The nef cone of toroidal compactifications of $\mathcal{A}_4$

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0 Introduction

The moduli space $\mathcal{A}_g$ of principally polarised abelian g-folds is a quasi-projective variety. It has a natural projective compactification, the Satake compactification, which has bad singularities at infinity. By the method of toroidal compactification we can construct other compactifications with milder singularities, at the cost of some loss of uniqueness. Two popular choices of toroidal compactification are the Igusa and the Voronoi compactifications: these agree for $g \leq 3$ but for $g = 4$ they are different.

In this paper, we shall be mainly interested in the Voronoi compactification $\mathcal{A}_4^{\mathrm{Vor}}$ of $\mathcal{A}_4$. This is a natural choice from the point of view of moduli in view of the results of Alexeev and Nakamura ([Al], [AN]), who show that $\mathcal{A}_g^{\mathrm{Vor}}$ represents a functor of geometric interest. The case $g = 4$ is also of particular interest as it is the first case where the Torelli map is not dominant and where we therefore cannot use results from the moduli space of curves.

In our main result, Theorem I.8, we describe the cones of nef divisors on $\mathcal{A}_4^{\mathrm{Igu}}$ and $\mathcal{A}_4^{\mathrm{Vor}}$. The proofs are inductive in the sense that they involve a reduction to the cases $g = 3$ and $g = 2$, where comparable results already exist; but some new techniques are also necessary for the proof.

However, the Voronoi compactification for $g > 4$ is rather complicated and for this reason we are not at present able to extend our results even to $g = 5$.

We also show (Theorem I.15) that the canonical bundle on $\mathcal{A}_4^{\mathrm{Igu}}(n)$ is ample for $n \geq 3$.

The paper is structured as follows. Section I covers the facts we need about the different toroidal compactifications that are available. We describe the Voronoi compactification, in particular, in some detail, and state the main results. In Section II we explain what is known about the partial compactification of Mumford, which we shall need later. In Section III we describe the fine structure of the Voronoi boundary in the case $g = 4$, which is largely a matter of understanding the behaviour over the lowest stratum of the Satake compactification $\mathcal{A}_4^{\mathrm{Sat}}$. The methods here are toric and much is deduced from the combinatorics of a single cone in a certain 10-dimensional real vector space. The main technical result is that each non-exceptional boundary divisor of $\mathcal{A}_4^{\mathrm{Vor}}(n)$, where $n \geq 3$ is a level structure, has a fibration over $\mathcal{A}_3^{\mathrm{Vor}}(n)$. This is the inductive step that allows us to deduce facts about $\mathcal{A}_4^{\mathrm{Vor}}$ from the cases where $g < 4$. Finally, in Section IV, we assign to a curve in $\mathcal{A}_4^{\mathrm{Vor}}$ an invariant called the depth, which is the stratum of $\mathcal{A}_4^{\mathrm{Sat}}$ that it comes from, and work through the five cases $0 \leq \mathrm{depth}(C) \leq 4$ that arise. No two of the cases turn out to be exactly alike.

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I Toroidal compactifications

The moduli space of principally polarised abelian varieties of dimension g is given (over the complex numbers $\mathbb{C}$) as the quotient

$$\mathcal{A}_g = \mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathbb{H}_g .$$

We shall also consider full (symplectic) level-n structures. The corresponding moduli spaces are

$$\mathcal{A}_g(n) = \Gamma_g(n) \backslash \mathbb{H}_g$$

where $\Gamma_g(n)$ is the principal congruence subgroup of level n, i.e. the set of all matrices $\gamma \in \mathrm{Sp}(2g, \mathbb{Z})$ that are congruent to the unit matrix $\mathbf{1}_{2g}$ mod n. The varieties $\mathcal{A}_g(n)$ are quasi-projective, but not projective, varieties with at most finite quotient singularities. The Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$ is the minimal compactification of $\mathcal{A}_g$. It is Proj of the ring of modular forms for $\mathrm{Sp}(2g, \mathbb{Z})$. Set-theoretically $\mathcal{A}_g^{\mathrm{Sat}}$ is the disjoint union

$$\mathcal{A}_g^{\mathrm{Sat}} = \mathcal{A}_g \amalg \mathcal{A}_{g-1} \amalg \ldots \amalg \mathcal{A}_0$$

where $\mathcal{A}_0$ is a point.

Mumford [Mu] introduced a partial compactification

$$\mathcal{A}_g' = \mathcal{A}_g \amalg D_g'$$

by adding the rank 1 degenerations. This is again a quasi-projective, but not projective, variety. There are several toroidal compactifications $\mathcal{A}_g^{\Sigma}$. These depend on the choice of a fan Σ in the cone of positive definite $g \times g$ matrices (see below, Remark I.2, for a more precise explanation). All of them contain $\mathcal{A}_g'$. The most important choices are:

- The perfect cone (or first Voronoi) decomposition: see [V1] or, for instance, [Co] for definitions and details.

- The central cone decomposition $\mathrm{Igu}(g)$. This leads to the Igusa compactification $\mathcal{A}_g^{\mathrm{Igu}}$.

- The second Voronoi decomposition $\mathrm{Vor}(g)$, defined in [V2b]. This leads to the Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}$.

For $g \leq 3$ all these fans coincide. For $g = 4$ the perfect cone and central cone decompositions coincide, but the second Voronoi decomposition is a refinement of the first two: this means that there is a birational morphism $\mathcal{A}_4^{\mathrm{Vor}} \to \mathcal{A}_4^{\mathrm{Igu}}$. For general g very little is known explicitly about the decompositions and their relation to each other. There is always a morphism $\mathcal{A}_g^{\Sigma} \to \mathcal{A}_g^{\mathrm{Sat}}$ for any toroidal compactification, and $\mathcal{A}_g'$ is the inverse image of $\mathcal{A}_g \amalg \mathcal{A}_{g-1}$ under this morphism.
For $g \leq 4$ the above decompositions are explicitly known (see e.g. [V2b], [ER1], [ER2]). Since the fan $\mathrm{Vor}(4)$ is basic the space $\mathcal{A}_4^{\mathrm{Vor}}$ has only finite quotient singularities and $\mathcal{A}_4^{\mathrm{Vor}}(n)$ is smooth for $n \geq 3$. The spaces $\mathcal{A}_4^{\mathrm{Igu}}(n)$ are always singular. We shall denote by D_g^{Vor} and D_g^{Igu} the closures of D_g' in $\mathcal{A}_g^{\mathrm{Vor}}$ and $\mathcal{A}_g^{\mathrm{Igu}}$ respectively. It is well known that $\mathcal{A}_g^{\mathrm{Igu}}$ is a blow-up of the Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$ and Alexeev [Al] has proved the same for $\mathcal{A}_g^{\mathrm{Vor}}$. In particular, D_g^{Vor} and D_g^{Igu} are $\mathbb{Q}$-Cartier divisors. In any case it is clear that D_4^{Vor} is $\mathbb{Q}$-Cartier, since $\mathcal{A}_4^{\mathrm{Vor}}$ is an orbifold and thus $\mathbb{Q}$-factorial. We can see directly that D_4^{Igu} is $\mathbb{Q}$-Cartier by exhibiting a suitable support function: see Remark I.4 below.
We denote by L the $\mathbb{Q}$-line bundle of modular forms of weight 1 on $\mathcal{A}_g^{\mathrm{Sat}}$, and also its pullback to $\mathcal{A}_g^{\mathrm{Vor}}$ or to $\mathcal{A}_g^{\mathrm{Igu}}$.

Proposition I.1 *$\mathrm{Pic}\,\mathcal{A}_g' \otimes \mathbb{Q} = \mathbb{Q}D_g' \oplus \mathbb{Q}L$ for $g \geq 2$.*

Proof. This is proved by Mumford ([Mu, p. 355]) for $g \geq 4$. It is also well known for $g = 2$ and $g = 3$: see for instance [vdG]. □

For what follows we shall need explicit descriptions of the perfect cone (=central cone) and the second Voronoi decompositions in the case $g = 4$.
We fix generators $x_1, \ldots, x_4$ for a free abelian group $\mathbb{L}_4 \cong \mathbb{Z}^4$, and we denote $\mathrm{Sym}_2(\mathbb{L}_4)$ by $\mathbb{M}_4$; so $\mathbb{M}_4 \cong \mathbb{Z}^{10}$ is the space of 4×4 integer symmetric matrices with respect to the basis x_i. We shall use the basis of $\mathbb{M}_4$ given by the matrices U_{ij}^*, $1 \leq i \leq j \leq 4$ given by

$$(U_{ij}^*)_{kl} = \delta_{\{i,j\},\{k,l\}}.$$

Thus U_{ii}^* is the diagonal matrix with 1 in the ith place, corresponding to the quadratic form x_i^2, and U_{ij}^* has 1 in the ij- and ji-places, corresponding to the quadratic form $2x_ix_j$ for $1 \leq i < j \leq 4$.

The cone $\mathrm{Sym}_2^+(\mathbb{L}_4 \otimes \mathbb{R})$ is defined to be the convex hull (that is, $\mathbb{R}_{\geq 0}$-span) of the positive semidefinite forms in $\mathbb{M}_4 \otimes \mathbb{Q}$. The perfect cone decomposition and the second Voronoi decomposition are decompositions of the cone $\mathrm{Sym}_2^+(\mathbb{L}_4 \otimes \mathbb{R}) \subset \mathbb{M}_4 \otimes \mathbb{R}$ into rational polyhedral cones; that is, polyhedral cones with generators in $\mathbb{M}_4$. These cones form fans $\mathrm{Igu}(4)$ (coming from the perfect cone decomposition) and $\mathrm{Vor}(4)$, which are invariant under the action of $\mathrm{GL}(\mathbb{L}_4) \cong \mathrm{GL}(4, \mathbb{Z})$.

Remark I.2 $\mathrm{Sym}_2^+(\mathbb{L}_4 \otimes \mathbb{R})$, or more generally $\mathrm{Sym}_2^+(\mathbb{L}_g \otimes \mathbb{R})$, is defined in terms of the lattice $\mathbb{L}_g$, and does not depend just on the vector space $\mathbb{L}_g \otimes \mathbb{R}$. The same is true of the torus embeddings $T_{\mathbb{M}_g} \operatorname{emb}(\Sigma)$ (see [Oda]) which are defined by fans Σ in $\mathrm{Sym}_2^+(\mathbb{L}_g \otimes \mathbb{R})$ and which are used to construct the compactifications $\mathcal{A}_g^{\Sigma}$. If there is no danger of confusion about which lattice (and hence which torus) is intended, we sometimes denote $T_{\mathbb{M}_g} \operatorname{emb}(\Sigma)$ by X_{Σ}.

In any real vector space V (usually $V = \mathbb{M}_4 \otimes \mathbb{R}$ or its dual) we denote the closed cone $\mathbb{R}_{\geq 0} q_1 + \cdots + \mathbb{R}_{\geq 0} q_k$ generated by $\{q_1, \ldots, q_k\} \subset V$ by $\langle q_1, \ldots, q_k \rangle$. In particular $\langle \pm q \rangle$ is the line $\mathbb{R} q$.
The perfect cone decomposition has, up to $\mathrm{GL}(\mathbb{L}_4)$-equivalence, two maximal, i.e. 10-dimensional, cones: the principal cone $\Pi_1(4)$ and the second perfect cone $\Pi_2(4)$. The principal cone is given by

$$\Pi_1(4) = \left\langle x_1^2, \ldots, x_4^2, (x_1 - x_2)^2, \ldots, (x_3 - x_4)^2 \right\rangle.$$

This cone is basic. The second perfect cone is given by

$$\begin{aligned} \Pi_2(4) \quad = \quad & \left\langle x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2, \right. \\ & \left. (x_3 - x_4)^2, (x_1 + x_2 - x_3)^2, (x_1 + x_2 - x_4)^2, (x_1 + x_2 - x_3 - x_4)^2 \right\rangle. \end{aligned}$$

$\Pi_2(4)$ has 64 9-dimensional faces, which fall into two $\mathrm{GL}(\mathbb{L}_4)$-equivalence classes called BF and RT: see [ER2] and the proof of Proposition III.6, below. Representatives of the orbits are given by setting the coefficients of x_1^2, x_3^2 and x_4^2, respectively of $(x_2 - x_3)^2$, $(x_2 - x_4)^2$ and $(x_1 + x_2 - x_3 - x_4)^2$, equal to 0. These cones are basic. Hence $\mathcal{A}_4^{\mathrm{Igu}}$ has exactly one singular point, which we denote P_{sing}.
In order to describe the second Voronoi decomposition we have to introduce another ray η, generated by the sum of the primitive generators of $\Pi_2(4)$. The primitive generator e of η in $\mathbb{M}_4$ is given by

$$\begin{aligned} e \quad = \quad & \frac{1}{3}\Big[x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ & + (x_1 - x_3)^2 + (x_1 - x_4)^2 + (x_2 - x_3)^2 + (x_2 - x_4)^2 + (x_3 - x_4)^2 \\ & + (x_1 + x_2 - x_3)^2 + (x_1 + x_2 - x_4)^2 + (x_1 + x_2 - x_3 - x_4)^2\Big] \\ = \quad & 2(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 x_2 - x_1 x_3 - x_1 x_4 - x_2 x_3 - x_2 x_4). \qquad (1) \end{aligned}$$

The second Voronoi decomposition of $\mathrm{Sym}_2^+(\mathbb{L}_4 \otimes \mathbb{R})$ is the refinement of the central cone decomposition given by adding all cones which arise as the span of the central ray η with the 9-dimensional faces of $\Pi_2(4)$ and the faces of these cones, together with their $\mathrm{GL}(\mathbb{L}_4)$-translates. Up to $\mathrm{GL}(\mathbb{L}_4)$ this defines two new 10-dimensional cones, both of which are basic. Hence $\mathcal{A}_4^{\mathrm{Vor}}$ is an orbifold, and there is a map $\pi\colon \mathcal{A}_4^{\mathrm{Vor}} \to \mathcal{A}_4^{\mathrm{Igu}}$ given by blowing up a certain ideal sheaf $\mathfrak{V}$ supported at the singular point $P_{\mathrm{sing}} \in \mathcal{A}_4^{\mathrm{Igu}}$. Let E be the exceptional divisor of this blow-up, i.e. the divisor corresponding to the ray η. Actually $\mathfrak{V}$ is the maximal ideal of $\mathcal{O}_{\mathcal{A}_4^{\mathrm{Igu}},P_{\mathrm{sing}}}$ and the singularity at P_{sing} is the cone on E, but we do not need this fact. It can be deduced, for instance, from [TE, Theorem I.10].
To simplify some calculations it is also useful to consider the Voronoi transformation $\Psi\colon \mathbb{L}_4 \to \mathbb{L}_4$, defined by

$$\Psi\colon (x_1,x_2,x_3,x_4) \longmapsto (x_1+x_2, x_1-x_2, x_1-x_3, x_1-x_4) \tag{2}$$

and the induced embedding

$$\Psi' = \mathrm{Sym}_2(\Psi)\colon \mathbb{M}_4 = \mathrm{Sym}_2(\mathbb{L}_4) \longrightarrow \mathbb{M}_4.$$

Note that Ψ and Ψ' are embeddings but not isomorphisms, since $\det\Psi = 2$. We have

$$\Psi'\big(\Pi_2(4)\big) = \big\langle \big\{(x_i \pm x_j)^2,\ 1 \le i < j \le 4\big\}\big\rangle,$$

so if we put $\mathbf{y} = \Psi_{\mathbb{Q}}^{-1}(\mathbf{x})$ we may express $\Pi_2(4)$ in the convenient form

$$\begin{aligned} \Pi_2(4) &= \big\langle (y_i \pm y_j)^2, 1 \le i < j \le 4 \big\rangle \\ &= \Big\{ \sum_{1\le i<j\le 4} \big(\beta_{ij}(y_i+y_j)^2 + \rho_{ij}(y_i-y_j)^2\big) \,\Big|\, \beta_{ij}, \rho_{ij} \in \mathbb{R}_{\ge 0} \Big\}. \end{aligned} \tag{3}$$

The generator e of η is mapped to

$$\Psi'(e) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2).$$

Now let $\pi\colon \mathcal{A}_4^{\mathrm{Vor}}(n) \to \mathcal{A}_4^{\mathrm{Igu}}(n)$ and let $E(n)$ be the exceptional divisor in $\mathcal{A}_4^{\mathrm{Vor}}(n)$. We set $D_4(n) = \pi^*\big(D_4^{\mathrm{Igu}}(n)\big)$.

Proposition I.3 $D_4(n) = \pi^*\big(D_4^{\mathrm{Igu}}(n)\big) = D_4^{\mathrm{Vor}}(n) + 4E(n)$.

Proof. The level structure plays no part here so we suppress it, taking $n=1$ without loss of generality and writing E for $E(1)$ and so on. We shall first consider the toric situation. Let

$$\mathrm{Tr}\colon \mathbb{M}_4 \cong \mathrm{Sym}_2(\mathbb{Z}^4) \to \mathbb{Z}$$

be the linear form given by the trace. Then $\mathrm{Tr}' = \mathrm{Tr} \circ \Psi'$ is an integral linear form on $\mathbb{M}_4$ which is 2-divisible. The form $\frac{1}{2}\mathrm{Tr}'$ assumes the value 1 on all

basic generators of the 1-dimensional rays of $\Pi_2(4)$ and the value 4 on the $\mathbb{M}_4$-primitive generator e of η.
Locally (analytically) near the singular point P_{sing} on $\mathcal{A}_4^{\text{Igu}}$ and near the exceptional locus E in $\mathcal{A}_4^{\text{Vor}}$, the moduli spaces $\mathcal{A}_4^{\text{Igu}}$ and $\mathcal{A}_4^{\text{Vor}}$ are isomorphic to finite quotients of the toric varieties $X_{\text{Igu}(4)}$ and $X_{\text{Vor}(4)}$ respectively. The finite group by which we take the quotient is the stabiliser of P_{sing}, respectively E. It is a subgroup of $\mathrm{GL}(\mathbb{L}_4)$, which acts on $\mathrm{Sym}_2^+(\mathbb{L}_4 \otimes \mathbb{R})$ by $M \mapsto {}^tQ^{-1}MQ^{-1}$. It is enough to compute the subgroup which fixes E pointwise. A straightforward calculation shows that this is $\pm\mathbf{1}_4$, which acts trivially. Together with the above toric calculation this shows that $\pi^*(D_4^{\text{Igu}}) = D_4^{\text{Vor}} + 4E$. □

Remark I.4 Considering $\frac{1}{2}\operatorname{Tr}'$ as a support function on the fan $\mathrm{Igu}(4)$ shows that the boundary D_4^{Igu} of $\mathcal{A}_4^{\text{Igu}}$ is a $\mathbb{Q}$-Cartier divisor and that the boundary of $\mathcal{A}_4^{\text{Igu}}(n)$ for $n \geq 3$ is a Cartier divisor.

Corollary I.5 *Let $n \geq 3$. Then $\mathcal{A}_4^{\text{Igu}}(n)$ is a Gorenstein variety with canonical singularities.*

Proof. For $n \geq 3$ the group $\Gamma_g(n)$ is neat. Hence we only have to consider singularities which come from the toric construction. The varieties $\mathcal{A}_4^{\text{Igu}}(n)$, $n \geq 3$ are normal varieties with finitely many singularities. Outside these singularities the canonical divisor is given by

$$K = \left(5L - D_4^{\text{Igu}}(n)\right)|_{\mathcal{A}_{4,\text{smooth}}^{\text{Igu}}(n)}$$

where $D_4^{\text{Igu}}(n)$ is the boundary. Both L and $D_4^{\text{Igu}}(n)$ are Cartier divisors on $\mathcal{A}_4^{\text{Igu}}(n)$ and hence

$$K_{\mathcal{A}_4^{\text{Igu}}(n)} = i_*K = 5L - D_4^{\text{Igu}}(n) \tag{4}$$

where i is the inclusion. This shows that these varieties are Gorenstein.
The varieties $\mathcal{A}_4^{\text{Vor}}(n)$, $n \geq 3$ are smooth and the canonical divisor is

$$K_{\mathcal{A}_4^{\text{Vor}}(n)} = 5L - D_4^{\text{Vor}}(n) - \sum_s E_s(n), \tag{5}$$

where the $E_s(n)$ are the irreducible exceptional divisors of the blow-up map $\pi\colon \mathcal{A}_4^{\text{Vor}}(n) \to \mathcal{A}_4^{\text{Igu}}(n)$. Since

$$\pi^*\left(K_{\mathcal{A}_4^{\text{Igu}}(n)}\right) = 5L - D_4^{\text{Vor}}(n) - \sum_s 4E_s(n)$$

it follows that $\mathcal{A}_4^{\text{Igu}}(n)$ has canonical, in fact terminal, singularities. □

We define the open set $\mathcal{A}_4^0 = \mathcal{A}_4^{\text{Igu}} \setminus P_{\text{sing}} = \mathcal{A}_4^{\text{Vor}} \setminus E$, common to both toroidal compactifications.

Proposition I.6 *The Picard groups satisfy*

$$\begin{aligned}\operatorname{Pic}\mathcal{A}_4^{\mathrm{Igu}}\otimes\mathbb{Q} &\cong \operatorname{Pic}\mathcal{A}_4^0\otimes\mathbb{Q} \cong \mathbb{Q}L\oplus\mathbb{Q}D_4^{\mathrm{Igu}},\\ \operatorname{Pic}\mathcal{A}_4^{\mathrm{Vor}}\otimes\mathbb{Q} &= \mathbb{Q}L\oplus\mathbb{Q}D_4^{\mathrm{Vor}}\oplus\mathbb{Q}E.\end{aligned}$$

Proof. Restricting line bundles defines maps

$$\operatorname{Pic}\mathcal{A}_4^{\mathrm{Igu}}\longrightarrow\operatorname{Pic}\mathcal{A}_4^0\longrightarrow\operatorname{Pic}\mathcal{A}_4'.$$

All the varieties involved are normal and since the codimensions of $\mathcal{A}_4^0\setminus\mathcal{A}_4'$ in $\mathcal{A}_4^0$ and of $\mathcal{A}_4^{\mathrm{Igu}}\setminus\mathcal{A}_4^0$ in $\mathcal{A}_4^{\mathrm{Igu}}$ are at least 2, these maps are injective. Since $\operatorname{Pic}\mathcal{A}_4'=\mathbb{Q}L\oplus\mathbb{Q}D_4'$ and since both L and D_4' extend to $\mathbb{Q}$-line bundles on $\mathcal{A}_4^{\mathrm{Igu}}$ these maps are also surjective.
The exceptional locus E is irreducible, being the image of the closure of a torus orbit. Hence the claim about $\operatorname{Pic}\mathcal{A}_4^{\mathrm{Vor}}\otimes\mathbb{Q}$ follows from the exact sequence of Chow groups

$$A_9(E)\otimes\mathbb{Q}\to A_9(\mathcal{A}_4^{\mathrm{Vor}})\otimes\mathbb{Q}\to A_9(\mathcal{A}_4^0)\otimes\mathbb{Q}\to 0$$

(see [Ful1, Proposition 1.8]). □

We are now in a position to state the main results of this paper. The first result is auxiliary and can be stated for general $g\geq 2$. Note that although $\mathcal{A}_g'$ is not a projective variety we can still speak about nef line bundles. By this we mean line bundles whose restriction to each complete curve has non-negative degree.

Proposition I.7 *The nef cone of $\mathcal{A}_g'$ for $g\geq 2$ is given by*

$$\operatorname{Nef}(\mathcal{A}_g')=\left\{aL-bD_g' \mid\ b\geq 0,\ a\geq 12b\right\}.$$

For the projective varieties $\mathcal{A}_4^{\mathrm{Igu}}$ and $\mathcal{A}_4^{\mathrm{Vor}}$ we obtain much better results.

