ON THE GENERALISED KUMMER FOURFOLD OF THE JACOBIAN OF A GENUS TWO CURVE

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Abstract. We construct a birational model of the generalised Kummer fourfold of the Jacobian of a genus two curve, based on a geometric interpretation of the addition law on this Jacobian, obtained by the properties of the linear system of conics on that curve. We show that our model has mild singularities and that it admits a finite ramified covering to the four-dimensional projective space.

Contents

1. Introduction

An irreducible holomorphic symplectic manifold, abbreviated to IHS manifold, is a simply-connected compact complex manifold X such that $\mathrm{H}^{2,0}(X)$ is generated by an everywhere nondegenerate 2-form. If X is an IHS manifold then $\mathrm{H}^2(X,\mathbb{Z})$ naturally carries a nondegenerate integral quadratic form q_X of signature $(3, b_2(X)-3)$ (see [5]), the *Beauville–Bogomolov–Fujiki quadratic form*. We denote by $\langle -,-\rangle_X$ the associated bilinear form. In all the known deformation types, the lattice $\left(\mathrm{H}^2(X,\mathbb{Z}), q_X\right)$ is even, but not unimodular, except when X is a K3 surface.

All the IHS manifolds that we consider will furthermore be projective. A *polarisation* of X is a primitive ample class L in the Néron–Severi group $\mathrm{NS}(X)$ of X. The *degree* of the polarisation is the positive integer $d \coloneqq q_X(L)$ and its *divisibility* is the integer γ such that $\langle \mathrm{H}^2(X,\mathbb{Z}), L\rangle_X = \gamma\mathbb{Z}$.

Gritsentko, Hulek and Sankaran [25] constructed coarse moduli spaces of polarised pairs (X, L) of a given deformation type: these moduli spaces are quasi-projective varieties. Our initial motivation in this paper is to search for concrete geometric descriptions of the generic elements in some of these moduli spaces $\mathcal{M}$. In practice, in most cases, such a description of a generic object can be used to construct a dominant rational map $\mathbb{P}^N \dashrightarrow \mathcal{M}$ for some integer N, so that $\mathcal{M}$ would be unirational.

In general it is hard to decide whether a given moduli space $\mathcal{M}$ is unirational or not. The general philosophy is that these moduli spaces may be unirational for low values of the numerical invariants but will be of general type when the numerical invariants are high. For example this is the case for K3 surfaces (see [24]) and for any putative class of IHS manifolds whose moduli space is of large dimension (see [38]), and analogous statements hold for moduli of curves and of abelian varieties.

Instead of unirationality one could ask for related properties such as being rational (stronger) or uniruled, stably rational or rationally connected (weaker). In Appendix A we summarise the currently known results about the birational types of moduli spaces of polarised IHS manifolds.

Except for K3 surfaces, each of the known moduli spaces is named after a codimension one family. For instance, the four-dimensional moduli spaces $\mathcal{M}^{d,\gamma}_{\mathrm{Kum}^2}$ that feature in this paper parametrise polarised IHS manifolds of Kummer type, *i.e.* deformation equivalent to the second generalised Kummer variety of an abelian surface, of degree d and divisibility γ.

Most of the unirationality results for moduli of polarised IHS manifolds concern the deformation class of Hilbert type. For the other known types, the question is relatively unexplored, apart from the recent results of Barros, Beri, Flapan and Williams [3] for the generalised Kummer and OG6 cases. In this paper, we focus on the deformation type of the second generalised Kummer variety of a polarised abelian surface. In order to attack the unirationality question in this deformation class, our first objective, which we achieve in the present paper, is to construct and to study a birational model of a generalised Kummer fourfold using only rational tools. Our second objective, which is still work in progress, will then be to understand how this construction may deform. Benedetti, Manivel and Tanturri [7] worked on a similar question, from a different point of view, using Coble hypersurfaces to get models of generalised Kummer fourfolds as flag varieties, but their construction does not deform. Very recently another construction of a similar nature has been given by Agostini, Beri, F. Giovenzana and Ríos Ortiz in [1].

By the general philosophy on the moduli spaces, we guess that to lower the discrete invariants it is wise to lower the polarisation. We therefore consider principally polarised abelian surfaces, and study the codimension 1 family of generalised Kummer fourfolds over Jacobians of a genus two curve. It might also be interesting to study the other principally polarised case, products of elliptic curves, which will give a codimension 2 family.

Our original intuition is to fix a genus 2 curve C and look at projective coordinates on $\mathrm{Jac}(C)$ in a model where addition is well described. Such models are used in cryptography, for instance by Flynn [19] and Leitenberger [37], whose works on the addition law inspired the present paper. Our main results are:

Theorem 1.1 (Corollary 5.2)**.** *Let C be a smooth genus two curve. The linear system of cubics embeds C in $(\mathbb{P}^4)^\vee$ and the dual variety $C^* \subset \mathbb{P}^4$ of C is a degree 14 irreducible hypersurface. The second generalised Kummer variety $\mathrm{Kum}^2(\mathrm{Jac}(C))$ of the Jacobian of C is birational to a degree 15 covering of $\mathbb{P}^4$ branched along C^*.*

We denote by $\mathscr{G}_C$ the degree 15 covering of $\mathbb{P}^4$ branched along C^* mentioned in the above statement, whose definition is given in Definition 2.1 and §4.3, and by:

$$\gamma_C \colon \mathscr{G}_C \dashrightarrow \mathrm{Kum}^2(\mathrm{Jac}(C))$$

the birational map in question, whose definition is given in Formula (2).

Proposition 1.2 (Proposition 4.4)**.** *The variety $\mathscr{G}_C$ is normal and Gorenstein, and with quotient singularities.*

In particular, $\mathscr{G}_C$ is Cohen-Macaulay.

Proposition 1.3 (Propositions 2.3 and 2.4)**.** *The birational map γ_C contracts one divisor to the noncurvilinear point of $\mathrm{Kum}^2(\mathrm{Jac}(C))$ supported at the origin of $\mathrm{Jac}(C)$, and a second divisor to the Kummer surface $\mathrm{Kum}^1(\mathrm{Jac}(C))$.*

Proposition 1.4 (Proposition 4.5)**.** *The Galois closure of the covering $\mathscr{G}_C \to \mathbb{P}^4$ is a local complete intersection scheme.*

In this paper, the term "variety" denotes an integral separated noetherian scheme of finite type over the field of complex numbers. The term "curve" means an irreducible projective variety of dimension 1.

In §2 we construct the variety $\mathscr{G}_C$ under more general assumptions, starting from any abelian surface A. We study the contraction to the generalised Kummmer fourfold $\mathrm{Kum}^2(A)$ in this general setup. Then in §3 we specialise to the case where A is the Jacobian of a genus two curve and we study the properties of the linear system of cubics on the curve. We apply this geometry in §4 to realise $\mathscr{G}_C$ as a finite cover of $\mathbb{P}^4$. Finally in §5 we study the branch locus of this cover. Several Appendices contain some backgrounds, alternative or complementary views and proofs of some results used in the main text, as such as some helpful computer algebra scripts.

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2. A rational contraction to the generalised Kummer fourfold

We consider a polarised abelian surface $(A, c_1(H))$, with origin $O_A \in A$, where $H \in \mathrm{Pic}^{2t}(A)$ is an ample divisor of degree $H^2 = 2t$ with $t \geq 1$, whose first Chern class $c_1(H)$ is primitive in $\mathrm{H}^2(A, \mathbb{Z})$.

2.1. **The second generalised Kummer variety of an abelian surface.** For any integer $m \geq 0$, we denote the Hilbert scheme of 0-dimensional subschemes of A of length m by $\mathrm{Hilb}^m(A)$, the Chow quotient by $\mathrm{Sym}^m(A)$, and the Hilbert–Chow morphism by:

$$h_A \colon \mathrm{Hilb}^m(A) \to \mathrm{Sym}^m(A).$$

The addition law on A defines the following morphisms:

$$\begin{array}{ccc} A^m & \xrightarrow{\alpha_A} & A \\ {\scriptstyle \pi_A}\downarrow & \nearrow_{\bar{\alpha}_A} & \\ \mathrm{Sym}^m(A) & & \end{array} \tag{1}$$

We restrict to the case $m = n+1$ for an $n \geq 0$ and denote by $\mathrm{Sym}_0^{n+1}(A) \coloneqq \alpha_A^{-1}(\{O_A\})$ the fibre over the origin of the addition map α_A and by $\mathrm{Kum}^n(A)$ the *n-th generalised Kummer variety of* A, defined as the fibre over the origin of $\alpha_A \circ h_A$:

$$\mathrm{Kum}^n(A) \coloneqq (\bar{\alpha}_A \circ h_A)^{-1}(O_A) \subset \mathrm{Hilb}^{n+1}(A),$$

The restriction h_A° of the Hilbert–Chow morphism h_A to the generalised Kummer variety is still birational. It is a resolution of the singularities of the Chow quotient $\mathrm{Sym}_0^{n+1}(A)$. It is well known ([5]) that the variety $\mathrm{Sym}_0^{n+1}(A)$ has symplectic singularities and that $\mathrm{Kum}^n(A)$ is an irreducible holomorphic symplectic manifold of dimension $2n$.

The variety $\mathrm{Kum}^1(A)$ is the classical Kummer surface associated to A, *i.e.* the minimal resolution of the quotient $A/\pm 1$.

In this paper, we are mostly interested in the second generalised Kummer variety $\mathrm{Kum}^2(A)$. Its second integral cohomology group decomposes as follows. There exists a natural injection $\mathrm{H}^2(A, \mathbb{Z}) \hookrightarrow \mathrm{H}^2(\mathrm{Kum}^2(A), \mathbb{Z})$ and we have:

$$\mathrm{H}^2(\mathrm{Kum}^2(A), \mathbb{Z}) = \mathrm{H}^2(A, \mathbb{Z}) \oplus \mathbb{Z}\delta,$$

where δ is half the class of the exceptional divisor of the Hilbert–Chow morphism intersected with $\mathrm{Kum}^2(A)$. This decomposition is orthogonal with respect to the lattice structure on $\mathrm{H}^2(\mathrm{Kum}^2(A), \mathbb{Z})$ given by the Beauville–Bogomolov–Fujiki (BBF) form, and the isometry class of the lattice is computed in [47]:

$$\mathrm{H}^2(\mathrm{Kum}^2(A), \mathbb{Z}) \cong U^{\oplus 3} \oplus \langle -6 \rangle,$$

where U is the hyperbolic plane.

By the decomposition above, we have a splitting of the Néron–Severi lattice:

$$\mathrm{NS}(\mathrm{Kum}^2(A)) = \mathrm{NS}(A) \oplus \mathbb{Z}\delta.$$

We denote by $h \in \mathrm{NS}(\mathrm{Kum}^2(A))$ the image of $c_1(H)$, which is a big and nef divisor. By a result of Debarre and Macrì [15, Corollary 4.11] the classes $ah - b\delta$, with $a, b > 0$, are ample when $b/a < 1/3$. If (A, H) is *generic*, meaning that $\mathrm{NS}(A) = \mathbb{Z}c_1(H)$, we have $\mathrm{NS}(\mathrm{Kum}^2(A)) = \mathbb{Z}h \oplus \mathbb{Z}\delta$. Furthermore, A. Mori [41], has shown that if H is a principal polarisation, *i.e.* $t = 1$, then the ample cone is precisely the interior of the cone generated by the classes h and $2h - \delta$. The smallest possible polarisation degree with respect to the Beauville–Bogomolov–Fujiki quadratic form is thus given by the smallest integer d such that $\ell \coloneqq ah - b\delta$ is an ample class with $q_{\mathrm{Kum}^2(A)}(\ell) = d = 2e$.

The smallest integer e such that $e = a^2 - 3b^2$ with $a, b \in \mathbb{N}$ and $b/a < 1/2$ is $e = 6$, obtained for $(a, b) = (3, 1)$, so the minimal polarisation is $\ell = 3h - \delta$, of degree $d = 12$. It is easy to check that $\langle \mathrm{NS}(\mathrm{Kum}^2(A)), \ell\rangle = 6\mathbb{Z}$. But since the embedding of $\mathrm{NS}(\mathrm{Kum}^2(A))$ in $\mathrm{H}^2(\mathrm{Kum}^2(A), \mathbb{Z})$ sends the class h to an element of the unimodular lattice $U^{\oplus 3}$, there exists $u \in U^{\oplus 3}$ such that $\langle u, h\rangle = 1$ and this implies that $\langle \mathrm{H}^2(\mathrm{Kum}^2(A), \mathbb{Z}), \ell\rangle = 3\mathbb{Z}$. The divisibility is thus $\gamma = 3$ and $(\mathrm{Kum}^2(A), \ell) \in \mathcal{M}^{12,3}_{\mathrm{Kum}^2}$. This space does not appear in [3] and nothing is known about its birational geometry.

2.2. **The birational model.** Consider the blowup of the origin of A:

$$\beta_A \colon \widetilde{A} \coloneqq \mathrm{Bl}_{O_A} A \longrightarrow A,$$

with exceptional divisor $E_A \coloneqq \beta_A^{-1}(O_A)$. First, using the summation maps defined in Diagram (1) we put:

$$A^3_0 \coloneqq \alpha_A^{-1}(O_A),$$

and we denote by π°_A the restriction of the Chow quotient:

$$\begin{array}{ccccc} A^3_0 & \hookrightarrow & A^3 & \xrightarrow{\alpha_A} & A \\ \downarrow{\scriptstyle \pi^\circ_A} & & \downarrow{\scriptstyle \pi_A} & \nearrow{\scriptstyle \bar{\alpha}_A} & \\ \mathrm{Sym}^3_0(A) & \hookrightarrow & \mathrm{Sym}^3(A) & & \end{array}$$

We do similarly starting with $\widetilde{A}$; we define $\widetilde{A}^3_0$ as the fibre over the origin of the morphism:

$$\widetilde{A}^3 \xrightarrow{\beta_A^{\times 3}} A^3 \xrightarrow{\alpha_A} A,$$

and we finally define the main object of interest in this paper:

Definition 2.1. We denote by $\mathscr{G}_A$ the scheme-theoretic fibre over the origin of the morphism:

$$\mathrm{Sym}^3(\widetilde{A}) \xrightarrow{\mathrm{Sym}^3(\beta_A)} \mathrm{Sym}^3(A) \xrightarrow{\bar{\alpha}_A} A,$$

that is: $\mathscr{G}_A \coloneqq \mathrm{Sym}^3_0(\widetilde{A}) \coloneqq (\bar{\alpha}_A \circ \mathrm{Sym}^3(\beta_A))^{-1}(O_A)$.

The morphism $\mathrm{Sym}^3(\beta_A)$ is clearly birational and its restriction to $\mathscr{G}_A$ is still birational since it is an isomorphism above the open subset of triples of nonzero points on A whose sum is zero. We are interested in the birational map:

$$\gamma_A \coloneqq h_A^{-1} \circ \mathrm{Sym}^3(\beta_A)\big|_{\mathscr{G}_A} : \mathscr{G}_A \dashrightarrow \mathrm{Kum}^2(A). \tag{2}$$

All the relevant maps are shown in the diagram (3) below:

$$
\begin{array}{ccccccc}
\widetilde{A}^3 & \hookleftarrow & \widetilde{A}^3_0 & \xrightarrow{\beta_A^{\times 3}|_{\widetilde{A}^3_0}} & A^3_0 & \hookrightarrow & A^3 \\
\downarrow{\scriptstyle \pi_{\widetilde{A}}} & & \downarrow{\scriptstyle \pi^\circ_{\widetilde{A}}} & & \downarrow{\scriptstyle \pi^\circ_A} & & \downarrow{\scriptstyle \pi_A} \\
\mathrm{Sym}^3(\widetilde{A}) & \hookleftarrow & \mathscr{G}_A & \xrightarrow{\mathrm{Sym}^3(\beta_A)|_{\mathscr{G}_A}} & \mathrm{Sym}^3_0(A) & \hookrightarrow & \mathrm{Sym}^3(A) \\
 & & & \overset{\gamma_A}{\searrow} & \uparrow{\scriptstyle h^\circ_A} & & \uparrow{\scriptstyle h_A} \\
 & & & & \mathrm{Kum}^2(A) & \hookrightarrow & \mathrm{Hilb}^3(A)
\end{array}
\tag{3}
$$

Proposition 2.1. *The scheme $\mathscr{G}_A$ is reduced. It is a normal projective variety of dimension 4, Cohen–Macaulay and $\mathbb{Q}$-factorial with quotient singularities, and it is birational to* $\mathrm{Kum}^2(A)$.

Proof. Let $\mathfrak{a} \to A$ be the natural cover where $\mathfrak{a}$ is the (abelian) Lie algebra of A. By choosing linear coordinates $(\boldsymbol{x}, \boldsymbol{y})$ on $\mathfrak{a}$, we get an identification $\mathfrak{a} \to \mathbb{C}^2$ and thus a cover $\mathbb{C}^2 \to A$ of groups (in the category of complex manifolds). The linear coordinates $(\boldsymbol{x}, \boldsymbol{y})$ induce local coordinates on A around each point, up to a choice of an element in the kernel of $\mathbb{C}^2 \to A$. Near O_A, we always make this choice such that the coordinates of O_A become $(0, 0)$. Around a point $a \in A$, with $a \neq O_A$, the coordinates $(\boldsymbol{x}, \boldsymbol{y})$ are also local coordinates of $\widetilde{A}$ at the point $\tilde{a} \coloneqq \beta_A^{-1}(a)$. Around the point $a = O_A$, local coordinates of $\widetilde{A}$ near the exceptional divisor $E_A = \beta_A^{-1}(O_A)$ are $\boldsymbol{x}, \boldsymbol{z}$ with relation $\boldsymbol{y} = \boldsymbol{x}\boldsymbol{z}$ (or with the roles of $\boldsymbol{x}$ and $\boldsymbol{y}$ exchanged). The map β_A is then locally given by $\beta_A(\boldsymbol{x}, \boldsymbol{z}) = (\boldsymbol{x}, \boldsymbol{x}\boldsymbol{z})$.

