Slopes of Siegel cusp forms and geometry of compactified Kuga varieties

Flora Poon, Riccardo Salvati Manni and Gregory Sankaran

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Introduction

The universal family $X(\Gamma)$ over a moduli space $\mathcal{A}(\Gamma) = \Gamma\backslash\mathbb{H}_g$ of abelian varieties associated with a finite index subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ is known as the Kuga variety. Such families were first studied systematically by M. Kuga, whose 1964 Chicago lecture notes on the subject [12] have been recently published. The construction is given in [12] for $\Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ a torsion-free subgroup of finite index, but the restriction to torsion-free can be removed. However, if $-1 \in \Gamma$ then the fibre of the family is no longer the abelian variety but instead the corresponding Kummer variety. One could also allow Γ to be a subgroup of $\mathrm{Sp}(2g, \mathbb{Q})$ commensurable with $\mathrm{Sp}(2g, \mathbb{Z})$, for example taking $\mathcal{A}(\Gamma)$ to be the moduli space of abelian varieties with some non-principal polarisation, but we shall not pursue this here.

A natural generalisation is to consider the n-fold Kuga varieties $X^n(\Gamma)$, whose general fibre is the n-fold direct product of the corresponding abelian variety A or, if $-1 \in \Gamma$, of the associated Kummer variety $A/\pm 1$.

Alternatively one may consider the universal family $\underline{X}^n(\Gamma)$ over the stack $\underline{\mathcal{A}}(\Gamma) := [\Gamma\backslash\mathbb{H}_g]$. In this case the fibre is the abelian variety in all cases, but if $-1 \in \Gamma$ then the base has non-trivial stabilisers generically. This is the object that is studied in a particular case in [9].

We shall be concerned with compactifications of the n-fold Kuga variety $X_g^n = X^n(\mathrm{Sp}(2g, \mathbb{Z}))$ associated with the coarse moduli space $\mathcal{A}_g$ of principally polarised abelian g-folds over $\mathbb{C}$. By convention, $X_g^0 = \mathcal{A}_g$.

The starting point for our work is Ma's study [13] of $X_g^n(\Gamma)$. We construct a special compactification, which we call a Namikawa compactification, of X_g^n and this, together with recent and less recent results about the slope of $\mathcal{A}_g$, allows us to determine the Kodaira dimension $\kappa(X_g^n)$ of X_g^n whenever $g \geq 2$ and $n \geq 1$.

Theorem 1 *Suppose that $g \geq 2$ and $n \geq 1$. Then the Kodaira dimension $\kappa(X_g^n)$ of X_g^n satisfies*

- $\kappa(X_g^n) = \frac{1}{2}g(g+1) = \dim \mathcal{A}_g$ *if* $g+n \geq 7$, *except for* $(g,n) = (4,3)$, $(3,5)$, $(3,4)$, $(2,7)$, $(2,6)$ *and* $(2,5)$.
- $\kappa(X_2^7) = \kappa(X_3^5) = \kappa(X_4^3) = 0$, *and*
- $\kappa(X_g^n) = -\infty$ *otherwise, in particular* $\kappa(X_2^6) = \kappa(X_2^5) = \kappa(X_3^4) = -\infty$.

In fact X_g^1 is unirational for $g \leq 5$: see [22] for $g = 3$ and $g = 4$ and [9] for $g = 5$. Note that X_g^1 is the boundary of the Mumford partial compactification $\mathcal{A}'_{g+1}$. One can therefore compactify X_g^1 by taking its closure in any toroidal compactification of $\mathcal{A}_{g+1}$, since these all contain $\mathcal{A}'_{g+1}$ as a dense open set, but it is not straightforward to control the singularities that then arise.

Our main technical result concerns the existence of a sufficiently good compactification of X_g^n. We will say that a compactification of X_g^n is a Namikawa compactification if it dominates a toroidal compactification of $\mathcal{A}_g$ and boundary divisors are mapped to boundary divisors: see Definition 1.1 for a full explanation and precise details. We prove (see Theorem 1.2 for the precise statement):

Theorem 2 *Suppose* $g \geq 2$ *and* $n \geq 1$. *There exists a Namikawa compactification* $\overline{X_g^n}$ *with canonical singularities as long as* $g+n \geq 6$.

There is some overlap between our results and those of [13]. The singularities of X_g^n are studied in [13, Section 10], but the singularities at the boundary $\overline{X_g^n}$ are not considered there. That is sufficient for computing the geometric genus, but not other plurigenera: on the other hand, Ma's approach gives precise information about the geometric genus and therefore some information about the Kodaira dimension.

Our original motivation came from the case $g = 6$ and $n = 1$, and the Kodaira dimension $\kappa(X_6^1)$. As we have seen, X_g^1 is unirational for $g < 6$. On the other hand, for $g \geq 7$ we have $\kappa(X_g) = \kappa(\mathcal{A}_g) = \frac{1}{2}g(g+1)$, since Iitaka's conjecture holds, cf. [11], and $\mathcal{A}_g$ is of general type.

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1 Namikawa compactifications

Suppose throughout this section that $g \geq 2$. Let $\mathcal{A}_g \subset \overline{\mathcal{A}}_g$ be the inclusion of the coarse moduli space $\mathcal{A}_g = \mathrm{Sp}(2g,\mathbb{Z})\backslash\mathbb{H}_g$ of principally polarised abelian varieties of dimension g into a toroidal compactification $\overline{\mathcal{A}}_g$. Denote by $X_g = (\mathbb{Z}^{2g} \ltimes \mathrm{Sp}(2g,\mathbb{Z}))\backslash(\mathbb{C}^g \times \mathbb{H}_g)$ the universal family $f\colon X_g \to \mathcal{A}_g$ and by X_g^n the n-fold Kuga family, as in [13].

Definition 1.1 *A* Namikawa compactification *of X_g^n is an irreducible normal projective variety $\overline{X_g^n}$ containing X_g^n as an open subset, together with a projective toroidal compactification $\overline{\mathcal{A}}_g$ of $\mathcal{A}_g$ for which the following conditions hold.*

1. *$f\colon X_g^n \to \mathcal{A}_g$ extends to a projective morphism $f\colon \overline{X_g^n} \to \overline{\mathcal{A}}_g$;*
2. *every irreducible component of $\Delta_X := \overline{X_g^n} \smallsetminus X_g^n$ dominates an irreducible component of $\Delta_{\mathcal{A}} := \overline{\mathcal{A}}_g \smallsetminus \mathcal{A}_g$;*

Compactifications satisfying these conditions were first given by Namikawa [16, 17], for the case $n = 1$. There is little difficulty in extending Namikawa's construction to arbitrary n, and it is essentially done in [7, Ch. VI.1]. The conditions in Definition 1.1 are the same as in [13, Theorem 1.2], but we also require $\overline{X_g^n}$ to be normal (rather than just smooth in codimension 1) and projective. In fact neither of these conditions presents any difficulty.