Theorem I.8 *The nef cone of $\mathcal{A}_4^{\mathrm{Igu}}$ is given by*

$$\operatorname{Nef}(\mathcal{A}_4^{\mathrm{Igu}})=\left\{aL-bD_4^{\mathrm{Igu}}\mid b\geq 0,\ a\geq 12b\right\}.$$

The nef cone of $\mathcal{A}_4^{\mathrm{Vor}}$ is given by

$$\operatorname{Nef}(\mathcal{A}_4^{\mathrm{Vor}})=\{aL-bD_4-cE\mid a\geq 12b,\ b\geq 2c\geq 0\}.$$

Remark I.9 If we work with D_4^{Vor} rather than D_4 then, in view of Proposition I.3, the nef cone of the Voronoi compactification has the following description:

$$\operatorname{Nef}(\mathcal{A}_4^{\mathrm{Vor}})=\left\{\alpha L-\beta D_4^{\mathrm{Vor}}-\gamma E\mid \beta\geq 0,\ \alpha\geq 12\beta,\ \gamma\geq 4\beta\geq \tfrac{8}{9}\gamma\right\}.$$

Remark I.10 We have Galois coverings

$$\alpha_{n,\mathrm{Igu}}\colon \mathcal{A}_4^{\mathrm{Igu}}(n) \to \mathcal{A}_4^{\mathrm{Igu}}, \qquad \alpha_{n,\mathrm{Vor}}\colon \mathcal{A}_4^{\mathrm{Vor}}(n) \to \mathcal{A}_4^{\mathrm{Vor}}$$

given by an action of $\mathrm{Sp}(8,\mathbb{Z}/n)$. These coverings, which extend the obvious covering $\mathcal{A}_4(n) \to \mathcal{A}_4$, exist because the definitions of perfect cone and Voronoi decomposition ([V1], [V2b], [Co], [ER1]) are purely lattice-theoretic and so the collections of fans that define the Igusa and Voronoi compactifications are $\mathrm{Sp}(8,\mathbb{Z})$-invariant. Compare [San, Proposition 5.1] for a similar situation in the $g = 2$ case.
The inverse images $D_4^{\mathrm{Igu}}(n)$ and $D_4^{\mathrm{Vor}}(n)$ of D_4^{Igu} and D_4^{Vor} will have several components, as will the inverse image $E(n)$ of E. The above Galois covers are ramified of order n along the boundary, i.e. $\alpha_{n,\mathrm{Igu}}^*\big(D_4^{\mathrm{Igu}}\big) = nD_4^{\mathrm{Igu}}(n)$ and $\alpha_{n,\mathrm{Vor}}^*\big(D_4^{\mathrm{Vor}}\big) = nD_4^{\mathrm{Vor}}(n)$; it then follows from Proposition I.3 that $\alpha_{n,\mathrm{Vor}}^*(E) = nE(n)$.

The Picard groups of $\mathcal{A}_4^{\mathrm{Igu}}(n)$ and $\mathcal{A}_4^{\mathrm{Vor}}(n)$ will be much bigger than those of $\mathcal{A}_4^{\mathrm{Igu}}$ and $\mathcal{A}_4^{\mathrm{Vor}}$, but we still obtain a description of part of the nef cone.

Corollary I.11 *A divisor $aL - bD_4^{\mathrm{Igu}}(n)$ on $\mathcal{A}_4^{\mathrm{Igu}}(n)$ is nef if and only if $b \geq 0$ and $a \geq 12b/n$.*
A divisor $aL - bD_4(n) - cE(n)$ on $\mathcal{A}_4^{\mathrm{Vor}}(n)$ is nef if and only if $a \geq 12b/n$ and $b \geq 2c \geq 0$.

This also allows us to draw a conclusion about the nefness of the canonical divisor.

Lemma I.12 *For any $n \in \mathbb{N}$*

$$\begin{aligned} K_{\mathcal{A}_4^{\mathrm{Igu}}(n)} &= 5L - D_4^{\mathrm{Igu}}(n), \\ K_{\mathcal{A}_4^{\mathrm{Vor}}(n)} &= 5L - D_4^{\mathrm{Vor}}(n) - E(n) = 5L - D_4(n) + 3E(n). \end{aligned}$$

Proof. For $n \geq 3$ this was shown above (equations (4) and (5)). To show that these equalities also hold for $n = 1$ and $n = 2$, it is enough to check that there are no elements in $\mathrm{Sp}(8,\mathbb{Z})$ whose fixed locus in $\mathbb{H}_4$ is a divisor. This follows easily from [Tai, Lemma 4.1]: if an element $\gamma \in \mathrm{Sp}(2g,\mathbb{Z})$ of order m fixes $\tau \in \mathbb{H}_g$ then it acts on the tangent space with eigenvalues $e^{2\pi i(t_j+t_k)/m}$, where $t_j,\ t_k \in \mathbb{Z}$ and $1 \leq j \leq k \leq g$. If τ is a general point of a fixed divisor then $t_j + t_k \equiv 0 \mod m$ for all but one pair of indices, say (j_0, k_0). But this is impossible if $g \geq 3$. To see this, we consider first the case $j_0 = k_0$. We may assume $j_0 = 1$, so $2t_1 \not\equiv 0$, but then $t_1 \equiv -t_2$ so $2t_2 \not\equiv 0$, and (j_0, k_0) is not unique. On the other hand, if $j_0 \neq k_0$, we may assume $j_0 = 1$ and $k_0 = 2$, so $t_1 + t_2 \not\equiv 0$; but in that case $t_3 \equiv -t_2 \equiv -t_1$ so $2t_3 \not\equiv 0$ and again (j_0, k_0) is not unique. □

Corollary I.13 *If $n \geq 3$, then the canonical bundle of $\mathcal{A}_4^{\mathrm{Igu}}(n)$ is nef. On the other hand, the canonical bundle of $\mathcal{A}_4^{\mathrm{Vor}}(n)$ is never nef.*

Remark I.14 $\mathcal{A}_4^{\mathrm{Igu}}(n)$ is a minimal model as defined in ([KM, Definition 2.13]), because the singularities are terminal; but they are not $\mathbb{Q}$-factorial because $\Pi_2(4)$ is not simplicial, and some authors prefer to reserve the term "minimal model" for the Mori category, whose objects are projective varieties with $\mathbb{Q}$-factorial terminal singularities. By toric methods, following the argument of Fujino [Fuj], a small $\mathbb{Q}$-factorialisation may be constructed, and this will be a $\mathbb{Q}$-factorial minimal model.

Theorem I.15 *If $n \geq 3$ then the canonical bundle of $\mathcal{A}_4^{\mathrm{Igu}}(n)$ is ample.*

Proof. By Lemma I.12 the canonical bundle satisfies the conditions of Corollary I.11, but with strict inequalities, $a > 12b/n > 0$. Hence $K_{\mathcal{A}_4^{\mathrm{Igu}}(n)}$ belongs to the interior of $\mathrm{Nef}\left(\mathcal{A}_4^{\mathrm{Igu}}(n)\right) \cap \left(\mathbb{R}L + \mathbb{R}D_4^{\mathrm{Igu}}(n)\right)$. Let H_0 be an ample class on $\mathcal{A}_4^{\mathrm{Igu}}(n)$ spanned by L and $D_4^{\mathrm{Igu}}(n)$: such an H_0 exists because $\mathcal{A}_4^{\mathrm{Igu}}(n)$ is projective and $\mathbb{R}L + \mathbb{R}D_4^{\mathrm{Igu}}(n)$ is the $\mathrm{Sp}(8, \mathbb{Z}/n)$-invariant part of $\mathrm{Pic}\left(\mathcal{A}_4^{\mathrm{Igu}}(n)\right) \otimes \mathbb{R}$, so if H is some ample line bundle class it is sufficient to take $H_0 = \sum_{\gamma \in \mathrm{Sp}(8,\mathbb{Z}/n)} \gamma(H)$. Now we copy the proof of Kleiman's criterion given in [KM, 1.39]: $tK_{\mathcal{A}_4^{\mathrm{Igu}}(n)} - H_0$ is nef for $t \gg 0$, so for any dimension d subscheme $Z \subset \mathcal{A}_4^{\mathrm{Igu}}(n)$ we have $(tK_{\mathcal{A}_4^{\mathrm{Igu}}(n)})^d \cdot Z \geq H_0^d \cdot Z > 0$ (this is a non-trivial step in the proof of [KM, 1.38]). Therefore $tK_{\mathcal{A}_4^{\mathrm{Igu}}(n)}$ is ample by the Nakai-Moishezon criterion, [KM, Theorem 1.37]. □

Thus $\mathcal{A}_4^{\mathrm{Igu}}(n)$ is the canonical model if $n \geq 3$.

II The nef cone of the partial compactification

We shall work with the partial compactification $\mathcal{A}'_g = \mathcal{A}_g \cup D'_g$, sometimes with an additional level-n structure $\mathcal{A}'_g(n) = \mathcal{A}_g(n) \cup D'_g(n)$. If $n \geq 2$, then $D'_g(n) = \sum_i D'_{g,i}(n)$ consists of several disjoint components, each of which has a natural fibration $D'_{g,i}(n) \to \mathcal{A}_{g-1}(n)$. For $n \geq 3$ this is the universal family over $\mathcal{A}_{g-1}(n)$, and for $n = 1, 2$ it is a family of Kummer varieties. Indeed

$$D'_{g,i}(n) = \left(\mathbb{Z}^{2g-2} \rtimes \Gamma_{g-1}(n)\right) \backslash \mathbb{C}^{g-1} \times \mathbb{H}_{g-1}.$$

To describe this action let $m = (m', m'')$ with $m', m'' \in \mathbb{Z}^{g-1}$ and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{g-1}(n)$. Then

$$(m, \gamma) \colon (z, \tau) \mapsto \left((z + nm' + nm''\tau)(C\tau + D)^{-1}, (A\tau + B)(C\tau + D)^{-1}\right).$$

If $n \geq 3$, then the fibre of the map $D'_{g,i}(n) \to \mathcal{A}_{g-1}(n)$ over a point $[\tau]$ is the abelian variety $A_{n,n\tau}$ whose period matrix is given by $(n\mathbf{1}_{g-1}, n\tau)$. For $n = 1, 2$ we obtain the Kummer variety $A_{n,n\tau}/(\pm 1)$.
Let $\Theta_{00}(z,\tau)\colon \mathbb{C}^{g-1} \times \mathbb{H}_{g-1} \to \mathbb{C}$ be the standard theta function. The automorphy factors of Θ_{00}^2 define a $\mathbb{Q}$-line bundle on $D'_{g,i}(n)$ which we shall denote by $M'(n)$. For $n \geq 3$, let $N' = N_{D'_{g,i}(n)/\mathcal{A}'_g(n)}$ be the normal bundle of the boundary component $D'_{g,i}(n)$ in $\mathcal{A}'_g(n)$.

Lemma II.1 *If $n \geq 3$ then $M'(n) = -nN' + L$.*

Proof. This is proved in [Hu, Proposition 2.3]. The proof consists of comparing the cocycles of $M'(n)$ and N'. □

Proposition II.2 *The nef cone of $\mathcal{A}'_g$ for $g \geq 2$ is given by*

$$\mathrm{Nef}(\mathcal{A}'_g) = \left\{ aL - bD'_g \mid b \geq 0, a \geq 12b \right\}.$$

Proof. The condition $b \geq 0$ is necessary, since L is trivial on the fibres of $D'_{g,i}(n) \to \mathcal{A}_{g-1}(n)$, whereas $-D'_g(n)$ is ample on the fibres (cf. [Mu, Proposition 1.8]). In order to prove that $a \geq 12b$ is a necessary condition we consider curves C of the form $X(1) \times \{A\}$ in $\mathcal{A}'_g$ where A is a fixed $(g-1)$-dimensional abelian variety and $X(1)$ is the modular curve of level 1, i.e. we consider a family of abelian varieties of type $E_\tau \times A$ where E_τ is an elliptic curve degenerating to a nodal curve. Such a family is indeed contained in $\mathcal{A}'_g$ and for general A, and $C.D'_g = 1$. This is because the corresponding family with a level-n structure ($n \geq 3$ as usual) meets the boundary transversally in a smooth point. Since the degree of the $\mathbb{Q}$-line bundle L on $X(1)$ is $1/12$ we find the necessary condition $a \geq 12b$.
Next we shall prove that these conditions are sufficient. Here we shall distinguish between curves C which meet $\mathcal{A}_g$ and curves C which are contained in the boundary D'_g. For curves of the first type the result was already proved in [Hu, Proposition 1.4]. Since the argument is very short we shall repeat it here. Assume that $b \geq 0$ and $a \geq 12b$. Since L is ample on the Satake compactification, it follows that $L.C > 0$, and we can assume that $b > 0$. Choose some $\varepsilon > 0$ with $a/b > 12 + \varepsilon$ and some point $[\tau] \in \mathcal{A}_g$ on C. By a result of Weissauer [Wei, p. 220] there exists a modular form F of weight k and vanishing order m such that $F(\tau) \neq 0$ and $m/k \geq 1/(12+\varepsilon)$. In terms of divisors this gives

$$kL = mD'_g + D_F,\ C \not\subset D_F$$

where D_F is the closure in $\mathcal{A}'_g$ of the divisor $\{F = 0\} \subset \mathcal{A}_g$. Hence

$$\left(\tfrac{k}{m}L - D'_g\right).C = \tfrac{1}{m}D_F.C \geq 0$$

and since $a/b > 12+\varepsilon \geq k/m$ and $L.C > 0$ we conclude that

$$\left(\tfrac{a}{b}L - D'_g\right).C > \left(\tfrac{k}{m}L - D'_g\right).C \geq 0.$$

Finally let C be a curve contained in D'_g. Here it is slightly easier to work with level structures: we choose some $n \in \mathbb{N}$ and assume that $C \subset D'_{g,i}(n)$ for some boundary component $D'_{g,i}(n)$ of $\mathcal{A}'_g(n)$. By Lemma II.1

$$\left(aL - bD'_{g,i}(n)\right)|_{D'_{g,i(n)}} = \left(a - \tfrac{b}{n}\right)L + \tfrac{b}{n}M'(n).$$

The condition $a \geq 12b$ for level 1 now becomes $a \geq 12b/n$. In any case $a - b/n \geq 0$ and hence it suffices to prove that $M'(n).C \geq 0$. Fix a prime p, and choose n so that $n \equiv 0 \mod 4p^2$. If $m', m'' \in \frac{1}{2p}\mathbb{Z}^{g-1}$ then the functions $\Theta^2_{m',m''}(z,\tau)$ define sections of $M'(n)$, by [Hu, Proposition 2.3]: the proof uses the theta transformation formula and the formulae (Θ1)–(Θ3) from [Ig] to show that $\Theta^2_{m',m''}(z,\tau)$ has the appropriate automorphy factor. But this shows that $M'(n)$ is generated by global sections and hence $M'(n).C \geq 0$. $\square$

III Structure of the Voronoi boundary

In this section we revert to the case $g = 4$ and examine the geometry of the Voronoi boundary in detail. Our chief purpose is to prove that the fibration $D'_{4,i}(n) \to \mathcal{A}_3(n)$ extends to the closure of $D'_{4,i}(n)$ in the Voronoi compactification of $\mathcal{A}_4(n)$. This results in a fibration of each non-exceptional boundary divisor in $\mathcal{A}_4^{\mathrm{Vor}}(n)$ over $\mathcal{A}_3^{\mathrm{Vor}}(n) = \mathcal{A}_3^{\mathrm{Igu}}(n)$. The proof involves careful study of the combinatorics of the cone $\Pi_2(4)$, and we also assemble in this section some other results of that nature which we shall need later.

Proposition III.1 *Let $n \geq 3$, and suppose $D_{4,i}^{\mathrm{Vor}}(n) \subset \mathcal{A}_4^{\mathrm{Vor}}(n)$ is the closure of a boundary divisor, not contracted by $\pi\colon \mathcal{A}_4^{\mathrm{Vor}}(n) \to \mathcal{A}_4^{\mathrm{Igu}}(n)$. Then there is a morphism*

$$p_i = p_{i,n}\colon D_{4,i}^{\mathrm{Vor}}(n) \longrightarrow \mathcal{A}_3^{\mathrm{Vor}}(n)$$

extending the fibration $D'_{4,i}(n) \to \mathcal{A}_3(n)$.

Proof. We work, without loss of generality, with $D_{4,1}^{\mathrm{Vor}}(n)$, corresponding to $\tau_{11} \to i\infty$: if $n \geq 3$ then $D_{4,1}^{\mathrm{Vor}}(n)$ is normal (see Remark III.9, below). Thus we fix a rank 3 sublattice $\mathbb{L}_3 = \mathbb{Z}x_2 + \mathbb{Z}x_3 + \mathbb{Z}x_4 \subset \mathbb{L}_4$ and set $\mathbb{M}_3 = \mathrm{Sym}_2(\mathbb{L}_3)$. The projection $\mathrm{pr}_1\colon \mathbb{L}_4 \to \mathbb{L}_3$ with kernel $\mathbb{Z}x_1$ induces a projection $\mathrm{Sym}_2 \mathrm{pr}_1\colon \mathbb{M}_4 \to \mathbb{M}_3$ with kernel spanned by the U^*_{1j}, $1 \leq j \leq 4$.

Consider the matrix

$$\tilde{\tau} = \left(\begin{array}{c|ccc} * & \tau_{12} & \tau_{13} & \tau_{14} \\ \hline \tau_{12} & \tau_{22} & \tau_{23} & \tau_{24} \\ \tau_{13} & \tau_{23} & \tau_{33} & \tau_{34} \\ \tau_{14} & \tau_{24} & \tau_{34} & \tau_{44} \end{array} \right).$$

Then

$$z = (\tau_{12}, \tau_{13}, \tau_{14}) \in \mathbb{C}^3, \ \tau = (\tau_{ij})_{2\le i,j\le 4} \in \mathbb{H}_3.$$

and the map $\tilde{\tau} \mapsto \tau$ is $\mathrm{Sp}(6,\mathbb{Z})$-equivariant and therefore induces a rational map $p_{1,n}\colon D_{4,1}^{\mathrm{Vor}}(n) \dashrightarrow \mathcal{A}_3^{\mathrm{Vor}}(n)$. The problem is to extend this map to the cusps of $\mathcal{A}_3^{\mathrm{Vor}}(n)$.
We first check that $p_{1,n}$ extends over the smallest cusps, i.e. over $\phi_n^{-1}(\mathcal{A}_0)$, where $\phi_n\colon \mathcal{A}_4^{\mathrm{Vor}}(n) \to \mathcal{A}_4^{\mathrm{Sat}}$. This is the only case which is nontrivial. Near a component of $\phi_n^{-1}(\mathcal{A}_0)$, the boundary divisor $D_{4,1}^{\mathrm{Vor}}(n)$ is given by the fan $\mathrm{Star}\big(\langle x_1^2\rangle, \mathrm{Vor}(4)\big)$ with respect to the lattice $\overline{\mathbb{M}}_4 = \mathbb{M}_4/\mathbb{Z}x_1^2$; see for example [Ful2, 3.1]. The map we are trying to extend, $p_{1,n}$, is given on the torus part of this toric variety by forgetting all coordinates involving x_1. More precisely, $D_{4,1}^{\mathrm{Vor}}(n)$ is locally isomorphic to an analytic open set in

$$X_{\mathrm{Star}\langle x_1^2\rangle} = T_{\overline{\mathbb{M}}_4}\mathrm{emb}\,\left(\mathrm{Star}\left(\langle x_1^2\rangle, \mathrm{Vor}(4)\right)\right).$$

The natural embedding $\mathbb{M}_3 \to \overline{\mathbb{M}}_4$ induces a map on the corresponding tori

$$T_{\overline{\mathbb{M}}_4} = \mathrm{Hom}(\overline{\mathbb{M}}_4, \mathbb{C}^*) \longrightarrow T_{\mathbb{M}_3}$$

which is $p_{1,n}$ on the torus part of $X_{\mathrm{Star}\langle x_1^2\rangle}$.
Now the result follows from Lemma III.2 below. The extension to the lower cusps, and the compatibility of the extensions, are immediate consequences of the straightforward fact that if σ is an Igusa (i.e. Voronoi) cone in $\mathrm{Sym}_2^+(\mathbb{Z}^g)$ for $g < 4$ then $\mathrm{Sym}_2\,\mathrm{pr}_1(\sigma)$ is an Igusa cone in $\mathrm{Sym}_2^+(\mathbb{Z}^{g-1})$. □

Lemma III.2 *The map $T_{\overline{\mathbb{M}}_4} \to T_{\mathbb{M}_3}$ extends to a $\mathrm{GL}(\mathbb{L}_3)$-equivariant map*

$$p_{1,n}\colon X_{\mathrm{Star}\langle x_1^2\rangle} \longrightarrow T_{\mathbb{M}_3}\mathrm{emb}\,\big(\mathrm{Vor}(3)\big)$$

of the corresponding torus embeddings.

Proof. We need to check that the dual of the embedding, which may be thought of as a projection $\overline{\mathbb{M}}_4 \to \mathbb{M}_3$ with kernel spanned by the classes of U_{1j}^*, is a map of fans (the $\mathrm{GL}(\mathbb{L}_3)$-equivariance is automatic). To do that we must show that the projection of any cone in $\mathrm{Star}\,\big(\langle x_1^2\rangle, \mathrm{Vor}(4)\big)$ lies in a cone of $\mathrm{Vor}(3)$. By the definition of Star, it is enough to show that if $\sigma \in \mathrm{Vor}(4)$ and $\sigma \succ \langle x_1^2\rangle$, then $\mathrm{Sym}_2\,\mathrm{pr}_1(\sigma) \subseteq \sigma'$ for some $\sigma' \in \mathrm{Vor}(3)$. Moreover, since

$\mathrm{Vor}(4)$ and $\mathrm{Vor}(3)$ are fans and $\mathrm{Sym}_2\,\mathrm{pr}_1$ preserves the relation $\succ$ among cones, it is only necessary to check this for top-dimensional cones in $\mathrm{Vor}(4)$ which have $\langle x_1^2\rangle$ as a face. The result therefore follows from Prop III.7 and Prop III.3, below. □

In verifying the assertion made in the above proof there are two cases to be considered separately. If $\eta \prec \sigma$ (up to $\mathrm{GL}(\mathbb{L}_4)$-equivalence) then σ corresponds to a point of the exceptional locus $E \subset \mathcal{A}_4^{\mathrm{Vor}}$. Otherwise σ corresponds to a point of $\mathcal{A}_4^{\mathrm{Igu}}$.

Proposition III.3 *Suppose that $\sigma \in \mathrm{Vor}(4)$ is of maximal dimension (i.e. dimension 10), that $\langle x_1^2\rangle \prec \sigma$ and that no $\mathrm{GL}(\mathbb{L}_4)$-translate of η is a face of σ. Then $\mathrm{Sym}_2\,\mathrm{pr}_1(\sigma) \in \mathrm{Vor}(3)$.*

Proof. In this case, σ is equivalent under $\mathrm{GL}(\mathbb{L}_4)$ to the first perfect domain $\Pi_1(4)$. (The level structure plays no role here.) More than that: the subgroup of $\mathrm{GL}(\mathbb{L}_4)$ that preserves $\Pi_1(4)$ permutes the generating rays transitively, so σ is even equivalent to $\Pi_1(4)$ under the stabiliser of x_1^2. To see that the rays are permuted transitively, note first that the permutation matrices are in the stabiliser of $\Pi_1(4)$ in $\mathrm{GL}(\mathbb{L}_4)$, so all four monomial generators x_i^2 are equivalent to one another and so are all six binomial generators $(x_i - x_j)^2$. The element of $\mathrm{GL}(\mathbb{L}_4)$ given by $x_i \mapsto x_i - x_2$ for $i \neq 2$ and $x_2 \mapsto -x_2$ preserves $\Pi_1(4)$ but does not preserve the distinction between monomial and binomial generators, so all the generators are in one orbit.
Since, for any g,

$$\Pi_1(g) = \langle x_1^2, \ldots, x_g^2, (x_i - x_j)^2\ (1 \le i < j \le g)\rangle,$$

the projection of $\Pi_1(g)$ to $\mathbb{M}_{g-1}$ is $\Pi_1(g-1)$. Since $\Pi_1(g) \in \mathrm{Vor}(g)$ for all g and $\mathrm{Vor}(g)$ is $\mathrm{GL}(\mathbb{L}_g)$-invariant, we certainly have $\mathrm{Sym}_2\,\mathrm{pr}_1\big(\Pi_1(g)\big) \in \mathrm{Vor}(g-1)$. □

This part of the argument is not restricted to $g = 4$, but it only applies to $\Pi_1(g)$. We want to mention an alternative proof, which uses the information we have in a slightly different way.