With this convention, around each point $a \in A$ the local coordinates are such that the addition law is the standard one. That is, around a point $(a_1, a_2, a_3) \in A^3$, the local coordinates $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2), (\boldsymbol{x}_3, \boldsymbol{y}_3)$ are such that the subvariety A^3_0 is locally given by the relations $\boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{x}_3 = 0$ and $\boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 = 0$. This allows us to analyse the singularities of $\widetilde{A}^3_0$ by computing local equations of $\widetilde{A}^3_0$ using the morphism $\beta_A^{\times 3}$. Again the local equations of $\widetilde{A}^3_0$ are $\boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{x}_3 = 0$ and $\boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 = 0$, but $\boldsymbol{y}_i$ is in degree 1 if $a_i \neq O_A$ and in degree 2 if $a_i = O_A$. Therefore, unless $a_1 = a_2 = a_3 = O_A$ the ideal defining $\widetilde{A}^3_0$ in the local ring is generated by two elements with independent linear parts, so $\widetilde{A}^3_0$ is smooth. If $a_1 = a_2 = a_3$ then $\widetilde{A}^3_0$ is locally a linear section (by $\boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{x}_3 = 0$) through a rank 3 quadric cone $\boldsymbol{x}_1\boldsymbol{z}_1 + \boldsymbol{x}_2\boldsymbol{z}_2 + \boldsymbol{x}_3\boldsymbol{z}_3 = 0$, which is again a quadric cone in $\mathbb{A}^5$, of rank 2. Indeed, a local equation is $\boldsymbol{x}_1\boldsymbol{w}_1 + \boldsymbol{x}_2\boldsymbol{w}_2 = 0$, in local coordinates $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{w}_1 \coloneqq \boldsymbol{z}_1 - \boldsymbol{z}_3, \boldsymbol{w}_2 \coloneqq \boldsymbol{z}_2 - \boldsymbol{z}_3, \boldsymbol{z}_3$.

It follows that $\widetilde{A}^3_0$ is normal and has hypersurface singularities, so it is Gorenstein [17, Corollary 21.19], and it is connected and irreducible by Zariski's Main Theorem.

Since $\mathscr{G}_A$ is the quotient of $\widetilde{A}^3_0$ by the action of the symmetric group $\mathfrak{S}_3$ acting by permutation of the factors, we deduce easily that $\mathscr{G}_A$ is reduced, normal, connected and irreducible. Moreover, $\mathscr{G}_A$ is geometrically Cohen-Macaulay by the Hochster–Roberts theorem [30, Main Theorem and Remark 2.3] and it is $\mathbb{Q}$-Gorenstein (see the argument in the proof of [35, Lemma 5.16]. We can even be more specific here: since $\widetilde{A}^3_0$ has transversal nodal singularities, it has in particular quotient singularities, so $\mathscr{G}_A$ too. It follows that $\mathscr{G}_A$ is $\mathbb{Q}$-factorial with rational singularities (see [35, Proposition 5.15]). □

Remark 2.1. The variety $\widetilde{A}_0^3$ is not locally factorial. Consider for instance the divisor $F_1 = \mathrm{Sym}^3(E_A)$ and its pre-image $\widetilde{F}_1 \coloneqq (\pi^\circ_{\widetilde{A}})^{-1}(F_1)$. In the local chart used in the proof above, the divisor E_A has equation $\boldsymbol{x} = 0$, so $\widetilde{F}_1$ has equations $\boldsymbol{x}_1 = \boldsymbol{x}_2 = 0$ inside $\widetilde{A}_0^3$: it is not a Cartier divisor. Computations with Macaulay2 [22] indicate that $\mathrm{Sym}^3_0(\tilde{A})$ and $\mathscr{G}_A$ are not local complete intersection schemes. They indicate also that $\mathscr{G}_A$ is Gorenstein: we will prove it in Proposition 4.4 under the assumption that A is the Jacobian of a genus two curve. The script is given in Remark C.1.

2.3. **The divisorial contraction to the Chow quotient.** To fully describe the geometric relation between $\mathscr{G}_A$ and $\mathrm{Sym}^3_0(A)$, we exhibit two meaningful divisors F_1, F_2 on $\mathscr{G}_A$ that parametrise special configurations of triples of points on $\widetilde{A}$.

Recall that $E_A \subset \widetilde{A}$ is the exceptional divisor of the blowup $\beta_A\colon \widetilde{A} \to A$. As in Remark 2.1 we define the prime divisor

$$F_1 \coloneqq \mathrm{Sym}^3(E_A) \subset \mathscr{G}_A. \tag{4}$$

Since $E_A \cong \mathbb{P}^1$, the divisor F_1 is isomorphic to $\mathbb{P}^3$.

Denote by $\tau\colon A \to A$, $a \mapsto -a$ the sign involution and by $\bar{a} \in A/\tau$ the class of $a \in A$. There is an embedding of A/τ in $\mathrm{Sym}^3_0(A)$ given by $\bar{a} \mapsto a + (-a) + O_A$. The surface A/τ contains in particular the point $3\,O_A$. The sign involution τ on A lifts to $\widetilde{A}$ as an involution denoted $\tilde{\tau}$, making the blowup morphism $\beta_A\colon \widetilde{A} \to A$ equivariant, that is $\beta_A \circ \tilde{\tau} = \tau \circ \beta_A$, and leaving the exceptional divisor E_A pointwise fixed. We then define the second prime divisor $F_2 \subset \mathscr{G}_A$ as the image of the morphism

$$\widetilde{A}/\tilde{\tau} \times E_A \to \mathscr{G}_A, \quad (\tilde{a}, e) \mapsto \tilde{a} + \tilde{\tau}(\tilde{a}) + e,$$

that is:

$$F_2 \coloneqq \left\{ \tilde{a} + \tilde{\tau}(\tilde{a}) + e \mid (\tilde{a}, e) \in \widetilde{A}/\tilde{\tau} \times E_A \right\} \subset \mathscr{G}_A. \tag{5}$$

Proposition 2.2. *The birational morphism* $\mathrm{Sym}^3(\beta_A)\colon \mathscr{G}_A \to \mathrm{Sym}^3_0(A)$ *contracts the divisor* F_1 *to the point* $3\,O_A$*, and it contracts the divisor* F_2 *to the surface* A/τ*. It is* $1:1$ *outside of these two divisors.*

Proof. The divisor F_1 is contracted to the point $3O_A$ since $\beta_A(E) = O_A$. Similarly, with the same notation as above,

$$\mathrm{Sym}^3(\beta_A)(\tilde{a} + \tilde{\tau}(\tilde{a}) + e) = a + (-a) + O_A,$$

so the divisor F_2 is contracted to the surface A/τ. Take a point $a+b+c \in \mathrm{Sym}^3(A)$. If none of these points is the origin of A, it has a unique preimage by $\mathrm{Sym}^3(\beta_A)$. If $c = 0$, then $b = -a$ and the fibre over this point belongs to the divisor F_2. If $b = c = 0$, then $a = 0$ and the fibre over this point is the divisor F_1. So $\mathrm{Sym}^3(\beta_A)$ is an isomorphism outside of these two divisors. □

Note that F_1 and F_2 intersect along the big diagonal of $\mathrm{Sym}^3(E_A)$ parametrising 0-cycles with at least one double point.

2.4. **The rational contraction.** We now study the birational map $\gamma_A\colon \mathscr{G}_A \dashrightarrow \operatorname{Kum}^2(A)$. Since the variety $\mathscr{G}_A$ is normal by Proposition 2.1 and since $\operatorname{Kum}^2(A)$ is a projective variety, the indeterminacy locus of γ_A is a subset of codimension at least two of $\mathscr{G}_A$ [29, Lemma V.5.1].

We first analyse the behaviour of γ_A around the divisor F_1. For this we introduce the 2-dimensional *Briançon variety* $B^3_{O_A}$ parametrising the locus of nonreduced subschemes in $\operatorname{Kum}^2(A)$ supported at O_A. As the indeterminacy locus of γ_A is of codimension at least 2, and since the morphism $\operatorname{Sym}^3(\beta_A)$ contracts F_1 to the point $3\,O_A$, by restriction we have a rational map

$$\gamma_A|_{F_1} : F_1 \dashrightarrow B^3_{O_A},$$

so the divisor F_1 is contracted by γ_A. Its rational image is the *centre* of F_1 for γ_A: our goal is to compute this centre.

For this, let us recall the geometry of the variety $B^3_{O_A}$, following [36, §2]. This depends only on the local geometry of A near O_A, so in the proof of Proposition 2.1 and Remark 2.1, we take local coordinates $\boldsymbol{x}, \boldsymbol{y}$ at the origin $O_A \in A$. This identifies the tangent space T_{A,O_A} with $\mathbb{C}^2$, compatibly with addition: hence for the purpose of local computation we may replace A by $\mathbb{C}^2$. The *curvilinear* subschemes supported at the origin arise as limit points of triples of points that move along a smooth curve. Their ideals have the form $\langle \boldsymbol{y} + \alpha\boldsymbol{x} + \beta\boldsymbol{x}^2, \boldsymbol{x}^3\rangle$ or similarly with $\boldsymbol{x}$ and $\boldsymbol{y}$ exchanged. They form a line bundle over $\mathbb{P}^1$, where the base $\mathbb{P}^1$ parametrises the tangent direction of the curve at the origin (encoded by the parameter α) and the fibre depends on the parameter β that encodes the curvature of the curve. The Briançon variety $B^3_{O_A}$ is obtained by compactifying this affine bundle by adding as point at infinity the non-curvilinear subscheme Z_∞ that arises as the limit of triples of points going to the origin from three different directions: its ideal is $I_\infty \coloneqq \langle \boldsymbol{x}^2, \boldsymbol{x}\boldsymbol{y}, \boldsymbol{y}^2\rangle$.

Proposition 2.3. *The birational map γ_A contracts the divisor F_1 to the point Z_∞.*

Proof. The behaviour of γ_A at the divisor F_1 is a local property over a neighbourhood of the origin O_A of A so we can study it by computing a local model of the variety $\mathscr{G}_A$ in the neighbourhood of the divisor F_1, as we did in the proof of Proposition 2.1. Instead of directly computing $\gamma_A|_{F_1}$, it is equivalent, but more convenient, to study the composite rational map

$$g\colon \widetilde{A}^3_0 \xrightarrow{\pi^\circ_{\widetilde{A}}} \mathscr{G}_A \overset{\gamma_A}{\dashrightarrow} \operatorname{Kum}^2(A) \hookrightarrow \operatorname{Hilb}^3(A).$$

We denote by $\widetilde{F}_1$ the preimage of F_1 in $\widetilde{A}^3_0$. The rational map g is defined at the generic point of $\widetilde{F}_1$, and we want to compute its image and the indeterminacy locus of the restriction of g to $\widetilde{F}_1$.

The local coordinates. As in the proof of Proposition 2.1 and Remark 2.1, we take local coordinates $\boldsymbol{x}, \boldsymbol{y}$ at the origin $O_A \in A$. This identifies the tangent space $\mathfrak{a} = T_{A,O_A}$ with $\mathbb{C}^2$, compatibly with addition: hence for the purpose of local computation near $\widetilde{F}_1$ we may replace A by $\mathbb{C}^2$, and in particular $\operatorname{Hilb}^3(A)$ by $\operatorname{Hilb}^3(\mathbb{C}^2)$.

Again as in Proposition 2.1 and Remark 2.1, with coordinates $(\boldsymbol{x}_1, \boldsymbol{y}_1)$, $(\boldsymbol{x}_2, \boldsymbol{y}_2)$, $(\boldsymbol{x}_3, \boldsymbol{y}_3)$ on $\mathbb{C}^6$, using the relations $\boldsymbol{y}_i = \boldsymbol{x}_i\boldsymbol{z}_i$ and introducing $\boldsymbol{w}_1 \coloneqq \boldsymbol{z}_1 - \boldsymbol{z}_3$ and $\boldsymbol{w}_2 \coloneqq \boldsymbol{z}_2 - \boldsymbol{z}_3$ we get down to five variables $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{z}_3$ where $\widetilde{A}^3_0$ is defined by the single relation $\boldsymbol{x}_1\boldsymbol{w}_1 + \boldsymbol{x}_2\boldsymbol{w}_2 = 0$ and $\widetilde{F}_1$ has local equations $\boldsymbol{x}_1 = \boldsymbol{x}_2 = 0$.

The open covering. In what follows, the notation $I \in \mathrm{Hilb}^3(\mathbb{C}^2)$ means that $I \subset \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$ is the ideal of the corresponding length three subscheme of $\mathrm{Hilb}^3(A)$. Following Haiman [28], the Hilbert scheme $\mathrm{Hilb}^3(\mathbb{C}^2)$ is covered by three affine charts labelled by the partitions of the integer 3, as follows. Let

$$\mathcal{B}_{(1,1,1)} \coloneqq \{1, \boldsymbol{x}, \boldsymbol{x}^2\}, \quad \mathcal{B}_{(2,1)} \coloneqq \{1, \boldsymbol{x}, \boldsymbol{y}\}, \quad \mathcal{B}_{(3)} \coloneqq \{1, \boldsymbol{y}, \boldsymbol{y}^2\},$$

and for any partition μ of the integer 3, define the subset

$$\mathcal{U}_\mu \coloneqq \left\{ I \in \mathrm{Hilb}^3(\mathbb{C}^2) \mid \mathcal{B}_\mu \text{ spans } \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]/I \right\}.$$

By [28, Proposition 2.1], the subsets $\mathcal{U}_\mu$ are open affine subvarieties that cover $\mathrm{Hilb}^3(\mathbb{C}^2)$. They cover the Briançon subvariety $B^3_{O_A}$ of $\mathrm{Hilb}^3(\mathbb{C}^2)$ parametrising length 3 subschemes supported at the origin, as follows: the curvilinear subschemes of the form $\langle \boldsymbol{y} + \alpha\boldsymbol{x} + \beta\boldsymbol{x}^2, \boldsymbol{x}^3 \rangle$ belong to $\mathcal{U}_{(1,1,1)}$, similarly the curvilinear subschemes of the form $\langle \boldsymbol{y} + \alpha\boldsymbol{x} + \beta\boldsymbol{x}^2, \boldsymbol{x}^3 \rangle$ belong to $\mathcal{U}_{(3)}$, whereas the noncurvilinear point Z_∞ belongs to $\mathcal{U}_{(2,1)}$, and since it is unique, we have $\mathrm{Hilb}^3(\mathbb{C}^2) \setminus \big(\mathcal{U}_{(1,1,1)} \cup \mathcal{U}_{(3)}\big) = \{Z_\infty\}$.

The Hilbert–Chow morphism in coordinates. Following [28, pp. 210–214], the coordinate ring of the chart $\mathcal{U}_{(1,1,1)}$ is $\mathbb{C}[e_1, e_2, e_3, a_0, a_1, a_2]$, and an ideal $I_{(e,a)} \in \mathcal{U}_{(1,1,1)}$, with coordinates $(e, a) \coloneqq (e_1, e_1, e_3, a_0, a_1, a_2)$ is given by

$$I_{(e,a)} \coloneqq \langle \boldsymbol{x}^3 - e_1\boldsymbol{x}^2 + e_2\boldsymbol{x} - e_3, \boldsymbol{y} - (a_0 + a_1\boldsymbol{x} + a_2\boldsymbol{x}^2) \rangle.$$

Whenever $e = (e_1, e_2, e_3) = 0$ and $a = (0, a_1, a_2)$, we get an element $I_{(0,a)} \in B^3_{O_A}$ of the Briançon variety. The meaning of these affine coordinates is that if the zero locus $\mathcal{V}(I_{(e,a)})$ consists of three points of $\mathbb{C}^2$ of coordinates $(\boldsymbol{x}_1, \boldsymbol{y}_1)$, $(\boldsymbol{x}_2, \boldsymbol{y}_2)$ and $(\boldsymbol{x}_3, \boldsymbol{y}_3)$, repeated with multiplicity, then in this chart we use the Viète formula

$$\boldsymbol{x}^3 - e_1\boldsymbol{x}^2 + e_2\boldsymbol{x} - e_3 = \prod_{i=1}^{3} (\boldsymbol{x} - \boldsymbol{x}_i), \tag{6}$$

so that the coordinates e_i are the elementary symmetric functions in the variables $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$, whereas the coordinates a_i are the coefficients of the Lagrange interpolation polynomial $\phi_a(\boldsymbol{x}) = a_0 + a_1\boldsymbol{x} + a_2\boldsymbol{x}^2$ such that $\boldsymbol{y}_i = \phi_a(\boldsymbol{x}_i)$ for $i = 1, 2, 3$. We see that the coordinates a_0, a_1, a_2 are well defined only when the three coordinates $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are different, that is when $I_{(e,a)}$ is a reduced subscheme.