Theorem 1.2 *Suppose $g \geq 2$ and $n \geq 1$. Then if $g + n \geq 6$, there exists a Namikawa compactification $\overline{X_g^n} \supset X_g^n$ such that $\overline{X_g^n}$ has canonical singularities. In particular $\overline{X_g^n}$ is $\mathbb{Q}$-Gorenstein.*

The proof will occupy the rest of this section. The first step is the following lemma. Although similar statements are already known (see for example [2]) we give a proof here as we have not found one elsewhere.

Suppose that a finite group G acts effectively on a variety X. A nontrivial element of G is called a *quasireflection* if it preserves a divisor on X.

Lemma 1.3 *Suppose that G is a finite group acting effectively and without quasireflections on a variety X that has canonical singularities. Let $f\colon \hat{X} \to X$ be a G-equivariant resolution of singularities, and suppose that $\hat{X}/G$ has canonical singularities. Then X/G also has canonical singularities.*

Proof. Since X has canonical singularities it is in particular $\mathbb{Q}$-Gorenstein (we do not require, nor expect, X or X/G to be $\mathbb{Q}$-factorial), and therefore X/G is also $\mathbb{Q}$-Gorenstein. Suppose that $rK_{X/G}$ is Cartier and $\sigma \in \mathcal{O}(rK_{X/G})$, so σ is a pluricanonical form on $(X/G)_{\mathrm{reg}}$. Therefore σ lifts to a G-invariant form $g^*\sigma$ on an open G-invariant subset $X_0 \subset X$, where $g\colon X \to X/G$ is the quotient map.

The complement $X \smallsetminus X_0$ consists of the fixed loci of elements of G, together with the singular locus of X; but the fixed loci have codimension at least 2 by assumption, so $g^*\sigma$ extends G-invariantly to X_{reg}. Therefore it lifts on $\hat{X}$ to a form defined away from the exceptional locus of f, but because X has canonical singularities, this extends to a G-invariant form $\hat{\sigma} = f^*g^*\sigma$ on $\hat{X}$ without poles, which in turns descends to the smooth part $(\hat{X}/G)_{\mathrm{reg}}$ of $\hat{X}/G$, because $\hat{g}\colon \hat{X} \to \hat{X}/G$ is étale in codimension 1.

This form $\hat{\sigma}$ agrees with $\bar{f}^*\sigma$ on the dense open set $\bar{f}^{-1}((X/G)_{\mathrm{reg}})$, where $\bar{f}\colon \hat{X}/G \to X/G$ is the map induced by f: therefore $\bar{f}^*\sigma$ extends to $(\hat{X}/G)_{\mathrm{reg}}$.

Now consider a resolution of singularities $h\colon Y \to X/G$ and form the pullback $\hat{h}\colon \hat{Y} \to \hat{X}/G$ as in the diagram.

$$\begin{tikzcd}
\hat{X} \arrow[rr, "f"] \arrow[dr, "\hat{g}"'] & & X \arrow[dr, "g"] & \\
& \hat{X}/G \arrow[rr, "\bar{f}"] & & X/G \\
\hat{Y} \arrow[ur, "\hat{h}"] \arrow[rr, "\hat{f}"] & & Y \arrow[ur, "h"'] &
\end{tikzcd}$$

This resolves the singularities of $\hat{X}/G$, so $\hat{h}^*\bar{f}^*\sigma$ extends without poles to the whole of $\hat{Y}$. But now if $h^*\sigma$ has poles along a divisor $E \subset Y$, then $\hat{f}^*h^*\sigma = \hat{h}^*\bar{f}^*\sigma$ has poles along $\hat{f}^{-1}E$, which is impossible. Therefore $h^*\sigma$ is holomorphic, and hence X/G has canonical singularities. ∎

Next, we construct Namikawa compactifications $\overline{X_g^n}$ using the methods of [16], choosing suitable fans Σ to determine $\overline{\mathcal{A}}_g$ and ensure projectivity.

We write $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$. For simplicity, and because it is enough for our purposes, we consider only this case, but allowing Γ to be a finite-index subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ does not change the argument. To fix notation, we choose a free $\mathbb{Z}$-module $\mathbb{W}$ of rank $2g$ equipped with a $\mathbb{Z}$-basis $e_1, \ldots, e_{2g}$, and fix a standard skew-symmetric form by requiring $\langle e_i, e_{i+g}\rangle = -1$ for $1 \le i \le g$ and $\langle e_i, e_j\rangle = 0$ if $|i-j| \neq g$, so that the matrix of the form with respect to the basis $\{e_i\}$ is $\begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix}$ (we denote the $r \times r$ identity matrix by $\mathbf{1}_r$ throughout). Then

$$\mathrm{Sp}(2g, \mathbb{Z}) = \left\{ \gamma \in \mathrm{GL}(2g, \mathbb{Z}) \mid {}^t\gamma \begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix} \right\}.$$

The compactifications constructed in [16] are smooth [16, Proposition 5.4(i)], and [7, Theorem VI.1.1(i)] also describes a smooth compactification. However, [16] makes the assumption that the monodromy is unipotent (in general it is only quasi-unipotent), and the smoothness of $\overline{Y}$ in [7, Theorem VI.1.1(i)] is smoothness as a stack: see the remarks at the top of [7, Page 195].

We follow the procedure in [16], and we have chosen notation as far as possible compatible with that of [16]. Our notation is therefore incompatible with [7], where X is used both for the lattice that we call $\mathbb{X}$ and for the moduli space $\mathcal{A}_g$, the universal family X_g^s is called Y and n refers to level, which is 1 for us.

Choose a cusp $F_{g'}$ of rank g' of $\mathbb{H}_g$, for some $0 \leq g' < g$, and put $g'' = g - g'$. Such a cusp corresponds to a rank g'' isotropic sublattice $\mathbb{X}$ of $\mathbb{W}$ up to the action of Γ, but $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$ acts transitively on such lattices. Therefore, without loss of generality, we may take $F_{g'}$ to have stabiliser

$$P(g') = \left\{ \begin{pmatrix} A & 0 & B & m' \\ m & u & n & M \\ C & 0 & D & n' \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g', \mathbb{R}),\ u \in \mathrm{GL}(g'', \mathbb{R}) \right\}$$

in $\mathrm{Sp}(2g, \mathbb{R})$, by choosing $\mathbb{X} = \mathbb{Z}e_{g'+1} + \ldots + \mathbb{Z}e_g \cong \mathbb{Z}^{g''}$.

