representation

$$\Sigma_{i,j} : G \longrightarrow \mathrm{Sym}(L_i \cap D_j).$$

The image $\Sigma_{i,j}(G)$ is a finite group, while the kernel $K_{i,j} := \ker(\Sigma_{i,j})$ fixes any point $p \in L_i \cap D_j$ and acts on D_j. It is well known that the holomorphic automorphisms $\mathrm{Aut}_p(D_j)$ of an elliptic curves D_j, which fix a point $p \in D_j$ form a cyclic group of order 2, 4 or 6. Therefore, $K_{i,j} \leq \mathrm{Aut}_p(D)$, G, $\ker(\Sigma_1)$ and $\mathrm{Aut}(X, D)$ are finite groups.

□

Definition 21. *A smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ with a blow down $\beta : X \to Y$ of $n \in \mathbb{N}$ smooth irreducible rational (-1)-curves onto a minimal surface Y is Galois non-primitive if there is a fixed point free automorphism $g \in \mathrm{Aut}(X, D) \setminus \{\mathrm{Id}_X\}$.*

Any Galois non-primitive $X = (\mathbb{B}/\Gamma)'$ is non-primitive, because the $\langle g \rangle$-Galois covering $\zeta : X \to \zeta(X) = X/\langle g \rangle$ is unramified and restricts to unramified coverings $\zeta : \mathbb{B}/\Gamma \to \zeta(\mathbb{B}/\Gamma)$ and $\zeta : E(\beta) = \coprod_{i=1}^{n} L_i \to \zeta(E(\beta))$ of degree $|\langle g \rangle| = \mathrm{ord}(g)$.

Note that the presence of an unramified covering $\varphi : Y \to \varphi(Y)$ implies the coincidence $\widetilde{Y} = \widetilde{\varphi(Y)}$ of the universal cover $\widetilde{Y}$ of Y with the universal cover $\widetilde{\varphi(Y)}$ of $\varphi(Y)$. The fundamental group $\pi_1(\varphi(Y))$ of $\varphi(Y)$ acts on $\widetilde{Y}$ by biholomorphic automorphisms without fixed points and contains the fundamental group $\pi_1(Y)$ of Y as a subgroup of index $[\pi_1(\varphi(Y)) : \pi_1(Y)] = d$.

Proposition 22. *Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification with toroidal compactifying divisor $D := X \setminus (\mathbb{B}/\Gamma)$, $\beta : X \to Y$ be a blow down of $n \in \mathbb{N}$ smooth irreducible rational (-1)-curves to a minimal surface Y and $N(\pi_1(Y))$ be the normalizer of the fundamental group $\pi_1(Y)$ of Y in the biholomorphism group $\mathrm{Aut}(\widetilde{Y})$ of the universal cover $\widetilde{Y}$ of Y. Then X is Galois non-primitive if and only if there exist a natural divisor $d > 1$ of $\mathrm{GCD}(\left|\beta(D)^{\mathrm{sing}}\right|, e(Y)) \in \mathbb{N}$ and an unramified covering $\varphi : Y \to \varphi(Y)$ of degree d, such that $\pi_1(\varphi(Y)) \cap N(\pi_1(Y)) \gneqq \pi_1(Y)$ and $\varphi : \beta(D) \to \varphi\beta(D)$ is an unramified covering of degree d.*

Proof. If $X = (\mathbb{B}/\Gamma)'$ is Galois non-primitive then there exists a fixed point free biholomorphism $g \in \mathrm{Aut}(X, D) \setminus \{\mathrm{Id}_X\}$ of X. By Proposition 19(iv), g induces a fixed point free biholomorphism $g_o = \Phi^{-1}(g) \in \mathrm{Aut}(Y, \beta(D)) \setminus \{\mathrm{Id}_Y\}$ of Y. The element g_o of the finite group $\mathrm{Aut}(Y, \beta(D))$ is of finite order $d \in \mathbb{N} \setminus \{1\}$ and the $\langle g_o \rangle$-Galois coverings $\zeta : Y \to Y/\langle g_o \rangle$, $\zeta : \beta(D) \to \zeta\beta(D)$ are unramified and of degree d. The automorphism g_o of Y lifts to an automorphism $\sigma \in \mathrm{Aut}(\widetilde{Y})$ of the universal cover $\widetilde{Y}$ of Y, which normalizes $\pi_1(Y)$ and belongs to

$$\pi_1(\zeta(Y)) = \pi_1(Y/\langle g_o \rangle) = \pi_1\left((\widetilde{Y}/\pi_1(Y))/\langle \sigma\pi_1(Y) \rangle\right) = \pi_1\left(\widetilde{Y}/\langle \sigma, \pi_1(Y) \rangle\right) = \langle \sigma, \pi_1(Y) \rangle.$$