The Chow quotient $\mathrm{Sym}^3(\mathbb{C}^2)$ is $A^3/\mathfrak{S}_3$, where any element σ of the symmetric group $\mathfrak{S}_3$ acts by $\sigma(\boldsymbol{x}_i, \boldsymbol{y}_i) = (\boldsymbol{x}_{\sigma(i)}, \boldsymbol{y}_{\sigma(i)})$ for any $i = 1, 2, 3$. The Hilbert–Chow morphism

$$h_{|\mathcal{U}_{(1,1,1)}} \colon \mathcal{U}_{(1,1,1)} \to \mathrm{Sym}^3(\mathbb{C}^2)$$

is defined by

$$h(e, a) = ((\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2), (\boldsymbol{x}_3, \boldsymbol{y}_3)),$$

where $\boldsymbol{x}_i$ are the roots (not necessarily distinct) of the polynomial $\boldsymbol{x}^3 - e_1\boldsymbol{x}^2 + e_2\boldsymbol{x} - e_3$, and $\boldsymbol{y}_i = \phi_a(\boldsymbol{x}_i)$ as above.

In this chart we can describe the birational inverse map

$$h^{-1} \colon \mathrm{Sym}^3(\mathbb{C}^2) \dashrightarrow \mathrm{Hilb}^3(\mathbb{C}^2):$$

the coordinates (e_1, e_2, e_3) are always defined by formula (6), even when the points $(\boldsymbol{x}_i, \boldsymbol{y}_i)$ are not distinct, but the coordinates (a_0, a_1, a_2) are not well defined when

$\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are not distinct. More precisely, the interpolation $\boldsymbol{y}_i = \phi_a(\boldsymbol{x}_i)$ means that these coordinates are defined by Cramer's rule:

$$a_0 = \frac{\begin{vmatrix} \boldsymbol{y}_1 & \boldsymbol{x}_1 & \boldsymbol{x}_1^2 \\ \boldsymbol{y}_2 & \boldsymbol{x}_2 & \boldsymbol{x}_2^2 \\ \boldsymbol{y}_3 & \boldsymbol{x}_3 & \boldsymbol{x}_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{x}_1^2 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{x}_2^2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{x}_3^2 \end{vmatrix}}, \quad a_1 = \frac{\begin{vmatrix} 1 & \boldsymbol{y}_1 & \boldsymbol{x}_1^2 \\ 1 & \boldsymbol{y}_2 & \boldsymbol{x}_2^2 \\ 1 & \boldsymbol{y}_3 & \boldsymbol{x}_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{x}_1^2 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{x}_2^2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{x}_3^2 \end{vmatrix}}, \quad a_2 = \frac{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{vmatrix}}{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{x}_1^2 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{x}_2^2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{x}_3^2 \end{vmatrix}}. \tag{7}$$

We get a very similar picture on the chart $\mathcal{U}_{(3)}$, simply by exchanging the roles of the variables $\boldsymbol{x}$ and $\boldsymbol{y}$. The coordinate ring of the third chart $\mathcal{U}_{(2,1)}$ is slightly different. Its coordinate ring is $\mathbb{C}[a_1, a_2, b_1, b_2, c_1, c_2]$, and an ideal $I_{(a,b,c)} \in \mathcal{U}_{(2,1)}$ with coordinates $(a, b, c) = (a_1, a_2, b_1, b_2, c_1, c_2)$ is given by

$$I_{(a,b,c)} \coloneqq \langle \boldsymbol{x}^2 - a_0 - a_1\boldsymbol{x} - a_2\boldsymbol{y}, \boldsymbol{x}\boldsymbol{y} - b_0 - b_1\boldsymbol{x} - b_2\boldsymbol{y}, \boldsymbol{y}^2 - c_0 - c_1\boldsymbol{x} - c_2\boldsymbol{y}\rangle.$$

with the following formulae, patiently deduced from [28, (2.16) and (2.17)]:

$$a_0 = a_2(b_1 - c_2) + b_2(b_2 - a_1), \quad b_0 = a_2c_1 - b_1b_2, \quad c_0 = c_1(b_2 - a_1) + b_1(b_1 - c_2).$$

Restriction to the fibre over the origin. Let us restrict the computation to the variety $\mathrm{Kum}^2(A)$. On the chart $\mathcal{U}_{(1,1,1)}$ the condition $\boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{x}_3 = 0$ gives $e_1 = 0$, and the condition $\boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 = 0$ gives $3a_0 + a_2(\boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 + \boldsymbol{x}_3^2) = 0$, so $3a_0 - 2a_2e_2 = 0$: the local equations of $\mathrm{Kum}^2(A)$ are thus $e_1 = 0$ and $a_0 = \frac{2}{3}a_2e_2$. On the variety $\widetilde{A}_0^3$, we have the relations

$$\begin{aligned} \boldsymbol{x}_3 &= -\boldsymbol{x}_1 - \boldsymbol{x}_2, \quad & \boldsymbol{y}_1 &= (\boldsymbol{w}_1 + \boldsymbol{z}_3)\boldsymbol{x}_1, \quad & \boldsymbol{y}_2 &= (\boldsymbol{w}_2 + \boldsymbol{z}_3)\boldsymbol{x}_2, \\ \boldsymbol{x}_2\boldsymbol{w}_2 &= -\boldsymbol{x}_1\boldsymbol{w}_1, \quad & \boldsymbol{y}_3 &= \boldsymbol{z}_3\boldsymbol{x}_3 = -(\boldsymbol{x}_1 + \boldsymbol{x}_2)\boldsymbol{z}_3, \end{aligned} \tag{8}$$

and we interpret the coordinates a_i as rational maps $\widetilde{a}_i \colon \widetilde{A}_0^3 \dashrightarrow \mathbb{C}$. An elementary computation, starting from Formula (7) gives:

$$\widetilde{a}_1(\boldsymbol{x}_1, \boldsymbol{w}_1, \boldsymbol{x}_2, \boldsymbol{w}_2, \boldsymbol{z}_3) = \boldsymbol{z}_3 + \boldsymbol{w}_1\boldsymbol{w}_2 \frac{\boldsymbol{w}_1^2 + \boldsymbol{w}_2^2 - 4\boldsymbol{w}_1\boldsymbol{w}_2}{(\boldsymbol{w}_1 - 2\boldsymbol{w}_2)(2\boldsymbol{w}_1 - \boldsymbol{w}_2)(\boldsymbol{w}_1 + \boldsymbol{w}_2)}, \tag{9}$$

$$\widetilde{a}_2(\boldsymbol{x}_1, \boldsymbol{w}_1, \boldsymbol{x}_2, \boldsymbol{w}_2, \boldsymbol{z}_3) = \frac{-3\boldsymbol{w}_1\boldsymbol{w}_2^2(\boldsymbol{w}_1 - \boldsymbol{w}_2)}{\boldsymbol{x}_1(\boldsymbol{w}_1 - 2\boldsymbol{w}_2)(2\boldsymbol{w}_1 - \boldsymbol{w}_2)(\boldsymbol{w}_1 + \boldsymbol{w}_2)}. \tag{10}$$

We see that $\widetilde{a}_1$ defines a rational function on $\widetilde{F}_1$, but $\widetilde{a}_2$ does not, because of its pole along $\boldsymbol{x}_1 = 0$. Let $G \subset \widetilde{F}_1$ be the support of the 1-cycle defined by the numerator of this function: that is,

$$G \coloneqq \mathcal{V}\left(\boldsymbol{w}_1\boldsymbol{w}_2^2(\boldsymbol{w}_1 - \boldsymbol{w}_2)\right)_{\mathrm{red}} \subset \widetilde{F}_1. \tag{11}$$

The fact that the coordinate function $\widetilde{a}_2$ cannot be extended to $\widetilde{F}_1 \setminus G$ means that the rational image of F_1 by γ_A does not land in the open subset $\mathcal{U}_{(1,1,1)} \cap \mathrm{Kum}^2(A)$. By exchanging the roles of the variables $\boldsymbol{x}$ and $\boldsymbol{y}$, we get that it does not land in the open subset $\mathcal{U}_{(3)} \cap \mathrm{Kum}^2(A)$ either. Since $\mathrm{Hilb}^3(A) \setminus \left(\mathcal{U}_{(1,1,1)} \cup \mathcal{U}_{(3)}\right) = \{Z_\infty\}$, the conclusion is that γ_A contracts the generic point of F_1 to Z_∞, so the restriction $\gamma_A|_{F_1}$ extends to the whole of F_1 and contracts it to the noncurvilinear point (however, γ_A itself is not defined on the whole F_1). This concludes the proof. □

In Appendix B we give three alternative arguments, the first one using saturation of ideals, the second one using the computation on the chart $\mathcal{U}_{(2,1)}$ to see more explicitly the contraction of the divisor F_1 and the third one using explicit projective coordinates.

We now analyse the behaviour of γ_A around the divisor F_2. The fixed locus of the natural involution $\mathrm{Kum}^2(\tau)$ induced by τ on $\mathrm{Kum}^2(A)$ consists of the Kummer surface $\mathrm{Kum}^1(A)$, embedded in $\mathrm{Kum}^2(A)$ as the locus of subschemes supported on 0-cycles of the form $a + (-a) + O_A \in \mathrm{Sym}^3_0(A)$, plus 36 isolated points (see for instance [9, §4.3.1]). This embedding of the Kummer surface yields the following diagram:

$$\begin{array}{ccc} \mathrm{Kum}^1(A) & \hookrightarrow & \mathrm{Kum}^2(A) \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle h_A} \\ A/\tau & \hookrightarrow & \mathrm{Sym}^3_0(A) \end{array}$$

Proposition 2.4. *The birational map γ_A contracts the divisor F_2 to the Kummer surface $\mathrm{Kum}^1(A)$ embedded in $\mathrm{Kum}^2(A)$.*

Proof. Under the embedding of $\mathrm{Kum}^1(A)$ in $\mathrm{Kum}^2(A)$, if $a \in A$ is a nonzero 2-torsion point with image $\bar{a} \in \mathrm{Kum}^1(A)$, the exceptional fibre $\varepsilon^{-1}(\bar{a})$ is sent to the curve $h_A^{-1}(2a + O_A)$ parametrising the nonreduced length two subschemes of A supported at a (the third support point being the origin). This is a rational curve as it is isomorphic to the Briançon variety $B^2_a \cong \mathbb{P}^1$. The embedding of the fibre $\varepsilon^{-1}(\overline{O_A})$ in $B^3_{O_A}$ is similar, and can be computed as follows, using the same method as in the proof of Proposition 2.3. We compute, locally over the origin, the image of the composite map on the first row:

$$\begin{array}{ccccc} \widetilde{A} & \xrightarrow{2:1} & \mathrm{Kum}^1(A) & \hookrightarrow & \mathrm{Kum}^2(A) \\ {\scriptstyle\beta_A}\downarrow & & {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle h_A} \\ A & \xrightarrow{2:1} & A/\tau & \hookrightarrow & \mathrm{Sym}^3_0(A) \end{array}$$

Let $\boldsymbol{x}, \boldsymbol{y}$ be local coordinates around the origin of A and $(\boldsymbol{x}, \boldsymbol{z})$ be local coordinates around $\beta_A^{-1}(O_A)$, with $\boldsymbol{y} = \boldsymbol{x}\boldsymbol{z}$. Using notation as above, we put $a = (\boldsymbol{x}_1, \boldsymbol{y}_1) = (\boldsymbol{x}, \boldsymbol{y})$, then $-a = (\boldsymbol{x}_2, \boldsymbol{y}_2) = (-\boldsymbol{x}, -\boldsymbol{y})$ and $(\boldsymbol{x}_3, \boldsymbol{y}_3) = (0, 0)$. Thus, on the chart $\mathcal{U}_{(1,1,1)}$, the coordinate functions are

$$e_1 = 0, \quad e_2 = \boldsymbol{x}^2, \quad e_3 = 0, \quad a_0 = 0, \quad a_1 = \boldsymbol{z}, \quad a_2 = 0.$$

Putting $\boldsymbol{x} = 0$, we see that the image of $\varepsilon^{-1}(\overline{O_A})$ in $\mathrm{Kum}^2(A)$ consists of the rational curve in $B^3_{O_A}$ parametrising the subschemes with zero curvature, *i.e.* where $\beta = 0$ in the description given above.

The minimal resolution morphism ε is the blowup of the classes of the sixteen 2-torsion points of A. Since the morphism β_A blows up the class of the origin, it factorises through the blowup ε' of the classes of the fifteen nonzero ones:

$$\begin{array}{ccc} \mathrm{Kum}^1(A) & & \\ {\scriptstyle\varepsilon'}\downarrow & \searrow{\scriptstyle\varepsilon} & \\ \widetilde{A}/\tilde{\tau} & \xrightarrow{\beta_A} & A/\tau \end{array} \tag{12}$$

To show that the map γ_A contracts the divisor F_2 to the Kummer surface, we first observe that this map is dominated by the embedding of $\mathrm{Kum}^1(A)$ in $\mathrm{Kum}^2(A)$, as shown in the following diagram, where pr_1 denotes the projection to the first factor:

$$\begin{array}{ccccc}
\mathrm{Kum}^1(A) \times E_A & \xrightarrow{\mathrm{pr}_1} & \mathrm{Kum}^1(A) & \hookrightarrow & \mathrm{Kum}^2(A) \\
\downarrow{\scriptstyle \tilde{\varepsilon}\times \mathrm{id}} & & & \nearrow{\scriptstyle \gamma_A|_{F_2}} & \downarrow{\scriptstyle h_A} \\
\widetilde{A}/\tilde{\tau} \times E_A & \longrightarrow & F_2 & \xrightarrow[\mathrm{Sym}^3(\beta_A)]{} & \mathrm{Sym}^3_0(A)
\end{array}$$

But this is not enough to show that the map γ_A contracts the divisor F_2, so we use a similar computation as in the proof of Proposition 2.3. That is, we compute the rational map g in a neighbourhood of the premiage $\widetilde{F_2}$ of F_2 in $\widetilde{A}^3_0$, which is given as the image of the morphism:

$$\widetilde{A} \times E_1 \to \widetilde{A}^3_0, \quad (\tilde{a}, e) \mapsto (\tilde{a}, \tilde{\tau}(\tilde{a}), e).$$

Taking local coordinates $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2), (\boldsymbol{x}_3, \boldsymbol{y}_3)$ of A^3 and $\boldsymbol{y}_i = \boldsymbol{x}_1\boldsymbol{z}_i$, the divisor F_2 is given by the relations $\boldsymbol{x}_1 + \boldsymbol{x}_2 = 0$ and $\boldsymbol{y}_1 + \boldsymbol{y}_2 = 0$, so in the local coordinates $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{z}_3$ of $\widetilde{A}^3_0$, it has local equations $\boldsymbol{x}_1 + \boldsymbol{x}_2 = 0$ and $\boldsymbol{w}_1 - \boldsymbol{w}_2 = 0$. In the chart $\mathcal{U}_{(1,1,1)}$, we observed that the local coordinates of the variety $\mathrm{Kum}^2(A)$ in $\mathrm{Hilb}^3(A)$ are (e_2, e_3, a_1, a_2). The coordinate $e_3 = 0$ is zero along $\widetilde{F_2}$ and we see with Equation (9) that the function $\widetilde{a_2}$ extends generically to zero along $\widetilde{F_2}$. This shows that γ_A contracts F_2 to the surface of local equations $e_1 = a_2 = 0$: these are the local equations of $\mathrm{Kum}^1(A)$ that we computed above. □

3. The genus two curves

Let C be a smooth projective curve of genus g. For any $n \geq 0$, we denote by $\mathrm{Sym}^n(C)$ the symmetric product of C, by $\Phi_n \colon C^n \to \mathrm{Sym}^n(C)$ the Chow quotient, by $\mathrm{Pic}^n(C)$ the moduli space of isomorphism classes of degree n line bundles on C and by $\mathrm{Hilb}^n(C)$ the Hilbert scheme parametrising length n zero-dimensional subschemes of C.

For $n \geq 1$ and for any $(p_1, \ldots, p_n) \in C^n$, we denote by $p_1 + \cdots + p_n \in \mathrm{Sym}^n(C)$ the formal sum, which we interpret, depending on the context, as an element of the Chow quotient, as a divisor on C or as a length n subscheme of C, since the Hilbert–Chow morphism $\mathrm{Hilb}^n(C) \to \mathrm{Sym}^n(C)$ is an isomorphism. We denote by $\Delta_n \subset \mathrm{Sym}^n(C)$ the locus of nonreduced subschemes.

For any $p_1 + \cdots + p_n \in \mathrm{Sym}^n(C)$, we denote by $\mathcal{O}_C(p_1 + \cdots + p_n) \in \mathrm{Pic}^n(C)$ the corresponding isomorphism class of line bundles, or equivalently linear equivalence class of divisors on C. We denote by $\sim$ the linear equivalence relation between divisors on C. We define the Jacobian $\mathrm{Jac}(C)$ of C as the group $\mathrm{Pic}^0(C)$ of isomorphism classes of degree zero line bundles on C.

From now on, we assume that C is a genus two curve and we put $A = \mathrm{Jac}(C)$. All notation introduced above and indexed by an abelian surface A will be indexed by C for more readibility. That is:

$$\mathscr{G}_C \coloneqq \mathscr{G}_{\mathrm{Jac}(C)}, \quad h_C \coloneqq h_{\mathrm{Jac}(C)}, \quad \gamma_C \coloneqq \gamma_{\mathrm{Jac}(C)}, \quad E_C \coloneqq E_{\mathrm{Jac}(C)}, \quad \bar{\alpha}_C = \bar{\alpha}_{\mathrm{Jac}(C)}.$$

3.1. **The Jacobian of a genus two curve.** Let C be a genus two curve. It is hyperelliptic: we denote the hyperelliptic involution by σ and the associate ramified canonical double covering by $\pi \colon C \to \mathbb{P}^1$. The ramification locus consists of six distinct points, the *Weierstrass points* of C. The curve C thus admits an equation of the form $z^2 = f(x, y)$, where f is a degree six homogeneous polynomial vanishing at the six branch points, so we may consider C as a curve in the weighted projective plane $\mathbb{P}_{113}$ with homogeneous coordinates $[x : y : z]$. We denote this embedding of C by $\iota\colon C \hookrightarrow \mathbb{P}_{113}$.