Next we consider the integral affine symplectic group $\widetilde{\Gamma}_g^n$. (This is what is referred to as the metaplectic group in [16], but it is not the double cover $\mathrm{Mp}(2g, \mathbb{Z})$ of $\mathrm{Sp}(2g, \mathbb{Z})$.) It is given by

$$\widetilde{\Gamma}^n = \mathbb{Z}^{2gn} \ltimes \mathrm{Sp}(2g, \mathbb{Z}) < \widetilde{\Gamma}_{\mathbb{R}}^n = \mathbb{R}^{2gn} \ltimes \mathrm{Sp}(2g, \mathbb{R}) < \mathrm{GL}(n + 2g, \mathbb{R}),$$

and consists of elements $\widetilde{\gamma}$ of the form

$$\widetilde{\gamma} = \begin{pmatrix} 1 & a & b \\ 0 & A_0 & B_0 \\ 0 & C_0 & D_0 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{R}),\ a,\, b \in M_{n \times g}(\mathbb{R}) \tag{1}$$

(cf. [16, Paragraph (2.7)]).

The integral affine symplectic group acts on $\mathbb{C}^{gn} \times \mathbb{H}_g$, and the Kuga variety, cf. [16, Equation (3.4.1)], is the quotient

$$X_g^n := \widetilde{\Gamma}^n \backslash (\mathbb{C}^{gn} \times \mathbb{H}_g).$$

The stabiliser of $F_{g'}$ in $\widetilde{\Gamma}_{\mathbb{R}}^n$ is (cf. [16, Example (2.8)])

$$\widetilde{P}(g') = \left\{ \begin{pmatrix} \mathbf{1}_n & a' & a'' & b' & b'' \\ 0 & A & 0 & B & m' \\ 0 & m & u & n & M \\ 0 & C & 0 & D & n' \\ 0 & 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \middle| \begin{array}{c} \gamma' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g', \mathbb{R}), \\ u \in \mathrm{GL}(g'', \mathbb{R}), \\ a',\, b' \in M_{n \times g'}(\mathbb{R}), \\ a'',\, b'' \in M_{n \times g''}(\mathbb{R}) \end{array} \right\} \tag{2}$$

where as before M, m and n, and m' and n' are subject to the symplecticity conditions, and its unipotent radical has centre

$$\widetilde{U}(g') = \{u(b'', M) \mid M = {}^t M\}, \text{ where } u(b'', M) = \begin{pmatrix} 1 & 0 & 0 & 0 & b'' \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & M \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Intersecting with $\widetilde{\Gamma}^n$ we obtain the group

$$\widetilde{\Upsilon}^n = \widetilde{U}(g') \cap \widetilde{P}(g') = \{u(b'', M) \mid b'' \in \mathbb{Z}^{ng''},\ M = {}^tM \in M_{g''\times g''}(\mathbb{Z})\},$$

which is identified with $\mathrm{Sym}^2(\mathbb{X}^\vee)\times(\mathbb{X}^\vee)^n$. (This means a symmetric bilinear function on $\mathbb{X}$ and n linear functions on $\mathbb{X}$. In other words, $\mathrm{Sym}^1(\mathbb{X}^\vee)$ is $\mathbb{X}^\vee$. It is what is called $B(X)\oplus(X^*)^n$ in [7].)

To obtain the partial compactification at the cusp $F_{g'}$, we first take the partial quotient by $\widetilde{\Upsilon}^n$. For this we use the Siegel domain realisation of $\mathbb{H}_g$: for $\tau\in\mathbb{H}_g$ we write

$$\tau = \begin{pmatrix} \tau' & \omega \\ {}^t\omega & \tau'' \end{pmatrix}$$

with $\tau'\in\mathbb{H}_{g'}$, $\omega\in M_{g'\times g''}(\mathbb{C})$ and $\tau''\in M^{\mathrm{sym}}_{g''\times g''}(\mathbb{C})$, and then

$$\mathbb{H}_g \cong \mathcal{D}_{g'} := \{(\tau',\omega,\tau'') \mid \mathrm{Im}\,\tau'' - (\mathrm{Im}\,{}^t\omega)(\mathrm{Im}\,\tau')^{-1}(\mathrm{Im}\,\omega) > 0\}.$$

Then $M\in\mathrm{Sym}^2(\mathbb{X})$ acts by translations in the imaginary directions in $M^{\mathrm{sym}}_{g''\times g''}(\mathbb{C})$, so near this boundary X_g^n is covered by

$$\mathcal{D}_{g'}\times\mathbb{C}^{ng'}\times(\mathbb{C}^*)^{ng''} \subset \mathbb{H}_{g'}\times\mathbb{C}^{g'g''}\times(\mathbb{C}^*)^{g''\times g''}_{\mathrm{sym}}\times(\mathbb{C}^{g'}\times(\mathbb{C}^*)^{g''})^n \qquad (3)$$

where the $((\mathbb{C}^*)^{g''})^n$ term is $\mathbb{C}^{ng''}/(\mathbb{X}^\vee)^n$, given by b'' acting by translations in the imaginary directions in $\mathbb{C}^{ng''}=\mathbb{X}^\vee\otimes\mathbb{C}$.

Now we compactify by replacing the torus part $(\mathbb{C}^*)^{g''\times g''}_{\mathrm{sym}}\times((\mathbb{C}^*)^{g''})^n$ (that is, $\mathbb{C}^{(g''^2+(2n+1)g'')/2}/\widetilde{\Upsilon}^n$) by a suitable torus embedding $\mathrm{Temb}(\Sigma(g'))$ corresponding to a fan $\Sigma(g')$ in $(\mathrm{Sym}^2(\mathbb{X}^\vee)\times(\mathbb{X}^\vee)^n)\otimes\mathbb{R}$. We first define two (non-polyhedral) cones, in $\mathrm{Sym}^2(\mathbb{X}^\vee)$ and $\mathrm{Sym}^2(\mathbb{X}^\vee)\times(\mathbb{X}^\vee)^n$ respectively, following [7].

The first cone is $C(\mathbb{X})\subset\mathrm{Sym}^2(\mathbb{X}^\vee\otimes\mathbb{R})$, which is defined to be the cone of positive semi-definite symmetric bilinear forms b on $\mathbb{X}^\vee\otimes\mathbb{R}$ with rational radical $\mathrm{rad}\,b$. Equivalently, $C(\mathbb{X})$ is the convex hull of the rank-1 forms in $\mathrm{Sym}^2(\mathbb{X}^\vee\otimes\mathbb{Q})$. Decomposing this cone for each g', using reduction theory for quadratic forms, is equivalent to giving toroidal compactifications $\overline{\mathcal{A}}_g$.

The second cone is $\widetilde{C}(\mathbb{X})\subset(\mathrm{Sym}^2(\mathbb{X}^\vee)\oplus(\mathbb{X}^\vee)^n)\otimes\mathbb{R}$, given by

$$\widetilde{C}(\mathbb{X}) = \{(b,\ell_1,\ldots,\ell_n)\in\mathrm{Sym}^2(\mathbb{X}^\vee\otimes\mathbb{R})\oplus(\mathbb{X}^\vee\otimes\mathbb{R})^n \mid \ell_i|_{\mathrm{rad}}=0 \text{ for all } i\}.$$

as described in [7, Definition VI.1.3]. The number n does not change so we suppress it in the notation, but observe that $\widetilde{C}(\mathbb{X})$ implicitly depends on n whereas $C(\mathbb{X})$ does not.