Conversely, suppose that $\varphi : Y \to \varphi(Y)$ is an unramified covering of degree $d > 1$, which restricts to an unramified covering $\varphi : \beta(D) \to \varphi\beta(D)$ of degree d and there exists $\sigma \in [\pi_1(\varphi(Y)) \cap N(\pi_1(Y))] \setminus \pi_1(Y)$. Then $g_o := \sigma\pi_1(Y) \in \mathrm{Aut}(Y) = N(\pi_1(Y))/\pi_1(Y)$ is a non-identical biholomorphism $g_o : Y \to Y$. Since $\langle \sigma, \pi_1(Y) \rangle$ is a subgroup of $\pi_1(\varphi(Y))$, the unramified covering $\varphi : Y \to \varphi(Y)$ factors through the $\langle g_o \rangle$-Galois covering $\zeta : Y \to Y/\langle g_o \rangle$ and a covering $\varphi_o : Y/\langle g_o \rangle \to \varphi(Y)$ along the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ \zeta\ } & Y/\langle g_o \rangle \\ & \underset{\varphi}{\searrow} & \downarrow \varphi_o \\ & & \varphi(Y) \end{array} \tag{6}$$

The finite coverings $\zeta : Y \to Y/\langle g_o \rangle$ and $\varphi_o : Y/\langle g_o \rangle \to \varphi(Y)$ are unramified, because their composition $\varphi = \varphi_o\zeta : Y \to \varphi(Y)$ is unramified. That is why, g_o has no fixed points on Y. If $\beta(D) \subset Y$ is not $\langle g_o \rangle$-invariant then there is an orbit $\mathrm{Orb}_{\langle g_o \rangle}(y_o) \subset Y$ of some $y_o \in \beta(D)$ which intersects, both, $\beta(D)$ and $Y \setminus \beta(D)$. Therefore, $\zeta : \beta(D) \to \zeta\beta(D)$ has a fibre $\zeta^{-1}(\zeta(y_o))$ of cardinality $\left|\zeta^{-1}(\zeta(y_o))\right| < \deg(\zeta) = |\langle g_o \rangle| = \mathrm{ord}(g_o)$ and $\zeta : \beta(D) \to \zeta\beta(D)$ is ramified. As a result, the composition $\varphi = \varphi_o\zeta : \beta(D) \to \varphi\beta(D)$ is ramified. The contradiction shows the $\langle g_o \rangle$-invariance of $\beta(D)$. According to Proposition 19 (iv), the fixed point free $g_o \in \mathrm{Aut}(Y, \beta(D)) \setminus \{\mathrm{Id}_Y\}$ corresponds to a fixed point free $g = \Phi(g_o) \in \mathrm{Aut}(X, D) \setminus \{\mathrm{Id}_X\}$ and X is Galois non-primitive.

□

Definition 23. *A covering $\varphi : Y \to \varphi(Y)$ by a smooth projective surface Y has Galois factorization if there exist $g_o \in \mathrm{Aut}(Y) \setminus \{\mathrm{Id}_Y\}$ and a covering $\varphi_o : Y/\langle g_o \rangle \to \varphi(Y)$, such that $\varphi = \varphi_o \zeta$ gactors through the $\langle g_o \rangle$-Galois covering $\zeta : Y \to Y/\langle g_o \rangle$ and a covering φ_o along the commutative diagram (6).*

Now, Proposition 22 can be reformulated in the form of the following

Corollary 24. *Let $X = (\mathbb{B}/\Gamma)'$ be a non-primitive smooth toroidal compactification with toroidal compactifying divisor $D := X \setminus (\mathbb{B}/\Gamma)$, $\beta : X \to Y$ be a blow down of $n \in \mathbb{N}$ smooth irreducible rational (-1)-curves onto a minimal surface Y and $\varphi : Y \to \varphi(Y)$ be an unramified covering of degree d, which restricts to an unramified covering $\varphi : \beta(D) \to \varphi\beta(D)$ of degree d. Then X is Galois non-primitive if and only if φ admits a Galois factorization.*

Corollary 25. *(i) Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification with abelian minimal model Y. Then X is not saturated and X is non-primitive if and only if it is Galois non-primitive.*

(ii) If $X = (\mathbb{B}/\Gamma)'$ is a smooth toroidal compactification with bi-elliptic minimal model Y then X is not saturated.

Proof. (i) Any abelian surface Y has non-trivial fundamental group $\pi_1(Y) \simeq (\mathbb{Z}^4, +)$. According to Corollary 9, that suffices for a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ with abelian minimal model Y to be non-saturated.

By Theorem 1.3 from Di Cerbo and Stover's article [DiCerboStover2], if a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ has abelian minimal model Y then there is a blow down $\beta : X \to Y$ of $n \in \mathbb{N}$ smooth irreducible rational (-1)-curves on X onto Y. Such X is non-primitive exactly when there exists an unramified covering $\varphi : Y \to \varphi(Y)$ of degree $d > 1$, which restricts to an unramified covering $\varphi : \beta(D) \to \varphi\beta(D)$ of degree d. Since Y and $\varphi(Y)$ have one and a same universal cover $\widetilde{\varphi(Y)} = \widetilde{Y} = \mathbb{C}^2$ and one and a same Kodaira dimension $\kappa(\varphi(Y)) = \kappa(Y) = 0$, the minimal smooth irreducible projective surface $\varphi(Y)$ is abelian or bi-elliptic.