Lemma III.4 *Let $\mathbb{L}$ be a lattice and $l_i\colon \mathbb{L} \to \mathbb{Z}$ be linear forms such that the quadratic form $\sum l_i^2$ is positive definite. Then the Delaunay decomposition for the quadratic form $\sum \alpha_i l_i^2$ is independent of the choice of positive constants α_i if, and only if, the forms l_i define a dicing; that is, the 0-skeleton of the cell decomposition defined by the hyperplanes $\{l_i(x) = n\}$ for $n \in \mathbb{Z}$ coincides with the original lattice $\mathbb{L}$.*

Proof. [ABH, Lemma 3.1]. □

Recall that it would be enough for our purposes to prove that $\mathrm{Sym}_2 \mathrm{pr}_1(\sigma)$ is contained in a cone of $\mathrm{Vor}(3)$. Every ray in $\mathrm{Vor}(4)$ is either a $\mathrm{GL}(\mathbb{L}_4)$-translate of η or spanned by the square of a linear form. So if $\sigma \in \mathrm{Vor}(4)$ satisfies the conditions of Proposition III.3, then $\sigma = \langle\{l_i^2\}\rangle$ for some linear forms $l_i \colon \mathbb{L}_4^* \to \mathbb{Z}$. Now we can apply the following proposition.

Proposition III.5 *Suppose $\sigma \in \mathrm{Vor}(g)$ is a cone of maximal dimension which is spanned by squares of linear forms. Then $\mathrm{Sym}_2 \mathrm{pr}_1(\sigma)$ is contained in a cone of $\mathrm{Vor}(g-1)$.*

Proof. If $\sigma = \langle\{l_i^2\}\rangle$ then, since σ is of maximal dimension, $\sum l_i^2$ is positive definite. Therefore, by Lemma III.4 the l_i define a dicing of $\mathbb{L}_g^* \otimes \mathbb{R}$. If $\xi' \in \mathbb{L}_{g-1} \otimes \mathbb{R}$ is a point of the 0-skeleton of the decomposition induced by the $\mathrm{pr}_1(l_i)$, then it is the projection of a cell in the dicing of $\mathbb{L}_g \otimes \mathbb{R}$ induced by the l_i. Any vertex ξ of this cell is in $\mathbb{L}_g$, so $\xi' = \mathrm{pr}_1(\xi)$ is in $\mathbb{L}_{g-1}$. Therefore the $\mathrm{pr}_1(l_i)$ induce a dicing of $\mathbb{L}_{g-1} \otimes \mathbb{R}$.
The projection of any positive definite form is again positive definite, so $\sum (\mathrm{pr}_1(l_i))^2 = \mathrm{Sym}_2 \mathrm{pr}_1(\sum l_i^2)$ is positive definite. Therefore, again by Lemma III.4, the Delaunay decompositions induced by any two forms in the interior of $\mathrm{Sym}_2 \mathrm{pr}_1(\sigma)$ are the same. Hence $\mathrm{Sym}_2 \mathrm{pr}_1(\sigma)$ is contained in a cone of $\mathrm{Vor}(g-1)$. $\square$

Now suppose that $\eta \prec \sigma$, and that $\langle x_1^2\rangle \prec \sigma$, so that σ gives rise to a cone in $\mathrm{Star}\big(\langle x_1^2\rangle, \mathrm{Vor}(4)\big)$. We need only consider 10-dimensional cones up to the action of the stabiliser $\tilde{G}_1$ in $\mathrm{GL}(\mathbb{L}_4)$ of $\langle x_1^2\rangle$. Such a cone is spanned by η and a 9-dimensional facet of the second perfect domain $\Pi_2(4)$. These facets are described in [ER2]. The authors of [ER2] have kept the coordinates x_i and work with the cone $\Psi'\big(\Pi_2(4)\big)$, but we prefer to work directly with $\Pi_2(4)$ and to display the symmetry instead by using the coordinates $y_i = \Psi^{-1}(x_i)$ as in equation (3) above. Facets of $\Pi_2(4)$ are then given by setting some of the β_{ij} and ρ_{ij} equal to zero.

Proposition III.6 *Every 10-dimensional cone $\sigma \in \mathrm{Vor}(4)$ with $\langle x_1^2, e\rangle \prec \sigma$ is equivalent under $\tilde{G}_1$ to one of the following three cones:*

$$\begin{aligned}
\Pi_2^1(4) &= \{\beta_{14} = \beta_{34} = \rho_{13} = 0\} + \eta;\\
\Pi_2^2(4) &= \{\beta_{13} = \beta_{14} = \beta_{34} = 0\} + \eta;\\
\Pi_2^3(4) &= \{\beta_{14} = \beta_{34} = \rho_{24} = 0\} + \eta.
\end{aligned}$$

Proof. Later (Corollary III.8) we shall show that $\Pi_2^1(4)$, $\Pi_2^2(4)$ and $\Pi_2^3(4)$ are inequivalent under $\tilde{G}_1$. For now, since we are only interested in subcones of $\Pi_2(4)$, we need not consider $\tilde{G}_1$ but only $G_1 = \tilde{G}_1 \cap G$, where $G \subset \mathrm{GL}(\mathbb{L}_4)$ is the subgroup that preserves $\Pi_2(4)$. Note that if σ is as above, $g \in \tilde{G}_1$ and $g(\sigma) \succ \langle x_1^2, e\rangle$ also, then $g \in G_1$ anyway. This is because $g(e)$ is the barycentre of $g\big(\Pi_2(4)\big)$, so if g does not preserve $\Pi_2(4)$ then e and $g(e)$ are in the interiors of different top-dimensional cones of $\mathrm{Igu}(4)$ and cannot both be generators of $g(\sigma)$, since $\mathrm{Vor}(4)$ is a refinement of $\mathrm{Igu}(4)$.

The symmetry group of $\Pi_2(4)$ is described in [ER2]. It is a reflection group of order 1152, isomorphic to the reflection group F_4, generated by elements k_i, $(1 \le i \le 4)$; s_{ij}, $(1 \le i < j \le 4)$; and an extra transformation w. These are given by

$$k_i(y_i) = -y_i, \quad k_i(y_j) = y_j \ (j \neq i);$$
$$s_{ij}(y_i) = y_j, \quad s_{ij}(y_j) = y_i, \quad s_{ij}(y_k) = y_k \ (k \neq i, j);$$
$$w(y_i) = -y_i + \frac{1}{2}\sum_{k=1}^{4} y_k.$$

We claim that this group is G; to show this, we must prove that it is a subgroup of $\mathrm{GL}(\mathbb{L}_4)$. Thus we need to check that the matrices $\Psi^{-1}K_i\Psi$, $\Psi^{-1}S_{ij}\Psi$ and $\Psi^{-1}W\Psi$ are all integral, where K_i, S_{ij} and W are the matrices of the above transformations and Ψ is the matrix of the Voronoi transformation defined by equation (2). Then

$$\Psi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \text{ so } 2\Psi^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}; \text{ and}$$
$$2W = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

so it is sufficient to check that $2\Psi^{-1}K_i\Psi$ and $2\Psi^{-1}S_{ij}\Psi$ are congruent to zero mod 2 and that $2\Psi^{-1}2W\Psi$ is congruent to zero mod 4. The first of these is trivial since $K_i \equiv \mathbf{1}_4$ mod 2. For S_{ij} it is enough to notice that any two columns of $2\Psi^{-1}$ are equivalent mod 2, so $2\Psi^{-1}S_{ij} \equiv 2\Psi^{-1}$ mod 2 and hence $2\Psi^{-1}S_{ij}\Psi \equiv 2\mathbf{1}_4 \equiv 0$. The case of W is checked directly.

All these elements of G preserve η: they must do, as it is spanned by the barycentre of $\Pi_2(4)$. There are 12 rays generating $\Pi_2(4)$, spanned by $(y_i \pm y_j)^2$, and G permutes them transitively because $k_j\colon (y_i + y_j)^2 \mapsto (y_i - y_j)^2$ and $s_{ii'}s_{jj'}\colon (y_i + y_j)^2 \mapsto (y_{i'} + y_{j'})^2$. Hence G_1, which is the stabiliser of one of the rays (generated by $x_1^2 = (y_1 + y_2)^2$) has order 96. The transformations k_3, k_4, k_1k_2, s_{12}, s_{34} and $w' = s_{14}s_{23}w$ all belong to G_1; and in fact they generate a group of order 96, which is therefore the whole of G_1. To see this, note that the elements k_3, k_4, k_1k_2, s_{12} and s_{34} generate a group of order 32, and the element k_3w' has order 3: hence the group generated has order at least 96.

The G-orbits of 9-dimensional facets of $\Pi_2(4)$ are also studied in [ER2]. There are exactly two such orbits, denoted RT and BF. A facet is RT if

it is G-equivalent to the facet $\rho_{12} = \rho_{23} = \rho_{13} = 0$, and BF if it is G-equivalent to $\beta_{12} = \beta_{13} = \beta_{14} = 0$: there are 16 RT and 48 BF facets. The names come from the following representation, which will also be useful to us. We construct a bicoloured graph on four vertices numbered 1 to 4: conventionally we think of these vertices as the four corners of a square, numbered clockwise starting from the top left. We join i and j with a red edge to represent the equation $\rho_{ij} = 0$ or with a black edge to represent $\beta_{ij} = 0$. The facets are then given by graphs with three edges that are forked (there is a vertex of valency 3) or triangular. An RT facet is G-equivalent to a facet described by a red triangular graph, and a BF facet is G-equivalent to a facet described by a black forked graph. The effect of k_i on the graphs is to change the colour of all edges having i as a vertex. s_{ij} is just the transposition (ij) on the vertices. w' interchanges a left black edge with a right black edge (i.e. β_{14} and β_{23}) and leaves other black edges alone: to red edges it does the opposite, leaving the left and right edges alone but interchanging top and bottom and the two diagonals.
We are interested in facets adjoining $\langle x_1^2 \rangle$ up to G_1-equivalence. The coefficient associated to x_1^2 is β_{12}, so we have $\beta_{12} \neq 0$: in other words, we look only at graphs that do not have a black edge joining vertices 1 and 2.
There are 48 facets adjoining $\langle x_1^2 \rangle$, of which 12 are RT and 36 are BF: this follows because $\Pi_2(4)$ has twelve edges, all equivalent under G, and each facet adjoins nine of them. G_1 preserves the property of being RT or BF, because G does. $\Pi_2^1(4)$ is RT, $\Pi_2^2(4)$ and $\Pi_2^3(4)$ are BF.
The rest of the proof consists of checking that every facet of $\Pi_2(4)$ that adjoins $\langle x_1^2 \rangle$ occurs in one of these three orbits, which is straightforward, using the description of the effects of the generators above. The details are shown in Figure 1. Red edges are shown as dotted lines and the vertices are numbered according to the above convention, clockwise starting from the top left. Only half the facets are shown, the others being their reflections (left-right) under $s_{12}s_{34}$. □

Proposition III.7 *Each of the three cones* $\Pi_2^i(4)$ *projects under* $\mathrm{Sym}_2\,\mathrm{pr}_1$ *to a cone contained in a cone of* $\mathrm{Vor}(3)$.

Proof. We simply check this for each case. (The representatives $\Pi_2^i(4)$ have been chosen so as to keep this part of the calculation fairly simple.) Note that the projection of e is given by $\mathrm{Sym}_2\,\mathrm{pr}_1(e) = \bar{e}$, where

$$\bar{e} = (x_2 - x_3)^2 + (x_2 - x_4)^2 + x_3^2 + x_4^2. \tag{6}$$

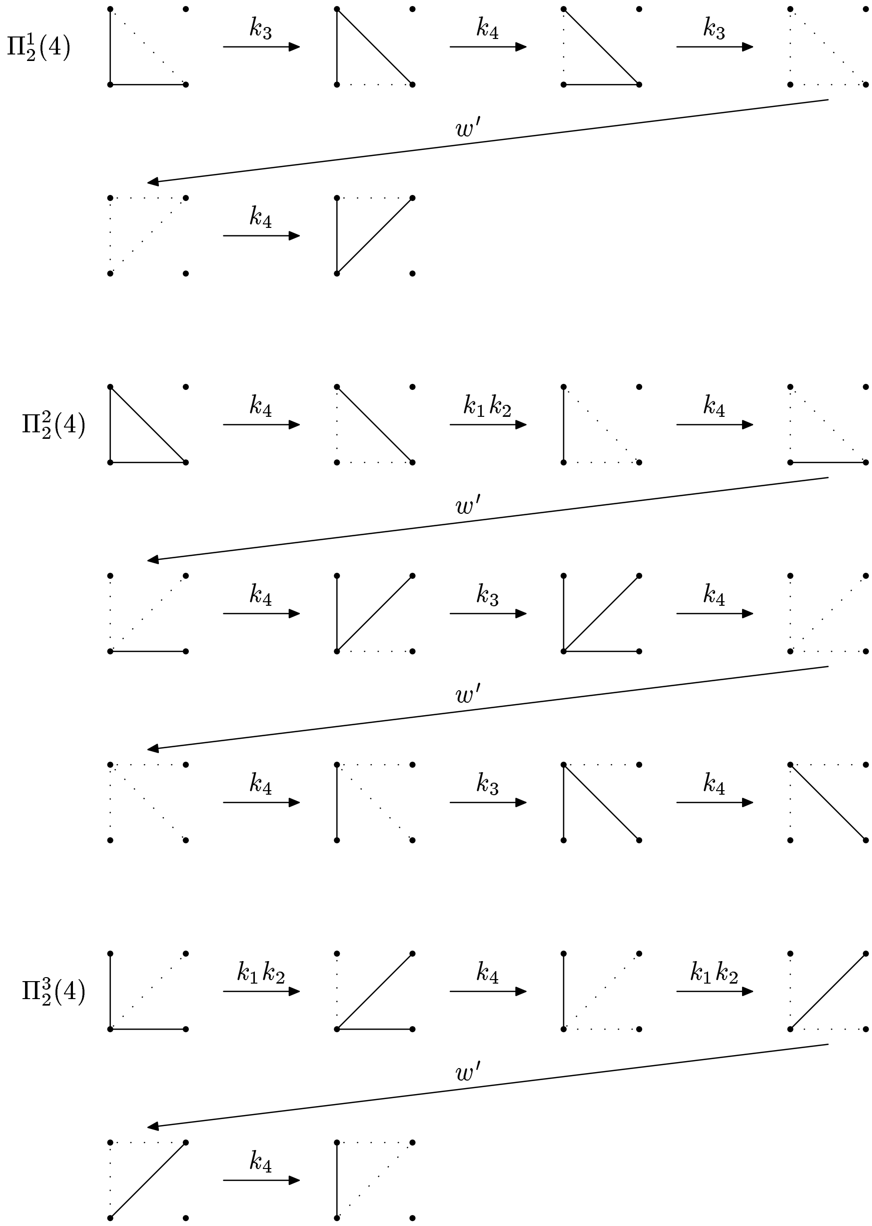

Figure 1: Orbits of facets of the second perfect domain

Using this we have

$$\begin{aligned}
\operatorname{Sym}_2 \operatorname{pr}_1\left(\Pi_2^1(4)\right) &= \operatorname{Sym}_2 \operatorname{pr}_1 \left\langle x_1^2, x_2^2, x_4^2, (x_1-x_3)^2, (x_1-x_4)^2,\right. \\
&\quad \left.(x_2-x_3)^2, (x_2-x_4)^2, (x_3-x_4)^2, (x_1+x_2-x_3)^2, e\right\rangle \\
&= \left\langle x_2^2, x_4^2, x_3^2, x_4^2, (x_2-x_3)^2, (x_2-x_4)^2, (x_3-x_4)^2,\right. \\
&\quad \left.(x_2-x_3)^2, (x_2-x_3)^2+(x_2-x_4)^2+x_3^2+x_4^2\right\rangle \\
&= \left\langle x_2^2, x_3^2, x_4^2, (x_2-x_3)^2, (x_2-x_4)^2, (x_3-x_4)^2\right\rangle \\
&= \Pi_1(3)
\end{aligned}$$

and $\Pi_1(3) \in \operatorname{Vor}(3)$.
The first BF case is given by

$$\begin{aligned}
\operatorname{Sym}_2 \operatorname{pr}_1\left(\Pi_2^2(4)\right) &= \operatorname{Sym}_2 \operatorname{pr}_1 \left\langle x_1^2, x_2^2, x_3^2, x_4^2, (x_1-x_3)^2, (x_1-x_4)^2,\right. \\
&\quad \left.(x_2-x_3)^2, (x_2-x_4)^2, (x_3-x_4)^2, e\right\rangle \\
&= \left\langle x_2^2, x_3^2, x_4^2, x_3^2, x_4^2, (x_2-x_3)^2, (x_2-x_4)^2, (x_3-x_4)^2,\right. \\
&\quad \left.(x_2-x_3)^2+(x_2-x_4)^2+x_3^2+x_4^2\right\rangle \\
&= \left\langle x_2^2, x_3^2, x_4^2, (x_2-x_3)^2, (x_2-x_4)^2, (x_3-x_4)^2\right\rangle \\
&= \Pi_1(3)
\end{aligned}$$

For the second BF case, $\Pi_2^3(4)$, we have

$$\operatorname{Sym}_2 \operatorname{pr}_1\left(\Pi_2^3(4)\right) = \left\langle x_2^2, x_3^2, x_4^2, (x_2-x_3)^2, (x_3-x_4)^2, \bar{e}\right\rangle$$

which by (6) is strictly contained in $\Pi_1(3)$. $\square$

Corollary III.8 *The $\tilde{G}_1$ orbits of $\Pi_2^1(4)$, $\Pi_2^2(4)$ and $\Pi_2^3(4)$ are distinct.*

Proof. Since $\Pi_2^1(4)$ is an RT facet it is in a different G-orbit from the other two, by [ER2]. As we have just seen, $\Pi_2^2(4)$ projects onto a maximal-dimensional cone of $\operatorname{Vor}(3)$ and $\Pi_2^3(4)$ does not, so they are inequivalent under $\tilde{G}_1$. $\square$

Now we want to investigate the pullbacks of line bundles on $\mathcal{A}_3^{\mathrm{Vor}}$ under the morphisms $p_i\colon D_{4,i}^{\mathrm{Vor}}(n) \longrightarrow \mathcal{A}_3^{\mathrm{Vor}}(n)$. We define

$$E_s(n)|_i = D_{4,i}^{\mathrm{Vor}}(n) \cap E_s(n). \tag{7}$$

This intersection is either empty or a divisor on $D_{4,i}^{\mathrm{Vor}}(n)$ which is contracted to a variety of codimension ≥ 2 under the map $D_{4,i}^{\mathrm{Vor}}(n) \to D_{4,i}^{\mathrm{Igu}}(n)$.

Remark III.9 The varieties $E_s(n)$ do not depend on n for $n \geq 3$: more precisely, all the $E_s(n)$ have the same normalisation (up to isomorphism), independently of s or n; and if $n \geq 3$ they are normal. The normalisation is $\bar{\mathcal{O}}(\eta) \subset X_{\mathrm{Vor}(4)}$. Since the edges of $\Pi_2(4)$ are all equivalent under G any two non-empty varieties $E_s(n)|_i$ are mutually isomorphic as well. If $n \geq 3$ no nontrivial cone of $\mathrm{Vor}(4)$ has nontrivial stabiliser, since the principal congruence subgroup of level n in $\mathrm{GL}(8, \mathbb{Z})$ is torsion-free. This also implies that the boundary divisors $D_{4,i}^{\mathrm{Vor}}(n)$ are normal.

We recall here some facts about the structure of toroidal compactifications. Recall from [Nam1] that any toroidal compactification of $\mathcal{A}_g$ is a disjoint union of strata of the form

$$Z_{h,\bar{\sigma}}(n) = \mathcal{P}_{g-h}(n) \backslash \mathbb{H}_h \times \mathbb{C}^{h(g-h)} \times \mathcal{O}(\bar{\sigma})$$

where $\mathcal{P}_{g-h}(n)$ is a group which acts properly discontinously, $\bar{\sigma}$ is a cone in some copy of $\mathrm{Sym}_2(\mathbb{R}^{g-h})$ containing some positive definite form, and $\mathcal{O}(\bar{\sigma})$ is the corresponding torus orbit.

Remark III.10 Suppose C is an irreducible curve in $\mathcal{A}_4^{\mathrm{Vor}}$. Then let σ be a maximal cone in $\mathrm{Vor}(4)$ such that C is contained in the image in $\mathcal{A}_4^{\mathrm{Vor}}$ of the closure of the torus orbit $\mathcal{O}(\sigma)$ (such a σ is unique up to the action of $\mathrm{GL}(\mathbb{L}_4)$). If we assume that C is not contained in the exceptional divisor E, then σ must be of the form $\langle l_1^2, \ldots, l_k^2 \rangle$ where the l_i are linear forms on $\mathbb{L}_4^*$, as in the proof of Proposition III.5.

The connection with the strata $Z_{h,\bar{\sigma}}(n)$ is the following. Let

$$U = \bigcap_{q \in \sigma} \operatorname{Ker} q \subset \mathbb{M}_4^* \otimes \mathbb{R},$$

and set $h = \dim_{\mathbb{R}} U$ and $V = \mathbb{M}_4^* \otimes \mathbb{R}/U \cong \mathbb{R}^{4-h}$. Then every form $q \in \sigma$ defines a form $\bar{q}$ on V and this defines an injective map $\sigma \to \bar{\sigma} \subset \mathrm{Sym}_2^+(V)$.

Lemma III.11 *$\bar{\sigma} \subset \mathrm{Sym}_2^+(V)$, as defined above, contains positive definite forms.*

Proof. If $C \subset E$ then $e \in \sigma$ and since e is positive definite there is nothing to prove. Otherwise we prove this by induction on the number m of generators of σ. Suppose $C \not\subset E$ and $\sigma = \langle q_1, \ldots, q_m \rangle$. We have $U = \bigcap_{i=1}^{m} \operatorname{Ker} q_i$, since if $q = \sum a_i q_i \in \sigma$ and $q_i(x) = 0 \in \mathbb{M}_4 \otimes \mathbb{R}$ for all i then $q(x) = 0$. Thus $\bigcap \operatorname{Ker} \bar{q}_i = 0$. Suppose $0 \neq x \in \operatorname{Ker}(\bar{q}_1 + t\bar{q}_2)$ for some $t > 0$. Then, evaluating at $x \in V$, we get $\bar{q}_1(x) + t\bar{q}_2(x) = 0$, and since both forms are positive semidefinite this implies $\bar{q}_1(x) = \bar{q}_2(x) = 0 \in \mathbb{R}$. But since $\bar{q}_i$ is semidefinite, $\bar{q}_i(x) = 0$ if and only if $x \in \operatorname{Ker} \bar{q}_i$, so $x \in \operatorname{Ker} \bar{q}_1 \cap \operatorname{Ker} \bar{q}_2$. So $\operatorname{Ker}(\bar{q}_1 + t\bar{q}_2) = \operatorname{Ker} \bar{q}_1 \cap \operatorname{Ker} \bar{q}_2$, and this reduces to the case of $\langle q_1 + tq_2, q_3, \ldots, q_m \rangle$. $\square$

C is then contained in a stratum of the form $\mathcal{P}_{4-h}(n) \backslash \mathbb{H}_h \times \mathbb{C}^{h(4-h)} \times \mathcal{O}(\bar{\sigma})$.