We then further decompose $\widetilde{C}(\mathbb{X})$ as in [7, Definition VI.1.3], for each g', obtaining a collection of fans $\widetilde{\Sigma}=\{\widetilde{\Sigma}(g')\mid 0\le g'<g\}$. Provided that we choose the fans compatibly for different cusps we obtain, by the standard toroidal compactification procedure from [3], a compactification $\overline{X_g^n}^{\widetilde{\Sigma}}$ that

is analytically locally isomorphic to the product of a smooth space with a quotient of $\mathrm{Temb}(\widetilde{\Sigma}(g'))$ by $\widetilde{\Gamma}^n \cap P(g')$.

Ultimately we shall choose $\widetilde{\Sigma}(g')$ to be regular, so that $\mathrm{Temb}(\widetilde{\Sigma}(g'))$ is smooth. Singularities will then arise when we move beyond $\widetilde{\Upsilon}^n$ and take the quotient by the rest of $\widetilde{\Gamma}^n$, which in our situation (but not in [16], where that has been avoided by a base change) may have fixed points.

We need more, however, because we seek a Namikawa compactification. These are alluded to in [7], but the equidimensional condition [7, Definition VI.1.3(v)], which is not used and therefore not examined in detail there, is crucial for us. To be precise, we require a weaker version of equidimensional, which we call equidimensional in codimension 1. For this, it is enough if every ray in $\widetilde{\Sigma}(g')$ maps onto a cone of the fans Σ that define $\overline{\mathcal{A}}_g$, whereas equidimensionality requires this for every cone in $\widetilde{\Sigma}(g')$ of any dimension for every ray τ.

These two conditions, smoothness and equidimensionality, are in general opposed to one another. Choosing $\widetilde{\Sigma}$ to give smooth covering spaces $\mathrm{Temb}(\widetilde{\Sigma}(g')$ typically involves blowing up, and thus instantly violates condition (iii) of the Namikawa compactification.

Therefore, to construct an appropriate $\overline{X_g^n}$, we need a slightly more indirect approach. Instead of taking a regular decomposition straight away, we first choose a decomposition $\widetilde{\Sigma}^\flat$ such that $\mathrm{Temb}(\widetilde{\Sigma}^\flat(g'))$ itself has canonical singularities. We do this by extending the perfect cone, or first Voronoi, compactification of $\mathcal{A}_g$.

Proposition 1.4 *There exist a Γ-admissible collection $\Sigma^\flat$ of fans $\Sigma^\flat(g')$, for $0 \leq g' \leq g$, and a $\widetilde{\Gamma}$-admissible collection $\widetilde{\Sigma}^\flat$ of fans $\widetilde{\Sigma}(g')$ such that*

(i) $|\widetilde{\Sigma}^\flat(g')| = \widetilde{C}(\mathbb{X})$ *and* $|\Sigma^\flat(g')| = C(\mathbb{X})$*;*

(ii) $\widetilde{\Sigma}^\flat(g')$ *is* $\mathrm{GL}(\mathbb{X}) \ltimes \mathbb{X}^n$*-admissible relative to* $\Sigma^\flat(g')$*, for each* g'*;*

(iii) $\mathrm{Temb}(\widetilde{\Sigma}^\flat(g'))$ *has canonical singularities and* $\widetilde{\Sigma}^\flat(g')$ *is equidimensional in codimension 1 over* $\Sigma^\flat(g')$*;*

(iv) $\sigma \times \{0\} \in \widetilde{\Sigma}^\flat(g')$ *for every* $\sigma \in \Sigma^\flat(g')$*.*

Remark. The uses of the word "admissible" in the preamble to Proposition 1.4 and in condition (ii) there are different. Γ-admissible refers to the property of a collections of fans, one for each cusp, being compatible under restriction (see [3]), whereas $\mathrm{GL}(\mathbb{X}) \ltimes \mathbb{X}^n$-admissible is a property of compatibility at each cusp separately with the the projection map $X_g^n \to \mathcal{A}_g$, defined in [7, Definition VI.1.3].

Proof. We take $\Sigma^\flat(g')$ to be defined by the perfect cone decomposition of $C(\mathbb{X})$. This is the same as taking the the cones of $\Sigma^\flat(g')$ to be the cones on the faces of the convex hull of the rank-1 forms in the closure of C with rational radical, by [4]. It is known to give an admissible decomposition and a polyhedral fundamental domain for the action of $\mathrm{GL}(\mathbb{X})$: see, for example, [17, Ch. 8] and the references there.

We can extend this to a decomposition of $\widetilde{C}(\mathbb{X})$ by taking the convex hull of all $(b,\ell) \in \widetilde{C}(\mathbb{X}) \cap \mathrm{Sym}^2(\mathbb{X}) \oplus (\mathbb{X}^\vee)^n$ with $\mathrm{rank}\, b = 1$. From the description of the action of $\mathrm{GL}(\mathbb{X}) \ltimes \mathbb{X}^n$, given for example in [7, VI.1.1], it follows immediately that $\mathrm{GL}(\mathbb{X}) \ltimes \mathbb{X}^n$ acts on $\widetilde{C}(\mathbb{X})$ with a polyhedral fundamental domain.

If $q = (b; \ell_1, \ldots, \ell_n) = (b; \ell_j) \in \widetilde{C}(\mathbb{X})$ then $b \in C(\mathbb{X})$ so we can write $b = \sum \lambda_i r_i$, where $r_i = \xi_i{}^t\xi_i$ is of rank 1, $\xi_i \in \mathbb{X}$, and $\lambda_i \in \mathbb{R}_+$. Since $\ell_j|_{\mathrm{rad}\, b}$ we may write $\ell_j = b(\mu_j, \bullet)$ with $\mu_j \in \mathrm{rad}\, b$ (notice that $\mathrm{rad}\, b \supseteq \bigcup_i \mathrm{rad}\, r_i$, with equality if the r_i are linearly independent). Hence

$$q = (b; \ell_j) = (b; b(\mu_j, \bullet)) = (b; \sum_i \lambda_i r_i(\mu_j, \bullet)) = \sum_i \lambda_i (r_i; r_i(\mu_j, \bullet)),$$

so that the cone over the convex hull of the integral q with rank 1 quadratic part b is indeed $\widetilde{C}(\mathbb{X})$, and every integral point of $\widetilde{C}(\mathbb{X})$ is in the convex hull.

The first of these conditions shows that the definition of $\widetilde{\Sigma}^\flat$ does give an admissible collection of fans. That is, it is $\widetilde{\Gamma}^n$-invariant, and chosen for different cusps $F_{g'}$ so as to be compatible with restriction (Siegel Φ-operator) to adjacent cusps. This therefore yields a compactification $\overline{X_g^n}^\flat$.

The second condition shows that the covering spaces $\mathrm{Temb}(\widetilde{\Sigma}^\flat(g'))$ have canonical singularities.