If $\varphi(Y)$ is an abelian surface then its fundamental group $\pi_1(\varphi(Y)) \simeq (\mathbb{Z}^4, +)$ is abelian and $\pi_1(Y) \simeq (\mathbb{Z}^4, +)$ is a normal subgroup of $\pi_1(\varphi(Y))$. As a result, $\varphi : Y \to \varphi(Y)$ is a $\pi_1(\varphi(Y))/\pi_1(Y)$-Galois covering and Y is Galois non-primitive.

Let us suppose that $\varphi(Y)$ is a bi-elliptic surface. According to Bagnera-de Franchis classification of the bi-elliptic surfaces from [BagneraDeFranchis], there is an abelian surface A and a cyclic subgroup $\langle g \rangle \leq \mathrm{Aut}(A)$ of order $d \in \{2, 3, 4, 6\}$ with a non-translation generator $g \in \mathrm{Aut}(A)$, such that $\varphi(Y) = A/\langle g \rangle$. Let $\mathrm{AffLin}(\mathbb{C}) := \mathcal{T}(\mathbb{C}^2) \rtimes \mathrm{GL}(2, \mathbb{C})$ be the group of the affine linear transformations of $\mathbb{C}^2 = \widetilde{Y} = \widetilde{\varphi(Y)} = \widetilde{A}$ and

$$\mathcal{L} : \mathrm{AffLin}(\mathbb{C}^2) \longrightarrow \mathrm{GL}(2, \mathbb{C})$$

be the group homomorphism, associating to $\sigma \in \mathrm{AffLin}(\mathbb{C}^2)$ its linear part $\mathcal{L}(\sigma) \in \mathrm{GL}(2, \mathbb{C})$. Then the fundamental group of A is the maximal translation subgroup

$$\pi_1(A) = \pi_1(\varphi(Y)) \cap \ker(\mathcal{L})$$

of $\pi_1(\varphi(Y))$. The translation subgroup $\pi_1(Y) \leq \pi_1(\varphi(Y)) \cap \ker(\mathcal{L})$ of $\pi_1(\varphi(Y))$ is contained in $\pi_1(A)$ and the unramified covering $\varphi : Y \to \varphi(Y)$ factors through unramified coverings $\varphi_1 : Y \to A$ and $\varphi_2 : A \to \varphi(Y)$, along the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi_1} & A \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \varphi_2} \\ & & \varphi(Y) \end{array} .$$

The covering $\varphi_1 : Y \to A$ is $\pi_1(A)/\pi_1(Y)$-Galois, so that $\varphi = \varphi_2\varphi_1$ is a Galois factorization of φ for $\pi_1(Y) \lneqq \pi_1(A)$. In the case of $\pi_1(Y) = \pi_1(A)$, there is an isomorphism $Y \simeq \mathbb{C}^2/\pi_1(Y) \simeq \mathbb{C}^2/\pi_1(A) = A$ and the covering $\varphi : Y \simeq A \to \varphi(Y) = A/\langle g \rangle$ is $\langle g \rangle$-Galois. Thus, X is Galois non-primitive and a co-abelian smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ is non-primitive if and only if it is Galois non-primitive.

(ii) The fundamental group $\pi_1(Y)$ of a bi-elliptic surface Y is subject to an exact sequence

$$1 \longrightarrow \pi_1(Y) \cap \ker(\mathcal{L}) \longrightarrow \pi_1(Y) \longrightarrow \langle g \rangle \longrightarrow 1$$

with a non-translation cyclic subgroup $\langle g \rangle$ of $\mathrm{Aut}\left(\mathbb{C}^2/\pi_1(Y) \cap \ker(\mathcal{L})\right) = \mathrm{Aut}(A_o)$ of order 2, 3, 4 or 6. In particular, Y is not simply connected and a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ with bi-elliptic minimal model Y is not saturated.

□

References

[BagneraDeFranchis] Sur les surfaces hyperelliptiques, C. R. Acad. Sci. 145 (1907), 747-749. Bagnera G., de Franchis M.,

[DiCerboStover1] De Cerbo L. F., Stover M., Multiple realizations of varieties as ball quotient compactifications, Michigan Mathematical Journal 65, 2 (2016), 441-447.

[DiCerboStover2] Di Cerbo L.F., Stover M., Punctures spheres in complex hyperbolic surfaces and bi-elliptic ball quotient compactifications, Transections of the American Mathematical Society, 372 (2019) 4627-4646.

[Groth4] Grothendieck A., Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Quatriéme partie, Publications mathématiques de l' I.H.E.S., tome 32 (1967), p. 5-361.

[Ha] Hartshorne R., *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer, 1977.

[Stover] Stover M., Cusp and b_1 growth for ball quotients and maps onto $\mathbb{Z}$ with finitely generated kernel, ArXiV : 1506.06126v3 [mathGT] 8 Aug 2018.

[Uludag] Uludag M., Covering relations between ball quotient orbifolds, Mathematische Annalen 328, 2 (2004), 503-523.