Proposition III.12 *Under the morphism p_i of Proposition III.1*

$$p_i^*\big(D_3(n)\big) = \sum_{j\neq i} D_{4,j}^{\mathrm{Vor}}(n)|_{D_{4,i}^{\mathrm{Vor}}(n)} + 4\sum_s E_s(n)|_i.$$

Proof. The behaviour away from $\phi_n^{-1}(\mathcal{A}_0)$ is clear and gives the coefficient 1 for the boundary components $D_{4,j}^{\mathrm{Vor}}$. It is necessary to check the coefficient of $E_s(n)$. The bundle $p_i^*\big(D_3(n)\big)$ is given on $X_{\mathrm{Star}\langle x_1^2\rangle}$ by the support function $\psi_3 \circ \mathrm{Sym}_2\,\mathrm{pr}_1$, where ψ_3 is the support function on $\mathrm{Vor}(3)$ that takes the value 1 on each primitive generator of a ray. Hence the coefficient of E is $\psi_3(\bar{e}) = 4$, by equation (6). □

We insert here some further details about the orbits of cones of $\mathrm{Vor}(4)$ that will be useful to us later on.

Lemma III.13 *The dimension* 2 *faces of* $\Pi_2(4)$ *fall into two orbits under the action of G, the symmetry group of* $\Pi_2(4)$*. Representatives for these orbits are* $\langle x_1^2, x_2^2\rangle$ *and* $\langle x_1^2, x_3^2\rangle$*.*

Proof. Any such face is equivalent under G to a face spanned by x_1^2 and one other generator of $\Pi_2(4)$. So we are interested in the G_1-orbits of the other eleven generators. We can represent such a generator by a bicoloured graph as we did for facets, only it is easier to use the complementary graph, so that a red (respectively black) edge joining vertices i and j represents $\rho_{ij} \neq 0$ (respectively $\beta_{ij} \neq 0$). A generator of $\Pi_2(4)$ is thus represented by a single edge. The generators of G_1 listed together with their action on the graphs in the proof of Proposition III.6 all preserve the property of an edge being horizontal. It is easy to see that the horizontal and non-horizontal edges each form a G_1-orbit: see Figure 2. The representatives given are defined by the non-vanishing of ρ_{12} and ρ_{13} respectively. □

Corollary III.14 *Any cone of* $\mathrm{Vor}(4)$ *spanned by two rank* 1 *forms and a form of maximal rank is* $\mathrm{GL}(\mathbb{L}_4)$ *equivalent to one of the cones*

$$\sigma_3 = \langle x_1^2, x_2^2, e\rangle \quad \textit{and} \quad \sigma_3' = \langle x_1^2, x_2^2, e'\rangle$$

where

$$e' = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_3 - x_1x_2 - x_1x_4 - x_2x_3 - x_3x_4).$$

Proof. Any such cone is equivalent to a cone spanned by e and a dimension 2 face of $\Pi_2(4)$, so we can apply Lemma III.13. However $\langle x_1^2, x_2^2, e\rangle = \sigma_3$, and $\langle x_1^2, x_3^2, e\rangle$ is equivalent under $\mathrm{GL}(\mathbb{L}_4)$ to σ_3'; the element of $\mathrm{GL}(\mathbb{L}_4)$ involved is simply the transposition $x_2 \leftrightarrow x_3$. Applying this to e gives the result. □

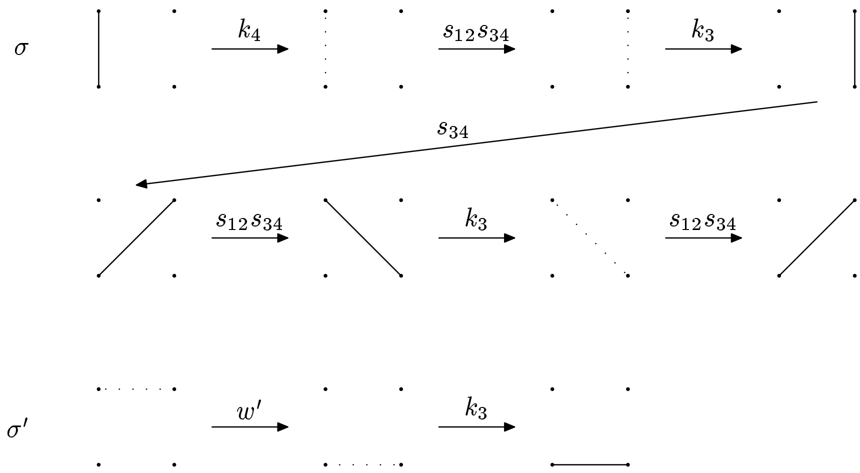

Figure 2: Orbits of dimension 2 faces of the second perfect domain

Lemma III.15 *The dimension 3 faces of $\Pi_2(4)$ fall into four orbits under the action of G, the symmetry group of $\Pi_2(4)$. These orbits are to be referred to as string, BF*, RT* and disconnected: they are represented by the cones $\langle x_1^2, x_2^2, x_3^2 \rangle$, $\langle x_1^2, x_3^2, x_4^2 \rangle$, $\langle x_1^2, x_4^2, (x_1 - x_4)^2 \rangle$ and $\langle x_1^2, x_2^2, (x_3 - x_4)^2 \rangle$ respectively.*

Proof. Such a face σ is determined by three generators, i.e. by a bicoloured graph with three edges. It is always possible to draw a forked or triangular graph on the complement of such a graph. Therefore any collection of three generators of $\Pi_2(4)$ spans a 3-dimensional face of $\Pi_2(4)$, since if we draw a triangular or forked graph on the complement we specify, according to [ER2], a facet containing all those generators; and the facets, again according to [ER2], are simplicial. (Note that the edges in the graphs in [ER2] represent a condition $\beta_{ij} = 0$ or $\rho_{ij} = 0$, whereas for us here they represent $\beta_{ij} \neq 0$ or $\rho_{ij} \neq 0$.)
If the graph representing σ is itself triangular or forked, then we may appeal directly to the argument of [ER2]. We conclude that there are two orbits of these types, which we may call RT* and BF*, represented by the same graphs as the facets of types RT and BF. Examples are $\langle x_1^2, x_4^2, (x_1 - x_4)^2 \rangle$ for RT* and $\langle x_1^2, x_3^2, x_4^2 \rangle$ for BF*. (A forked or triangular graph is RT if is triangular and has an even number of red sides: otherwise it is BF.)
So suppose that the graph is neither forked nor triangular (this means that the rays not spanning σ do not span a facet of $\Pi_2(4)$). Then the vertices must have valencies 0, 1 or 2, and at least one of them has valency 1. The possibilities are that all four vertices have valency 1; one has valency 0, one has valency 1, and the other two have valency 2; or two have valency 1 and two have valency 2. These possibilities are illustrated in Figure 3.

We claim that the first two of these cases together form one G-orbit, and that the third forms another. In the first case, of valencies all equal to 1, the graph is a string. By applying s_{ij} we may assume that the string consists of edges joining 1 to 2, 2 to 3 and 3 to 4. We may change the colour of the outside edges by applying k_1 or k_4, and we may change the colour of the central edge by applying k_3k_4, in each case without changing anything else. Any graph of the second type (valencies 0, 1, 2, 2) may be converted to a string by moving the double edge to join 1 to 2 and then applying w'. So the first two types form a single orbit. In the last type, where the graph is disconnected, we may always move the double edge to join 1 and 2 and we may change the colour of the remaining edge (necessarily joining 3 and 4) by applying k_3. It is also easy to see that this type cannot be converted into a string.
Examples of these two possibilities, which we call "string" and "disconnected", are $\langle x_1^2, x_2^2, x_3^2 \rangle$ and $\langle x_1^2, x_2^2, (x_3 - x_4)^2 \rangle$ respectively. $\square$

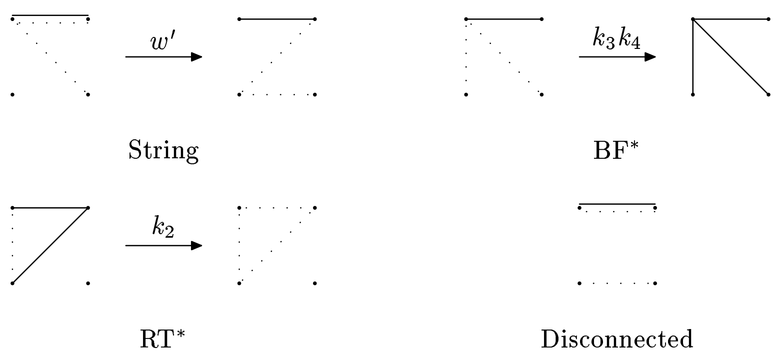

Figure 3: Orbits of dimension 3 faces of the second perfect domain

Lemma III.16 *Suppose $\sigma \prec \Pi_2(4)$ and that $\sigma = \langle l_1^2, l_2^2, l_3^2, l_4^2, l_5^2, l_6^2 \rangle$, and suppose that the l_i span a subspace of dimension 3. Then σ is G-equivalent to $\langle x_1^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_3 - x_4)^2 \rangle$.*

Proof. Without loss of generality we may assume that l_1, l_2 and l_3 are linearly independent. Since every face of $\Pi_2(4)$ is simplicial, $\langle l_1^2, l_2^2, l_3^2 \rangle \prec \Pi_2(4)$, so according to Lemma III.15 it is equivalent to either $\langle x_1^2, x_2^2, x_3^2 \rangle$ or $\langle x_1^2, x_3^2, x_4^2 \rangle$. Type RT* is excluded by the linear independence condition, and disconnected type is excluded because for any other generator l_4^2 of $\Pi_2(4)$ the linear forms x_1, x_2, $x_3 - x_4$ and l_4 span a space of dimension 4. We prefer to replace $\langle x_1^2, x_2^2, x_3^2 \rangle$ by the equivalent face $\langle x_1^2, x_3^2, (x_3 - x_4)^2 \rangle$ (also of string type – apply w' followed by $s_{12}s_{34}$).
Now the result follows from the observation that any seven linear forms whose squares are generators of $\Pi_2(4)$ span a linear space of dimension 4.

Therefore the six l_i are all the linear forms whose squares are generators of $\Pi_2(4)$ and which lie in the linear span of l_1, l_2 and l_3. In both cases this gives $\langle x_1^2, x_3^2, x_4^2, (x_1-x_3)^2, (x_1-x_4)^2, (x_3-x_4)^2 \rangle$. □

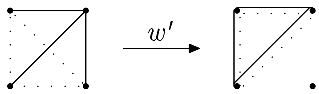

Figure 4: An orbit of faces of the second perfect domain spanned by six generators

Remark III.17 The graph corresponding to this example is a coloured complete graph, as shown on the left of Figure 4. Applying w' gives a graph which is a bicoloured triangle. Notice that each edge of this graph has a distinguished opposite edge, which shares an even number of vertices with it but is of the opposite colour: this will be used below in Proposition IV.29.

IV Proof of the main result

Recall that $D_g^{\mathrm{Vor}}(n)$ denotes the closure in $\mathcal{A}_g^{\mathrm{Vor}}(n)$ of the boundary $D_g'(n)$. We refer to the irreducible components of $D_g^{\mathrm{Vor}}(n)$ as $D_{g,i}^{\mathrm{Vor}}(n)$, or simply as $D_{g,i}(n)$ if $g \leq 3$ (when there is only one toroidal compactification we need consider). On $D_{g,i}^{\mathrm{Vor}}$ we define the line bundle

$$M_{g,i}(n) = -nN_{D_{g,i}^{\mathrm{Vor}}(n)/\mathcal{A}_g^{\mathrm{Vor}}(n)} + L. \tag{8}$$

Clearly, in view of Lemma II.1, $M_{g,i}(n)$ is an extension of the line bundle $M'(n)$ introduced in section II.
In the case $g = 4$ the boundary of $\mathcal{A}_4^{\mathrm{Vor}}(n)$ decomposes as

$$\mathcal{A}_4^{\mathrm{Vor}}(n) \setminus \mathcal{A}_4(n) = D_4^{\mathrm{Vor}}(n) + E(n) = \sum_i D_{4,i}^{\mathrm{Vor}}(n) + \sum_s E_s(n).$$

We define the line bundle on $D_{4,i}^{\mathrm{Vor}}$

$$J_4(n) = J_{4,i}(n) = M_{4,i}(n) - \sum_s nE_s(n)|_i. \tag{9}$$

The proof of Proposition II.2 shows that certain theta functions define sections of $M'_{4,i}(n)$. The first technical result of this section, Proposition IV.1, is that these sections extend to sections of $J_4(n)$.
The following notation will be used throughout the rest of the paper: if I is a set of indices then we write $D_{g,I}^{\mathrm{Vor}}(n)$ for $\bigcap_{i \in I} D_{g,i}^{\mathrm{Vor}}(n)$, and if $\mathcal{F}$ is a bundle

(or sheaf, etc.) on some variety containing $D^{\mathrm{Vor}}_{g,I}(n)$ we denote the restriction $\mathcal{F}|_{D^{\mathrm{Vor}}_{g,I}(n)}$ more simply by $\mathcal{F}|_I$. We have already used this convention above (equation (7)): as we did there, we normally abuse notation by writing $D^{\mathrm{Vor}}_{4,ij}(n)$ and $\mathcal{F}|_{ij}$ rather than $D^{\mathrm{Vor}}_{4,\{i,j\}}(n)$ and $\mathcal{F}|_{\{i,j\}}$, etc.

Proposition IV.1 *Let p be a prime and $n \equiv 0 \bmod 4p^2$. If the characteristics $m', m'', \bar{m}', \bar{m}'' \in \frac{1}{2p}\mathbb{Z}^{g-1}$, then the functions $\Theta_{m',m''}(z,\tau)\Theta_{\bar{m}',\bar{m}''}(z,\tau)$ define sections of the line bundle $J_4(n)$.*

Proof. Since the group $\mathrm{Sp}(8,\mathbb{Z}/n)$ acts transitively on the boundary components $D^{\mathrm{Vor}}_{4,i}(n)$ we can again restrict ourselves to the standard boundary component $D^{\mathrm{Vor}}_{4,1}(n)$, given by the line $\mathbb{Q}\mathbf{e}_1$ in $\mathbb{Q}^8 = \mathbb{Q}\mathbf{e}_1 \oplus \cdots \oplus \mathbb{Q}\mathbf{e}_8$. We write $D^{\mathrm{Vor}}_{4,1}(n) \cap \mathcal{A}'_4(n) = D'_{4,1}(n)$. We have already observed that the functions $\Theta_{m'm''}(z,\tau)\Theta_{\bar{m}'\bar{m}''}(z,\tau)$ have the correct transformation behaviour. We shall have to study how the sections defined by these functions extend to the generic point of the intersections $D^{\mathrm{Vor}}_{4,1j}(n) = D^{\mathrm{Vor}}_{4,1}(n) \cap D^{\mathrm{Vor}}_{4,j}(n)$, $j \neq 1$ and to the generic point of the divisors $E_s(n)|_1$.
A standard calculation shows that $\mathrm{Sp}(8,\mathbb{Z}/n)$ acts transitively on pairs $\big(D^{\mathrm{Vor}}_{4,i}(n), D^{\mathrm{Vor}}_{4,j}(n)\big)$ with $D^{\mathrm{Vor}}_{4,ij}(n) \neq \emptyset$. Hence we can work with the standard cusp corresponding to the isotropic subspace $\mathbb{Q}(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4)$ and we can, moreover, take $j = 2$ and assume that $D^{\mathrm{Vor}}_{4,1}(n)$ and $D^{\mathrm{Vor}}_{4,2}(n)$ correspond to the rays $\langle x_1^2 \rangle$ and $\langle x_2^2 \rangle$ in $\mathbb{M}_4 \otimes \mathbb{R}$. The cone $\langle x_1^2, x_2^2 \rangle \in \mathrm{Vor}(4)$ has dual cone given by

$$\langle x_1^2, x_2^2 \rangle^\vee = \langle U_{11}, U_{22}, \pm U_{33}, \pm U_{44}, \pm U_{ij}\ (i \neq j) \rangle,$$

where $\{U_{ij}\}$ is the dual basis to $\{U^*_{ij}\}$.
We have the partial quotient

$$\begin{array}{rcl} \mathbb{H}_4 & \longrightarrow & \mathbb{C} \times \mathbb{C} \times (\mathbb{C}^*)^8 = T_{\mathbb{M}_4}\mathrm{emb}\,\big(\langle x_1^2, x_2^2 \rangle\big) \\ (\tau_{ij}) & \longmapsto & (t_{11} = e^{2\pi i \tau_{11}/n}, t_{22} = e^{2\pi i \tau_{22}/n}, t_{ij} = e^{2\pi i \tau_{ij}/n}). \end{array}$$

in which $D^{\mathrm{Vor}}_{4,1}(n)$ corresponds to $\{t_{11} = 0\}$ and $D^{\mathrm{Vor}}_{4,12}(n)$ corresponds to $\{t_{11} = t_{22} = 0\}$.
We now have to study the theta functions $\Theta_{m'm''}(z,\tau)$ where $m', m'' \in (1/2p)\mathbb{Z}^{g-1}$. The transformation of these functions with respect to $\mathrm{Sp}(8,\mathbb{Z})$ is given by the theta transformation formula [Ig, pp. 84, 85]. Note that characteristics of the form (m', m'') with $m', m'' \in \frac{1}{2p}\mathbb{Z}^{g-1}$ are transformed to characteristics of the same type.
The connection between the variables $(z,\tau) \in \mathbb{C}^3 \times \mathbb{H}_3$ and $\mathbb{H}_4$ is the following. Recall that $D^{\mathrm{Vor}}_{4,1}(n)$ corresponds to $\tau_{11} \to i\infty$. In terms of the

coordinates $t_{ij} = e^{2\pi i \tau_{ij}/n}$ we have

$$\Theta_{m'm''}(z,\tau) = \left(\sum_{q\in\mathbb{Z}^3} t_{22}^{\frac{1}{2}(q_2+m_2')^2 n} t_{33}^{\frac{1}{2}(q_3+m_3')^2 n} t_{44}^{\frac{1}{2}(q_4+m_4')^2 n} \right) \cdot \prod_{2\le i<j\le 4} t_{ij}^{(q_i+m_i')(q_j+m_j')n} t_{12}^{(q_2+m_2')n} t_{13}^{(q_3+m_3')n} t_{14}^{(q_4+m_4')n} e^{2\pi i^t(q+m')m''}. \tag{10}$$

The claim that the sections of $M'_{4,i}(n)$ defined by the products of theta functions $\Theta_{m'm''}(z,\tau)\Theta_{\bar{m}'\bar{m}''}(z,\tau)$ can be extended over $D_{4,12}^{\mathrm{Vor}}(n)$ now follows from the observation that the exponent of t_{22} in $\Theta_{m'm''}(z,\tau)$ is equal to $(q_2+m_2')^2 n/2$ and, in particular, non-negative.
Next we study the extension to the generic point of a divisor $E_s(n)|_1$. Again we claim that $\mathrm{Sp}(8,\mathbb{Z}/n)$ acts transitively on the pairs $(D_{4,i}^{\mathrm{Vor}}(n), E_s(n)|_i)$. By the action of the group $\mathrm{GL}(\mathbb{L}_4)$ on $\mathbb{M}_4$ we can assume that $E_s(n)$ corresponds to the central ray η in the second perfect cone. The transitivity now follows from the observation at the start of the proof of Proposition III.6 that G, the stabiliser of $\Pi_2(4)$ in $\mathrm{GL}(4,\mathbb{Z})$, permutes the generators of $\Pi_2(4)$ transitively and preserves η.
We can, therefore, restrict our attention to the 2-dimensional cone $\langle x_1^2, e\rangle$. But

$$e = 2U_{11}^* + 2U_{22}^* + 2U_{33}^* + 2U_{44}^* + U_{12}^* - U_{13}^* - U_{14}^* - U_{23}^* - U_{24}^*$$

and, using this, a straightforward calculation shows that the dual cone is given by

$$\begin{aligned} \langle x_1^2, e\rangle^\vee &= \langle (U_{11}-2U_{12}), U_{12}, \pm(U_{22}-U_{33}), \pm(U_{22}-U_{44}) \\ &\quad \pm(U_{24}+U_{12}), \pm(U_{24}-U_{13}), \pm(U_{24}-U_{14}), \pm(U_{24}-U_{23}) \\ &\quad \pm(2U_{12}-U_{22}), \pm U_{34}\rangle . \end{aligned}$$

Hence $T_{\mathbb{M}_4}\mathrm{emb}\left(\langle x_1^2, e\rangle\right) \cong \mathbb{C}^2 \times (\mathbb{C}^*)^8$ and the torus embedding is given by

$$\begin{array}{rcl} T_{\mathbb{M}_4} & \longrightarrow & T_{\mathbb{M}_4}\mathrm{emb}\left(\langle x_1^2, e\rangle\right) \cong \mathbb{C}\times\mathbb{C}\times(\mathbb{C}^*)^8 \\ (t_{ij}) & \longmapsto & (t_{11}t_{12}^{-1}, t_{12}, t_{22}t_{33}^{-1}, t_{22}t_{44}^{-1}, t_{24}t_{12}, t_{24}t_{13}^{-1}, t_{24}t_{14}^{-1}, t_{24}t_{23}^{-1}, t_{12}^2 t_{22}^{-1}, t_{34}). \end{array}$$

Let $T_1,\ldots,T_{10}$ be the obvious coordinates on $\mathbb{C}^2\times(\mathbb{C}^*)^8$, corresponding to $U_{11}-2U_{12}, U_{12}, U_{22}-U_{33}, \ldots, 2U_{12}-U_{22}, U_{34}$. The hyperplane $\{T_1 = 0\}$ describes $D_{4,1}^{\mathrm{Vor}}(n)$ and $\{T_1 = T_2 = 0\}$ defines $E_s(n)|_1$. A straightforward calculation shows

$$\begin{array}{l} t_{11} = T_1T_2^2, \quad t_{12} = T_2, \quad t_{13} = T_2^{-1}T_5T_6^{-1}, \quad t_{14} = T_2^{-1}T_5T_7^{-1}, \\ t_{22} = T_2^2T_9^{-1}, \quad t_{23} = T_2^{-1}T_5T_8^{-1}, \quad t_{24} = T_2^{-1}T_5, \\ t_{33} = T_2^2T_3^{-1}T_9^{-1}, \quad t_{34} = T_{10}, \quad t_{44} = T_2^2T_4^{-1}T_9^{-1}. \end{array} \tag{11}$$

Combining this with formula (10), we can write the function $\Theta_{m',m''}(z,\tau)$ in terms of these new coordinates $T_1,\ldots,T_{10}$. We are interested in the exponent of T_2. If

$$w_2 = (q_2 + m_2'),\ w_3 = (q_3 + m_3'),\ w_4 = (q_4 + m_4')$$

then this exponent is given by

$$\ell(w_2, w_3, w_4) = n(w_2 - w_2w_4 - w_3 - w_4 - w_2w_3 + w_2^2 + w_3^2 + w_4^2).$$

Another straightforward calculation shows that this function assumes its minimum for $w_2 = 0$, $w_3 = w_4 = 1/2$, where we find that $\ell(0, 1/2, 1/2) = -n/2$. Altogether for products of the form $\Theta_{m'm''}(z,\tau)\Theta_{\bar{m}'\bar{m}''}(z,\tau)$ we pick up poles of order at most n. On the other hand $t_{11} = T_1T_2^2$ shows that we have a zero of order $2n$ and this means that, in total, we have a zero of order at least n. $\square$

Before we give the proof of the main theorem we want to introduce the notion of depth of an irreducible curve C. Recall that we have a morphism

$$\phi_n\colon \mathcal{A}_4^{\mathrm{Vor}}(n) \longrightarrow \mathcal{A}_4^{\mathrm{Vor}} \longrightarrow \mathcal{A}_4^{\mathrm{Sat}} = \mathcal{A}_4 \amalg \mathcal{A}_3 \amalg \mathcal{A}_2 \amalg \mathcal{A}_1 \amalg \mathcal{A}_0.$$

The depth of an irreducible curve $C \subset \mathcal{A}_4^{\mathrm{Vor}}(n)$ is defined by

$$\mathrm{depth}(C) := \min\{k \mid \phi_n(C) \cap \mathcal{A}_{4-k} \neq \emptyset\}.$$

Obviously $0 \leq \mathrm{depth}(C) \leq 4$ and $\mathrm{depth}(C) = 0$ if and only if C is not contained in the boundary. In the rest of the paper we shall treat each case in turn, starting with depth 4 (subsection IV.1) and then going on to depth 0 in subsection IV.2, depth 1 in subsection IV.3, depth 2 in subsection IV.4 and finally depth 3 in subsection IV.5.