With respect to these coordinates, for $[x : y : z] \in C$ we have $\pi([x : y : z]) = [x : y]$ and $\sigma([x : y : z]) = [x : y : -z]$. We may choose coordinates on $\mathbb{P}^1$ such that the branch points of π are $[1 : 0], [0 : 1], [1 : 1]$ and three other distinct points $[\lambda_i : 1]$. Then the equation of C in $\mathbb{P}_{113}$ is

$$z^2 = xy(x - y)(x - \lambda_1 y)(x - \lambda_2 y)(x - \lambda_3 y) =: f(x, y). \tag{13}$$

We denote by $\infty := [1 : 0 : 0] \in C$ the ramification point over $[1 : 0]$.

Every element of $\mathrm{Jac}(C)$ has a unique representative, called a *reduced divisor* (see [43, Chapter 3, §2]), which is one of

(1) $\mathcal{O}_C(p_1 + p_2 - 2\infty)$ with $p_i \in C \setminus \{\infty\}$, $p_1 \neq p_2$ and $p_1 \neq \sigma(p_2)$;
(2) $\mathcal{O}_C(2p - 2\infty)$ with $p \in C \setminus \{\infty\}$ and $p \neq \sigma(p)$;
(3) $\mathcal{O}_C(p - \infty)$ with $p \in C \setminus \{\infty\}$;
(4) $\mathcal{O}_C$.

In particular, $p + \sigma(p) \sim 2\infty$ for all $p \in C$. It is a classical result, see for instance [42, Lecture 3], that the Abel–Jacobi map

$$\mathrm{AJ}_2\colon \mathrm{Sym}^2(C) \to \mathrm{Jac}(C), \quad p_1 + p_2 \mapsto \mathcal{O}_C(p_1 + p_2 - 2\infty) \tag{14}$$

is the blowup β_C of the origin of $\mathrm{Jac}(C)$, that is $\widetilde{A} = \widetilde{\mathrm{Jac}(C)} \cong \mathrm{Sym}^2(C)$. The exceptional divisor of the blowup is thus

$$E_C := \{p + \sigma(p) \mid p \in C\} \subset \mathrm{Sym}^2(C). \tag{15}$$

The sign involution on $\mathrm{Jac}(C)$ is given by $\tau(L) = L^{-1}$. For any $p_1, p_2 \in C$ we have

$$p_1 + p_2 + \sigma(p_1) + \sigma(p_2) \sim 4\infty,$$

so $\mathcal{O}_C(p_1 + p_2 - 2\infty)^{-1} = \mathcal{O}_C(\sigma(p_1) + \sigma(p_2) - 2\infty) \in \widetilde{\mathrm{Jac}(C)}$. It follows that $\tau = \sigma^*$ and that the involution $\tilde{\tau}$ on $\mathrm{Sym}^2(C) = \widetilde{\mathrm{Jac}(C)}$ introduced in 2.3 is given by $\tilde{\tau}(p_1 + p_2) = \sigma(p_1) + \sigma(p_2)$.

By the Poincaré formula [8, 11.2.1], the theta divisor giving the principal polarisation of $\mathrm{Jac}(C)$ is the image of the Abel–Jacobi map $C \to \mathrm{Jac}(C)$, that is $\Theta := \{p - \infty \mid p \in C\}$, and $\Theta^2 = 2$. If C is generic in its moduli space, then $\mathrm{NS}(\mathrm{Jac}(C)) = \mathbb{Z}\Theta$. Following §2.1, we denote by h the image of Θ in $\mathrm{Kum}^2(\mathrm{Jac}(C))$ and we observed above that the minimal possible polarisation degree of $\mathrm{Kum}^2(\mathrm{Jac}(C))$ is $q_{\mathrm{Kum}^2(\mathrm{Jac}(C))}(3h - \delta) = 12$.

Remark 3.1. In this setup, Diagram (12) has rich and famous geometric properties. Recall the situation:

$$\begin{array}{ccc} \mathrm{Kum}^1(\mathrm{Jac}(C)) & & \\ {\scriptstyle\tilde{\varepsilon}}\downarrow & \searrow^{\varepsilon} & \\ \mathrm{Sym}^2(C)/\tilde{\tau} & \xrightarrow{\mathrm{AJ}_2} & \mathrm{Jac}(C)/\tau \end{array}$$

The double covering $\pi\colon C \to \mathbb{P}^1$ induces a $4 : 1$ covering

$$\mathrm{Sym}^2(\pi)\colon \mathrm{Sym}^2(C) \to \mathrm{Sym}^2(\mathbb{P}^1) \cong \mathbb{P}^2,$$

which factors through $\tilde{\tau}$ to a double covering $\mathrm{Sym}^2(C)/\tilde{\tau} \to \mathbb{P}^2$ whose branch locus is the union of the six lines

$$\{w + x \mid x \in \mathbb{P}^1\} \in \mathrm{Sym}^2(\mathbb{P}^1),$$

where w is one the six branch points of π. Each of these curves defines a line in $\mathbb{P}^2$ and the branch locus is thus a sextic curve in $\mathbb{P}^2$ that consists of six lines tangent to a conic, meeting in 15 points that are blown up by $\tilde{\varepsilon}$ (see for instance [20, §10.2], with a complementary point of view).

3.2. **The linear system of cubics.** The linear system of cubics in $\mathbb{P}_{113}$ is 5-dimensional:

$$\mathrm{H}^0(\mathbb{P}_{113}, \mathcal{O}_{\mathbb{P}_{113}}(3)) = \mathrm{Span}(x^3, x^2y, xy^2, y^3, z).$$

We put $\mathcal{L} := \iota^*\mathcal{O}_{\mathbb{P}_{113}}(3)$, so $|\mathcal{L}|$ is the complete linear system of cubics on the curve C. We have $h^0(C, \mathcal{L}) = 5$ by Riemann–Roch (see also the simple part of Mattuck's argument in the proof below) and so $\mathrm{H}^0(C, \mathcal{L}) = \mathrm{H}^0(\mathbb{P}_{113}, \mathcal{O}_{\mathbb{P}_{113}}(3))$ with $|\mathcal{L}| \cong \mathbb{P}^4$.

Any divisor $D \in |\mathcal{L}|$ has an equation of the form:

$$\alpha_0 x^3 + \alpha_1 x^2 y + \alpha_2 x y^2 + \alpha_3 y^3 + \alpha_4 z = 0, \tag{16}$$

with $\alpha_i \in \mathbb{C}$, so a generic D cuts the curve C in six distinct points (see Figure 1). It follows that $\deg(\mathcal{L}) = 6$, so the line bundle $\mathcal{L}$ is very ample [29, Corollary IV.3.2].

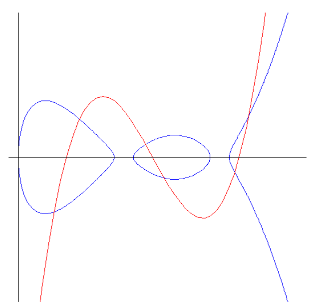


FIGURE 1. A genus 2 curve (in blue) and a cubic interpolation (in red) intersecting in 6 points on the affine chart $y = 1$ of coordinates x (abscissa) and z (ordinate).

Lemma 3.1. *We have* $|\mathcal{L}| = |6\infty|$ *and* $\pi_*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3)$.

Proof. The equality $|\mathcal{L}| = |6\infty|$ is an application to our situation of classical results of Mattuck and Mumford. If C is a smooth curve of genus g, and if $n > 2g - 2$, then Mattuck [39] shows that the divisor class map

$$\delta\colon \mathrm{Sym}^n(C) \to \mathrm{Pic}^n(C), \quad p_1 + \cdots + p_n \mapsto \mathcal{O}_C(p_1 + \cdots + p_n)$$

is a $\mathbb{P}^{n-g}$ bundle. For completeness we give the part of Mattuck's argument that we need: we do not need the local triviality, which is harder. Writing $\xi := p_1 + \cdots + p_n \in \mathrm{Sym}^n(C)$, the fibre of δ over $\mathcal{O}_C(\xi)$ is by definition the complete linear system $|\xi|$. Since $\deg(K_C - \xi) = 2g - 2 - n < 0$, we have $h^0(X, K_C - \xi) = 0$, hence $h^0(C, \xi) = n - g + 1$ by the Riemann–Roch theorem. Thus $|\xi| \cong \mathbb{P}^{n-g}$ and this dimension does not depend on ξ.

We apply this result with $g = 2$, for the genus two curve C introduced above, and we keep the same notation as above.

Secondly, by a result of Mumford [43, Chapter 3,§2], if D is a degree d irreducible curve D in $\mathbb{P}_{113}$ and by $p_1, \dots, p_k$ are the not necessarily distinct intersection points of D with the genus two curve C, then the divisor $p_1 + \cdots + p_k$ is linearly equivalent to $(2d)\infty$. In particular, we have $k = 2d$. Again we recall the proof for completeness. The curve D has equation $g(x, y, z) = 0$, with g a weighted homogeneous polynomial of degree d, and the curve C is given by the equation (13). The quotient $\frac{g(x,y,z)}{y^d}$ thus defines a rational function on C whose divisor is:

$$p_1 + \cdots + p_k - (2d)\infty.$$

Since this divisor has degree zero, we have $k = 2d$. We apply this result with $g = 2$ and $d = 3$ and $k = 6$, this gives $|\mathcal{L}| = |6\infty|$.

The computation of the direct image $\pi_*\mathcal{L}$ is a straightforward application of the cyclic covering methods [4, §I.17]. Since the canonical sheaf of C is $\omega_C \cong \mathcal{O}_C(2\infty)$, by the first assertion we get that $\mathcal{L} \cong \omega_C^{\otimes 3}$. The double covering π is branched at the six Weierstrass points, so it is determined by the line bundle $L = \mathcal{O}_{\mathbb{P}^1}(3)$ such that $L^{\otimes 2} \cong \mathcal{O}_{\mathbb{P}^1}(6)$ and $\pi_*\mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus L^{-1}$. We have $\omega_C \cong \pi^*(\omega_{\mathbb{P}^1} \otimes L) \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$, so by the projection formula

$$\pi_*\omega_C^{\otimes 3} \cong \pi_*\pi^*\mathcal{O}_{\mathbb{P}^1}(3) \cong \mathcal{O}_{\mathbb{P}^1}(3) \otimes \pi_*\mathcal{O}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3). \qquad \square$$

The Abel–Jacobi map is the map

$$\text{(17)} \quad \mathrm{AJ}_n \colon \mathrm{Sym}^n(C) \to \mathrm{Jac}(C), \qquad p_1 + \cdots + p_n \mapsto \mathcal{O}_C(p_1 + \cdots + p_n - n\infty),$$

By Lemma 3.1 we have $|\mathcal{L}| \cong \mathrm{AJ}_6^{-1}(\{\mathcal{O}_C\})$. Let us rephrase this: all length 6 zero-dimensional subschemes of C that admit a cubic interpolation belong to the fibre over $\mathcal{O}_C$ of the Abel–Jacobi map AJ_6. But since the linear system on C cut out by the cubics has the correct dimension, this is a characterisation of this fibre.

3.3. **Interpretation of the linear system of cubics as a degeneracy locus.** Let us describe the linear system $|\mathcal{L}|$ differently. Consider a length 6 subscheme $\xi \subset C$, that is $\xi \in \mathrm{Sym}^6(C)$. Start with the exact sequence

$$0 \longrightarrow \mathcal{I}_\xi \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_\xi \longrightarrow 0,$$

and tensor by $\mathcal{L}$:

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{I}_\xi \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_\xi \longrightarrow 0.$$

Since $\deg(\mathcal{L}) > \deg(K_C)$, we have by Serre duality $h^1(C, \mathcal{L}) = 0$, so we get a four terms exact sequence in cohomology:

$$\text{(18)} \quad 0 \to \mathrm{H}^0(C, \mathcal{L} \otimes \mathcal{I}_\xi) \to \mathrm{H}^0(C, \mathcal{L}) \xrightarrow{\mathrm{res}_\xi} \mathrm{H}^0(C, \mathcal{L} \otimes \mathcal{O}_\xi) \to \mathrm{H}^1(C, \mathcal{L} \otimes \mathcal{I}_\xi) \to 0.$$

Since $I_\xi \cong \mathcal{O}_C(-\xi)$, the sheaf $\mathcal{L}(-\xi) \coloneqq \mathcal{L} \otimes \mathcal{I}_\xi$ is a degree zero line bundle. We are interested in the restriction morphism res_ξ, whose rank is at most 5:

$$\mathrm{res}_\xi \colon \mathbb{C}^5 \cong \mathrm{H}^0(C, \mathcal{L}) \to \mathrm{H}^0(\xi, \mathcal{L}|_\xi) \cong \mathbb{C}^6.$$

Proposition 3.2. *We have* $|\mathcal{L}| = \{\xi \in \mathrm{Sym}^6(C) \mid \mathrm{rk}(\mathrm{res}_\xi) = 4\}$, *and there is no* $\xi \in \mathrm{Sym}^6(C)$ *such that* $\mathrm{rk}\,\mathrm{res}_\xi < 4$.

Proof. For any $\xi \in \mathrm{Sym}^6(C)$, the exact sequence (18) says that

$$\mathrm{rk}(\mathrm{res}_\xi) = 5 - h^0(C, \mathcal{L}(-\xi)),$$

so $\mathrm{rk}(\mathrm{res}_\xi) = 4$ means that $h^0(C, \mathcal{L}(-\xi)) = 1$. Since $\mathcal{L}(-\xi)$ has degree zero, this is equivalent to $\mathcal{L}(-\xi) \cong \mathcal{O}_C$, or equivalently to $\xi \in |\mathcal{L}|$. Moreover, if $\mathrm{rk}\,\mathrm{res}_\xi < 4$ then $h^0(C, \mathcal{L}(-\xi)) \geq 2$: this is not possible since $\mathcal{L}(-\xi)$ has degree zero. □

Proposition 3.2 means that for every length 6 subscheme of C, there exists at most one cubic interpolation. In what follows, we will frequently interpret the identification $\mathrm{AJ}_6^{-1}(\{\mathcal{O}_C\}) = |\mathcal{L}|$ as the isomorphism

$$\begin{aligned} \mathrm{AJ}_6^{-1}(\{\mathcal{O}_C\}) &\longrightarrow |\mathcal{L}| \cong \mathbb{P}\mathrm{H}^0(C, \mathcal{L}) \\ \xi &\longmapsto \ker(\mathrm{res}_\xi) \end{aligned} \tag{19}$$

that sends a length 6 subscheme admitting a cubic interpolation to the equation of this uniquely defined cubic.

Remark 3.2. By Proposition 3.2, there exists at most one cubic interpolation for every given length 6 subscheme of C. Although this is expected for general subschemes, it is remarkable that it holds for all of them. The basic general observation is that a cubic interpolation can never factor as conic and a "nonvertical" line (this expression makes sense in the affine chart $y = 1$ with coordinates (x, z), see Figure 2), because of shape of the equation and the fact that the variable z has degree 3. If the cubic contains a "vertical line" $x = \alpha$, then its equation does not contain the variable z and it thus factors as a product of three vertical lines.

3.4. **The linear system of conics.** Similarly as in §3.2, the linear system of conics in $\mathbb{P}_{113}$ is 3-dimensional:

$$\mathrm{H}^0(\mathbb{P}_{113}, \mathcal{O}_{\mathbb{P}_{113}}(2)) = \mathrm{Span}(x^2, xy, y^2).$$

We put $\mathcal{C} \coloneqq \iota^*\mathcal{O}_{\mathbb{P}_{113}}(2)$. Let $\xi \in \mathrm{Sym}^4(C)$. Similarly as in §3.3, to study the cubic interpolations that pass through four points we make use of the restriction morphism

$$\mathrm{res}_\xi \colon \mathbb{C}^5 \cong \mathrm{H}^0(C, \mathcal{L}) \to \mathrm{H}^0(\xi, \mathcal{L}|_\xi) \cong \mathbb{C}^4.$$

This time, $\ker(\mathrm{res}_\xi)$ is never zero so there exists at least one cubic interpolation at ξ, and it is unique if and only if $h^0(C, \mathcal{L}(-\xi)) = 1$.

We now characterise those length four subschemes of C that admit a conic interpolation, and we relate this to the extent to which the cubic interpolations passing through these points fail to be unique.

Lemma 3.3. *Let $\xi \coloneqq p_1 + p_2 + p_3 + p_4 \in \mathrm{Sym}^4(C)$. The following assertions are equivalent:*

(1) $\xi \sim 4\infty$.
(2) There exists $K \in |\mathcal{C}|$ such that $K \cap C = p_1 + \cdots + p_4$.
(3) Up to permutation of the points, $p_2 = \sigma(p_1)$ and $p_4 = \sigma(p_3)$.
(4) $h^0(C, \mathcal{L}(-\xi)) = 2$.