Finally, $\overline{X_g^n}^\flat$ is a Namikawa compactification because there are no rays of $\widetilde{\Sigma}^\flat(g')$ in the interior of $\widetilde{C}(\mathbb{X})$, and the ray spanned by $q = (b; \ell_j)$ projects onto the ray spanned by b. ■

Shepherd-Barron showed in [20] (see also the correction [2]) that the perfect cone compactification of $\mathcal{A}_g$ has canonical singularities for $g \geq 5$. We are not constrained to use a specific compactification: rather, we choose a suitable one, as in [21, Section 5]. We choose smooth subdivisions $\widetilde{\Sigma}^\sharp(g')$ of the fans $\widetilde{\Sigma}^\flat(g')$ (that is, toric resolutions of $\mathrm{Temb}(\widetilde{\Sigma}^\flat(g'))$) in a $\widetilde{\Gamma}$-equivariant way, and denote the resulting compactification by $\overline{X_g^n}^\sharp$. This is of course no longer a Namikawa compactification, nor is it smooth in general since $\widetilde{\Gamma}$ is not neat. However, we have the following easy consequence of Lemma 1.3.

Corollary 1.5 *Suppose that $\overline{X_g^n}^\sharp$ has canonical singularities and that the action of $P(g') \cap \widetilde{\Gamma}$ on $\mathrm{Temb}(\widetilde{\Sigma}^\sharp(g'))$ has no quasireflections. Then $\overline{X_g^n}^\flat$ has canonical singularities.*

Proof. It is enough to apply Lemma 1.3 to the resolutions $\mathrm{Temb}(\widetilde{\Sigma}^\sharp(g')) \to \mathrm{Temb}(\widetilde{\Sigma}^\flat(g'))$ for each g'. ■

2 Quotient singularities

In this section we shall verify the conditions of Corollary 1.5 for the values of g and n that concern us.

Recall that any toroidal compactification $\overline{\mathcal{A}}_g$ of $\mathcal{A}_g$ comes with a surjective map to the Satake compactification $\mathcal{A}_g^*$, and that the Satake compactification has a stratification

$$\mathcal{A}_g^* = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \ldots \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_0$$

where $\mathcal{A}_0$ is a point. Hence any compactification of X_g^n that dominates a Namikawa compactification has a map to $\mathcal{A}_g^*$ and in particular there is a natural map $\pi\colon \overline{X_g^n}^\sharp \to \mathcal{A}_g^*$.

If $p \in \overline{X_g^n}^\sharp$ is such that $\pi(p) \in \mathcal{A}_{g'} \subset \mathcal{A}_g^*$, then near p the toroidal compactification is a quotient of an invariant open subset

$$\mathcal{D}^\sharp \subset \mathbb{H}_{g'} \times \mathbb{C}^{g'g''} \times \mathbb{C}^{g'n} \times \mathrm{Temb}(\widetilde{\Sigma}^\sharp) \tag{4}$$

(where $g' + g'' = g$) by an action of $\widetilde{P}(g') \cap \widetilde{\Gamma}^n$ that preserves the product structure, extending the decomposition (3). To determine the singularity at $p \in \overline{X_g^n}^\sharp$ we therefore have to examine the action of the stabiliser G of a preimage $\tilde{p} \in \mathcal{D}^\sharp$ of p on the tangent space $T_{\mathcal{D}^\sharp,\tilde{p}}$.

We recall some basic facts from [21] and [18]. Suppose that $\rho\colon G \to \mathrm{GL}(m,\mathbb{C})$ is a finite-dimensional representation of a finite group G, and suppose that $h \in G$ and that $\rho(h)$ has order $k > 1$. If the eigenvalues of $\rho(h)$ are $\zeta^{a_1}, \ldots, \zeta^{a_m}$ (where $\zeta = e^{2\pi i/k}$ is a primitive k-th root of unity and $0 \le a_i < k$) then the *Reid-Tai sum* of h, also called the *age* of h, is

$$\mathrm{RT}(h) = \sum_{i=1}^{m} \frac{a_i}{k}.$$

The RST (Reid–Shepherd-Barron–Tai) criterion states that if $\rho(G)$ has no quasireflections then the quotient $\mathbb{C}^m/\rho(G)$ has canonical singularities if and only if $\mathrm{RT}(h) \ge 1$ for every $h \in G$.

If $\rho(h)$ is a quasireflections then exactly one of the a_i is non-zero, so $\mathrm{RT}(h) < 1$. It follows that in any case if $\mathrm{RT}(h) \ge 1$ for every $h \in G$ then the quotient has canonical singularities.

We apply this to the action of $\tilde{\gamma} \in \widetilde{P}(g')$. We use the block decomposition given in (1) and (2) and the notation for submatrices in the rest of this section is taken from there.

To check that the singularity at p is canonical it is enough to verify that G contains no quasireflection on the tangent space (which we need to do anyway in order to apply Lemma 1.3) and that $\mathrm{RT}(\widetilde{\gamma}) \ge 1$ for any nontrivial $\widetilde{\gamma} \in G$.

Note that the decomposition (4) is G-invariant, so that $\mathrm{RT}(\widetilde{\gamma})$ is the sum of the age of $\widetilde{\gamma}$ restricted to each factor.

Proposition 2.1 *If $g'' = 0$ then $\overline{X_g^n}$ has a canonical singularity at p and the stabiliser G of $\tilde{p}$ has no quasireflections unless $g = 2$ and $n \leq 2$ or $g = 3$ and $n = 1$.*

Proof. Recall that we are assuming $g \geq 2$ anyway. If $g'' = 0$ then the local cover in (4) becomes $\mathcal{D}^\sharp = \mathbb{H}_g \times \mathbb{C}^{gn}$, which is just the covering space of X_g^n. But X_g^n has canonical singularities by [13, Proposition 10.3], and there are no quasireflections by [13, Lemma 7.1], for these values of g and n. ∎

In view of the above, we may assume for the rest of this section that $g'' > 0$.

In order to use the results of [21] we need to verify the condition that $\tilde{\gamma}$ should act trivially on each cone of the fan (see [21, p. 438]).