IV.1 Curves of depth 4

For a depth 4 curve, the question of whether it meets a given divisor negatively is a purely toric one, depending only on facts about Vor(4).

Proposition IV.2 *A divisor $aL - bD_4 - cE$ on $\mathcal{A}_4^{\mathrm{Vor}}$ has non-negative intersection with all irreducible curves C of depth 4 if and only if $b \geq 2c \geq 0$.*

Proof. Since C is mapped to a point in the Satake compactification it follows that $L.C = 0$. First we suppose that $C = C_E$ is a curve in E. It is known that $\mathcal{A}_4^{\mathrm{Vor}}$ is projective: this was proved by Alexeev in [Al] for all $\mathcal{A}_g^{\mathrm{Vor}}$ but seems to have been known for much longer for $g = 4$; see for instance [Nam2]. Therefore there is an ample line bundle on E which is the restriction of a line bundle on $\mathcal{A}_4^{\mathrm{Vor}}$. But we saw in Proposition I.6 that $\mathrm{Pic}(\mathcal{A}_4^{\mathrm{Vor}})$ is generated by L, D_4^{Vor} and E; and the first two of these are pulled back from $\mathcal{A}_4^{\mathrm{Igu}}$ and

hence zero on E. So either $E|_E$ or $-E|_E$ is ample, and it is easy to see that in fact $-E|_E$ is ample. This follows because $\Pi_2(4)$ is contained in a half-space. Hence on the toric variety $T_{\mathbb{M}_4}\mathrm{emb}\big(\mathrm{Vor}(4)\cap\Pi_2(4)\big)$ got by subdividing the second perfect domain into Voronoi cones, with torus-invariant divisors $E = \bar{\mathcal{O}}(\eta)$ (this is an abuse of notation as it is the normalisation of $E \subset \mathcal{A}_4^{\mathrm{Vor}}$) and $D_1,\ldots,D_{12}$ given by the generators, there is a linear relation involving E and all the D_i with positive coefficients. So, on the toric variety, $-E|_E$ is effective; and this remains true on $\mathcal{A}_4^{\mathrm{Vor}}$. So $H.C \geq 0$ if and only if $c \geq 0$.
Actually we can do better. If we work instead with $\Psi'\big(\Pi_2(4)\big)$ and use the linear relation induced by the linear form $\sum U_{ii}$, we see that on the toric variety $-4E|_E = K|_E$, so that $E \subset X_{\mathrm{Vor}(4)}$ is a toric Fano variety.
Next we consider the case of a curve C of depth 4 that does not meet E. Then C is contained in the boundary D_4^{Igu} of $\mathcal{A}_4^{\mathrm{Igu}}$, and the same considerations as above, applied to $\mathcal{A}_4^{\mathrm{Igu}} \to \mathcal{A}_4^{\mathrm{Sat}}$, show that $-D_4^{\mathrm{Igu}}|_{D_4^{\mathrm{Igu}}}$ is ample.
We remark that the morphism $\mathcal{A}_4^{\mathrm{Vor}} \to \mathcal{A}_4^{\mathrm{Sat}}$ is a normalised blow-up of some sheaf of ideals, (see [Ch] and [SC, Section IV]), and this is also sufficient for our purposes.
For other curves of depth 4, it is convenient to work on $\mathcal{A}_4^{\mathrm{Vor}}(n)$ for some $n \geq 3$: pulling back by $\alpha_{n,\mathrm{Vor}}$, we must show that $\big(bD_4(n) + cE(n)\big).C \leq 0$ for every irreducible curve C of depth 4 if and only if $b \geq 2c \geq 0$. Notice that this will also prove that these conditions $b \geq 2c \geq 0$ are necessary for $aL - bD_4 - cE$ to be nef, as claimed in Theorem I.8.
It is enough to consider the curves C corresponding to codimension 1 cones $\sigma \in \mathrm{Vor}(4)$. This is because we need only consider irreducible curves, and any such curve in $E(n)$ lifts to a single irreducible component E of the boundary of $X_{\mathrm{Vor}(4)}$. But such a component is a toric variety, and the curves corresponding to codimension 1 cones generate the cone of effective curves in E. Indeed, such curves generate the whole of $A_1(E)$, by for instance [Dan, Proposition 10.3], and rational equivalence implies numerical equivalence.
We have already dealt with such curves in the case where $\sigma \succ \langle g(e)\rangle$ for some $g \in \mathrm{GL}(\mathbb{L}_4)$, because they are contained in $E(n)$. So it remains to deal with $\sigma \prec \Pi_2(4)$. Up to $\mathrm{GL}(\mathbb{L}_4)$-action there are two such cones (RT and BF facets in the notation of [ER2]). We choose to work with the cones $\sigma_0 = \{\beta_{13} = \beta_{14} = \rho_{34} = 0\}$ and $\sigma_1 = \{\beta_{13} = \beta_{14} = \beta_{34} = 0\}$. Each of these is a 9-dimensional face of the second perfect cone $\Pi_2(4)$ and defines a rational curve $C \cong \mathbb{P}^1$ in $X_{\mathrm{Vor}(4)}$.
Let $\Pi_2^1(4)'$ be the 10-dimensional cone $\langle\sigma_0, e\rangle$. In the language of [ER2], $\Pi_2^1(4)'$ is a type III domain: in the classification of Proposition III.6 it is equivalent to $\Pi_2^1(4)$. The facet σ_0 is an RT facet of $\Pi_2(4)$. The transformation $x_1 \leftrightarrow x_3, x_2 \leftrightarrow x_4$ leaves σ_0 invariant, but maps η to a ray $\eta' = \langle e'\rangle$ and $\Pi_2^1(4)'$ to another 10-dimensional cone, a part of a translate of $\Pi_2(4)$, which is again a type III domain.

The geometric situation is this: the curve $C_0 \subset X_{\mathrm{Vor}(4)}$ corresponding to σ_0 is contained in nine boundary components, which we call $D_{4,2}^{\mathrm{Vor}}, \ldots, D_{4,10}^{\mathrm{Vor}}$ (here 10 means 'ten', not $\{1,0\}$: the reason for the indexing will appear below), out of the twelve boundary components $D_{4,1}^{\mathrm{Vor}}, \ldots, D_{4,12}^{\mathrm{Vor}}$ corresponding to 1-dimensional faces of $\Pi_2(4)$. These are the ones belonging to the 1-dimensional faces of σ_0. It is met (transversely) by two exceptional divisors E, E' corresponding to the rays η and η'. No other invariant divisors meet C, and because of the level structure E and E' give distinct disjoint components of $E(n)$.

The form $\frac{1}{2}\operatorname{Tr}'$ which we introduced in the proof of Proposition I.3 takes the value 1 on each primitive generator of σ_0, while $\frac{1}{2}\operatorname{Tr}'(e) = 4$ and $\frac{1}{2}\operatorname{Tr}'(e') = 5$. This shows that

$$D_{4,2}^{\mathrm{Vor}} + \ldots + D_{4,10}^{\mathrm{Vor}} + 4E + 5E' + R \sim 0$$

where R is a divisor in $X_{\mathrm{Vor}(4)}$ which does not meet C_0. This implies

$$D_4(n).C_0 = (D_{4,2}^{\mathrm{Vor}} + \ldots + D_{4,10}^{\mathrm{Vor}} + 4E + 4E').C_0 = -E'.C_0 = -1.$$

We also have $L.C_0 = 0$ and $E(n).C_0 = (E + E').C_0 = 2$. Hence

$$\big(-bD_4(n) - cE(n)\big).C_0 = b - 2c \geq 0,$$

which is the desired inequality.

Finally we do the same calculation for σ_1. This cone has $\langle \sigma_1, e\rangle = \Pi_2^2(4)$ which is a type II domain: σ_1 itself is a BF facet, since $w(\sigma_1) = \{\beta_{12} = \beta_{23} = \beta_{24} = 0\}$. It forms the boundary between $\Pi_2(4)$ and $\Pi_1(4)$ and shares an 8-dimensional face with σ_0.

Choosing the numbering suitably, we have the following geometric picture: the curve C_1 lies in the intersection of the nine boundary divisors $D_{4,1}^{\mathrm{Vor}}, D_{4,2}^{\mathrm{Vor}}, \ldots, D_{4,9}^{\mathrm{Vor}}$, and is met transversely by E and another boundary component $D_{4,0}^{\mathrm{Vor}}$ corresponding to $\big\langle (x_1 - x_2)^2 \big\rangle$. (With this choice of indexing the 1-dimensional faces of $\Pi_1(4)$ correspond to $D_{4,0}^{\mathrm{Vor}}, \ldots, D_{4,9}^{\mathrm{Vor}}$.) Now $\frac{1}{2}\operatorname{Tr}'\big((x_1 - x_2)^2\big) = 2$ so

$$D_{4,1}^{\mathrm{Vor}} + \cdots + D_{4,9}^{\mathrm{Vor}} + 4E + 2D_{4,0}^{\mathrm{Vor}} + R \sim 0$$

for some R not meeting C_1; and hence

$$D_4(n).C_1 = (D_{4,0}^{\mathrm{Vor}} + D_{4,1}^{\mathrm{Vor}} + \cdots + D_{4,9}^{\mathrm{Vor}} + 4E).C_1 = -D_{4,0}^{\mathrm{Vor}}.C_1 = -1.$$

We also have $E(n).C_1 = E.C_1 = 1$ so

$$\big(-bD_4(n) - cE(n)\big).C_1 = b - c \geq b - 2c \geq 0,$$

which completes the proof. $\square$

IV.2 Curves of depth 0

The method we use in this case, of curves that are not contained in the boundary of $\mathcal{A}_4^{\mathrm{Vor}}$, is analogous to the proof of Proposition II.2, but rather more complicated. We have to produce a modular form vanishing to sufficiently high order along each boundary component of $\mathcal{A}_4^{\mathrm{Vor}}$. It is relatively easy to supply the modular form: our proof that it does indeed have the required vanishing is neither simple nor elegant.

Proposition IV.3 *Let C be a depth 0 curve and let $H = aL - bD_4 - cE$ be a divisor on $\mathcal{A}_4^{\mathrm{Vor}}$ with $a \geq 0$, $a - 12b \geq 0$ and $b \geq 2c \geq 0$. Then $H.C \geq 0$.*

Proof. It is simpler to work this time with D_4^{Vor} rather than D_4, so we write $H = \alpha L - \beta D_4^{\mathrm{Vor}} - \gamma E$ (as in Remark I.9) and assume that $\beta \geq 0$, $\alpha - 12\beta \geq 0$ and $\gamma \geq 4\beta \geq \frac{8}{9}\gamma$.
We note that $L.C > 0$ since C maps to a curve in the Satake compactification and L is ample on $\mathcal{A}_4^{\mathrm{Sat}}$. It is enough to prove that $H.C > 0$ if $a - 12b > 0$. Choose some $\varepsilon > 0$ with $a/b > 12 + \varepsilon$, and let F be a modular form of weight k, with $F|_C \not\equiv 0$, vanishing of order $m \geq k/(12+\varepsilon)$ on D_4^{Vor} and order $r \geq 9m/2$ on E: such a form exists by Proposition IV.4, below.
Now we can write

$$kL = mD_4^{\mathrm{Vor}} + rE + D_F,\ C \not\subset D_F$$

where D_F is the zero divisor of F on $\mathcal{A}_4^{\mathrm{Vor}}$ (that is, the closure in $\mathcal{A}_4^{\mathrm{Vor}}$ of the set $\{F = 0\} \subset \mathcal{A}_4$). So

$$\left(\tfrac{k}{m}L - D_4^{\mathrm{Vor}} - \tfrac{r}{m}E\right).C = \tfrac{1}{m}D_F.C \geq 0.$$

Since $a/b > 12 + \varepsilon \geq k/m$ and $r/m \geq 9/2$, and $E.C \geq 0$ since C is not of depth 4, it follows that

$$\left(\tfrac{\alpha}{\beta}L - D_4^{\mathrm{Vor}} - \tfrac{\gamma}{\beta}E\right).C > \left(\tfrac{k}{m}L - D_4^{\mathrm{Vor}} - \tfrac{r}{m}E\right).C \geq 0.$$

□

It remains to establish that the modular form F exists.

Proposition IV.4 *Given an irreducible curve $C \subset \mathcal{A}_4^{\mathrm{Vor}}$ of depth 0, There exists a $k \in \mathbb{N}$ and a modular form F for $\mathrm{Sp}(8,\mathbb{Z})$ of weight k with $F|_C \not\equiv 0$, such that F vanishes of order m on D_4^{Vor} and order r on E, and $m/k \geq 1/(12+\varepsilon)$ and $r \geq 9m/2$.*

Proof. Apart from the inequality $r \geq 9m/2$ this is the result of Weissauer [Wei, p. 220] that we used in the proof of Proposition II.2. We shall prove that the forms that Weissauer constructs also fulfill the inequality $r \geq 9m/2$. For this purpose we need to recall his construction.

Let $l = 2p$ with p prime and consider the set $\mathcal{M}$ of all characteristics in $(\frac{1}{l}\mathbb{Z}/\mathbb{Z})^8$ of the form

$$m = (m_{(p)}, m_{(2)}) \in (\tfrac{1}{p}\mathbb{Z}/\mathbb{Z})^8 \oplus (\tfrac{1}{2}\mathbb{Z}/\mathbb{Z})^8,\ m_{(p)} \neq 0.$$

For a characteristic $m = (m', m'')$ with $m', m'' \in \mathbb{R}^4$ the associated theta constant is defined by

$$\Theta_m(\tau, 0) = \sum_{q \in \mathbb{Z}^g} e^{2\pi i [\frac{1}{2}(q+m')\tau^t(q+m') + (q+m')^t m'']}$$

For $\tilde{\mathcal{M}} \subset \mathcal{M}$, define

$$\Theta_{\mathcal{M},\tilde{\mathcal{M}}}(\tau) = \prod_{m \in \mathcal{M}\setminus\tilde{\mathcal{M}}} \Theta_m(\tau, 0)^l$$

and

$$F_r(\tau) = \sum_{M \in \Gamma_g/\Gamma_g(l)} \Theta_{\mathcal{M},\tilde{\mathcal{M}}}(\tau)^r | M$$

where $\Theta_{\mathcal{M},\tilde{\mathcal{M}}}(\tau)^r|M$ denotes the usual slash operator and M runs through a set of representatives of $\Gamma_4/\Gamma_4(2l)$. Weissauer then shows that for given $\varepsilon > 0$ and $\tau \in \mathbb{H}_4$ there is a subset $\tilde{\mathcal{M}} \subset \mathcal{M}$ such that $\Theta_{\mathcal{M},\tilde{\mathcal{M}}}(\tau) \neq 0$ and such that the resulting form F_r has the property that $m/k \geq 1/(12+\varepsilon)$.
We have to compare the vanishing order of such a form F_r on D_4^{Vor} with its vanishing order on E. In order to do this, we consider the 2-dimensional cone $\langle x_1^2, e\rangle$. The dual cone was computed in subsection IV.1 above, equation (11). We write $m' = (m_1, m_2, m_3, m_4)$ and assume that we have normalised in such a way that $-1/2 \leq m_i \leq 1/2$. In order to compute the vanishing order of $F_r(\tau)$ we have to compute the Fourier expansions of the theta constants

$$\Theta_{m,0}(\tau, 0) = \sum_{q \in \mathbb{Z}^4} \prod_{i,j} t_{ij}^{\frac{1}{2}(m_i+q_i)(m_j+q_j)} e^{2\pi i (q+m')^t m''}.$$

We can rewrite this in terms of the coordinates $T_1, T_2, \ldots, T_{10}$. The vanishing order of $\Theta_{m,0}(\tau, 0)$ along the divisor corresponding to x_1^2 is then the minimum of the exponent of $T_1^{\frac{1}{2}(q_1+m_1)^2}$ for $q_1 \in \mathbb{Z}$ and equals $m_1^2/2$.
The divisor E is given by $T_2 = 0$, so the vanishing order of $F_r(\tau)$ along E is the minimum over all $q \in \mathbb{Z}^4$ of the exponents of T_2 in the summand given by q. For fixed q this order can easily be computed to be $\frac{1}{2}e(q+m)$, where e is the familiar quadratic form given in Section I, equation (1). That is,

$$\begin{aligned} \tfrac{1}{2}e(q+m) \quad = \quad & \sum_i (q_i+m_i)^2 + (q_1+m_1)(q_2+m_2) \\ & - (q_1+m_1)(q_3+m_3) - (q_1+m_1)(q_4+m_4) \\ & - (q_2+m_2)(q_3+m_3) - (q_2+m_2)(q_4+m_4). \end{aligned}$$

For $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ we define

$$e_{\min}(x) = \min_{q \in \mathbb{Z}^4} e(q+x);$$

then $F_r(\tau)$ will have the required vanishing as long as

$$\sum_{m \in \mathcal{M} \setminus \tilde{\mathcal{M}}} \left(e_{\min}(m) - \tfrac{9}{2} m_1^2\right) \geq 0.$$

We claim that this is true for l large enough. Given $\varepsilon > 0$ we have $\#\tilde{\mathcal{M}} < \varepsilon \#\mathcal{M}$ for $l \gg 0$, so (cf. [Wei, pp. 218–219])

$$\begin{aligned} &\lim_{l \to \infty} \frac{1}{\#(\mathcal{M} \setminus \tilde{\mathcal{M}})} \sum_{m \in \mathcal{M} \setminus \tilde{\mathcal{M}}} \left(e_{\min}(m) - \tfrac{9}{2} m_1^2\right) \\ = &\iiiint_{[-\frac{1}{2}, \frac{1}{2}]^4} \left(e_{\min}(x) - \tfrac{9}{2} x_1^2\right) dx_1\, dx_2\, dx_3\, dx_4 \\ = &-\frac{3}{16} + \iiiint_{[-\frac{1}{2}, \frac{1}{2}]^4} e_{\min}(x)\, dx_1\, dx_2\, dx_3\, dx_4. \end{aligned}$$

The integral is not easy to evaluate, even though its value is rational. The region of $\mathbb{R}^4$ for which the minimum is achieved by some particular value of q is a Delaunay cell for the quadratic form e, but these are complicated: there is a complete description in [V2a]. Instead of attempting to evaluate the integral precisely, we chose to estimate it by calculating $\frac{1}{\#\mathcal{M}'} \sum_{m \in \mathcal{M}'} e_{\min}(m)$ using a computer, for a suitable set of points $\mathcal{M}'$. Taking $\mathcal{M}'$ to be the set of points with coordinates of the form $\frac{2k+1}{158}$ gave us the estimate

$$\frac{1}{\#\mathcal{M}'} \sum_{m \in \mathcal{M}'} e_{\min}(m) \approx 0.2166667$$

and by bounding the derivatives of the piecewise differentiable continuous function $e_{\min}$ one easily checks that the difference between this and the actual value of the integral is less than 0.025. Therefore $\int e_{\min} > \frac{3}{16} = \int \frac{9}{2} x_1^2$, and this proves the result. □

Remark IV.5 The numerical evidence is overwhelming that $\int e_{\min} = \frac{13}{60}$, so that

$$\lim_{l \to \infty} \frac{1}{\#(\mathcal{M} \setminus \tilde{\mathcal{M}})} \sum_{m \in \mathcal{M} \setminus \tilde{\mathcal{M}}} \left(e_{\min}(m) - \tfrac{9}{2} m_1^2\right) = \frac{7}{240}.$$

IV.3 Curves of depth 1

In this subsection we want to prove a result (Proposition IV.7) giving conditions for a divisor to have non-negative intersection with every curve of

depth 1. If $C \subset \mathcal{A}_4^{\mathrm{Vor}}$ is a curve of depth 1 we denote the irreducible components of $\alpha_{n,\mathrm{Vor}}^{-1}(C) \subset \mathcal{A}_4^{\mathrm{Vor}}(n)$ by $C_j(n)$. Since C is of depth 1 it is contained in D_4^{Vor}: any component $C_j(n)$ is therefore contained in a boundary component $D_{4,i}^{\mathrm{Vor}}(n)$.
The crucial point of the proof is the following lemma.