Proof. Since the equation of a conic $K \in |\mathcal{C}|$ does not contain the variable z, within the affine chart $y = 1$ it consists of two vertical lines (or one double line). Therefore it cuts C in four points, which form two orbits under the action of the involution σ, so up to reordering the points, we conclude that $p_2 = \sigma(p_1)$ and $p_4 = \sigma(p_3)$. Since the converse is clear, this proves (2)⇔(3). Moreover, any line of equation $ax + by = 0$, with $(a : b) \in \mathbb{P}^1$, completes this conic to cubic interpolation that consisting of three vertical lines. This means that there is a pencil of cubics intersecting C at ξ, so $h^0(C, \mathcal{L}(-\xi)) = 2$. This proves (2)⇒(4). Since $p+\sigma(p) \sim 2\infty$ for any $p \in C$ (see [43, Chapter 3, §2]), we get

$$p_1 + \cdots + p_4 \sim 4\infty.$$

Conversely, if four points $p_1, \ldots, p_4$ satisfy the relation $p_1 + \cdots + p_4 \sim 4\infty$, arguing similarly as in the proof of Lemma 3.1, we see that $|\mathcal{C}| = |4\infty|$ and we deduce that these four points admit a unique conic interpolation. This proves (1)⇔(2).

It remains to show that (4)⇒(2). Suppose that (2) does not hold, so at least three of the x-coordinates of the points p_i are distinct.

First assume that the x-coordinates of the four points are all different from one another and from ∞, so that we may write them as $p_i = (x_i : 1 : z_i)$. There exists a unique cubic interpolation $g(x, z)$ at these points, defined by the four conditions $g(x_i) = z_i$, so $h^0(C, \mathcal{L}(-\xi)) = 1$. If instead one of the points is ∞, there is still a cubic interpolation, but it has no x^3-term and it is uniquely determined by the interpolation at the three remaining points.

If only three of the x-coordinates are distinct, we may assume that $p_2 = \sigma(p_1)$ and that the coordinates x_1, x_3, x_4 are distinct. To construct a cubic interpolation at these four points, we first need to take the line joining p_1 and p_2. Then the only way to interpolate C at p_3 and p_4 with a conic is to take the lines joining p_3 to $\sigma(p_3)$ and p_4 to $\sigma(p_4)$, so these points admit a unique cubic interpolation and $h^0(C, \mathcal{L}(-\xi)) = 1$. □

4. A degree 15 covering of the linear system of cubics

4.1. **Covering maps and Galois closure.** Inequivalent notions of covering map coexist in the literature. We follow [23, §3] and make the definition below. Note that we do not require a covering map to be étale.

Definition 4.1. A *covering map* is a finite surjective morphism $f : X \to X'$ between normal projective varieties. A covering map f is called *Galois* if there exits a finite group $G \subset \mathrm{Aut}(X)$ such that f is isomorphic to the quotient map $X \to X/G$.

Let $f : X \to X'$ be a morphism between normal projective varieties. The support R of the sheaf of relative Kähler differentials

$$\Omega_{X/X'} := \mathrm{coker}\,(f^*\Omega_{X'} \to \Omega_X)\,,$$

endowed with its structure of a closed subscheme of X, is the *ramification scheme* of f. Its image $B := f(R)$, defined as a closed subscheme of X', is the *branch scheme* of f. When X is normal and X' is nonsingular, by the Zariski–Nagata purity theorem [44, 53], R and B are divisors on X and X' respectively (see also [54, Theorem 2.4]). The terms *ramification locus* and *branch locus* refer to the underlying sets of closed points.

Theorem 4.1. [23, Theorem 3.7] *Let $f : X \to X'$ be a covering map between quasi-projective varieties. There exists a normal, quasi-projective variety $\widehat{X}$ and a finite surjective morphism $\hat{f} : \widehat{X} \to X$, called the* Galois closure *of f, such that*

1. *there exist finite groups $H \subset G$ such that the morphisms $F := f \circ \hat{f}$ and $\hat{f}$ are Galois coverings with respective groups G and H.*
2. *The branch loci of F and f are equal.*

4.2. **Organising six points on the curve into three pairs.**

Definition 4.2. We denote by H the subgroup of the symmetric group $\mathfrak{S}_6$ generated by the permutations $(1,2)$, $(1,3)(2,4)$ and $(1,5)(2,6)$.

The group H has order 48 and hence index 15 in $\mathfrak{S}_6$, but it is not normal.

Lemma 4.2. *The Chow quotient* $\Phi_6\colon C^6 \to \mathrm{Sym}^6(C)$ *factorises through a finite morphism* $\varphi\colon \mathrm{Sym}^3(\mathrm{Sym}^2(C)) \to \mathrm{Sym}^6(C)$ *of degree* 15, *as follows:*

$$\begin{array}{rccccc} \Phi_6\colon & C^6 & \xrightarrow{\widehat{\varphi}} & \mathrm{Sym}^3(\mathrm{Sym}^2(C)) & \xrightarrow{\varphi} & \mathrm{Sym}^6(C) \\ & (p_1,\dots,p_6) & \longmapsto & [p_1+p_2]+[p_3+p_4]+[p_5+p_6] & \longmapsto & p_1+\cdots+p_6. \end{array}$$

The morphism $\widehat{\varphi}$ *is the Galois closure of* φ.

I have to admit that I do not really like the notation with the parentheses. The deeper problem is that we overload the "+". In some instances it is associative, in others, it isn't. Do we want to use something else such as $[p_1+\cdots+p_k]$ for elements of $\mathrm{Sym}^k(C)$ throughout, then? If so, we must do it carefully and fully. It may be the best thing. I changed $[p_1+p_2]$ in the definition, but not the notation for 0-cycle everywhere. Later we use $p_{i,j}=p_i+p_j$, maybe this could be better?

Proof. The morphism φ is quasi-finite, hence finite by Stein factorisation since all varieties involved are projective. Its degree is $15=\frac{1}{3!}\binom{6}{2}\binom{4}{2}$ and it is clearly surjective. It is also flat since $\mathrm{Sym}^6(C)$ is nonsingular and $\mathrm{Sym}^3(\mathrm{Sym}^2(C))$ is Cohen–Macaulay (see [17, 18.17]). The Chow quotient $\Phi_6\colon C^6\to\mathrm{Sym}^6(C)$ is a Galois covering with group the symmetric group $\mathfrak{S}_6$. It is easy to check that $\widehat{\varphi}$ is the quotient of C^6 by the group H introduced in Definition 4.2. This group permutes the three pairs and the position of the points in each pair. Since $\mathrm{Sym}^3(\mathrm{Sym}^2(C))$ is normal, the morphism φ is a covering map in the sense of Definition 4.1 but it is not a Galois covering since H is a nonnormal subgroup of $\mathfrak{S}_6$. The morphism $\widehat{\varphi}$ is the Galois closure of φ in the sense of Theorem 4.1. □

The ramification scheme of Φ_6 is the big diagonal $\mathcal{D}_6\subset C^6$, *i.e.* the union of the closed subschemes defined by the equalities $p_i=p_j$ for $p=(p_1,\dots,p_6)\in C^6$. It is a reduced and reducible divisor. The branch scheme $\Phi_6(\mathcal{D}_6)=\Delta_6\subset\mathrm{Sym}^6(C)$ is the locus of nonreduced subschemes: it is a reduced and irreducible divisor, and it is clearly the branch scheme of φ. We study now the ramification scheme of φ.

Remark 4.1. Let us describe the fibres of φ explicitly. For any point $(p_1,\dots,p_6)\in C^6$, we write for short $p_{i,j}\coloneqq p_i+p_j\in\mathrm{Sym}^2(C)$. The fibre of φ over a generic point $p_1+\cdots+p_6\in\mathrm{Sym}^6(C)$ is the following set of 15 points in $\mathrm{Sym}^3(\mathrm{Sym}^2(C))$:

$$\begin{array}{lll} p_{1,2}+p_{3,4}+p_{5,6} & p_{1,2}+p_{3,5}+p_{4,6} & p_{1,2}+p_{3,6}+p_{4,5} \\ p_{1,3}+p_{2,4}+p_{5,6} & p_{1,3}+p_{2,5}+p_{4,6} & p_{1,3}+p_{2,6}+p_{4,5} \\ p_{1,4}+p_{2,3}+p_{5,6} & p_{1,4}+p_{2,5}+p_{3,6} & p_{1,4}+p_{2,6}+p_{3,5} \\ p_{1,5}+p_{2,3}+p_{4,6} & p_{1,5}+p_{2,4}+p_{3,6} & p_{1,5}+p_{2,6}+p_{3,4} \\ p_{1,6}+p_{2,3}+p_{4,5} & p_{1,6}+p_{2,4}+p_{3,5} & p_{1,6}+p_{2,5}+p_{3,4} \end{array}$$

Over a generic point of Δ_6, say for instance when $p_1 = p_2$, this fibre contains only nine closed points:

$$\begin{array}{lll} p_{1,1}+p_{3,4}+p_{5,6} & p_{1,1}+p_{3,5}+p_{4,6} & p_{1,1}+p_{3,6}+p_{4,5} \\ p_{1,3}+p_{1,4}+p_{5,6} & p_{1,3}+p_{1,5}+p_{4,6} & p_{1,3}+p_{1,6}+p_{4,5} \\ p_{1,4}+p_{1,5}+p_{3,6} & p_{1,4}+p_{1,6}+p_{3,5} & p_{1,5}+p_{1,6}+p_{3,4} \end{array} \tag{20}$$

In the scheme-theoretic fibres, the first three points, on the top line of the list (20), are nonreduced subschemes with a length two subscheme supported at p_1. The remaining six points are (generically) reduced subschemes obtained when the point p_1 is in the support of two different summands $p_{1,i}$ and $p_{1,j}$ at the same time.

From Remark 4.1 we deduce that a generic fibre of φ consists of nine distinct points that belong to two distinct divisors R_1 and R_2, defined below. We will see shortly that only R_2 is indeed a ramification divisor. Consider first the Chow quotient:

$$q\colon \mathrm{Sym}^2(C)\times\mathrm{Sym}^2(C)\times\mathrm{Sym}^2(C)\to\mathrm{Sym}^3(\mathrm{Sym}^2(C))$$

and define

$$R_1 \coloneqq q\left(\Delta_2\times\mathrm{Sym}^2(C)\times\mathrm{Sym}^2(C)\right). \tag{21}$$

It is an irreducible divisor parametrising unordered triples of effective degree 2 divisors on C such that at least one of them is nonreduced. Over a generic point of Δ_6, three points of the fibre of φ belong to R_1 (for instance those on the top line of (20)).

Let us now introduce the *double incidence* subvariety:

$$\Xi \coloneqq \{(x,\xi_1,\xi_2)\in C\times\mathrm{Sym}^2(C)\times\mathrm{Sym}^2(C)\mid x\in\mathrm{Supp}(\xi_1)\cap\mathrm{Supp}(\xi_2)\},$$

where $\mathrm{Supp}(\xi)$ is the set-theoretic support of the subscheme ξ. The locus Ξ is a codimension 2 irreducible subvariety of the product variety. Denote the projection by

$$\pi\colon C\times\mathrm{Sym}^2(C)\times\mathrm{Sym}^2(C)\to\mathrm{Sym}^2(C)\times\mathrm{Sym}^2(C),\quad (x,\xi_1,\xi_2)\mapsto(\xi_1,\xi_2)$$

and define

$$R_2 \coloneqq q\left(\pi(\Xi)\times\mathrm{Sym}^2(C)\right). \tag{22}$$

Over a generic point of Δ_6, six points of the fibre of φ belong to R_2 (for instance those not on the top line of (20)).

Lemma 4.3. *We have $\varphi^*\Delta_6 = R_1 + 2R_2$. The ramification scheme of φ is the reduced and irreducible divisor R_2.*

Proof. The ramification divisor of φ decomposes as a sum of irreducible components with multiplicity as $r_1R_1 + r_2R_2$. Denote by e_i, $i = 1, 2$ the local degrees (or branching orders) of φ at generic points of R_i, so that $\varphi^*\Delta_6 = e_1R_1 + e_2R_2$. By symmetry, these degrees do not depend on the choice of one of the three (for $i = 1$), respectively six (for $i = 2$) preimage points in R_i. Since φ is generically 15 : 1, we have $3e_1 + 6e_2 = 15$, and this gives two possibilities: either $(e_1, e_2) = (1, 2)$ or $(e_1, e_2) = (3, 1)$.

Let us exclude the second possibility. We use the notation of Remark 4.1. It is enough to consider one point in R_2, say for instance $p_{1,3} + p_{1,4} + p_{5,6}$. This point is obtained as the limit point of the two reduced subschemes $p_{1,3} + p_{2,4} + p_{5,6}$ or $p_{2,3} + p_{1,4} + p_{5,6}$ when p_2 goes to p_1, so the local degree is two. In comparison, a point $p_{1,1} + p_{3,4} + p_{5,6}$ in R_1 is the limit of $p_{1,2} + p_{3,4} + p_{5,6}$ when p_2 goes to p_1, but there is only one limit direction since C is a smooth curve, so there is no ramification there. It follows that $e_1 = 1$ and $e_2 = 2$. We conclude using [4, Lemma I.16.1] that the ramification divisor is R_2. □

Let us rephrase the upshot of the above proof: the preimage by φ of the branch divisor Δ_6 is the union of the divisors R_1 and R_2, but the preimage points in R_1 are not ramification points, whereas the premiage points in R_2 are ramification points with branching order 2. The ramification divisor is thus the reduced divisor R_2.

4.3. **The degree 15 covering.** Recall that we denote by $\mathscr{G}_C \coloneqq \mathscr{G}_{\mathrm{Jac}(C)}$ the variety $\mathrm{Sym}^3_0(\mathrm{Sym}^2(C))$ introduced in Definition 2.1, using the identification $\widetilde{\mathrm{Jac}(C)} \cong \mathrm{Sym}^2(C)$ explained in (14).

Proposition 4.4. *The morphism $\varphi\colon \mathrm{Sym}^3(\mathrm{Sym}^2(C)) \to \mathrm{Sym}^6(C)$ restricts to a degree 15 finite morphism $\psi\colon \mathscr{G}_C \to |\mathcal{L}| \cong \mathbb{P}^4$. The variety $\mathscr{G}_C$ is normal, geometrically Cohen–Macaulay, $\mathbb{Q}$-factorial and Gorenstein, with quotient singularities.*

Proof. The isomorphism (19) can be formulated equivalently as a closed embedding $|\mathcal{L}| \hookrightarrow \mathrm{Sym}^6(C)$ sending a cubic D to the subscheme $D \cap C$ considered as a formal sum of points, where the multiplicity of $D \cap C$ at a point p is the length of the artinian ring $\mathcal{O}_{p,D\cap C}$. It is easy to check that the morphism $\mathrm{AJ}_6 \circ \varphi$ factorises by $\mathrm{Sym}^3(\beta_C)$, that is:

$$\begin{array}{ccccc} \mathrm{Sym}^3(\mathrm{Sym}^2(C)) & \xrightarrow{\varphi} & \mathrm{Sym}^6(C) & \xrightarrow{\mathrm{AJ}_6} & \mathrm{Jac}(C) \\ \Big\downarrow{\scriptstyle \mathrm{Sym}^3(\beta_C)} & & & \nearrow{\scriptstyle \bar{\alpha}_C} & \\ \mathrm{Sym}^3(\mathrm{Jac}(C)) & & & & \end{array}$$

Since $|\mathcal{L}| = |6\infty| \cong \mathrm{AJ}_6^{-1}(\mathcal{O}_C)$, this shows that $\mathscr{G}_C$ is the fibre of φ over $|\mathcal{L}|$. We denote by ψ the restriction of φ to $\mathscr{G}_C$. Since a generic cubic interpolation cuts C in 6 points, the generic fibre of ψ is reduced, so the morphism $\psi\colon \mathscr{G}_C \to |\mathcal{L}| = \mathbb{P}^4$ is finite of degree 15.

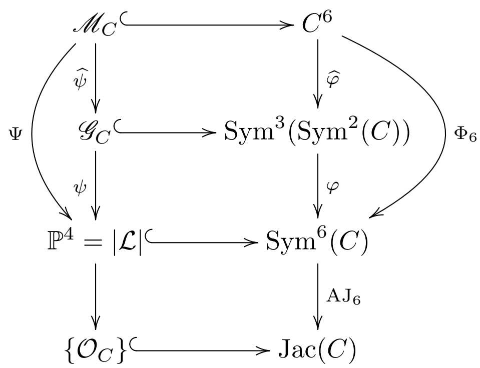
The surface $\mathrm{Sym}^2(C)$ is nonsingular, so the symmetric quotient $\mathrm{Sym}^3(\mathrm{Sym}^2(C))$ has rational singularities. Since the group $\mathfrak{S}_3$ acts on it without quasi-reflections, the quotient is Gorenstein (see for instance [33] and references therein). We know by Proposition 2.1 that $\mathscr{G}_C$ is normal, Cohen–Macaulay, $\mathbb{Q}$-factorial with quotient singularities. Since $\mathrm{Sym}^6(C)$ is nonsingular, the finite morphism φ is Gorenstein, meaning that its relative dualising sheaf is locally free. Since its formation commutes with base change, the fibre $\mathscr{G}_C$ over $|\mathcal{L}|$ is Gorenstein (see for instance [11, §1]). □

Definition 4.3. We denote by $\mathscr{M}_C$ the scheme-theoretic fibre over the origin of the morphism:

$$C^6 \xrightarrow{\Phi_6} \mathrm{Sym}^6(C) \xrightarrow{\mathrm{AJ}_6} \mathrm{Jac}(C),$$

that is, $\mathscr{M}_C \coloneqq (\mathrm{AJ}_6 \circ \Phi_6)^{-1}(\{\mathcal{O}_C\})$.