Lemma 2.2 *If $\tilde{\gamma}$ fixes a cusp then it acts trivially on each cone of $\widetilde{\Sigma}^\sharp(g')$.*

Proof. The eigenvectors in $\operatorname{Sym}^2 V$ are the rank 1 forms $f_i f_j$, where $f_i \in \mathbb{X}^\vee \otimes \mathbb{R}$ are the real eigenvectors of u (if any) and thus there are no eigenvectors in the interior of the cone $\widetilde{C}$. Now the result follows: for if $\tilde{\gamma}$ acts nontrivially on $\sigma \in \widetilde{\Sigma}^\sharp(g')$ then it acts on some subset $\{\rho_1, \ldots, \rho_k\}$ of the rays spanning σ by a free permutation, and then by the Brouwer fixed-point theorem it has an eigenvector in the closed cone spanned by $\{\rho_1, \ldots, \rho_k\}$; but this eigenvector is not a generator of any of the ρ_i, so it is in the interior of $\widetilde{C}$. ∎

Proposition 2.3 *Suppose that $p \in \overline{X_g^n}$ and $\pi(p) \in \mathcal{A}_{g'}$, and that $\widetilde{\gamma} \neq \mathbf{1}_{n+2g}$ belongs to the stabiliser G of $\tilde{p} \in \mathcal{D}^\sharp$ in $\widetilde{P}(g') \cap \widetilde{\Gamma}^n$. Suppose that $u \neq \pm\mathbf{1}_{g''}$, or that $u = -\mathbf{1}_{g''}$ and $(g'', n) \neq (1, 1)$. Then the action of $\widetilde{\gamma}$ on $T_{\mathcal{D}^\sharp, \tilde{p}}$ has $\mathrm{RT}(\widetilde{\gamma}) \geq 1$, and in particular it is not a quasireflection.*

Proof. For this, it is enough to look at the $\mathrm{Temb}(\widetilde{\Sigma}^\sharp)$ factor. This is a toric variety with torus $\mathbb{T}$ whose lattice of 1-parameter subgroups (the dual of the character lattice) is $\operatorname{Sym}^2(\mathbb{X}^\vee) \times (\mathbb{X}^\vee)^n$.

If the eigenvalues of $\tilde{\gamma}$ on $V = \mathbb{X}^\vee \otimes \mathbb{C}$, which are the eigenvalues of u, are $\mu_1, \ldots, \mu_{g''}$ then the eigenvalues on $\operatorname{Sym}^2 V \times V^n$ are $\mu_i\mu_j$ and n copies of each μ_i. The eigenvalues of $\tilde{\gamma}$ on the toric boundary component containing $\tilde{p}$ belongs include n copies of each μ_i and, as in [21, Lemma 5.2], all the $\mu_i\mu_j$ that are different from 1. Since $u \in \mathrm{GL}(g'', \mathbb{Z})$ is of finite order, its eigenvalues include all the primitive d-th roots of unity for some degree, and that gives $\mathrm{RT}(\tilde{\gamma}|_V) \geq 1$ on the V unless $d = 1$ or $d = 2$.

If $d = 1$ then $u = \mathbf{1}_{g''}$. If $d = 2$ then $\mu_i = \pm 1$ and we may assume $\mu_1 = -1$: then if $g'' > 1$ we either have $\mu_2 = -1$ so again $\mathrm{RT}(\tilde{\gamma}|_V) \geq 1$, or $\mu_2 = 1$ and then the eigenvalue $\mu_1\mu_2 = -1$ occurs on $\operatorname{Sym}^2(V)$. If $g'' = 1$ then the eigenvalue $\mu_1 = -1$ occurs n times on V^n, so $\mathrm{RT}(\tilde{\gamma}|_{V^n}) \geq 1$ unless $g'' = n = 1$. ∎

Next, we examine the action of $\tilde{\gamma}$ on the $\mathbb{C}^{ng'}$ factor. Because of Proposition 2.3 we may assume that $u = \epsilon \mathbf{1}_{g''}$ with $\epsilon = \pm 1$.

For any $r \leq g$, we let $M_{r\times g}(\mathbb{C})^*$ be the set of matrices of rank r in $M_{r\times g}(\mathbb{C})$. Then the Grassmannian $\mathrm{Gr}(r,g)$, of r-dimensional linear subspaces in $\mathbb{C}^g$, is $M_{r\times g}(\mathbb{C})^*/\operatorname{GL}(r,\mathbb{C})$, with $\operatorname{GL}(r,\mathbb{C})$ acting by right multiplication. Since $\mathbb{C}^{ng}$ is identified with $M_{n\times g}(\mathbb{C})$ by the choice of basis $e_1,\dots,e_{2g}$, we may regard $\mathbb{H}_g \times \mathbb{C}^{ng}$ as a subset of subset of $\mathrm{Gr}(g, n+2g)$ by sending an element $(\tau, Z) \in \mathbb{H}_g \times \mathbb{C}^{ng}$ to the equivalence class of block matrices:

$$\begin{bmatrix} Z \\ \tau \\ \mathbf{1}_g \end{bmatrix} \in M_{(n+2g)\times g}(\mathbb{C})/\operatorname{GL}(g,\mathbb{C}).$$

Recall that in this representation a boundary component of $\mathbb{H}_g$ is a subset of the closure $\overline{\mathbb{H}}_g$ of $\mathbb{H}_g$ in $\mathrm{Gr}(g,2g)$, so this description extends to the boundary. So the image of $\widetilde{p}$ in $\overline{\mathbb{H}}_g \times \mathbb{C}^{ng}$ may also be written in this way, with $\tau \in F_{g'} \subset \overline{\mathbb{H}}_g$ given by

$$\tau = \begin{pmatrix} \tau' & \omega \\ {}^t\omega & \tau'' \end{pmatrix}$$

where $\tau' \in \mathbb{H}_{g'}$, $\omega \in M_{g'\times g''}$ and $\tau'' \in \mathrm{Sym}^2(\mathbb{X}^\vee \otimes \mathbb{C}) = M^{\mathrm{sym}}_{g''\times g''}(\mathbb{C})$.

Then the action of $\tilde{\gamma}$ is given (notation from (1)) by

$$\tilde{\gamma} \cdot \begin{bmatrix} Z \\ \tau \\ \mathbf{1}_g \end{bmatrix} = \begin{bmatrix} \multicolumn{2}{c}{Z + a\tau + b} \\ A\tau' + B & Aw + m' \\ m\tau' + \epsilon {}^t w + n & mw + \epsilon\tau'' + M \\ C\tau' + D & Cw + n' \\ 0 & \mathbf{1}_{g''} \end{bmatrix}.$$

Because τ is preserved by γ, this simplifies to

$$\tilde{\gamma} \cdot \begin{bmatrix} Z \\ \tau \\ \mathbf{1}_g \end{bmatrix} = \begin{bmatrix} (Z + a\tau + b)\cdot N \\ \tau \\ \mathbf{1}_g \end{bmatrix}$$

where

$$N = \begin{pmatrix} (C\tau' + D)^{-1} & -(Cw + n')(C\tau' + D)^{-1} \\ 0 & \mathbf{1}_{g''} \end{pmatrix},$$

and therefore the action of $\tilde{\gamma}$ on the tangent space to $\mathbb{C}^{ng}$ at Z is by right multiplication by N.

Lemma 2.4 *The eigenvalues of $\tilde{\gamma}$ on $\mathbb{C}^{ng}$ are exactly the eigenvalues of N, which are* 1 *and the eigenvalues of $(C\tau' + D)^{-1}$. Moreover, γ' fixes τ', i.e. $(A\tau' + B)(C\tau' + D)^{-1} = \tau'$.*

Proof. Immediate from the discussion above. ■

The next lemma and its corollaries apply to the usual action of $\mathrm{Sp}(2g,\mathbb{Z})$ on $\mathbb{H}_g$, for arbitrary g. If $\gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{Z})$ then denote the eigenvalues of γ by $\Lambda = (\lambda_1 \ldots, \lambda_g\}$ and $\overline{\Lambda} = (\bar\lambda_1, \ldots, \bar\lambda_g)$: we will regard such a sequence as a diagonal matrix.