Lemma IV.6 *For any curve C as above, given $\varepsilon > 0$, there exist integers k and n and a boundary component $D_{4,i}^{\mathrm{Vor}}(n)$, for which we can find a section $\mathrm{s} \in H^0\big(kJ_4(n)\big)$ and a component $C_j(n) \subset D_{4,i}^{\mathrm{Vor}}$ such that*

(i) $\mathrm{s}|_{C_j(n)} \not\equiv 0$,

(ii) s *vanishes on $p_{i,n}^*\big(D_3(n)\big)$ to order λ, with $\lambda/k \geq n/(12+\varepsilon)$.*

Proof. The proof of this lemma is a version of the argument of Weissauer which we have already used for curves of depth 0. The argument given in [Hu, Proposition 4.1] for $g = 2$ is valid verbatim for all $g \geq 2$. □

Proposition IV.7 *Let C be a depth 1 curve and let $H = aL - bD_4 - cE$ be a divisor on $\mathcal{A}_4^{\mathrm{Vor}}$ with $a \geq 0$, $a - 12b \geq 0$ and $b \geq c \geq 0$. Then $H.C \geq 0$.*

Proof. It is sufficient to prove the result with the stronger condition $a - 12b > 0$, since we can then take the limit as $a - 12b \to 0$. Furthermore, because the cover $\alpha_{n,\mathrm{Vor}}$ is Galois, it is enough to prove that $\alpha_{n,\mathrm{Vor}}^*(H).C_j(n) \geq 0$ for some n and some component $C_j(n)$. We first of all choose some $\varepsilon > 0$ such that

$$(a - 12b) - b\left(1 - \frac{12}{12+\varepsilon}\right) > 0.$$

We also choose a point $(z, \tau) \in \mathbb{C}^3 \times \mathbb{H}_3$ whose image

$$[(z,\tau)] \in D_4' = \big((\mathbb{Z}^3 \times \mathbb{Z}^3) \rtimes \mathrm{Sp}(6,\mathbb{Z})\big)\backslash \mathbb{C}^3 \times \mathbb{H}_3$$

lies on the curve C. Choose a boundary component $D_{4,1}'(n)$ and consider the point

$$[(z,\tau)]_{n,1} \in D_{4,1}'(n) \cong \big((n\mathbb{Z}^3 \times n\mathbb{Z}^3) \rtimes \Gamma_3(n)\big)\backslash \mathbb{C}^3 \times \mathbb{H}_3 .$$

It lies on some component $C_j(n)$ of the preimage of C in $D_{4,1}^{\mathrm{Vor}}(n)$. We shall prove that $\alpha_{n,\mathrm{Vor}}^*(H).C_j(n) > 0$ for this component.
Recall from (8) and (9) that

$$\begin{aligned} -D_{4,1}^{\mathrm{Vor}}(n)|_{D_{4,1}^{\mathrm{Vor}}(n)} = -D_{4,1}^{\mathrm{Vor}}(n)|_1 &= \tfrac{1}{n}M_{4,1}(n) - \tfrac{1}{n}L \\ &= \tfrac{1}{n}J_4(n) + E(n)|_1 - \tfrac{1}{n}L, \end{aligned}$$

where $E(n)|_1 = \sum_s E_s(n)|_1$. It follows (see Remark I.10 and Proposition III.12) that

$$\begin{aligned}\alpha^*_{n,\mathrm{Vor}}(H)|_1 &= aL - bnD^{\mathrm{Vor}}_{4,1}(n)|_1 - bn\sum_{i\neq 1} D^{\mathrm{Vor}}_{4,i}(n)|_1 \\ &\quad - bnE(n)|_1 - cnE(n)|_1 \\ &= (a-b)L + bJ_4(n) - bnp^*_{1,n}\big(bD_3(n)\big) + n(b-c)E(n)|_1.\end{aligned}$$

In terms of divisors, Lemma IV.6 means that for some divisor $B \not\supset C$

$$J_4(n) \sim \tfrac{1}{k}B + \tfrac{\lambda}{k}p^*_{i,n}\big(D_3(n)\big).$$

Taking also into account that the divisor L on $D^{\mathrm{Vor}}_{4,1}(n)$ is a pullback from $\mathcal{A}^{\mathrm{Vor}}_3(n)$ we find that

$$\alpha^*_{n,\mathrm{Vor}}(H)|_1 = p^*_{1,n}\left((a-b)L - b\Big(n-\tfrac{\lambda}{k}\Big)D_3(n)\right) + \tfrac{b}{k}B + n(b-c)E(n)|_1.$$

By construction $B.C_j(n) \geq 0$ and since C is a curve of depth 1 and $b \geq c$ we also have $n(b-c)E(n)|_1.C \geq 0$. The result now follows from the corresponding result, [Hu, Theorem 0.2], on $\mathcal{A}^{\mathrm{Vor}}_3(n)$, provided

$$a - b > 12\frac{b}{n}\left(n - \frac{\lambda}{k}\right).$$

Since $\lambda k/n \geq 1/(12+\varepsilon)$ this follows from our choice of ε. $\square$

IV.4 Curves of depth 2

Let $C \subset \mathcal{A}^{\mathrm{Vor}}_4$ be a curve of depth ≥ 2 which is not contained in the exceptional divisor E. Then (see Remark III.10) there are at least two different linear forms $l \neq l'$ such that C is contained in the divisors corresponding to the rays $\langle l^2\rangle$ and $\langle l'^2\rangle$. This leads us to study the intersection of two boundary divisors in $\mathcal{A}^{\mathrm{Vor}}_4(n)$. Assume that $n \geq 3$ and that $D^{\mathrm{Vor}}_{4,ij}(n) \neq \emptyset$. We have seen that each of the boundary components admits a fibration

$$p_i \colon D^{\mathrm{Vor}}_{4,i}(n) \longrightarrow \mathcal{A}^{\mathrm{Vor}}_3(n).$$

Recall also (see e.g. [Hu]) that each boundary component $D_{3,k}(n)$ of $\mathcal{A}^{\mathrm{Vor}}_3(n)$ admits a fibration

$$q_k \colon D_{3,k}(n) \to \mathcal{A}^{\mathrm{Vor}}_2(n).$$

Indeed, this is the universal family over $\mathcal{A}^{\mathrm{Vor}}_2(n)$. Since $D^{\mathrm{Vor}}_{4,ij}(n)$ is mapped to the boundary of $\mathcal{A}^{\mathrm{Vor}}_3(n)$ this gives rise to the following situation, for some $k=k(i,j)$ determined by the ordered pair (i,j):

$$\begin{array}{ccc} D_{4,ij}^{\mathrm{Vor}}(n) & \subset & D_{4,i}^{\mathrm{Vor}}(n) \\ \big\downarrow{\scriptstyle p_i|_j} & & \big\downarrow{\scriptstyle p_i} \\ D_{3,k}(n) & \subset & \mathcal{A}_3^{\mathrm{Vor}}(n) \\ \big\downarrow{\scriptstyle q_k} & & \\ \mathcal{A}_2^{\mathrm{Vor}}(n). & & \end{array}$$

Let $r_{ij}=q_k\circ p_i|_j$. By Proposition III.12 we have that

$$\begin{aligned} -D_4(n)|_{ij} &= -D_{4,i}^{\mathrm{Vor}}(n)|_{ij} - p_i|_j^*\big(D_3(n)\big) \\ &= -D_{4,i}^{\mathrm{Vor}}(n)|_{ij} - p_i|_j^*\big(D_{3,k}(n)|_k\big) - r_{ij}^*\big(D_2(n)\big) \end{aligned}$$

where $D_2(n)$ is the boundary of $\mathcal{A}_2^{\mathrm{Vor}}(n)$, since

$$D_3(n)=D_{3,k}(n)+q_k^*\big(D_2(n)\big). \tag{12}$$

Lemma IV.8 *For $H=aL-bD_4(n)-cE(n)$ we have*

$$\begin{aligned} H|_{ij} &= \left(a-2\tfrac{b}{n}\right)L+\tfrac{b}{n}p_i|_j^*M_{3,k}(n)+\tfrac{b}{n}p_j|_i^*M_{3,k}(n)-br_{ij}^*\big(D_2(n)\big) \\ &\quad -br_{ji}^*\big(D_2(n)\big)+b\sum_{m\neq i,j}D_{4,m}^{\mathrm{Vor}}|_{ij}+(4b-c)E(n)|_{ij}. \end{aligned}$$

Proof. It follows from Proposition III.12 that

$$p_i|_j^*\big(D_3(n)|_k\big)=\sum_{l\neq i}D_{4,l}^{\mathrm{Vor}}(n)|_{ij}+4E(n)|_{ij}$$

and hence

$$-D_{4,j}^{\mathrm{Vor}}(n)|_{ij}=-p_i|_j^*\big(D_3(n)|_k\big)+\sum_{l\neq i,j}D_{4,l}^{\mathrm{Vor}}(n)|_{ij}+4E(n)|_{ij}.$$

Again using equation (12), this implies

$$-D_{4,j}^{\mathrm{Vor}}(n)|_{ij}=-p_i|_j^*\big(D_{3,k}(n)|_k\big)-r_{ij}^*(D_2(n))+\sum_{l\neq i,j}D_{4,l}^{\mathrm{Vor}}(n)|_{ij}+4E(n)|_{ij}.$$

Using

$$-D_{3,k}(n)|_k=\tfrac{1}{n}M_{3,k}(n)-\tfrac{1}{n}L$$

we find that

$$\begin{aligned} -D_{4,j}^{\mathrm{Vor}}(n)|_{ij} &= \tfrac{1}{n}p_i|_j^*\big(M_{3,k}(n)\big)-\tfrac{1}{n}L-r_{ij}^*\big(D_2(n)\big) \\ &\quad +\sum_{l\neq i,j}D_{4,l}^{\mathrm{Vor}}(n)|_{ij}+4E(n)|_{ij} \end{aligned} \tag{13}$$

Applying this formula also with i and j interchanged, and using the fact that $-D_4(n) = -\sum_l D^{\mathrm{Vor}}_{4,l}(n) - 4E(n)$, we obtain

$$\begin{aligned}-D_4(n)|_{ij} &= \tfrac{1}{n}p_i|_j^*\big(M_{3,k}(n)\big) + \tfrac{1}{n}p_j|_i^*\big(M_{3,k}(n)\big) - \tfrac{2}{n}L - r_{ij}^*\big(D_2(n)\big)\\ &\quad - r_{ji}^*\big(D_2(n)\big) + \sum_{m\neq i,j} D^{\mathrm{Vor}}_{4,m}(n)|_{ij} + 4E(n)|_{ij}.\end{aligned}$$

(There is a minor abuse of notation here, since the ks are not the same; but this does not matter.) The result follows from this immediately. □

We need to understand the last term in the expression in Lemma IV.8. A component $E_s(n)|_{ij}$ of the restriction of $E(n)$ to $D^{\mathrm{Vor}}_{4,ij}(n)$ corresponds to a 3-dimensional Voronoi cell $\langle l_i^2, l_j^2, e_s\rangle \in \mathrm{Vor}(4)$ where l_i and l_j are linear forms and e_s is $\mathrm{GL}(\mathbb{L}_4)$-equivalent to e. These were classified up to $\mathrm{GL}(\mathbb{L}_4)$-equivalence in Corollary III.14.
Let E and E' be the components of the exceptional divisor corresponding to the elements e and e' of $\mathbb{M}_4$ of Corollary III.14. As usual we assume that $D^{\mathrm{Vor}}_{4,1}(n)$ and $D^{\mathrm{Vor}}_{4,2}(n)$ correspond to x_1^2 and x_2^2 respectively. Note that the divisor $D_{3,k}(n) \subset \mathcal{A}_3^{\mathrm{Vor}}(n)$, the image of $p_1|_2$, corresponds to $x_2^2 \in \mathbb{M}_3$.

Proposition IV.9 *Let $E_s(n)|_i$ be a component of the restriction of $E(n)$ to some divisor $D^{\mathrm{Vor}}_{4,i}(n)$. Then $p_i\big(E_s(n)|_i\big)$ is contained in exactly four boundary components of $\mathcal{A}_3^{\mathrm{Vor}}(n)$.*

Proof. Since we now have only one non-exceptional boundary component to consider, it is enough to do this for the standard exceptional divisor E, corresponding to η. This is because the stabiliser of $\Pi_2(4)$ fixes η and permutes the generators. But $\bar{e} = \mathrm{Sym}_2\,\mathrm{pr}_1(e)$ is a sum of four forms of rank 1 in $\mathbb{M}_3$, by equation (6) in the proof of Proposition III.7. These rank 1 forms together span a cone of $\mathrm{Vor}(3)$ and $p_i(E)$ is contained in the intersection of the four corresponding boundary components. □

Remark IV.10 In the two cases E and E' above we find two different kinds of behaviour after projecting twice. The image $p_1(E|_1)$ is not contained in $D_{3,k}(n)$, but $p_1(E'|_1)$ is. This is because

$$\bar{e} = (x_2 - x_3)^2 + (x_2 - x_4)^2 + x_3^2 + x_4^2$$

does not have x_2^2 as a summand. In this case $r_{12}(E|_{12})$ is contained in two boundary components of $\mathcal{A}_2^{\mathrm{Vor}}(n)$.
On the other hand

$$\bar{e}' = (x_2 - x_3)^2 + (x_3 - x_4)^2 + x_2^2 + x_4^2$$

does have x_2^2 as a summand, so $p_1(E'|_1) \subseteq D_{3,k}(n)$. So $r_{12}(E'|_{12})$ is contained in three different boundary components of $\mathcal{A}_2^{\mathrm{Vor}}(n)$. In other words, $E'|_{12}$ is mapped under r_{12} to a deepest point in $\mathcal{A}_2^{\mathrm{Vor}}(n)$, whereas $E|_{12}$ is mapped to the intersection of two boundary components, i.e. to a $\mathbb{P}^1$ in $\mathcal{A}_2^{\mathrm{Vor}}(n)$ if $n \geq 3$.

Corollary IV.11 *Let $E_s(n)|_{ij}$ be a component of $E(n)|_{ij}$. Then the coefficient of $E_s(n)|_{ij}$ in $r_{ij}^*\big(D_2(n)\big)$ is equal to 3 if $E_s(n)$ is mapped under r_{ij} to a deepest point in $\mathcal{A}_2^{\mathrm{Vor}}(n)$ and equal to 4 otherwise.*

Proof. $D_2(n)$ is given by the support function ψ_2 on $\mathrm{Vor}(2)$ that takes the value 1 on the primitive generator of every ray. In toric terms, q_k is given by the projection $\mathrm{Sym}_2\,\mathrm{pr}_2\colon \mathbb{M}_3 \to \mathbb{M}_2$, which maps $\bar{e}$ to $2(x_3^2+x_4^2)$ and $\bar{e}'$ to $x_3^2+x_4^2+(x_3-x_4)^2$. So $\psi_2(\bar{e}) = 2\big(\psi_2(x_3^2)+\psi_2(x_4^2)\big) = 4$ and similarly $\psi_2(\bar{e}') = 3$. □

We are now in a position to begin checking nefness for depth 2 curves. Let C be such a curve. We choose a maximal cone σ such that C is contained in the closure of (the image in the moduli space) of $\mathcal{O}(\sigma)$. Since C is a depth 2 curve, it is not contained in the exceptional divisor and hence σ must be of the form $\langle l_1^2,\ldots,l_k^2\rangle$, where the l_i are linear forms on $\mathbb{L}_4$ and where $\{l_1=\cdots=l_k=0\}$ is a plane in $\mathbb{L}_4\otimes\mathbb{R}$. In particular, we may assume that l_1 and l_2 are linearly independent and that the other linear forms l_k, $k\geq 3$ are linear combinations of l_1 and l_2. Up to $\mathrm{GL}(\mathbb{L}_4)$-equivalence we may assume that $D_{4,i}^{\mathrm{Vor}}(n)$ and $D_{4,j}^{\mathrm{Vor}}(n)$ are $D_{4,1}^{\mathrm{Vor}}(n)$ and $D_{4,2}^{\mathrm{Vor}}(n)$, corresponding to $\langle l_1^2\rangle = \langle x_1^2\rangle$ and $\langle l_2^2\rangle = \langle x_2^2\rangle$ respectively. We may regard σ as a cone in $\mathrm{Vor}(2)$ and from the known description of $\mathrm{Vor}(2)$ it follows that we need only consider the cases $\sigma = \langle x_1^2, x_2^2\rangle$ or $\sigma = \langle x_1^2, x_2^2, (x_1-x_2)^2\rangle$.
In particular C is in either two or three (necessarily non-exceptional, since $\mathrm{depth}(C)\neq 4$) irreducible components of the boundary of $\mathcal{A}_4^{\mathrm{Vor}}(n)$. For any curve C with $0<\mathrm{depth}(C)<4$ we define the boundary multiplicity of C to be the number $\mu(C)$ of irreducible components of the boundary that contain C: it is the multiplicity of the generic point of C as a point of $D_4(n)$. First suppose that $\mathrm{depth}(C)=2$ and $\mu(C)=2$: that is, there are just two such components, which with the assumptions above are $D_{4,1}^{\mathrm{Vor}}$ and $D_{4,2}^{\mathrm{Vor}}$. We shall use the expression for $H|_{ij}$ which we derived in Lemma IV.8.
Notice (see Remark IV.10) that if a component of $E(n)|_{12}$ is contracted to a (deepest) point by r_{12} then it is also contracted to a point by r_{21}. Correspondingly we decompose $E(n)|_{12}$ as

$$E(n)|_{12} = E_+(n) + E_-(n),$$

where $E_+(n)$ consists of all the components that are not contracted to points.

Lemma IV.12 *Suppose $n\geq 3$ and $C\subset D_{4,12}^{\mathrm{Vor}}(n)$ is a depth 2 curve of boundary multiplicity 2. Then, given $\varepsilon>0$, we can write*

$$\frac{1}{n}p_1|_2^*\big(M_{3,k}(n)\big) \sim R_1 + \lambda_1 r_{12}^*\big(D_2(n)\big) + E_+(n)$$

where R_1 is an effective $\mathbb{Q}$-divisor with $C \not\subset \operatorname{Supp} R_1$ and $\lambda_1 \geq 1/(12+\varepsilon)$. A similar statement holds for $\frac{1}{n}p_2|_1^*\big(M_{3,k}(n)\big)$.

Proof. We have already explained that one can construct suitable sections of some power of $M_{3,k}(n)$ by taking products of theta functions of the form $\Theta_{m'm''}(z,\tau)$, where

$$\tau = \begin{pmatrix} \tau_{33} & \tau_{34} \\ \tau_{34} & \tau_{44} \end{pmatrix} \in \mathbb{H}_2 \text{ and } z = (z_1, z_2) = (\tau_{23}, \tau_{24}).$$

The claim about the vanishing along $r_{12}^*\big(D_2(n)\big)$ follows by Weissauer's argument as in [Hu, Proposition 4.1]. To check the contribution along the exceptional divisors it is sufficient to check the divisors given by the rays η: the terms corresponding to $\eta' = \langle e' \rangle$ (see Lemma III.14) can be absorbed into R_1. This works as in the proof of Proposition IV.1. The Fourier expansion, for $q = (q_3, q_4) \in \mathbb{Z}^2$, reads

$$\Theta_{m'm''}(z,\tau) = \sum_{q\in\mathbb{Z}^2} t_{33}^{\frac{1}{2}(q_3+m_3')^2 n} t_{44}^{\frac{1}{2}(q_4+m_4')^2 n} t_{34}^{(q_3+m_3')(q_4+m_4')n} t_{23}^{(q_3+m_3')n} t_{24}^{(q_4+m_4')n} e^{2\pi i (q+m')^t m''}.$$

Let $w_3 = (q_3 + m_3')$ and $w_4 = (q_4 + m_4')$. A computation analogous to that in the proof of Proposition IV.1 shows that the vanishing order along $E_+(n)$ is equal to the minimum value of $2 + 2(w_3^2 + w_4^2 - w_3 - w_4)$ for these values of w_3 and w_4. The real function $w_3^2 + w_4^2 - w_3 - w_4$ assumes its minimum at $w_3 = w_4 = 1/2$ where its value is $-1/2$ and this gives the result. $\square$

Proposition IV.13 *Let C be a depth 2 curve of boundary multiplicity 2, and let $H = aL - bD_4(n) - cE(n)$ be a divisor on $\mathcal{A}_4^{\mathrm{Vor}}(n)$ with $a - 12b/n \geq 0$, $b \geq c \geq 0$. Then $H.C \geq 0$.*

Proof. Using Proposition IV.8 and Lemma IV.12 we find that

$$\begin{aligned} H|_{12} \;=\;& (a - 2\tfrac{b}{n})L + b(R_1 + R_2) + b\lambda_1 r_{12}^*\big(D_2(n)\big) + b\lambda_2 r_{21}^*\big(D_2(n)\big) \\ & + 2bE_+(n) - br_{12}^*\big(D_2(n)\big) - br_{21}^*\big(D_2(n)\big) \\ & + b\sum_{i\neq 1,2} D_{4,i}^{\mathrm{Vor}}|_{12} + (4b-c)E(n)|_{12}. \end{aligned}$$

We can rewrite this in the form

$$\begin{aligned} H|_{12} \;=\;& \sum_{i=1,2} r_{ii'}^*\Big(\left(\tfrac{a}{2} - \tfrac{b}{n}\right) L - b\left(\tfrac{1}{2} - \lambda_i\right) D_2(n)\Big) + b(R_1 + R_2) \\ & - \sum_{i=1,2} \tfrac{b}{2} r_{ii'}^*\big(D_2(n)\big) + b\sum_{i\neq 1,2} D_{4,i}^{\mathrm{Vor}}(n)|_{12} \\ & + 2bE_+(n) + (4b-c)E(n)|_{12}, \end{aligned}$$

where $i' = 3 - i$. As before (Proposition IV.7), we may assume that in fact $a - 12b/n > 0$. The first two summands then have non-negative intersection with C. This follows by induction from our knowledge of the nef cone of $\mathcal{A}_2^{\mathrm{Vor}}(n)$ ([Hu, Theorem 0.2]) and the inequality $\lambda_i \geq 1/(12+\varepsilon)$, where we can assume $\varepsilon > 0$ arbitrarily small.
By construction also $(R_1 + R_2).C \geq 0$. By Corollary IV.11 we have

$$r_{12}^*\big(D_2(n)\big) = \sum_i D_{4,i}^{\mathrm{Vor}}(n)|_{12} + 4E_+(n) + 3E_-(n)$$

where $D_{4,i}^{\mathrm{Vor}}(n)$ runs through all components such that $r_{12}|_i$ is not dominant, and a similar formula for r_{21}^*. Altogether we see that the coefficients of $D_{4,i}^{\mathrm{Vor}}(n)|_{12}$ for $i \neq 1, 2$ and those of $E_+(n)$ and $E_-(n)$ are all non-negative. Since C is not contained in any of these divisors we have proved the assertion.
□

We now move on to the case where the curve C has depth 2 and boundary multiplicity 3. Such a curve is contained in the closure of the image in $\mathcal{A}_4^{\mathrm{Vor}}(n)$ of $\mathcal{O}(\sigma)$ where σ has dimension 3, and thus in $D_{4,I}^{\mathrm{Vor}}(n)$ for some set I of three indices. For convenience we take $I = \{1, 2, 3\}$, and we denote by $\mathfrak{S}_3$ the symmetric group on three elements acting as the symmetry group of I.
For each $\xi \in \mathfrak{S}_3$ we define the bundles

$$\mathcal{M}_\xi = \Big(p_{\xi(1)}|_{\xi(2)}^*\big(M_{3,k(\xi(1),\xi(2))}\big)\Big)\,|_{\xi(3)}$$

and

$$\mathcal{D}_\xi = \Big(r_{\xi(1)\xi(2)}^*\big(D_2(n)\big)\Big)\,|_{\xi(3)}$$

on $D_{4,I}^{\mathrm{Vor}}(n)$.

Lemma IV.14 *With the above notation*

$$-4D_4(n)|_I = \tfrac{1}{n}\sum_\xi \mathcal{M}_\xi - \tfrac{6}{n}L - \sum_\xi \mathcal{D}_\xi + 2\sum_{i\notin I} D_{4,i}^{\mathrm{Vor}}(n)|_I + 8E(n)|_I.$$

Proof. Apply equation (13) in the proof of Lemma IV.8 with $i = \xi(1)$ and $j = \xi(2)$ and restrict to $D_{4,\xi(3)}^{\mathrm{Vor}}(n)$. Rearranging this gives

$$-D_{4,\xi(2)}^{\mathrm{Vor}}(n)|_I - D_{4,\xi(3)}^{\mathrm{Vor}}(n)|_I = \tfrac{1}{n}\mathcal{M}_\xi - \tfrac{1}{n}L - \mathcal{D}_\xi + \sum_{i\notin I} D_{4,i}^{\mathrm{Vor}}(n)|_I + 4E(n)|_I;$$

taking the sum over $\xi \in \mathfrak{S}_3$ gives

$$-4\sum_{i\in I} D_{4,i}^{\mathrm{Vor}}(n)|_I = \tfrac{1}{n}\sum_\xi \mathcal{M}_\xi - \tfrac{6}{n}L - \sum_\xi \mathcal{D}_\xi + 6\sum_{i\notin I} D_{4,i}^{\mathrm{Vor}}(n)|_I + 24E(n)|_I.$$

Since $D_4(n)|_I = \sum_i D_{4,i}^{\mathrm{Vor}}(n)|_I + 4E(n)|_I$ we have the formula stated. $\square$

In this case we again decompose $E(n)|_I$ as $E_+(n) + E_-(n)$ by assigning a component to E_- if it is contracted to a deepest point by the r_{ij}. Again a component is either contracted by all of the r_{ij} or by none of them. This is easy to see, since σ is $\mathrm{GL}(\mathbb{L}_4)$-equivalent to $\langle x_1^2, x_2^2, (x_1 - x_2)^2 \rangle$ and hence r_{ij} corresponds to the linear map $\mathbb{L}_4 \to \mathbb{L}_2$ with kernel spanned by x_1 and x_2, independently of i and j.

Lemma IV.15 *For any $\xi \in \mathfrak{S}_3$, given $\varepsilon > 0$ we can write*

$$\tfrac{1}{n}\mathcal{M}_\xi \sim R_\xi + \lambda_\xi \mathcal{D}_\xi + E_+(n)$$

where R_ξ is an effective $\mathbb{Q}$-divisor on $D_{4,I}^{\mathrm{Vor}}(n)$ such that $C \not\subseteq \operatorname{Supp} R_\xi$, and $\lambda_\xi > 1/(12+\varepsilon)$.

Proof. Immediately from Lemma IV.12. $\square$

Note that, by Corollary IV.11, the coefficient of $E_+(n)$ in $\mathcal{D}_\xi$ is equal to 4 and the coefficient of $E_-(n)$ is equal to 3.