Let us summarise the situation in the following diagram, where, as we proved above, we have $\mathscr{G}_C = (\mathrm{AJ}_6 \circ \varphi)^{-1}(\{\mathcal{O}_C\})$:

(23)


Proposition 4.5.

(1) *The scheme $\mathscr{M}_C$ is a local complete intersection. It is a normal variety of dimension 4, Cohen–Macaulay and Gorenstein.*
(2) *The morphism $\widehat{\psi}\colon \mathscr{M}_C \to \mathscr{G}_C$ is the Galois closure of ψ and $\widehat{\psi}$ is the quotient map by the group H, that is $\mathscr{G}_C = \mathscr{M}_C/H$.*
(3) *The morphism $\Psi\colon \mathscr{M}_C \to |\mathcal{L}|$ is syntomic.*

We recall that *syntomic* means a flat local complete intersection morphism of locally finite presentation, see [50, Definition 29.30.1].

Proof. For any $p \coloneqq (p_1, \dots, p_6) \in C^6$, we consider the restriction morphism $\mathrm{res}_{\Phi_6(p)}$ introduced above:

$$\mathrm{res}_{\Phi_6(p)}\colon \mathbb{C}^5 \cong \mathrm{H}^0(C, \mathcal{L}) \to \mathrm{H}^0(\Phi_6(p), \mathcal{L}|_{\Phi_6(p)}) \cong \mathbb{C}^6.$$

By definition, the closed points of $\mathscr{M}_C$ are those sextuples of points on C that are interpolated (with multiplicity if necessary) by a cubic. The finite surjective morphism $\Psi\colon \mathscr{M}_C \to |\mathcal{L}|$ maps any $p \in \mathscr{M}_C(\mathbb{C})$ to $\ker(\mathrm{res}_{\Phi_6(p)})$, using the isomorphism (19). It follows that $\mathscr{M}_C$ is equidimensional of dimension 4. By Proposition 3.2, the cubic interpolation is always unique whenever it exists since the locus of points $p \in C^6$ such that $\mathrm{rk}(\mathrm{res}_{\phi_6(p)}) < 4$ is empty, so we have:

$$\begin{aligned} \mathscr{M}_C &= \{p \in C^6 \mid \exists s \in \mathrm{H}^0(C, \mathcal{L}),\ \mathcal{Z}(s) = \Phi_6(p)\} \\ &= \{p \in C^6 \mid \mathrm{rk}(\mathrm{res}_{\Phi_6(p)}) \leq 4\}, \end{aligned}$$

where $\mathcal{Z}(s)$ means the zero scheme of s (we refer to [2] for the definition of this scheme structure).

The local equations of $\mathscr{M}_C$ at a point p are thus the six 5×5 minors of any 6×5 matrix R associated to $\mathrm{res}_{\Phi_6(p)}$. At least one 4×4 minor, say the determinant $|R'|$ of the 4×4 submatrix R', is nonzero at p since the matrix has rank 4.

The determinants of the two 5×5 submatrices containing R' vanish at p and by basic linear algebra, this forces the vanishing at p of all the 5×5 minors. So $\mathscr{M}_C$ is locally given by two equations in the nonsingular variety C^6, hence it is a local complete intersection. It thus satisfies Serre's condition S_k for any $k \geq 1$.

Using these local equations, we show that $\mathscr{M}_C$ is regular in codimension zero (Serre's condition R_0). Since a generic cubic cuts C in six points whose images under the double covering $\pi\colon C \to \mathbb{P}^1$ are distinct, each irreducible component of

$\mathscr{M}_C$ contains a dense open subset of points $p = (p_1, \ldots, p_6) \in \mathscr{M}_C$ such that the x-coordinates of all p_i are distinct. Let us show that $\mathscr{M}_C$ is nonsingular at these points. Without loss of generality, we may assume that none of these points is ∞, so we denote their coordinates by $p_i = [x_i : 1 : z_i]$ with x_i distinct. The matrix of $\operatorname{res}_{\Phi_6(p)}$ is the 6×5 matrix

$$R := \begin{pmatrix} x_1^3 & x_1^2 & x_1 & 1 & z_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_6^3 & x_6^2 & x_6 & 1 & z_6 \end{pmatrix}.$$

We may take R' to be the top left 4×4 submatrix, so that $|R'|$ is the Vandermonde determinant $V(x_1, x_2, x_3, x_4)$, which is nonzero because the x_i are distinct. Hence, as discussed above, the two local equations of $\mathscr{M}_C$ at p are

$$M_5 := \begin{vmatrix} x_1^3 & x_1^2 & x_1 & 1 & z_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_4^3 & x_4^2 & x_4 & 1 & z_4 \\ x_6^3 & x_6^2 & x_6 & 1 & z_6 \end{vmatrix} = 0 \text{ and } M_6 := \begin{vmatrix} x_1^3 & x_1^2 & x_1 & 1 & z_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_4^3 & x_4^2 & x_4 & 1 & z_4 \\ x_5^3 & x_5^2 & x_5 & 1 & z_5 \end{vmatrix} = 0.$$

These satisfy

$$\frac{\partial M_5}{\partial z_5} = \frac{\partial M_6}{\partial z_6} = 0, \qquad \frac{\partial M_5}{\partial z_6} = \frac{\partial M_6}{\partial z_5} = |R'| \neq 0,$$

so the Jacobian matrix of (M_5, M_6) at p has rank 2. This shows that $\mathscr{M}_C$ is nonsingular at p. Since $\mathscr{M}_C$ is R_0 and S_1, it is reduced.

The same argument as in Lemma 4.3 (the computation of the local degrees) shows that the ramification scheme $R_2 \cap \mathscr{M}_C$ of ψ is reduced, so the branch scheme $B := \psi(R_2 \cap \mathscr{M}_C)$ is reduced (see [50, Lemma 29.6.7]). Locally over $\mathbb{P}^4$, the variety $\mathscr{M}_C$ is given by a polynomial equation of the form

$$P(x, y_1, \ldots, y_4) = x^{15} + \sum_{i=0}^{14} a_i(y_1, \ldots, y_4) x^i = 0,$$

where a_i are regular functions on affine charts of $\mathbb{P}^4$ with coordinates $(y_1, \ldots, y_4)$. The branch scheme B is the vanishing locus of the discriminant $D(y_1, \ldots, y_4)$ of the polynomial P. Since B is reduced, its singularities are given by the equations

$$D(y_1, \ldots, y_4) = 0, \quad \frac{\partial D}{\partial y_i}(y_1, \ldots, y_4) = 0, \quad \forall i = 1, \ldots, 4.$$

A point with local coordinates $(x, y_1, \ldots, y_4)$ is a singular point of $\mathscr{M}_C$ if

$$P(x, y_1, \ldots, y_4) = 0, \quad \frac{\partial P}{\partial x}(x, y_1, \ldots, y_4) = 0$$
$$\text{and } \frac{\partial P}{\partial y_i}(x, y_1, \ldots, y_4) = 0, \quad \forall i = 1, \ldots, 4.$$

An explicit computation shows that these conditions imply that $(y_1, \ldots, y_4)$ is a singular point of B, see [49, Theorem 4.2]. This means that $\mathscr{M}_C$ is regular in codimension one (condition R_1). Since it is S_2, it is normal by Serre's criterion [29, Proposition II.8.23].

Since $\mathscr{M}_C$ is a local intersection scheme, it admits a Koszul complex providing a locally free finite resolution $K_\bullet \to \mathcal{O}_{\mathscr{M}_C}$ of its structure sheaf considered as a $\mathcal{O}_{C^6}$-module. A standard argument (see for instance [2]) produces a spectral sequence

$$E_1^{p,q} = \mathrm{H}^q(C^6, K_p) \Rightarrow H^{p+q}(\mathscr{M}_C, \mathcal{O}_{\mathscr{M}_C}),$$

from which we deduce that $\mathrm{H}^0(\mathscr{M}_C, \mathcal{O}_{\mathscr{M}_C})$ is a quotient of $\mathrm{H}^0(C^6, \mathcal{O}_{C^6})$, which is 1-dimensional since C is connected. It follows that $\mathscr{M}_C$ is connected. Since $\mathscr{M}_C$ is normal and connected, using Zariski's Main Theorem we deduce that it is irreducible. Finally, $\mathscr{M}_C/\mathfrak{S}_6 \cong \mathbb{P}^4$. As $\mathscr{M}_C$ is a local complete intersection scheme, it is Cohen–Macaulay and Gorenstein [17, Corollary 21.19]. This proves assertion (1).

The variety $\mathscr{G}_C$ is normal by Proposition 4.4 and it follows from Lemma 4.2 and Diagram (23) that the morphism $\hat{\psi}\colon \mathscr{M}_C \to \mathscr{G}_C$ is the Galois closure of ψ, that is, $\mathscr{G}_C = \mathscr{M}_C/H$; this proves assertion (2). The morphism $\Psi\colon \mathscr{M}_C \to \mathbb{P}^4$ is a finite, hence flat, morphism from a normal local complete intersection variety to a regular variety, by "magic flatness" (see [17, 18.17]). It follows that Ψ is also a local complete intersection morphism, hence syntomic (see [50, Lemma 37.62.8 and Lemma 37.62.12]); this proves assertion (3). □

4.4. **The rational contraction, revisited.** We know from Propositions 2.3 and 2.4 that the birational map $\gamma_C\colon \mathscr{G}_C \dashrightarrow \mathrm{Kum}^2(\mathrm{Jac}(C))$ contracts the divisor F_1 defined in Equation (4) to the noncurvilinear point $3\mathcal{O}_C$, and contracts the divisor F_2 defined in Equation (5) to the Kummer surface $\mathrm{Kum}^1(\mathrm{Jac}(C))$ naturally embedded in $\mathrm{Kum}^2(\mathrm{Jac}(C))$. In this setup, these two divisors have very nice and concrete descriptions since they parametrise the possible configurations of triples of pairs of points that are interpolated by cubics consisting of three "vertical" lines (see Figure 2).

(1) *The comb.* Recall that the exceptional divisor $E_C \subset \mathrm{Sym}^2(C)$ consists of the 0-cycles of the form $p + \sigma(p)$ for $p \in C$, so

$$F_1 = \{[p_1 + \sigma(p_1)) + [p_2 + \sigma(p_2)] + [p_3 + \sigma(p_3)] \mid p_1, p_2, p_3 \in C\}.$$

(2) *The cross.* Similarly

$$F_2 = \{[p_1 + \sigma(p_2)] + [p_2 + \sigma(p_1)] + [p_3 + \sigma(p_3))] \mid p_1, p_2, p_3 \in C\}.$$

Given $p_1, p_2 \in C$, consider the curve:

$$\ell_{p_1,p_2} \coloneqq \{[p_1 + \sigma(p_2)] + [p_2 + \sigma(p_1)] + [p_3 + \sigma(p_3)] \mid p_3 \in C\}.$$

Clearly $\ell_{p_1,p_2} \cong E_C$ is a rational curve and the divisor F_2 is ruled by these rational curves.

Consider the birational inverse $\gamma_C^{-1}\colon \mathrm{Kum}^2(\mathrm{Jac}(C)) \dashrightarrow \mathscr{G}_C$. A general point of the exceptional divisor Ξ of the Hilbert–Chow morphism h_C° is a nonreduced subscheme ξ of length 3, consisting of a length 2 subscheme supported at $L_1 \in \mathrm{Jac}(C)$ and a reduced point $L_2 \in \mathrm{Jac}(C)$. Taking them general enough, these points are represented in reduced forms as $L_1 = \mathcal{O}_C(p_1 + p_2 - 2\infty)$ and $L_2 = \mathcal{O}_C(p_3 + p_4 - 2\infty)$. The condition of defining a point in the generalised Kummer fourfold of $\mathrm{Jac}(C)$ is that $L_1^{\otimes 2} \otimes L_2 \cong \mathcal{O}_C$: that is, $2p_1 + 2p_2 + p_3 + p_4 \sim 6\infty$. This means that the cubic interpolation at these four points is bitangent to C.

This shows that the rational map γ_C^{-1} sends the general element of Ξ to the big diagonal Δ of $\mathrm{Sym}_0^3(\mathrm{Sym}^2(C))$, consisting of nonreduced 0-cycles: in the case above

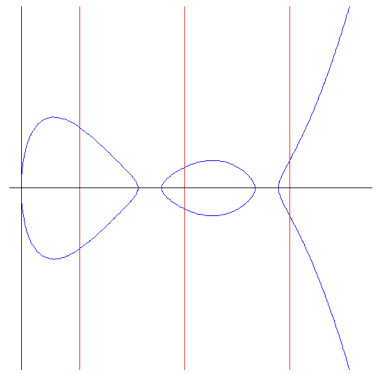


FIGURE 2. A genus 2 curve (in blue) and a cubic interpolation (in red) intersecting consisting in three vertical lines intersecting in 6 points on the affine chart $y = 1$ of coordinates x (abscissa) and z (ordinate).

ξ is mapped to $2[p_1 + p_2] + [p_3 + p_4]$, and so this is the limit point when a general nonreduced subcheme, defining a cubic interpolation whose image under π consists six different points, goes to ξ (the point ξ is not sent, say to $[p_1+p_3]+[p_1+p_4]+2p_2$).

The conclusion of this analysis is that γ_C^{-1} contracts the exceptional divisor Ξ to the codimension two locus Δ in $\mathscr{G}_C$.

4.5. **The Galois closure, revisited.** Consider the projection to the first four factors $\mathrm{pr}_1 \colon C^6 = C^4 \times C^2 \to C^4$ and its restriction to $\mathscr{M}_C$, still denoted $\mathrm{pr}_1 \colon \mathscr{M}_C \to C^4$. The quotient $C^4 \times C^2 \to C^4 \times \mathrm{Sym}^2(C)$, restricted to $\mathscr{M}_C$, induces a double covering $\mathscr{M}_C \to \overline{\mathscr{M}}_C$, which is the Stein factorisation of the projection map

$$pr_1 \colon \mathscr{M}_C \xrightarrow{2:1} \overline{\mathscr{M}_C} \to C^4.$$

To see this, observe that any four points $(p_1, \ldots, p_4) \in C^4$ admit a cubic interpolation, as we observed in Section 3.4, so there exist p_5, p_6 such that $p_1+\cdots+p_6 \sim 6\infty$. The projection $\mathrm{pr}_1 \colon \mathscr{M}_C \to C^4$ is thus surjective. If $p_1 + \cdots + p_4 \not\sim 4\infty$, then by Lemma 3.3 these four points are interpolated by a unique cubic so the remaining intersection points p_5, p_6 with C are uniquely determined. This shows that $(p_1, \ldots, p_4) \in C^4$ has a unique preimage in $\overline{\mathscr{M}_C}$. If $p_1+\cdots+p_4 \sim 4\infty$, by Lemma 3.3 these four points admit a conic interpolation and there is a pencil of cubic interpolations obtained by adding a line joining a point $p \in C$ to the point $\sigma(p)$. The fibre of $(p_1, \ldots, p_4) \in C^4$ in $\overline{\mathscr{M}}_C$ is thus $C/\langle\sigma\rangle = \mathbb{P}^1$.

Consider the composition of the Chow quotient with the Abel–Jacobi map:

$$\mathrm{AJ}_4 \circ \Phi_4 \colon C^4 \to \mathrm{Sym}^4(C) \to \mathrm{Jac}(C)$$

and define the closed subscheme $W \subset C^4$ parametrising quatruples of points that admit a conic interpolation:

$$W \coloneqq (\mathrm{AJ}_4 \circ \Phi_4)^{-1} (\{\mathcal{O}_C\}) = \left\{p \in C^4 \mid \Phi_4(p) \sim 4\infty\right\}.$$

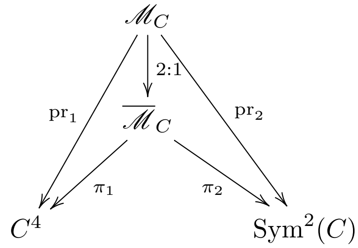
The locus W is exactly where the fibre of the morphism $\overline{\mathscr{M}}_C \to C^4$ is $\mathbb{P}^1$. In fact, it is is simply the blowup of W.

Proposition 4.6. *There is an isomorphism* $\overline{\mathscr{M}}_C \cong \operatorname{Bl}_W C^4$.