Lemma 2.5 *If γ fixes $\tau \in \mathbb{H}_g$, then γ is diagonalisable. Moreover $C_\gamma\tau + D_\gamma$ is diagonalisable and has $\bar\Lambda$ as sequence of eigenvalues.*

Proof. For every $\beta \in \mathrm{Sp}(2g,\mathbb{R})$ and $\tau \in \mathbb{H}_g$ we set $J(\beta,\tau) = C_\beta\tau + D_\beta$. Then J is a cocycle, i.e. $J(\beta_1\beta_2,\tau) = J(\beta_1,\beta_2\tau)J(\beta_2,\tau)$ for every β_1, $\beta_2 \in \mathrm{Sp}(2g,\mathbb{R})$ and $\tau \in \mathbb{H}_g$.

It is well known, cf. [21, Lemma 4.1], that there exists $\alpha \in \mathrm{Sp}(2g,\mathbb{R})$ such that

$$\alpha\tau = i\mathbf{1}_g \qquad \text{and} \qquad \alpha\gamma\alpha^{-1} = \begin{pmatrix} \delta_1 & \delta_2 \\ -\delta_2 & \delta_1 \end{pmatrix}$$

with δ_1, δ_2 real diagonal matrices and $\delta_1 + i\delta_2 \in \mathrm{U}(g,\mathbb{C})$.

Obviously γ is diagonalisable, with eigenvalues $\Lambda = \delta_1 + i\delta_2$ and $\overline{\Lambda} = \delta_1 - i\delta_2$. Now using the cocycle property we have

$$\begin{aligned} \overline{\Lambda} &= J(\alpha\gamma\alpha^{-1}, i\mathbf{1}_g) \\ &= J(\alpha, \gamma\alpha^{-1}, i\mathbf{1}_g)J(\gamma\alpha^{-1}, i\mathbf{1}_g) \\ &= J(\alpha, \gamma\cdot\tau)J(\gamma,\tau)J(\alpha^{-1}, i\mathbf{1}_g) \end{aligned}$$

Now, since γ fixes τ and $J(\alpha^{-1}, i\mathbf{1}_g) = J(\alpha,\tau)^{-1}$, we get that $\overline{\Lambda}$ and $J(\gamma,\tau)$ are conjugate. ■

We have the following immediate corollary.

Corollary 2.6 *If $\gamma \in \mathrm{Sp}(2g,\mathbb{Z})$ is nontrivial and fixes $\tau \in \mathbb{H}_g$, then $C_\gamma\tau + D_\gamma$ has an eigenvalue that is not 1.*

Now we return to the singularities of $\overline{X_g^n}$.

Proposition 2.7 *If $g'' \neq 0$ then $\overline{X_g^n}$ has a canonical singularity at p, as long as $g + n \geq 6$.*

Proof. As before, we take $\tilde\gamma \in \widetilde{P}(g')$, fixing a point $\tilde p$, and write it using the block decomposition given in (1) and (2). Again, because of Proposition 2.3 we may assume that $u = \epsilon\mathbf{1}_{g''}$, where $\epsilon = \pm 1$ and $\epsilon = 1$ unless $g'' = n = 1$.

For any g, if $\gamma' = \mathbf{1}_{2g'}$ and $u = \mathbf{1}_{g''}$ then $\tilde\gamma \in \tilde U(g')$ (see [16, Example 2.8]) and acts trivially at the boundary $F_{g'}$. In particular this holds if $g' = 0$, unless $g'' = n = 1$ but then $g = 1$ which is excluded.

If $\gamma' = -\mathbf{1}_{2g'}$ and $\epsilon = 1$, or $\gamma' = \mathbf{1}_{2g'}$ and $\epsilon = -1$, then there are $g - 1$ eigenvalues $\lambda_i\epsilon = -1$ on the $\mathbb{C}^{g'g''}$ factor, giving $\mathrm{RT}(\tilde\gamma) > 1$.

If $\gamma' = -\mathbf{1}_{2g'}$ and $\epsilon = -1$ then the eigenvalues on the $\mathrm{Temb}(\widetilde{\Sigma}^\sharp)$ factor include n copies of ϵ, and, by Corollary 2.6, there are also n copies of -1 occurring on the $\mathbb{C}^{ng}$ factor. Even for $n = 1$, this gives $\mathrm{RT}(\gamma) \geq 1$.

Therefore we may assume that $\gamma' \neq \pm\mathbf{1}_{2g'}$, and thus does not act trivially on $\mathbb{H}_{g'}$.

If $g' \geq 5$ and $\gamma' \neq \pm\mathbf{1}_{2g'}$ then the contribution to $\mathrm{RT}(\gamma)$ from γ' acting on the $\mathbb{H}_{g'}$ factor (the $F_{g'}$ factor) is already at least 1, and $\mathcal{A}_{g'}$ itself has canonical singularities: this is [21, Lemma 4.5].

If $g' < 5$ then the eigenvalues not coming from the action of γ' on $\mathbb{H}_{g'}$ include g'' copies of $\epsilon\lambda_i$ on the $\mathbb{C}^{g'g''}$ factor and, by Corollary 2.6, a further n copies of $\epsilon\lambda_i^{\pm 1}$ on the $\mathbb{C}^{ng}$ factor. If the order of $\tilde{\gamma}$ on the tangent space at $\tilde{p}$ is d, then some λ_i is a nontrivial d-th root of unity and so this gives a contribution of at least $(n + g'')\frac{1}{d}$ to $\mathrm{RT}(\tilde{\gamma})$.

Moreover, according to [21, Lemma 4.4] we may assume $d \leq 6$, and in each case the action of γ' contributes at least $\frac{g'}{d}$ to $\mathrm{RT}(\tilde{\gamma})$, so in any case we have $\mathrm{RT}(\tilde{\gamma}) \geq (g' + n + g'')\frac{1}{d} \geq \frac{n+g}{6}$. So if $n + g \geq 6$ we are done. ■

The condition $n + g \geq 6$ cannot be strengthened: it is needed if $g' = 1$ and $d = 6$.

Theorem 1.2 now follows immediately.

3 Slope of $\mathcal{A}_g$

We assume throughout that $g > 1$, since if $g = 1$ the fibres of X_1^n are rational so $\kappa(X_1^n) = -\infty$.

We shall construct differential forms by using Siegel modular forms, so we begin with some elementary definitions concerning them.