Proposition IV.16 *Let C be a curve of depth 2 and boundary multiplicity 3, and let $H - aL - bD_4(n) - cE(n)$ be a divisor on $\mathcal{A}_4^{\mathrm{Vor}}(n)$ such that $a - 12b/n \geq 0$ and $b \geq 2c \geq 0$. Then $H.C \geq 0$.*

Proof. As usual we may assume $a - 12b/n > 0$. By Lemma IV.14 and Lemma IV.15 we have

$$\begin{aligned} H|_I &= \left(a - \tfrac{3b}{2n}\right)L + \tfrac{b}{4}\sum_\xi R_\xi - \tfrac{b}{4}\sum_\xi (1-\lambda_\xi)\mathcal{D}_\xi + \tfrac{b}{2}\sum_{i \notin I} D_{4,i}^{\mathrm{Vor}}(n)|_I \\ &\quad + 2bE(n)|_I + \tfrac{3b}{2}E_+(n) - cE(n)|_I \\ &= \sum_\xi r^*_{\xi(1)\xi(2)}\left(\left(\tfrac{a}{6} - \tfrac{b}{4n}\right)L - \tfrac{b}{2}\left(\tfrac{1}{3} - \tfrac{\lambda_\xi}{2}\right)D_2(n)\right)|_{\xi(3)} - \tfrac{b}{12}\sum_\xi \mathcal{D}_\xi \\ &\quad + \tfrac{b}{4}\sum_\xi R_\xi + \tfrac{b}{2}\sum_{i \notin I} D_{4,i}^{\mathrm{Vor}}(n)|_I + (2b - c)E_-(n) + (\tfrac{7}{2}b - c)E_+(n). \end{aligned}$$

Furthermore

$$\mathcal{D}_\xi = \sum_{l \in J(\xi)} D_{4,l}^{\mathrm{Vor}}(n)|_I + 4E_+(n) + 3E_-(n),$$

where $l \in J(\xi)$ runs through all boundary components such that the image $r_{\xi(1)\xi(2)}\big(D_{4,l}^{\mathrm{Vor}}(n)|_I\big)$ is contained in the boundary of $\mathcal{A}_2^{\mathrm{Vor}}(n)$. Note that $I \cap J(\xi) = \emptyset$.

Using this we get

$$\begin{aligned} H|_I &= \sum_\xi r^*_{\xi(1)\xi(2)}\left(\left(\tfrac{a}{6}-\tfrac{b}{4n}\right)L-\tfrac{b}{2}\left(\tfrac{1}{3}-\tfrac{\lambda_\xi}{2}\right)D_2(n)\right)|_{\xi(3)}+\tfrac{b}{4}\sum_\xi R_\xi \\ &\quad+\sum_{i\notin I} d_i D^{\mathrm{Vor}}_{4,i}(n)|_I+(\tfrac{b}{2}-c)E_-(n)+(\tfrac{3b}{2}-c)E_+(n). \end{aligned}$$

Moreover the coefficients d_i are non-negative. By induction, i.e. by our knowledge of the nef cone of $\mathcal{A}_2^{\mathrm{Vor}}(n)$ from [Hu], we can assume that the divisor $\left(\frac{a}{6}-\frac{b}{4n}\right)L-b\left(\frac{1}{6}-\frac{\lambda_\xi}{4}\right)D_2(n)$ is nef and therefore that its pullback has non-negative intersection with C. Also $\sum_{\xi\in\mathfrak{S}_3} R_\xi.C\geq 0$. Since C is not contained in any of the divisors $D^{\mathrm{Vor}}_{4,i}(n)$ for $i\notin I$, nor in any component of $E(n)|_I$, and since $d_i\geq 0$ and the coefficients of the components of $E_-(n)$ and $E_+(n)$ are all non-negative, the result follows. $\square$

IV.5 Curves of depth 3

Supppose C is an irreducible depth 3 curve of boundary multiplicity $\mu=\mu(C)$. Then $3\leq\mu\leq 6$ and C is contained in $D^{\mathrm{Vor}}_{4,i}(n)$ if and only if $i\in I$ for some index set I of size $\mu(C)$. We assume without loss of generality that $I=\{1,2,\dots,\mu\}$, and denote by $\mathfrak{S}_\mu$ the symmetric group on μ symbols acting as the symmetry group of I. For any $\xi\in\mathfrak{S}_\mu$ we have the following diagram:

$$\begin{array}{ccccc} D^{\mathrm{Vor}}_{4,I}(n) & \subset & D^{\mathrm{Vor}}_{4,\xi(1)\xi(2)}(n) & \subset & D^{\mathrm{Vor}}_{4,\xi(1)}(n) \\ \Big\downarrow p_{\xi(1)}|_I & & \Big\downarrow p_{\xi(1)}|_{\xi(2)} & & \Big\downarrow p_{\xi(1)} \\ D_{3,K(\xi)}(n) & \subset & D_{3,k(\xi)}(n) & \subset & \mathcal{A}_3^{\mathrm{Vor}}(n) \\ \Big\downarrow q_{k(\xi)}|_{K(\xi)} & & \Big\downarrow q_{k(\xi)} & & \\ D_{2,m(k(\xi))}(n) & \subset & \mathcal{A}_2^{\mathrm{Vor}}(n) & & \\ \Big\downarrow s_{m(k(\xi))} & & & & \\ \mathcal{A}_1^{\mathrm{Vor}}(n). & & & & \end{array}$$

Here the index $k(\xi)$ and the set of indices $K(\xi)$ are all determined by the choice of $\xi\in\mathfrak{S}_\mu$: we define $K(\xi)$ to be the set of indices k such that $p_{\xi(1)}|_I\big(D^{\mathrm{Vor}}_{4,I}(n)\big)\subset D_{3,k}(n)$. In fact $k(\xi)$ depends only on $\xi(1)$ and $\xi(2)$, and $K(\xi)$ only on $\xi(1)$. For any $k\in K(\xi)$, we define $m(k)$ so that $q_k|_{K(\xi)}$ maps $D_{3,K(\xi)}(n)$ to $D_{2,m(k)}(n)$. It is not quite immediate that $m(k)$ is well defined: in principle the image of $q_k|_{K(\xi)}$ could be in the intersection of two boundary components of $\mathcal{A}_2^{\mathrm{Vor}}(n)$. However, this does not happen: see Corollary IV.19 below.

To reduce the amount of notation we write $p_\xi=p_{\xi(1)}|_{\xi(2)}$, $r_\xi=r_{\xi(1)\xi(2)}=q_{k(\xi)}\circ p_{\xi(1)}|_{\xi(2)}$, and $r_\xi|_I=q_{k(\xi)}|_{K(\xi)}\circ p_{\xi(1)}|_I$. Note that p_ξ and r_ξ depend only on $\xi(1)$ and $\xi(2)$. As in the case of depth 2 and boundary multiplicity 3 we define bundles

$$\mathcal{D}_\xi = \left(r_\xi^*(D_2(n))\right)|_I$$

on $D_{4,I}^{\mathrm{Vor}}(n)$. We also write

$$\mathcal{S}_\xi = \left(p_\xi^*(D_{3,k(\xi)}(n))\right)|_I$$

Lemma IV.17 *If $H = aL - bD_4(n) - cE(n)$ and $a \geq 12b/n \geq 0$, then there is a nef $\mathbb{Q}$-divisor A_1 such that*

$$\begin{aligned} H|_I &= A_1 - \tfrac{b}{(\mu-1)\mu!} \sum_{\xi \in \mathfrak{S}_\mu} \mathcal{D}_\xi \\ &\quad - \tfrac{b}{(\mu-1)(\mu-1)!} \sum_{\xi \in \mathfrak{S}_\mu} \mathcal{S}_\xi + \tfrac{b}{(\mu-1)} \sum_{i \notin I} D_{4,i}^{\mathrm{Vor}}(n)|_I + \left(\tfrac{4b}{\mu-1} - c\right) E(n)|_I. \end{aligned}$$

Proof. Replacing $\mathfrak{S}_3$ by $\mathfrak{S}_\mu$ in the proof of Lemma IV.14, we see that

$$\begin{aligned} &-(\mu-1)(\mu-1)!\, D_4(n)|_I = \\ &\quad - \sum_{\xi \in \mathfrak{S}_\mu} \mathcal{S}_\xi - \sum_{\xi \in \mathfrak{S}_\mu} \mathcal{D}_\xi + (\mu-1)! \sum_{i \notin I} D_{4,i}^{\mathrm{Vor}}(n)|_I + 4(\mu-1)!\, E(n)|_I. \end{aligned}$$

From this it follows that

$$\begin{aligned} H|_I &= aL - \tfrac{b}{(\mu-1)(\mu-1)!} \sum_{\xi \in \mathfrak{S}_\mu} \mathcal{S}_\xi - \tfrac{b}{(\mu-1)(\mu-1)!} \sum_{\xi \in \mathfrak{S}_\mu} \mathcal{D}_\xi \\ &\quad + \tfrac{b}{(\mu-1)} \sum_{i \notin I} D_{4,i}^{\mathrm{Vor}}(n)|_I + \left(\tfrac{4b}{\mu-1} - c\right) E(n)|_I \\ &= \sum_{\xi \in \mathfrak{S}_\mu} r_\xi^* \left(\tfrac{a}{\mu!} L - \tfrac{b}{\mu!} D_2(n)\right)|_I - \tfrac{b}{(\mu-1)\mu!} \sum_{\xi \in \mathfrak{S}_\mu} \mathcal{D}_\xi \\ &\quad - \tfrac{b}{(\mu-1)(\mu-1)!} \sum_{\xi \in \mathfrak{S}_\mu} \mathcal{S}_\xi + \tfrac{b}{(\mu-1)} \sum_{i \notin I} D_{4,i}^{\mathrm{Vor}}(n)|_I + \left(\tfrac{4b}{\mu-1} - c\right) E(n)|_I, \end{aligned}$$

which gives the formula required. The first term is nef because it is the pullback of a nef bundle from $\mathcal{A}_2^{\mathrm{Vor}}$. □

For any boundary component $D_{3,k}(n)$ of $\mathcal{A}_3^{\mathrm{Vor}}(n)$ we have as before (equation (12))

$$q_k^*(D_2(n))|_k = \sum_{j \neq k} D_{3,j}(n)|_k, \tag{14}$$

so

$$r_\xi^*(D_2(n))|_I = \sum_{j \neq k(\xi)} p_{\xi(1)}^*(D_{3,j}(n))|_I.$$

Lemma IV.18 *For any $\xi \in \mathfrak{S}_\mu$ (with $3 \leq \mu \leq 6$) we have $\#K(\xi) = 2$ or $\#K(\xi) = 3$. If $\mu = 3$ then $\#K(\xi) = 2$; if $\mu = 6$ then $\#K(\xi) = 3$. If $\mu = 4$ or $\mu = 5$ then both cases can occur, depending on σ and ξ.*

Proof. The cone σ is of the form $\langle l_1^2, \ldots, l_\mu^2 \rangle$ and has three special properties: it is a Voronoi cone, $\sigma \in \mathrm{Vor}(4)$; it contains no rank 4 forms (otherwise $\mathrm{depth}(C) = 1$); and the linear span of the l_i is of dimension 3. Then

$$K(\xi) = \{\mathrm{Sym}_2 \mathrm{pr}_{\xi(1)}(l_i) \mid \sigma = \langle l_1, \ldots, l_\mu \rangle,\ i \neq \xi(1)\}.$$

By using the $\mathrm{GL}(\mathbb{L}_4)$-action we may assume that $\sigma \prec \Pi_1(4)$, that is, that the generators of σ are x_i^2 or $(x_i - x_j)^2$; and we may assume that x_4 does not occur at all, so that $\sigma \prec \Pi_1(3)$.
If $\mu = 3$ then the l_i are linearly independent, so the projections of $l_{\xi(2)}$ and $l_{\xi(3)}$ are also linearly independent. So $\#K(\xi) = 2$.
If $\mu = 4$, there are two possibilities. If three of the l_i are linearly dependent then $\#K(\xi) = 2$ if $\xi(1)$ is one of those three, since the projection identifies the other two; but $\#K(\xi) = 3$ if $\xi(1)$ is the fourth generator. On the other hand, if no three of the l_i are linearly dependent, then any two projections are distinct, so $\#K(\xi) = 3$.
We remark that both these cases occur: examples are $\langle x_1^2, x_2^2, (x_1 - x_2)^2, x_3^2 \rangle$ and $\langle x_1^2, (x_1 - x_2)^2, x_3^2, (x_2 - x_3)^2 \rangle$.
If $\mu = 5$ then σ is a codimension 1 face of $\Pi_1(3)$ and thus equivalent to $\langle x_1^2, x_2^2, x_3^2, (x_1 - x_2)^2, (x_1 - x_3)^2 \rangle$. There are two linear relations involving three of the l_i and one generator (namely $l_1^2 = x_1^2$) occurs in both of them. Each linear relation involving $l_{\xi(1)}$ reduces $\#K(\xi)$ by 1, starting from $\mu - 1$ (i.e. if there were no relations all the other generators would give different elements of $K(\xi)$, so we should have $\#K(\xi) = \mu - 1$); so $\#K(\xi) = 2$ if $\xi(1) = 1$ and $\#K(\xi) = 3$ otherwise.
If $\mu = 6$ then $\sigma = \Pi_1(3)$ and because $\mathrm{GL}(\mathbb{L}_3)$ permutes the generators we have $\#K(\xi) = \#K(\mathrm{id}) = 3$ for all ξ. □

Corollary IV.19 *If $k \in K(\xi)$ then $m(k)$ is unique: that is, the image of $q_k|_{K(\xi)}$ is contained in exactly one boundary component $D_{2,m(k)}$ of $\mathcal{A}_2^{\mathrm{Vor}}(n)$.*

Proof. The linear span of $K(\xi)$ has dimension 2, since it is the projection of the linear span of the l_i. Projecting again from an element of $K_(\xi)$ therefore gives a space of dimension 1. □

We can characterise $m(k)$ by saying that it gives the unique boundary component that contains the image of C. We denote the elements of $K(\xi)$ by $k(\xi)$, $k'(\xi)$ and (if $\#K(\xi) = 3$) $k''(\xi)$.
We also know from the case $g = 2$ that $-D_{2,m(k)}(n)|_{m(k)}$ is nef. We write

$$D_2(n) = D_{2,m(k)}(n) + D_{2,\hat{m}(k)}(n),$$

so $D_{2,\hat{m}(k)}(n)$ is the union of all the boundary components except for the unique one that contains the image of C.

Lemma IV.20 *Let $D_{3,1}(n)$ and $D_{3,2}(n)$ be two boundary components of $D_3^{\mathrm{Vor}}(n)$. Suppose $q_1\colon D_{3,1}(n) \to \mathcal{A}_2^{\mathrm{Vor}}(n)$ and $q_2\colon D_{3,2}(n) \to \mathcal{A}_2^{\mathrm{Vor}}(n)$ are the associated projection maps, and write $D_{2,m(1)}$ for the image $q_1\big(D_{3,12}(n)\big)$ and similarly $D_{2,m(2)} = q_2\big(D_{3,12}(n)\big)$. Then*

(i) $q_2^* D_{2,\hat{m}(2)}(n) = q_1^* D_{2,\hat{m}(1)}(n)$,

(ii) $D_{3,1}(n)|_{12} = D_{3,2}(n)|_{12} + q_2^* D_{2,m(2)}(n)|_{12} - q_1^* D_{2,m(1)}(n)|_{12}$.

Proof. (i) We may assume that $D_{3,1}(n)$ and $D_{3,2}(n)$ correspond to $\langle x_1^2\rangle$ and $\langle x_2^2\rangle$ in $\mathrm{Vor}(3)$. Then q_1 is given by $\mathrm{Sym}_2\,\mathrm{pr}_1$ so boundary components of the image of q_1 (which is abstractly $\mathcal{A}_2^{\mathrm{Vor}}(n)$) may be thought of as cones in the Voronoi decomposition of quadratic forms in the variables x_2 and x_3, while for q_2 one should consider quadratic forms in x_1 and x_3. In particular $\hat{m}(1)$ is $\langle x_2^2\rangle$ and $\hat{m}(2)$ is $\langle x_1^2\rangle$. Conside an arbitrary boundary component given by $\big\langle (a_1x_1 + a_2x_2 + a_3x_3)^2\big\rangle \in \mathrm{Vor}(3)$. Under q_1 it maps to $\big\langle (a_2x_2 + a_3x_3)^2\big\rangle$, which is different from $m(1) = \langle x_2^2\rangle$ if and only if $a_3 \neq 0$. Similarly the image under q_2 is different from $m(2)$ if and only if $a_3 \neq 0$.
(ii) Applying (i) and using the equation (compare equation (12))

$$\begin{aligned}
-D_{3,2}(n) &= \sum_{i\neq 1,2} D_{3,i}(n)|_{12} - q_1^* D_2(n)|_{12}\\
&= \sum_{i\neq 1,2} D_{3,i}(n)|_{12} - q_1^* D_{2,m(1)}(n)|_{12} - q_1^* D_{\hat{m}(1)}(n)|_{12}
\end{aligned}$$

and the same equation with the indices interchanged, we obtain

$$\begin{aligned}
\big(-D_{3,2}(n) + q_1^* D_{2,m(1)}\big)|_{12} &= \sum_{i\neq 1,2} D_{3,i}(n)|_{12} - q_1^* D_{2,\hat{m}(1)}|_{12}\\
&= \sum_{i\neq 1,2} D_{3,i}(n)|_{12} - q_2^* D_{2,\hat{m}(2)}|_{12}\\
&= \big(-D_{3,1}(n) + q_2^* D_{2,m(2)}\big)|_{12}
\end{aligned}$$

as required. □

Proposition IV.21 *There is a nef $\mathbb{Q}$-divisor A_2 such that the following* *expression for $H|_I$ holds:*

$$\begin{aligned}
H|_I &= A_2 - \tfrac{b}{(\mu-1)\mu!} \sum_{\xi\in\mathfrak{S}_\mu} r_\xi^* D_{2,\hat{m}(k)}(n)|_I \\
&\quad - \sum_{\xi\in\mathfrak{S}_\mu} \tfrac{b}{(\#K(\xi)-1)(\mu-1)(\mu-1)!} r_\xi^* D_{2,\hat{m}(k)}(n)|_I \\
&\quad + \sum_{\xi\in\mathfrak{S}_\mu} \tfrac{b}{(\#K(\xi)-1)(\mu-1)(\mu-1)!} p_\xi^* \Big(\sum_{l\notin K(\xi)} D_{3,l}(n) \Big) \\
&\quad + \tfrac{b}{\mu-1} \sum_{i\notin I} D_{4,i}^{\mathrm{Vor}}(n)|_I + \left(\tfrac{4b}{\mu-1} - c\right) E(n)|_I .
\end{aligned}$$

Proof. From equation (14) we have

$$-D_{3,k'}|_{K(\xi)} = \sum_{l\notin K(\xi)} D_{3,l}|_{K(\xi)} - q_k^* D_2(n)|_{K(\xi)} \tag{15}$$

if $\#K = 2$ and

$$-D_{3,k'}|_{K(\xi)} - D_{3,k''}|_{K(\xi)} = \sum_{l\notin K(\xi)} D_{3,l}|_{K(\xi)} - q_k^* D_2(n)|_{K(\xi)} \tag{16}$$

if $\#K = 3$. We also have, by Lemma IV.20

$$D_{3,k}|_{K(\xi)} = D_{3,k'}|_{K(\xi)} + q_{k'}^* D_{2,m(k')}(n)|_{K(\xi)} - q_k^* D_{2,m(k)}(n)|_{K(\xi)} \tag{17}$$

and similarly for k'' in place of k' if $\#K = 3$. In the latter case we add the two equations to obtain

$$\begin{aligned}
D_{3,k}|_{K(\xi)} &= \tfrac{1}{2} D_{3,k'}|_{K(\xi)} + \tfrac{1}{2} D_{3,k''}|_{K(\xi)} + \tfrac{1}{2} q_{k'}^* D_{2,m(k')}(n)|_{K(\xi)} \\
&\quad + \tfrac{1}{2} q_{k''}^* D_{2,m(k'')}(n)|_{K(\xi)} - q_k^* D_{2,m(k)}(n)|_{K(\xi)} .
\end{aligned} \tag{18}$$

We use these equations to eliminate $\mathcal{S}_\xi$ from the formula in Lemma IV.17. From (15) and (17) we have

$$\mathcal{S}_\xi = - \sum_{l\notin K(\xi)} p_\xi^* D_{3,l}(n)|_{K(\xi)} + \mathcal{D}_\xi + p_\xi^* q_{k'}^* D_{2,m(k')} - p_\xi^* q_k^* D_{2,m(k)}$$

if $\#K = 2$, and from (16) and (18) we have

$$\begin{aligned}
\mathcal{S}_\xi &= -\tfrac{1}{2} \sum_{l\notin K(\xi)} p_\xi^* D_{3,l}(n)|_{K(\xi)} + \tfrac{1}{2}\mathcal{D}_\xi \\
&\quad + p_\xi^* \left(\tfrac{1}{2} q_{k'}^* D_{2,m(k')} + \tfrac{1}{2} q_{k''}^* D_{2,m(k'')} - q_k^* D_{2,m(k)} \right) .
\end{aligned}$$

The term $\frac{b}{\mu-1}\sum_{i\notin I} D_{4,i}^{\mathrm{Vor}}(n)|_I + \left(\frac{4b}{\mu-1} - c\right) E(n)|_I$ plays no role at this point and we temporarily denote it by $*$. So for $\# K = 2$ we have

$$\begin{aligned} H|_I &= A_1 + * - \tfrac{b}{(\mu-1)\mu!}\sum_{\xi\in\mathfrak{S}_\mu}\mathcal{D}_\xi - \tfrac{b}{(\mu-1)(\mu-1)!}\sum_{\xi\in\mathfrak{S}_\mu}\mathcal{D}_\xi \\ &\quad + \tfrac{b}{(\mu-1)(\mu-1)!}\sum_{\xi\in\mathfrak{S}_\mu}\sum_{l\notin K(\xi)} p_\xi^* D_{3,l}(n)|_{K(\xi)} \qquad (19) \\ &\quad + \sum_{\xi\in\mathfrak{S}_\mu} p_\xi^*\left(q_{k'}^* D_{2,m(k')} - q_k^* D_{2,m(k)}\right) \end{aligned}$$

and for $\# K = 3$

$$\begin{aligned} H|_I &= A_1 + * - \tfrac{b}{(\mu-1)\mu!}\sum_{\xi\in\mathfrak{S}_\mu}\mathcal{D}_\xi - \tfrac{b}{(\mu-1)(\mu-1)!}\sum_{\xi\in\mathfrak{S}_\mu}\tfrac{1}{2}\mathcal{D}_\xi \\ &\quad + \tfrac{b}{(\mu-1)(\mu-1)!}\sum_{\xi\in\mathfrak{S}_\mu}\sum_{l\notin K(\xi)} \tfrac{1}{2}p_\xi^* D_{3,l}(n)|_{K(\xi)} \qquad (20) \\ &\quad - \tfrac{b}{(\mu-1)(\mu-1)!}\sum_{\xi\in\mathfrak{S}_\mu} p_\xi^*\left(\tfrac{1}{2}q_{k'}^* D_{2,m(k')} + \tfrac{1}{2}q_{k''}^* D_{2,m(k'')} - q_k^* D_{2,m(k)}\right). \end{aligned}$$

Since $-D_{2,m(k)}(n)|_{m(k)}$ is nef we may add those terms to A_1, obtaining a nef $\mathbb{Q}$-divisor A_2 where

$$A_2 = A_1 - \tfrac{b}{(\mu-1)(\mu-1)!}\sum_{\xi\in\mathfrak{S}_\mu} r_\xi^* D_{2,m(k(\xi))}(n)|_{m(k(\xi))}.$$

In view of Lemma IV.20 this allows us to replace $\mathcal{D}_\xi$ by $r_\xi^* D_{2,\hat{m}(k)}(n)|_I$. This gives the desired coefficients for the $D_{2,\hat{m}(k)}(n)$ and $D_{3,l}(n)$ terms. Finally, the terms

$$\sum_{\xi\in\mathfrak{S}_\mu}\left(p_\xi^* q_{k'}^* D_{2,m(k')} - p_\xi^* q_k^* D_{2,m(k)}\right)$$

and

$$\sum_{\xi\in\mathfrak{S}_\mu}\left(\tfrac{1}{2}p_\xi^* q_{k'}^* D_{2,m(k')} + \tfrac{1}{2}p_\xi^* q_{k''}^* D_{2,m(k'')} - p_\xi^* q_k^* D_{2,m(k)}\right)$$

vanish. Indeed, if we fix $\xi(1) = a$ say, and take $\mathfrak{S}_{\mu-1} = \{\xi\in\mathfrak{S}_\mu \mid \xi(1) = a\}$ then

$$\sum_{\xi\in\mathfrak{S}_{\mu-1}}\left(p_a|_{\xi(2)}^* q_{k'}^* D_{2,m(k')} - p_a|_{\xi(2)}^* q_k^* D_{2,m(k)}\right) = 0$$

and similarly for $\# K = 3$. □

In fact only the part of $H|_I$ supported on the exceptional divisors could have negative intersection with C. We have

$$p_\xi^*\Big(\sum_{l\notin K(\xi)} D_{3,l}(n)\Big)|_I = \sum_{j\in L(\xi)} D_{4,j}^{\mathrm{Vor}}(n)|_I + \sum_s \gamma_s(\xi) E_s(n) \qquad (21)$$

and

$$r_\xi^* D_{2,\hat{m}(k)}(n)|_I = \sum_{j\in M(\xi)} D_{4,j}^{\mathrm{Vor}}(n)|_I + \sum_s \delta_s(\xi) E_s(n), \tag{22}$$

where $L(\xi) \supseteq M(\xi)$, $L(\xi)\cap I = \emptyset$ and γ_s and δ_s depend on the cone σ, the permutation ξ and the component $E_s(n)$ of the exceptional divisor.