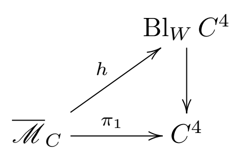
Proof. A similar argument as in §3.3 and Proposition 3.2 shows that

$$W = \{p \in C^4 \mid \operatorname{rk}\operatorname{res}_{\Phi_4(\xi)} \leq 2\}.$$

As in the proof of Proposition 4.5, it follows that W is locally given by two equations in C^4. Consider the two projections from $\overline{\mathscr{M}}_C$:

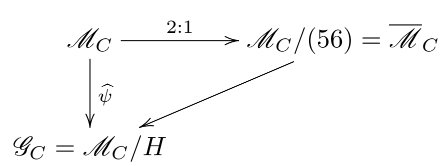


Recall from (15) that $\operatorname{Sym}^2(C)$ contains the exceptional divisor $E_C = \{p + \sigma(p) \mid p \in C\}$. The inverse ideal sheaf $\pi_1^{-1}\mathcal{I}_W \cdot \mathcal{O}_{\overline{\mathscr{M}}_C}$ defines the locus of those points $(p_1, \ldots, p_4, p_5{+}p_6) \in C^4 \times \operatorname{Sym}^2(C)$ such that $\{p_1, \ldots, p_4\}$ admits a conic interpolation and $\{p_1, \ldots, p_6\}$ admits a cubic one. As observed above, the only possibility is that $p_6 = \sigma(p_5)$: that is, $p_5 + p_6 \in E_C$. This means that $\pi_1^{-1}\mathcal{I}_W \cdot \mathcal{O}_{\overline{\mathscr{M}}_C} \cong \pi_2^*\mathcal{I}_{E_C} \cong \pi_2^*\mathcal{O}_{\operatorname{Sym}^2(C)}(-E_C)$ is an invertible sheaf. By the universal property of blowup, there exists a morphism $h\colon \overline{\mathscr{M}}_C \to \operatorname{Bl}_W C^4$ factoring π_1 through the blowup morphism:



Concretely, every point $p = (p_1, \ldots, p_4, p_5{+}p_6) \in \overline{\mathscr{M}}_C$ such that $\pi_2(p) \in W$ encodes the equation of a vertical line $ux - vy = 0$ intersecting C at p_5 and p_6. The morphism h locally maps p to $(\pi_2(p), (u : v)) \in C^4 \times \mathbb{P}^1$. The inverse morphism is clear. The morphism h is thus birational and bijective, and $\overline{\mathscr{M}}_C$ is normal since $\mathscr{M}_C$ is normal by Proposition 4.5, so by Zariski's Main Theorem, h is an isomorphism. □

Remark 4.2. The quotient $\mathscr{M}_C \to \overline{\mathscr{M}}_C$ is the quotient by the involution $(56) \in H$, so the quotient morphism $\widehat{\psi}\colon \mathscr{M}_C \to \mathscr{G}_C$ factorises as:



The morphism $\overline{\mathscr{M}}_C \to \mathscr{G}_C$ is not a Galois covering since $\langle(56)\rangle$ is not normal in H.

Remark 4.3. The morphism $W \to \mathbb{P}^2 = |\mathcal{C}|$ that sends four points admitting a conic interpolation to the equation of this conic is $24 : 1$ and it is ramified when the conic is tangent to C at one of its Weierstrass points. At each such point w, the conic interpolation is given by the tangent line to C at w and a second vertical line intersecting C at some points q and $\sigma(q)$. The branch locus in $\mathbb{P}^2$ is thus a sextic defined by six rational curves, each of them corresponding to the tangent line to C at a Weierstrass point.

5. The branch locus of the covering of the linear system of cubics

5.1. Computation of the branch locus. We focus on the $15:1$ covering map $\psi\colon \mathscr{G}_C \to |\mathcal{L}|$. By the Zariski–Nagata purity theorem, the branch scheme $B \coloneqq \Delta_6 \cap |\mathcal{L}|$ is a divisor in $|\mathcal{L}| \cong \mathbb{P}^4$. It parametrises those cubics that intersect the curve C with at least one multiple point. Said differently, identifying $|\mathcal{L}|$ with $\mathbb{P}H^0(C, \mathcal{L})$ we have

$$B = \{[s] \in |\mathcal{L}| \mid \mathscr{Z}(s) \text{ is nonreduced}\},$$

where $\mathscr{Z}(s) \subset C$ is the zero scheme of the section s of $\mathcal{L}$.

Proposition 5.1. *The linear system of cubics embeds C in $(\mathbb{P}^4)^\vee$ and the branch locus $B \subset \mathbb{P}^4$ of ψ is the dual variety of C, which is a reduced and irreducible hypersurface of degree* 14.

Proof. Since the line bundle $\mathcal{L}$ on C is very ample, it defines an embedding $C \hookrightarrow |\mathcal{L}|^\vee \cong (\mathbb{P}^4)^\vee$ in the dual projective space. The *conormal variety* [34] of C for this embedding is

$$V_C \coloneqq \{(p, D) \mid T_pC \subset D\} \subset |\mathcal{L}|^\vee \times |\mathcal{L}|.$$

The hyperplanes D in the projective space $|\mathcal{L}|$ are the cubics in $\mathbb{P}_{113}$ and it is easy to check that the condition $T_pC \subset D$ means that the cubic defined by D is tangent to the curve C at the point p, so in the definition of the branch locus B, the cubics that intersect the curve C with at least one multiple point correspond here to the hyperplane sections of $|\mathcal{L}|^\vee$ that are tangent to the embedding of C in $|\mathcal{L}|^\vee$. By definition, the *dual variety* C^* of C is the projection of V_C to $|\mathcal{L}|$, so with respect to this embedding the branch locus B is the *dual variety* $C^* \subset \mathbb{P}^4$ of C. It is thus reduced (we already observed this in the proof of Proposition 4.5). Its irreducibility is proved in [51, p.7]. Its degree can be computed using the general formula [51, Theorem 6.2(i)], which reduces in our case to

$$\deg B = \int_C c_1(T_C^\vee) + 2\int_C \mathcal{L}.$$

Here $T_C^\vee = K_C$ is a divisor of degree 2, and $\int_C \mathcal{L} = 6$, so $\deg B = 14$ (see also [51, Example 10.3]). □

Remark 5.1. It may be instructive to compute the degree of the branch divisor B with elementary tools. To do so, we compute the number of intersection points of B with the pencil of cubics $D_{[a:b]}$ with equation $ax^3 - bz = 0$, which is the number of points $[a:b] \in \mathbb{P}^1$ such that $D_{[a:b]}$ intersects the curve C with at least one multiple point. Clearly $[a:b] = [1,0]$ is a solution that counts with multiplicity 4 since the cubic $x^3 = 0$ cuts C at two points of multiplicity 3 (so each point counts twice). For the other solutions we may put $b = 1$. We may also restrict to the chart $y = 1$, avoiding the point $\infty = [1:0:0] \in C$, because $\infty \in D_{[0:1]}$ and $D_{[0:1]} \cap C$ consists of the six Weierstrass points, which are distinct.

Substituting $y = 1$ and $z = ax^3$ into the equation (13) of C, we find that the points $[x : 1 : ax^3]$ of $C \cap D_{[a:1]}$ satisfy an equation of the form $P(x) = 0$, where

$$P := a^2x^6 - x^5 + \varepsilon_1 x^4 + \cdots + \varepsilon_4 x + \varepsilon_5.$$

We need the number of values of a such that this polynomial has repeated roots. This is given by the degree of the discriminant of P as a polynomial in a. The resultant of P and P' is the following 11×11 determinant:

$$\begin{vmatrix}
\varepsilon_5 & \varepsilon_4 & \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & -1 & a^2 & & & & \\
 & \ddots & & & & & & \ddots & & & \\
 & & \ddots & & & & & & \ddots & & \\
 & & & \ddots & & & & & & \ddots & \\
 & & & & \varepsilon_5 & \varepsilon_4 & \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & -1 & a^2 \\
\varepsilon_4 & 2\varepsilon_3 & 3\varepsilon_2 & 4\varepsilon_1 & -5 & 6a^2 & & & & & \\
 & \ddots & & & & & \ddots & & & & \\
 & & \ddots & & & & & \ddots & & & \\
 & & & \ddots & & & & & \ddots & & \\
 & & & & \ddots & & & & & \ddots & \\
 & & & & & \varepsilon_4 & 2\varepsilon_3 & 3\varepsilon_2 & 4\varepsilon_1 & -5 & 6a^2
\end{vmatrix}$$

This determinant has degree 12 in a, so the discriminant of P has degree 10. In total we have 14 intersection points, so the hypersurface B has degree 14.

Corollary 5.2. *Let C be a smooth genus two curve. The linear system of cubics embeds C in $(\mathbb{P}^4)^\vee$ and the dual variety $C^* \subset \mathbb{P}^4$ of C is a degree 14 irreducible hypersurface. The second generalised Kummer variety $\mathrm{Kum}^2(\mathrm{Jac}(C))$ of the Jacobian of C is birational to a degree 15 covering of $\mathbb{P}^4$ branched along C^*.*

This corollary is simply a summary of Propositions 2.1, 4.4 and 5.1.

Proposition 5.3. *Let $z^2 = f(x)$ be the equation of the curve C in the chart $y = 1$ of $\mathbb{P}_{113}$, as in Equation (13). The branch locus B in $\mathbb{P}^4$ of coordinates $(\alpha_0 : \cdots : \alpha_4)$ has equation:*

$$\frac{1}{\alpha_4^6}\,\mathrm{Discr}_x\left(\alpha_4^2 f(x) - \left(\alpha_0 x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3\right)^2\right).$$

Proof. We compute on the affine chart $y = 1$ of $\mathbb{P}_{113}$. The curve C has equation $z^2 = f(x)$ and we consider a cubic D with equation as in (16):

$$g(x, z) = \alpha_0 x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3 + \alpha_4 z.$$

Let $p = [a : 1 : b] \in \mathbb{P}_{113}$. The intersection multiplicity of C and D at p is by definition the dimension as a complex vector space of the localisation of the quotient $\mathbb{C}[x, z]/\langle z^2 - f(x), g(x, z)\rangle$ at the maximal ideal of the point p. It is well known that if $\alpha_4 \neq 0$, this number is the order of vanishing at $x = a$ of the resultant $R(x) := \mathrm{Res}_z(z^2 - f(x), g(x, z))$. In fact $R(x) = \alpha_4^2 f(x) - \left(\alpha_0 x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3\right)^2$. We deduce that D is tangent to C when R has a multiple root, so the branch locus B is an irreducible component of the locus of vanishing of the discriminant $\mathrm{Discr}_x(R)$. Computation shows that $\mathrm{Discr}_x(R)$ has a factor of α_4^6. When $\alpha_4 = 0$, the cubic D consists of three vertical lines, and R has three double roots that correspond to tangencies between C and D only when the lines pass through one of the Weierstrass points. So the factor α_4^6 is irrelevant for the branch locus B, hence the result. □

Remark 5.2. An explicit computation of the equation of B is given in Remark C.3. We proved that the branch locus B is an irreducible component of the locus in $\mathbb{P}^4$ where the polynomial $R(x) = \alpha_4^2 f(x) - \left(\alpha_0 x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3\right)^2$ has a multiple root in the variable x. Here R is not monic but we may simply consider $\bar{R}(x) = \alpha_4^2 f(x) - \left(x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3\right)^2$, compute the discriminant of R, divide by α_4^6 and homogeneise with respect to the variable α_0 to recovering the branch locus B. Following Arnold (see [45] and references therein), the branch locus is stratified in closed subschemes B^λ, where λ is a partition of the integer 6. The main strata are the caustic stratum $B^{3,1,1,1}$ and the Maxwell stratum $B^{2,2,1,1}$.

Recalling the divisors R_1, R_2 defined in Equations (21) and (22), that describe the ramification of the morphism ϕ, let us introduce the following divisors on $\mathscr{G}_C$:

$$R_i^0 := R_i \cap \mathscr{G}_C, \quad i = 1, 2. \tag{24}$$

We denote by H the pullback $H := \psi^* L$, where $L \subset \mathbb{P}^4$ is a hyperplane. Since ψ is finite and L is (very) ample, the divisor H is ample. From the properties of the covering $\psi\colon \mathscr{G}_C \to \mathbb{P}^4$, we deduce some useful geometric information on $\mathscr{G}_C$:

Corollary 5.4.

1. *The divisor R_2° is very ample.*
2. $14H = R_1^\circ + 2R_2^\circ$.
3. *The canonical divisor is $K_{\mathscr{G}_C} = -5H + R_2^\circ$.*

Proof. By Proposition 4.4, the morphism ψ is Gorenstein so we can apply the general theory of [11, Theorem 2.1(ii)]: the Tschirnhausen bundle $\mathcal{E}^\vee$ of ψ gives an embedding $j\colon \mathscr{G}_C \hookrightarrow \mathbb{P}(\mathcal{E})$ such that the ramification divisor R_2° of ψ satisfies

$$\mathcal{O}_{\mathscr{G}_C}(R_2^\circ) \cong \omega_{\mathscr{G}_C/\mathbb{P}^4} \cong \mathcal{O}_{\mathscr{G}_C}(1) := j^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

It follows that R_2° is very ample. By Lemma 4.3, we have $\psi^* B = R_1^\circ + 2R_2^\circ$ and by Proposition 5.1 we have $B = 14L$, so $14H = R_1^\circ + 2R_2^\circ$. By Lemma 4.3 again, ψ has simple ramification along R_2°, so $K_{\mathscr{G}_C} = \psi^* K_{\mathbb{P}^4} + R_2^\circ = -5H + R_2^\circ$. □

Remark 5.3. The covering $\psi\colon \mathscr{G}_C \to \mathbb{P}^4$ satisfies some of the assumptions defining a *general* multiple space in [18, Definition 2.2]. It would thus be interesting to know whether R_2° is nonsingular and whether the restriction $\psi_{|R_2^\circ}\colon R_2^\circ \to B$ is the normalisation map. This covering is not Galois and has relatively high degree, making it hard to understand its possible deformations. Moreover, we noted in Remark 2.1 that $\mathscr{G}_C$ is not a local complete intersection, so the relative cotangent complex of ψ is not perfect, making it harder to compute the deformations of the covering ψ. Our interest in proving Proposition 4.5 is that the Galois closure $\Psi\colon \mathscr{M} \to \mathbb{P}^4$ of the covering has perfect cotangent complex, so the study of its deformations should behave more nicely: this is work in progress.

Appendix A. Moduli spaces of polarised IHS manifolds

For a more detailed survey, see [14].

A.1. **The moduli spaces $\mathcal{M}^d_{\mathrm{K3}}$:** they parametrise degree d polarised K3 surfaces, and their dimension is 19. One easy example is the projective family of smooth quartic surfaces in $\mathbb{P}^3$, which is also 19-dimensional. Since the moduli space is irreducible, this shows that $\mathcal{M}^4_{\mathrm{K3}}$ is unirational. The known results can be summarised as follows: the moduli spaces $\mathcal{M}^{2e}_{\mathrm{K3}}$ are unirational for $1 \leq e \leq 12$ and $e = 15, 16, 17, 19$ (see [27, §4] and references therein). Unirationality properties can be also obtained for moduli of lattice-polarised K3 surfaces, see for instance [48, Proposition 3.9]. In the other direction, it was shown in [24]I don't find it. Sure that it is $e \geq 40$, not $2e \geq 20$? that $\mathcal{M}^{2e}_{\mathrm{K3}}$ has non-negative Kodaira dimension if $e \geq 40$, with four possible exceptions, and is of general type for $e \geq 62$ and a few smaller numbers.

A.2. **The moduli spaces $\mathcal{M}^{d,\gamma}_{\mathrm{Hilb}^n}$:** they parametrise polarised IHS manifolds of Hilbert type, *i.e.* deformation equivalent to the Hilbert scheme of n points on a K3 surface, with $n \geq 2$, of degree d and divisibility γ. Their dimension is 20, whereas the families of Hilbert schemes of n points on polarised $K3$ surfaces have dimension 19. Gritsenko, Hulek and Sankaran [25, Theorem 4.1] proved that $\mathcal{M}^{2e,1}_{\mathrm{Hilb}^2}$ is of general type if $e \geq 12$. Otherwise:

- $\mathcal{M}^{2,2}_{\mathrm{Hilb}^2}$ is unirational since it contains a 20-dimensional family using double coverings of EPW-sextics (see O'Grady [46] and [27, Example 4.3]).
- $\mathcal{M}^{6,1}_{\mathrm{Hilb}^2}$ is unirational since it contains a 20-dimensional family using Fano varieties of lines on cubic fourfolds (see Beauville and Donagi [6] and [27, Example 4.2]).
- $\mathcal{M}^{38,2}_{\mathrm{Hilb}^2}$ is unirational: Iliev and Ranestad [32] constructed a 20-dimensional family (see [27, Example 4.4] and [40, Proposition 1.4.16]).
- $\mathcal{M}^{22,2}_{\mathrm{Hilb}^2}$ is unirational: Debarre and Voisin [16] constructed a 20-dimensional family (see [27, Example 4.5]).
- $\mathcal{M}^{4,2}_{\mathrm{Hilb}^3}$ is unirational: Iliev, G. Kapustka, M. Kapustka and Ranestad [31] constructed a 20-dimensional family called *EPW cubes*.

A.3. **The moduli spaces $\mathcal{M}^{d,\gamma}_{\mathrm{Kum}^n}$:** they parametrise polarised IHS manifolds of Kummer type, *i.e.* deformation equivalent to the n-th generalised Kummer variety of an abelian surface, with $n \geq 2$, of degree d and divisibility γ. Their dimension is 4, whereas the families of polarised abelian surfaces have dimension 3. By Dawes [13, Theorem 3.6] we know that $\mathcal{M}^{2d,1}_{\mathrm{Kum}^2}$ is of general type if $d \gg 0$. See also [12]. Otherwise:

- $\mathcal{M}^{2,1}_{\mathrm{Kum}^n}$ is uniruled if $n \geq 15$ or $n = 17, 20$ (see [3, Theorem 7.5]).
- $\mathcal{M}^{2,2}_{\mathrm{Kum}^n}$ is uniruled if $n = 4t - 2$ with $t \leq 11$ or $t = 13, 15, 17, 19$ (see [3, Theorem 7.5]).
- $\mathcal{M}^{2,2}_{\mathrm{Kum}^2}$ is rational (see [52, Theorem 5.4] and [3, Theorem 7.6]).