Definition 3.1 *A* modular form of weight k *is a holomorphic function* $f\colon \mathbb{H}_g \to \mathbb{C}$ *on the Siegel upper half-plane*

$$\mathbb{H}_g = \{Z \in M_{g\times g}(\mathbb{C}) \mid Z = {}^tZ,\ \operatorname{Im} Z > 0\}$$

such that

$$f(\gamma \cdot \tau) = \det(C\tau + D)^k f(\tau) \text{ for any } \gamma \in \mathrm{Sp}(2g, \mathbb{Z}).$$

Note that we need no extra condition at infinity when $g > 1$.

A Siegel modular form has a Fourier expansion

$$f(\tau) = \sum_T a(T) \exp(\pi i \operatorname{tr}(T\tau))$$

where the sum runs over all even integral symmetric matrices T.

Definition 3.2 *If f is a Siegel modular form, the* vanishing of f at the boundary *is*

$$b := \frac{1}{2}\min\{x^t T x \mid a(T) \neq 0,\ x \in \mathbb{Z}^g \smallsetminus \{0\}\}.$$

If $b > 0$, i.e. if $a(T) \neq 0$ implies $T > 0$, we say that f is a cusp form.

We recall, mainly from [14], some facts about $\mathcal{A}_g = \mathrm{Sp}(2g, \mathbb{Z})\backslash\mathbb{H}_g$ and its compactifications.

Modular forms of weight 1 determine a $\mathbb{Q}$-line bundle L, the Hodge line bundle. The Satake compactification $\mathcal{A}_g^*$ is Proj of the ring of modular forms, and the Mumford partial compactification $\mathcal{A}_g'$ is the blow-up of $\mathcal{A}_g \sqcup \mathcal{A}_{g-1} \subset \mathcal{A}_g^*$ along $\mathcal{A}_{g-1}$. Every toroidal compactification of $\mathcal{A}_g$ dominates $\mathcal{A}_g^*$ and contains $\mathcal{A}_g'$ as a Zariski open subset: the toroidal compactifications differ from one another only above the deeper strata $\mathcal{A}_{g'}$ for $g' < g - 1$.

If $g \geq 2$ then $\mathrm{Pic}(\mathcal{A}_g') \otimes \mathbb{Q} = \mathbb{Q}\lambda \oplus \mathbb{Q}\delta$, where λ is the class of L and δ is the class of the boundary divisor $\Delta_{\mathcal{A}'}$, the proper transform of $\mathcal{A}_{g-1} \subset \mathcal{A}_g^*$. This is proved in [14, Corollary 1.6] for $g \geq 4$, and for $g = 2$ and $g = 3$ it follows from the studies of the Chow ring in [15] and [6] respectively: see [10].

Definition 3.3 *The* slope *of an effective divisor $E = a\lambda - b\delta$ on $\mathcal{A}_g'$ with a, $b > 0$ is defined to be $s(E) = a/b$. In particular, if E is the zero divisor of a cusp form f of weight k and vanishing order b then $s(E) = k/b$.*

The Kodaira dimension of the Kuga varieties X_g^n is related to the transcendence degree of the field generated by cusp forms of slope less or equal to $g + n + 1$: see for example [13, Theorem 1.3].

Definition 3.4 *The* minimal slope $s_{\min}(g)$ *is the infimum of the slopes of all effective divisors on $\mathcal{A}_g'$.*

An upper bound $s_{\min}(g)$ is provided in small genera by the Andreotti-Mayer divisor N_0, the locus of principally polarised abelian varieties with singular theta divisor [1]. The divisor N_0 has two components: Θ_{null}, the locus where the theta divisor has a singular point of order 2, and N_0', the locus where the theta divisor has a singular point not of order 2. The classes of Θ_{null}, for $g \geq 1$, and N_0', for $g \geq 4$, are computed in [14]:

$$[\Theta_{\mathrm{null}}] = 2^{g-2}(2^g + 1)\lambda - 2^{2g-5}\delta.$$
$$[N_0'] = \left(\frac{(g+1)!}{4} + \frac{g!}{2} - 2^{g-3}(2^g + 1)\right)\lambda - \left(\frac{(g+1)!}{24} - 2^{2g-6}\right)\delta.$$

For $g \leq 3$ the minimal slope is achieved at $[\Theta_{\mathrm{null}}]$, giving the values $s_{\min}(1) = 12$, $s_{\min}(2) = 10$ and $s_{\min}(3) = 9$.

For $g = 4$ we have $s_{\min}(4) = 8$, achieved by $s([N_0'])$, and for $g = 5$ we have $s_{\min}(5) = 54/7$, also achieved by $s([N_0'])$: see [19] and [8], respectively.

In all these cases the divisors that minimise the slope are rigid.

For $g = 6$ we have $s_{\min}(6) \leq 7$: see [5]. However, the fact that $s_{\min}(6) \leq s([N_0']) = 550/73 < 8$ will suffice for our purposes.

Theorem 3.5 *Suppose that $g \geq 2$ and $\overline{X_g^n}$ is a Namikawa compactification of X_g^n with canonical singularities. Then*

1. *$\kappa(X_g^n) = \frac{1}{2}g(g+1)$ if $s_{\min}(g) < g+n+1$;*
2. *$\kappa(X_g^n) = 0$ if $s_{\min}(g) = s(D) = g+n+1$ and D is rigid;*
3. *$\kappa(X_g^n) = -\infty$ if $s_{\min}(g) > g+n+1$ (even if the singularities are not canonical).*

Proof: The first case (what one might call relatively general type) follows easily from [13, Proposition 9.2]. Pulling back via the morphism $f\colon \overline{X_g^n}^\sharp \to \overline{\mathcal{A}}_g$, this implies that for sufficiently divisible k one has $kK_X \geq f^*(k(g+n+1)L - k\Delta_{\mathcal{A}})$. So it is enough to show that the $\mathbb{Q}$-divisor $(g+n+1)L - \Delta_{\mathcal{A}}$ is big: however, since it has slope strictly greater than $s_{\min}(g)$ it is in the interior of the effective cone and can therefore be written as the sum of an effective divisor and an ample divisor.

In the second case, the same argument shows that K_X is effective, since it dominates the pullback of the effective divisor $(g+n+1)L - \Delta_{\mathcal{A}}$. Therefore $\kappa(X_g^n) \geq 0$. On the other hand, if some multiple of K_X moves, then so does some multiple of $(g+n+1)L - \Delta_{\mathcal{A}}$, which is to say that some multiple of D moves, but D is rigid.

In the third case, if $K_X \geq 0$ then $f_*(K_X) \geq 0$, but $s(f_*(K_X)) = g+n+1 < s_{\min}(g)$. So K_X is not effective, and $\kappa(X_g^n) = -\infty$. ∎

Theorem 1 follows immediately from this.

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Flora Poon, Gregory Sankaran
Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
Flora Poon: wkp25@bath.ac.uk
Gregory Sankaran: G.K.Sankaran@bath.ac.uk

Riccardo Salvati Manni
Dipartimento di Matematica "Guido Castelnuovo", Università di Roma "La Sapienza", P.le Aldo Moro 5, 00185 Roma, Italy
Riccardo Salvati Manni: salvati@mat.uniroma1.it