Corollary IV.22 *$H.C \geq 0$ provided that*

$$\begin{aligned} 0 \quad \leq \quad & \sum_{\xi\in\mathfrak{S}_\mu} \Big(-\tfrac{b}{(\mu-1)\mu!}\delta_s(\xi) - \tfrac{b}{(\#K(\xi)-1)(\mu-1)(\mu-1)!}\delta_s(\xi) \\ & + \tfrac{b}{(\#K(\xi)-1)(\mu-1)(\mu-1)!}\gamma_s(\xi)\Big) + \tfrac{4b}{\mu-1} - c \end{aligned}$$

for every component $E_s(n)$ such that $E_s(n)\cap C \neq \emptyset$.

Proof. Away from the exceptional divisors $E_s(n)$, by Proposition IV.21 and equations (21) and (22) we can write

$$-\tfrac{b}{(\mu-1)\mu!}\sum_{\xi\in\mathfrak{S}_\mu} r_\xi^* D_{2,\hat{m}(k)}(n)|_I + \tfrac{b}{\mu-1}\sum_{i\notin I} D_{4,i}^{\mathrm{Vor}}(n)|_I = \sum_{i\notin I}\alpha_i D_{4,i}^{\mathrm{Vor}}(n)|_I$$

and

$$\begin{aligned} & -\sum_{\xi\in\mathfrak{S}_\mu} \tfrac{b}{(\#K(\xi)-1)(\mu-1)(\mu-1)!} r_\xi^* D_{2,\hat{m}(k)}(n)|_I \\ & \quad + \sum_{\xi\in\mathfrak{S}_\mu} \tfrac{b}{(\#K(\xi)-1)(\mu-1)(\mu-1)!} p_\xi^* \Big(\sum_{l\notin K(\xi)} D_{3,l}(n)\Big) = \sum_{i\notin I} \beta_i D_{4,i}^{\mathrm{Vor}}(n)|_I \end{aligned}$$

with $\alpha_i \geq 0$ and $\beta_i \geq 0$. Hence

$$\begin{aligned} H|_I \quad = \quad & A_2 + \sum_{i\notin I}(\alpha_i+\beta_i) D_{4,i}^{\mathrm{Vor}}(n)|_I \\ & + \sum_s \Bigg(\sum_{\xi\in\mathfrak{S}_\mu} \Big(-\tfrac{b}{(\mu-1)\mu!}\delta_s(\xi) - \tfrac{b}{(\#K(\xi)-1)(\mu-1)(\mu-1)!}\delta_s(\xi) \\ & + \tfrac{b}{(\#K(\xi)-1)(\mu-1)(\mu-1)!}\gamma_s(\xi)\Big) + \tfrac{4b}{\mu-1} - c\Bigg) E_s(n). \end{aligned}$$

Since A_2 is nef and $D_{4,i}^{\mathrm{Vor}}.C \geq 0$ for $i\notin I$ we have the result claimed. □

We shall fix s so that $E_s = E$, the component corresponding to the ray η, and drop the suffix s from the notation $\gamma_s(\xi)$, $\delta_s(\xi)$. Since we may assume that $C \cap E \neq \emptyset$, we can take σ to be a face of $\Pi_2(4)$. Obviously we need only consider the faces up to G-equivalence.
Unfortunately the inequality of Corollary IV.22 does not always hold. We shall need an extra argument, applied in Proposition IV.26 and Proposition IV.29, to handle the cases where it fails.

We begin by calculating the values of $\gamma(\xi)$ and $\delta(\xi)$ for three of the four cases of Lemma III.15. We shall work with the representatives given in Proposition III.15, and with the generators in the order given there. Note that the values of $\gamma(\xi)$ and $\delta(\xi)$ do depend on ξ, not just σ.

Proposition IV.23 *If σ is a 3-dimensional face of $\Pi_2(4)$, not of type RT^*, and $\xi \in \mathfrak{S}_3$ then the values of $\gamma(\xi)$ and $\delta(\xi)$ are:*

$\gamma(\xi) = 3,\ \delta(\xi) = 2$	*if σ is of type string and $\xi(1) \neq 3$;*
$\gamma(\xi) = 2,\ \delta(\xi) = 2$	*if σ is of type string and $\xi(1) = 3$;*
$\gamma(\xi) = 2,\ \delta(\xi) = 2$	*if σ is of type BF^*;*
$\gamma(\xi) = 4,\ \delta(\xi) = 4$	*if σ is of type disconnected.*

Proof. We consider the support functions ψ_γ and ψ_δ on (suitable copies of) $\mathrm{Vor}(3)$ and $\mathrm{Vor}(2)$ respectively that determine the divisors $\sum_{l\notin K(\xi)} D_{3,l}(n)$ and $D_{2,\hat{m}(k)}(n)$: namely, ψ_γ (respectively ψ_δ) takes the value 0 on a generator v of a ray in $\mathrm{Vor}(3)$ (respectively $\mathrm{Vor}(2)$) if $v \in K(\xi)$ (respectively $v = m(\xi)$) and 1 otherwise. Observe in particular that $\gamma(\xi)$ depends only on $\xi(1)$, since that determines $K(\xi)$, but $\delta(\xi)$ depends on the unordered pair $\{\xi(1), \xi(2)\}$.
String type: $\langle x_1^2, x_2^2, x_3^2\rangle$. Since x_1^2 and x_2^2 are interchanged by $k_3 \in G$, which preserves σ, we need only consider the cases $\xi(1) = 1$ and $\xi(1) = 3$. If $\xi(1) = 1$ then the projection gives, as in (6)

$$\bar{e} = (x_2 - x_3)^2 + (x_2 - x_4)^2 + x_3^2 + x_4^2. \tag{23}$$

In this case $K(\xi) = \{x_2^2, x_3^2\}$ so $\gamma(\xi) = 3$. Taking the further projection from $\xi(2)$ we get either $\bar{\bar{e}} = 2x_3^2 + 2x_4^2$, if $\xi(2) = 2$, or $\bar{\bar{e}} = x_2^2 + (x_2 - x_4)^2 + x_4^2$, if $\xi(2) = 3$. In the first case $m(\xi) = x_3^2$ and in the second case $m(\xi) = x_2^2$: in either case $\delta(\xi) = \psi_\delta(\bar{e}) = 2$.
On the other hand, if $\xi(1) = 3$ then the projection $\mathrm{Sym}_2 \mathrm{pr}_3$ gives

$$\bar{e} = x_1^2 + x_2^2 + x_4^2 + (x_1 + x_2 - x_4)^2.$$

$K(\xi) = \{x_1^2, x_2^2\}$ so $\gamma(\xi) = \psi_\gamma(\bar{e}) = 2$. Applying k_1 if necessary we may take $\xi(2) = 1$, so $\bar{\bar{e}} = x_2^2 + x_4^2 + (x_2 - x_4)^2$ and $m(\xi) = x_2^2$ so $\delta(\xi) = 2$
BF^* type: $\langle x_1^2, x_3^2, x_4^2\rangle$. In this case the subgroup of G that preserves σ acts transitively on the generators ($s_{23}k_2k_3$ interchanges x_1^2 and x_3^2; $s_{24}k_2k_4$ interchanges x_1^2 and x_4^2 – more simply, consider the symmetries of a genuinely black forked graph), so we need only consider $\xi = \mathrm{id}$. Then $\bar{e}$ is as in (23) and and $K(\xi) = \{x_3^2, x_4^2\}$, and $\bar{\bar{e}} = x_2^2 + (x_2 - x_4)^2 + x_4^2$: $m(\xi) = x_4^2$ so $\gamma(\xi) = \delta(\xi) = 2$.
Disconnected type: $\langle x_1^2, x_2^2, (x_3 - x_4)^2\rangle$. In this case x_1^2 and x_2^2 are interchanged by k_1 and x_1^2 and $(x_3 - x_4)^2$ are interchanged by w', both of which preserve σ, so it is enough to consider $\xi = \mathrm{id}$. Then $\bar{e}$ is as in (23)

and $K(\xi) = \{x_2^2, (x_3 - x_4)^2\}$, and $\bar{\bar{e}} = 2x_3^2 + 2x_4^2$: $m(\xi) = (x_3 - x_4)^2$ so $\gamma(\xi) = \delta(\xi) = 4$. $\square$

Corollary IV.24 *If C is a depth 3 curve with $\mu = 3$ contained in the closure of the image of the orbit of σ, and σ is not of type disconnected, suppose $H = aL - bD_4(n) - cE(n)$ is a divisor on $\mathcal{A}_4^{\mathrm{Vor}}(n)$ with $a - 12b/n \geq 0$, $b \geq 2c \geq 0$. Then $H.C \geq 0$.*

Proof. Note first that σ is not of type RT*, since the linear forms whose squares span σ then span a space of dimension 2, not 3. Since $\mu = 3$ we have $\#K(\xi) = 2$ always, by Lemma IV.18. According to Corollary IV.22 we need to show that if $b \geq 2c$ then

$$\sum_{\xi \in \mathfrak{S}_3} \left(-\tfrac{b}{12}\delta(\xi) - \tfrac{b}{4}\delta(\xi) + \tfrac{b}{4}\gamma(\xi)\right) + 2b - c \geq 0.$$

If σ is of string type then $\sum_{\xi \in \mathfrak{S}_3} \delta(\xi) = 12$ and $\sum_{\xi \in \mathfrak{S}_3} \gamma(\xi) = 16$, so we get $2b - c \geq 0$; if σ is of type BF* then $\sum_{\xi \in \mathfrak{S}_3} \delta(\xi) = \sum_{\xi \in \mathfrak{S}_3} \gamma(\xi) = 12$, so we get $b - c \geq 0$. $\square$

If σ is of disconnected type then $\sum_{\xi \in \mathfrak{S}_3} \delta(\xi) = \sum_{\xi \in \mathfrak{S}_3} \gamma(\xi) = 24$ and we get $-c$ which will not be positive. We need to deal with this case separately. We do this by examining the contribution from $D_{2,m(k)}(n)$. If $n \geq 3$ then the boundary component $D_{2,m(k)}(n)$ of $\mathcal{A}_2^{\mathrm{Vor}}(n)$ is isomorphic to the Shioda modular surface (or universal elliptic curve) of level n, which we call $S(n)$: the projection to the modular curve is $s_{m(k)} \colon D_{2,m(k)}(n) \to \mathcal{A}_1^{\mathrm{Vor}}(n) = X(n)$.

Lemma IV.25 *Let $N_2 = -D_{2,m(k)}(n)|_{D_{2,m(k)}}(n)$ be the normal bundle of $D_{2,m(k)}(n)$ in $\mathcal{A}_2^{\mathrm{Vor}}(n)$. Then*

$$N_2 = \tfrac{2}{n} s_{m(k)}^* L_{X(n)} + \tfrac{2}{n} \sum L_{ij}$$

where the L_{ij} are the sections of $S(n)$.

Proof. This was proved in [Hu, p. 271]. $\square$

Proposition IV.26 *Assume C is a depth 3 curve with $\mu = 3$ contained in the closure of the image of the orbit of σ, and σ is of disconnected type. Suppose $H = aL - bD_4(n) - cE(n)$ is a divisor on $\mathcal{A}_4^{\mathrm{Vor}}(n)$ with $a - 12b/n \geq 0$, $b \geq 2c \geq 0$. Then $H.C \geq 0$.*

Proof. We may take $\sigma = \langle x_1^2, x_2^2, (x_3 - x_4)^2 \rangle$ and as in the proof of Proposition IV.23 we may assume that $\xi = \mathrm{id}$. Then the exceptional divisor is mapped by r_ξ to the line $\bar{\bar{E}}$ given by $\langle x_3^2, x_4^2 \rangle$ and $D_{2,m(k)}(n)$ intersects this line transversally. If $r_\xi(C) \subset D_{2,m(k)}(n)$ is a curve that intersects $\bar{\bar{E}}$ then

it does so at a singular point of a fibre of $s_{m(k)}$, since the two components containing the image of E also meet there. In particular, $r_\xi(C)$ cannot be a section of $s_{m(k)}$.
We now have to remember that the term A_2 in Proposition IV.21 contains copies of the nef line bundle $-D_{2,m(k)}(n)|_{D_{2,m(k)}}(n)$. In fact in view of the explicit description of this line bundle given in Lemma IV.25 we know that it is represented by an effective $\mathbb{Q}$-divisor, namely $\frac{2}{n}(\sum L_{ij}) + \frac{2}{n}s^*_{m(k)}L_{X(n)} = \frac{2}{n}(\sum L_{ij}) + s^*_{m(k)}\big(\frac{1}{6}X_\infty(n)\big)$, where $X_\infty(n)$ is the set of cusps on the modular curve $X(n) = \mathcal{A}_1^{\mathrm{Vor}}(n)$. In particular, it follows that this line bundle has positive degree on all curves that are not sections. In view of equation (19) and the subsequent reasoning, it will be enough to prove that

$$\Big(-\tfrac{b}{12}\sum_{\xi\in\mathfrak{S}_\mu} r^*_\xi D_{2,m(\xi)}(n) - \tfrac{b}{4}\sum_{\xi\in\mathfrak{S}_\mu} r^*_\xi D_{2,m(\xi)}(n) - cE(n)_{\xi(1)\xi(2)}\Big).C \geq 0,$$

where $E(n)_{\xi(1)\xi(2)}$ denotes the union of those components of $E(n)$ that map to $\bar{\bar{E}}$. This simplifies to

$$\Big(-\tfrac{b}{3}\sum_{\xi\in\mathfrak{S}_\mu} r^*_\xi D_{2,m(\xi)}(n) - cE(n)_{\xi(1)\xi(2)}\Big).C \geq 0.$$

A priori the curve C can meet several components of $E(n)_{\xi(1)\xi(2)}$ and each of these in several points. Let P be some such point of intersection and let C_P be the curve germ defined by C at P.
Since $r_\xi(C)$ is not a section it follows that $\frac{2}{n}(\sum L_{ij}).r_\xi(C) \geq 0$. Now as we have said before the curve $r_\xi(C)$ must intersect a singular fibre of $D_{2,m(\xi)}(n) \cong S(n)$ in a singular point over some cusp. Denote the fibre of $D_{2,m(\xi)}(n)$ over this cusp by F_0. Note that $D_{2,\hat{m}(\xi)}(n)$ intersects $D_{2,m(\xi)}(n)$ in two lines which are contained in F_0 and which meet $\bar{\bar{E}}$. Pulling back $-D_{2,\hat{m}(\xi)}(n)$ and using $\delta = 4$, we obtain

$$r^*_\xi(\tfrac{1}{6}F_0).C_P = \tfrac{1}{6}r^*_\xi(D_{2,\hat{m}(\xi)}).C_P \geq \tfrac{2}{3}E.C_P.$$

But now the claim follows since

$$-\tfrac{b}{3}\sum_{\xi\in\mathfrak{S}_\mu} r^*_\xi D_{2,m(\xi)}(n).C_P \geq \sum_{\xi\in\mathfrak{S}_\mu} \tfrac{b}{18} r^*_\xi F_0.C_P \geq \tfrac{4b}{3}E.C_P.$$

So here $b \geq \frac{3}{4}c$ is enough. □

Lemma IV.27 *If $3 \leq \mu \leq 6$ and σ is a cone such that the image of the closure of its orbit contains a depth 3 curve, then $\gamma(\xi) \geq \delta(\xi) = 2$ for all $\xi \in \mathfrak{S}_\mu$.*

Proof. $\mathrm{Sym}_2\,\mathrm{pr}_{\xi(2)}$ maps $K(\xi)$ to $m(\xi)$. If $\bar{e} = \mathrm{Sym}\,\mathrm{pr}_{\xi(1)}\, e = \bar{l}_1^2 + \bar{l}_2^2 + \bar{l}_3^2 + \bar{l}_4^2$ then $\gamma(\xi) = \psi_\gamma(\bar{e})$ is equal to the number of the $\bar{l}_i^2$ that are not in $K(\xi)$. But if $\bar{l}_i^2 \in K(\xi)$, so $\psi_\gamma(\bar{l}_i^2) = 0$, then $\bar{\bar{l}}_i^2 = (\mathrm{pr}_{\xi(2)}\,\bar{l}_i)^2 = m(\xi)$ so $\psi_\delta(\bar{\bar{l}}_i^2) = 0$, so $\psi_\delta(\bar{\bar{e}}) \leq \psi_\gamma(\bar{e})$. Hence $\gamma(\xi) \geq \delta(\xi)$.
But $\delta(\xi)$ is determined by $\xi(1)$ and $\xi(2)$ only, since these determine $m(\xi)$ and hence ψ_δ as well as $\bar{\bar{e}}$. Choose a non-RT* face τ of σ spanned by $\xi(1) = l_1^2$, $\xi(2) = l_2^2$ and some other generator of σ. These exist, since the third generator of an RT* face is in the linear span of l_1 and l_2, but the generators of σ span a rank 3 space. Moreover, such a non-RT* face cannot be of disconnected type, because any cone having a proper face of disconnected type contains forms of rank 4. The choice of $\xi(1)$, $\xi(2)$ and τ determines an element $m(\xi_\tau) \in \mathbb{M}_2$, namely $m(\xi_\tau) = \mathrm{Sym}_2\,\mathrm{pr}_{\xi(1)\xi(2)}(\tau)$; and since τ is not RT* it is non-zero. Therefore $m(\xi_\tau) = m(\xi)$, since both of them are the square of a generator of the same 1-dimensional space (namely, the projection of the linear space spanned by the generators of σ).
Now $\delta(\xi)$ is calculated exactly as one calculates δ for the cone $\langle \xi(1), \xi(2), \tau \rangle$ with the generators in that order. This is a non-RT* cone with $\mu = 3$ so by Lemma IV.23 that value of δ is equal to 2. □

Proposition IV.28 *Suppose C is a depth 3 curve with $\mu = 4$ or $\mu = 5$ and $H = aL - bD_4(n) - cE(n)$ is a divisor on $\mathcal{A}_4^{\mathrm{Vor}}(n)$ with $a - 12b/n \geq 0$, $b \geq 2c \geq 0$. Then $H.C \geq 0$.*

Proof. Since $\gamma(\xi) \geq \delta(\xi)$ we need only check that

$$\sum_{\xi \in \mathfrak{S}_\mu} -\tfrac{b}{(\mu-1)\mu!}\delta_s(\xi) + \tfrac{4b}{\mu-1} - c \geq 0.$$

But $\delta(\xi) = 2$ so

$$\sum_{\xi \in \mathfrak{S}_\mu} -\tfrac{b}{(\mu-1)\mu!}\delta_s(\xi) + \tfrac{4b}{\mu-1} - c = \tfrac{2b}{\mu-1} - c \geq 0.$$

This is always fulfilled for $\mu = 4$ or $\mu = 5$ and $b \geq 2c$. □

Proposition IV.29 *Let C be a depth 3 curve with $\mu = 6$ and let $H = aL - bD_4 - cE$ be a divisor on $\mathcal{A}_4^{\mathrm{Vor}}$ with $a \geq 0$, $a - 12b \geq 0$ and $b \geq 2c \geq 0$. Then $H.C \geq 0$.*

Proof. In this case, by Lemma III.16, we can always work with the cone $\sigma = \langle x_1^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_3 - x_4)^2 \rangle$. However, we cannot restrict ourselves to $\xi = \mathrm{id}$.
There are two essentially different cases, depending on whether the edges $\xi(1)$ and $\xi(2)$ in the bicoloured graphs given in the proof of Lemma III.16 (Figure 4) are opposites (3 cases) or not (12 cases): see Remark III.17. A

representative of the non-opposite case is $(\xi(1),\xi(2))=(x_1^2,x_3^2)$, of the other $(\xi(1),\xi(2))=(x_1^2,(x_3-x_4)^2)$. The geometry of these cases is as follows. In the non-opposite case we obtain

$$\begin{aligned}\bar{e} &= (x_2-x_3)^2+(x_2-x_4)^2+x_3^2+x_4^2,\\ \bar{\bar{e}} &= x_2^2+(x_2-x_4)^2+x_4^2,\\ m(\xi) &= x_4^2.\end{aligned}$$

In this case the exceptional divisor is mapped to a point $\bar{\bar{E}}$ which is a singular point of a singular fibre of $D_{2,m(\xi)}(n)$ and if C meets this divisor then $r_\xi(C)$ cannot be a section. In the opposite case we have

$$\begin{aligned}\bar{e} &= (x_2-x_3)^2+(x_2-x_4)^2+x_3^2+x_4^2,\\ \bar{\bar{e}} &= 2(x_2-x_4)^2+2x_4^2,\\ m(\xi) &= x_4^2.\end{aligned}$$

In this case $\bar{\bar{E}}$ is a line which is contained in $D_{2,m(\xi)}(n)$ and we cannot a priori exclude that $r_\xi(C)$ is a section.
It is straightforward to check by hand that this distinction coincides with the distinction in terms of opposite and non-opposite edges above, and to verify directly that there are 12 non-opposite cases.
We shall now argue as in the case $\mu=3$ of disconnected type (Proposition IV.26), but taking into account only the contribution from the non-opposite cases. The other contributions are non-negative. Using the same notation as in the proof of Proposition IV.26 and using that $\delta=2$ gives us $-r_\xi^*D_{2,m(\xi)}(n).C_P\geq r_\xi^*(\frac{1}{6}F_0).C_P\geq\frac{1}{3}E.C_P$. In view of formula (20) it will be enough to prove that

$$\Big(-\tfrac{b}{5\cdot6!}\sum_{\xi\in\mathfrak{S}'_\mu}r_\xi^*D_{2,m(\xi)}(n)-\tfrac{b}{10\cdot5!}\sum_{\xi\in\mathfrak{S}'_\mu}r_\xi^*D_{2,m(\xi)}(n)+(\tfrac{2b}{5}-c)E(n)\Big).C_P\geq0,$$

where $\mathfrak{S}'_\mu$ is the set of ξ giving rise to the non-opposite case, so $\#\mathfrak{S}'_\mu=\frac{4}{5}6!$. This simplifies to

$$-\tfrac{4b}{5\cdot6!}\sum_{\xi\in\mathfrak{S}'_\mu}r_\xi^*D_{2,m(\xi)}(n).C_P+(\tfrac{2}{5}b-c)E.C_P\geq0.$$

This leads to $b\geq\frac{75}{46}c$, and $\frac{75}{46}<2$ so we are done. □

Proposition IV.30 *Let C be a depth 3 curve and let $H=aL-bD_4-cE$ be a divisor on $\mathcal{A}_4^{\mathrm{Vor}}$ with $a\geq0$, $a-12b\geq0$ and $b\geq2c\geq0$. Then $H.C\geq0$.*

Proof. This follows from the cases dealt with above, in Propositions IV.24, IV.26, IV.28 and IV.29. □

Theorem I.8 now follows from Propositions IV.2, IV.3, IV.7, IV.13, IV.16 and IV.30.

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