A.4. **The moduli spaces $\mathcal{M}^{d,\gamma}_{\mathrm{OG6}}$:** these 6-dimensional spaces parametrise polarised IHS manifolds of type OG6, *i.e.* deformation equivalent to an O'Grady IHS sixfold, of degree d and divisibility γ. Following [3, Theorem 7.2] we have:

- $\mathcal{M}^{2d,1}_{\mathrm{OG6}}$ is uniruled if $d \leq 12$;
- $\mathcal{M}^{4t-1,2}_{\mathrm{OG6}}$ is uniruled if $t \leq 10$ or $t = 12$;
- $\mathcal{M}^{4t-2,2}_{\mathrm{OG6}}$ is uniruled if $t \leq 9$ or $t = 11, 13$.

- The moduli space $\mathcal{M}^{6,2}_{\mathrm{OG6}}$ and $\mathcal{M}^{2,1}_{\mathrm{OG6}}$ are rational (see [52, Theorem 5.4] and [3, Theorem 7.6]).

A.5. **The moduli spaces $\mathcal{M}^{d,\gamma}_{\mathrm{OG10}}$:** these 21-dimensional spaces parametrise polarised IHS manifolds of type OG10, *i.e.* deformation equivalent to an O'Grady IHS tenfold, of degree d and divisibility γ. As far as the authors know, no unirationality result is known for these spaces. It is conjectured in [26] that $\mathcal{M}^{24,3}_{\mathrm{OG10}}$, $\mathcal{M}^{60,3}_{\mathrm{OG10}}$ and $\mathcal{M}^{96,3}_{\mathrm{OG10}}$ are uniruled, and it is shown that $\mathcal{M}^{d,1}_{\mathrm{OG10}}$ is of general type unless d is a power of 2.

Appendix B. Alternative views on the proof of Proposition 2.3

B.1. **Saturation.** In this proof, we are interested in the locus inside $\widetilde{F}_1$ where the rational function $\tilde{a}_2$ can be extended, keeping the Cramer relation true. In the polynomial ring $\mathbb{C}[\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{z}_3, \tilde{a}_2]$, we consider the ideal I defining the Cramer relation satisfied by the rational function $\tilde{a}_2$ and the ideal J defining the divisor F_1. The locus inside F_1 where $\tilde{a}_2$ extends is the intersection with the Zariski closure $\overline{\mathcal{V}(I) \setminus \mathcal{V}(J)} \cap \mathcal{V}(J)$.

It is a classical result that this is the zero locus of the saturation $(I \colon J^\infty) + J$ of the ideal I with respect to J. A Gröbner basis computation (see Remark C.4) gives

$$(I \colon J^\infty) + J = \langle \boldsymbol{w}_1 \boldsymbol{w}_2^2 (\boldsymbol{w}_1 - \boldsymbol{w}_2) \rangle.$$

We recover the locus G defined in (11).

B.2. **Computations in the chart $\mathcal{U}_{(2,1)}$.** Following the notation of the proof, in the chart $\mathcal{U}_{(2,1)}$ the coordinate functions are $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2$ with the relations given in the proof. To compute them in terms of a triple of points $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2), (\boldsymbol{x}_3, \boldsymbol{y}_3)$ we use the generators of the ideal $I_{(a,b,c)}$. The generator $\boldsymbol{x}^2 - a_0 - a_1 \boldsymbol{x} - a_2 \boldsymbol{y}$ means that:

$$\begin{pmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{pmatrix},$$

so the coordinates a_i are given by Cramer's rule:

$$a_0 = \frac{\begin{vmatrix} \boldsymbol{x}_1^2 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ \boldsymbol{x}_2^2 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ \boldsymbol{x}_3^2 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{vmatrix}}{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{vmatrix}}, \quad a_1 = \frac{\begin{vmatrix} 1 & \boldsymbol{x}_1^2 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2^2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3^2 & \boldsymbol{y}_3 \end{vmatrix}}{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{vmatrix}}, \quad a_2 = \frac{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{x}_1^2 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{x}_2^2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{x}_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{vmatrix}},$$

and similarly

$$b_0 = \frac{\begin{vmatrix} \boldsymbol{x}_1\boldsymbol{y}_1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ \boldsymbol{x}_2\boldsymbol{y}_2 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ \boldsymbol{x}_3\boldsymbol{y}_3 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{vmatrix}}{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{vmatrix}}, \quad b_1 = \frac{\begin{vmatrix} 1 & \boldsymbol{x}_1\boldsymbol{y}_1 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2\boldsymbol{y}_2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3\boldsymbol{y}_3 & \boldsymbol{y}_3 \end{vmatrix}}{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{vmatrix}}, \quad b_2 = \frac{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{x}_1\boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{x}_2\boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{x}_3\boldsymbol{y}_3 \end{vmatrix}}{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{vmatrix}},$$

and

$$c_0 = \frac{\begin{vmatrix} \boldsymbol{y}_1^2 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ \boldsymbol{y}_2^2 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ \boldsymbol{y}_3^2 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{vmatrix}}{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{vmatrix}}, \quad c_1 = \frac{\begin{vmatrix} 1 & \boldsymbol{y}_1^2 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{y}_2^2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{y}_3^2 & \boldsymbol{y}_3 \end{vmatrix}}{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1^2 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2^2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{vmatrix}}.$$

The relations three between these nine coordinates, given in the proof, are easy to check. Substituting as in the proof, we get the following formulas:

$$\begin{aligned} \tilde{a}_1 &= \frac{x_1 A_1(w_1, w_2, z_3)}{w_1 x_2^2 (w_1 - w_2)}, & \tilde{a}_2 &= \frac{x_1 A_2(w_1, w_2, z_3)}{w_1 x_2^2 (w_1 - w_2)}, \\ \tilde{b}_1 &= \frac{x_1 B_1(w_1, w_2, z_3)}{w_1 x_2^2 (w_1 - w_2)}, & \tilde{b}_2 &= \frac{x_1 B_2(w_1, w_2, z_3)}{w_1 x_2^2 (w_1 - w_2)}, \\ \tilde{c}_1 &= \frac{x_1 C_1(w_1, w_2, z_3)}{w_1 x_2^2 (w_1 - w_2)}, & \tilde{c}_2 &= \frac{x_1 C_2(w_1, w_2, z_3)}{w_1 x_2^2 (w_1 - w_2)}, \end{aligned}$$

where A_i, B_i, C_i are polynomial expressions. This shows that all the coordinate functions vanish at $x_1 = 0$ when $w_1 x_2^2 (w_1 - w_2) \neq 0$, so the generic point of the divisor F_1 is sent to the ideal $I_\infty = \langle \boldsymbol{x}^2, \boldsymbol{xy}, \boldsymbol{y}^2 \rangle$. This gives a different proof that γ_A contracts the divisor F_1 to the point Z_∞.

B.3. **The projective embedding.** We use an explicit projective embedding of $\mathrm{Hilb}^3(A)$ following the presentation given by Haiman [28] of the original and general construction due to Grothendieck. We first recall this construction. Let M be the set of monomials in the variables $\boldsymbol{x}, \boldsymbol{y}$ of degree at most 3. For any ideal $I \in \mathrm{Hilb}^3(A)$, by Gordan [21] the quotient $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]/I$ is generated by M (at this point, monomials of degree at most two would suffice, but we need degree three monomials for the projective embedding). Denote by $V \coloneqq \mathrm{Span}(M) \subset \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$ the vector subspace generated by M. For any $I \in \mathrm{Hilb}^3(A)$, the linear map

$$\pi_I \colon V \to \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]/I$$

is surjective, and its kernel $\ker(\pi_I)$ has codimension three in V. Instead of working with a basis of this kernel, it is more convenient to work with its equations, so we consider its annihilator $\ker(\pi_I)^\perp \subset V^*$, which has dimension three. By a result of Grothendieck, we get an embedding in the Grassmannian of 3-dimensional subspaces of V^*:

$$\mathrm{Hilb}^3(A) \hookrightarrow \mathrm{Grass}(3, V^*), \quad I \mapsto \ker(\pi_I)^\perp.$$

The projective embedding of $\mathrm{Hilb}^3(A)$ inside which we will study the behaviour of the map g is the Plücker embedding (where we use here the projective space of lines in $\bigwedge^3 V^*$):

$$\wp \colon \mathrm{Hilb}^3(A) \hookrightarrow \mathbb{P}\left(\textstyle\bigwedge^3 V^*\right), \quad I \mapsto \textstyle\bigwedge^3 \left(\ker(\pi_I)^\perp\right).$$

Let $\mathcal{E} \subset \mathrm{Hilb}^3(A)$ be the exceptional divisor, parametrising non-reduced subschemes, and $\Delta \subset \mathrm{Sym}^3(A)$ the big diagonal. The Hilbert–Chow morphism h_A restricts to an isomorphism between the open subsets $\mathrm{Hilb}^3(A) \setminus \mathcal{E}$ and $\mathrm{Sym}^3(A) \setminus \Delta$. The rational map from $\widetilde{A}^3$ to $\mathbb{P}\left(\bigwedge^3 V^*\right)$ is regular on the following open subsets:

$$\widetilde{A}^3 \setminus \Delta'' \xrightarrow{\pi_{\widetilde{A}}} \mathrm{Sym}^3(\widetilde{A}) \setminus \Delta' \xrightarrow{\mathrm{Sym}^3(\beta_A)} \mathrm{Sym}^3(A) \setminus \Delta \xrightarrow{h_A^{-1}} \mathrm{Hilb}^3(A) \setminus \mathcal{E} \xrightarrow{\wp} \mathbb{P}\left(\textstyle\bigwedge^3 V^*\right)$$

where $\Delta' := \left(\mathrm{Sym}^3(\beta_A)\right)^{-1}(\Delta)$ and $\Delta'' := (\pi_{\widetilde{A}})^{-1}(\Delta')$. After restricting to the fibres over the origin, we recover the restriction of the rational map $\wp \circ g$, and we see that it is regular on an open subset containing $(\widetilde{A}^3 \setminus \Delta'') \cap \widetilde{A}^3_0$, but this subset is not optimal since it contains divisors. To compute the image of the divisor F_1 contracted by γ_A, we compute for each partition λ of the integer 3, the morphism

$$\wp \circ g \colon g^{-1}\left(\mathrm{Kum}^2(A) \cap \mathcal{U}_\lambda \cap \left(\mathrm{Hilb}^3(A) \setminus \mathcal{E}\right)\right) \to \mathbb{P}\left(\textstyle\bigwedge^3 V^*\right).$$

Let us compute on the chart $\mathcal{U}_{(1,1,1)}$. The vector space V has basis

$$(1, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}^2, \boldsymbol{xy}, \boldsymbol{y}^2, \boldsymbol{x}^3, \boldsymbol{x}^2\boldsymbol{y}, \boldsymbol{xy}^2, \boldsymbol{y}^3).$$

For any ideal $I := I_{(e,a)} \in \mathcal{U}_{(1,1,1)}$ generated by:

$$I_{(e,a)} := \langle \boldsymbol{x}^3 - e_1\boldsymbol{x}^2 + e_2\boldsymbol{x} - e_3, \boldsymbol{y} - (a_0 + a_1\boldsymbol{x} + a_2\boldsymbol{x}^2)\rangle,$$

the quotient space $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]/I$ has basis $(1, \boldsymbol{x}, \boldsymbol{x}^2)$ modulo I. The morphism π_I defines a 3×10 matrix A whose coefficients depend on e and a. The kernel of this matrix is organised as a (7×10)-matrix B whose rows are the coordinates of the generators of $\ker(\pi_I)$. Interpreting duality as a canonical scalar product, the kernel of B is organised as a (3×10)-matrix whose rows are the coefficients of the equations of $\ker(\pi_I)^\circ$ in the dual basis of V^*. We restrict to $\mathrm{Kum}^2(A)$ by inserting the equations of $\mathrm{Kum}^2(A)$ in $\mathrm{Hilb}^3(A)$ in our chart, that is $e_1 = 0, a_0 = \frac{2}{3}a_2e_2$, and we arrive at the following matrix:

$$C = \frac{1}{27}\begin{pmatrix} 0 & 0 & 27a_2 & 27 & 27a_1 & 9a_2^2e_2 + 27a_1^2 & 0 \\ 0 & 27 & 27a_1 & 0 & -9a_2e_2 & -18a_1a_2e_2 + 27a_2^2e_3 & -27e_2 \\ 27 & 0 & 18a_2e_2 & 0 & 27a_2e_3 & 12a_2^2e_2^2 + 54a_1a_2e_3 & 27e_3 \end{pmatrix}\cdots$$

$$\cdots \begin{matrix} -9a_2e_2 & -18a_1a_2e_2 + 27a_2^2e_3 \\ -27a_1e_2 + 27a_2e_3 & 3a_2^2e_2^2 - 27a_1^2e_2 + 54a_1a_2e_3 \\ 27a_1e_3 & 9a_2^2e_2e_3 + 27a_1^2e_3 \end{matrix}\cdots$$

$$\cdots \left.\begin{matrix} 9a_2^3e_2^2 - 27a_1^2a_2e_2 + 81a_1a_2^2e_3 \\ 9a_1a_2^2e_2^2 - 27a_1^3e_2 + 81a_1^2a_2e_3 \\ 27a_1a_2^2e_2e_3 + 8a_2^3e_2^3 + 27a_1^3e_3 + 27a_2^3e_3^2 \end{matrix}\right)$$

We now use formulas (7) and (8) to express the 120 Plücker coordinates $p_{i,j,k}$ of I in $\mathbb{P}\left(\bigwedge^3 V^*\right)$ as rational functions of the variables $\boldsymbol{x}_1, \boldsymbol{w}_1, \boldsymbol{x}_2, \boldsymbol{w}_2, \boldsymbol{z}_3$. Those are all the (3×3) minors of the matrix C. After some simplifications, we see that $p_{1,2,3}$ is divisible by x_1^{26} and that all the others are divisible by x_1^{27}. We obtain the rational image of the divisor F_1 by putting $x_1 = 0$, which gives the point $[1 : 0 : \ldots : 0] \in \mathbb{P}^{120}$. A similar method shows that these are the Plücker coordinates of the point Z_∞. The computation on the chart $\mathcal{U}_{(2,1)}$ is similar.

Appendix C. The scripts used in this work

We reproduce below the scripts used in this paper. None of our proofs actually needed computer algebra tools: these only served as a guidance. We used **Macaulay2** [22] and **Magma** [10].

Remark C.1. Here is a **Macaulay2** script used in Remark 2.1 to check that the variety $\mathscr{G}_A$ is not a local complete intersection scheme:

```
loadPackage "InvariantRing"
R = QQ [x1, x2, x3, y1, y2, y3]
M12 = matrix{{0, 1, 0, 0, 0, 0},
```

```
 {1, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0},
 {0, 0, 0, 0, 1, 0},
 {0, 0, 0, 1, 0, 0},
 {0, 0, 0, 0, 0, 1}}
M123 = matrix{{0, 0, 1, 0, 0, 0},
 {1, 0, 0, 0, 0, 0},
 {0, 1, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 1},
 {0, 0, 0, 1, 0, 0},
 {0, 0, 0, 0, 1, 0}}
L = {M12, M123}
S3 = finiteAction(L,R)
g = invariants S3
netList g
A = QQ[x1, x2, x3, y1, y2, y3, f0, f1, f2, f3, f4, f5, f6, f7, f8]
inv = {substitute(g#0, A), substitute(g#1, A), substitute(g#2, A),
       substitute(g#3, A), substitute(g#4, A), substitute(g#5, A),
       substitute(g#6, A), substitute(g#7, A), substitute(g#8, A)}
I = ideal{f0 - inv#0, f1 - inv#1, f2 - inv#2, f3 - inv#3, f4 - inv#4,
           f5 - inv#5, f6 - inv#6, f7 - inv#7, f8 - inv#8}
loadPackage "Elimination"
J = eliminate({x1, x2, x3, y1, y2, y3}, I)
-- Computation of Sym^3_0 A : the equations are f0, f1
sym = J + ideal{f0, f1}
loadPackage "TorAlgebra"
isGorenstein sym
isCI sym
-- Computation of Sym^3_0 hat A : the equations are f1, f3
model = J + ideal{f1, f3}
isGorenstein model
isCI model
```

Remark C.2. Here is a **Magma** script to compute the group H defined in §4.2:

```
G := SymmetricGroup(6);
H := sub<G | [(1, 2), (1, 3)(2, 4), (1, 5)(2, 6)]>;
Order(H);
IsNormal(G, H);
```

Remark C.3. Here is a **Magma** script to compute the branch locus B in §5.1:

```
R <e1, e2, e3, e4, e5> := PolynomialRing(Rationals(), 5);
A <x, a0, a1, a2, a3, a4> := PolynomialRing(R, 6);
f := x ^ 5 - e1 * x ^ 4 - e2 * x ^ 3 - e3 * x ^ 2 - e4 * x - e5;
p := a0 * x ^ 3 + a1 * x ^ 2 + a2 * x + a3;
r :=  a4 ^2 * f - p ^ 2;
b := Discriminant(r, x) div a4^6;
```

```
Degree(b);
```

Remark C.4. Here is the **Magma** script used in §B.1:

```
A<a2,x1,x2,w1,w2,z3> := PolynomialRing(Rationals(), 6, "elim", 1);
x3 := - x1 - x2;
v1 := w1 + z3;
v2 := w2 + z3;
y1 := x1 * v1;
y2 := x2 * v2;
y3 := x3 * z3;
D := (x1-x2) * (x1 - x3) * (x2 - x3);
I := ideal<A | x1 * w1 + x2 * w2,
     D * a2 - (x2 - x3) * y1 - (x3 - x1) * y2 - (x1 - x2) * y3>;
J := ideal<A | x1, x2>;
C := Saturation(I, J) + J;
GroebnerBasis(C);
```

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