Abelianisation of orthogonal groups and the fundamental group of modular varieties

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Abstract

We study the commutator subgroup of integral orthogonal groups belonging to indefinite quadratic forms. We show that the index of this commutator is 2 for many groups that occur in the construction of moduli spaces in algebraic geometry, in particular the moduli of K3 surfaces. We give applications to modular forms and to computing the fundamental groups of some moduli spaces.

Many moduli spaces in algebraic geometry can be described via period domains as quotients of a symmetric space by a discrete group, or modular group. We shall be concerned with the case of the symmetric space $\mathcal{D}_L$ associated with a lattice L of signature $(2, n)$, and discrete subgroups of the orthogonal group $O(L)$ that act on $\mathcal{D}_L$. Such groups arise in the study of the moduli of K3 surfaces and of other irreducible symplectic manifolds (see [GHS1], [GHS3] and the references there), and of polarised abelian surfaces. Orthogonal groups of indefinite forms also appear elsewhere in geometry, for instance in the theory of singularities (see [Br], [Eb]). In this paper we study the commutator subgroups and abelianisations of orthogonal modular groups of this kind, especially for lattices of signature $(2, n)$.

Notation. For definitions and notation concerning locally symmetric varieties and toroidal compactification we refer to [GHS2].

We write $\langle X \rangle$ for the group generated by a subset X of some group. If n is an integer $\langle n \rangle$ means the rank-1 lattice generated by an element of square n.

For a group G, we write $[G, G]$ for the commutator subgroup (derived subgroup) of G (not G' because we want to keep the notation $O'(L)$ from [Kn2]) and we use G^{ab} for the abelianisation, i.e. the quotient $G/[G, G]$, which is also the group $\mathrm{Hom}(G, \mathbb{C}^\times)$ of characters of G.

The commutator subgroup and the abelianisation of a modular group carry important information about modular forms. For example the fact that $\mathrm{SL}_2(\mathbb{Z})^{\mathrm{ab}} \cong \mathbb{Z}/12\mathbb{Z}$ reflects the existence of the Dedekind η-function.

Its square $\eta(\tau)^2$ is a modular form with respect to $\mathrm{SL}_2(\mathbb{Z})$ with a character of order 12. However, the commutator subgroup of orthogonal modular groups is generally not known. We are aware of two previous studies.

In [GH1] two of us analysed the commutator of the *paramodular group* Γ_t (the integral symplectic group of a symplectic form with elementary divisors $(1, t)$), which is the modular group of the moduli space $\mathcal{A}_t$ of polarised abelian surfaces with a polarisation of type $(1, t)$. According to [GH1, Theorem 2.1]

$$\Gamma_t^{\mathrm{ab}} \cong (\mathbb{Z}/(t, 12)\mathbb{Z}) \times (\mathbb{Z}/(2t, 12)\mathbb{Z}). \tag{1}$$

We note that $\Gamma_1 = \mathrm{Sp}_4(\mathbb{Z})$. The projectivised paramodular group $\Gamma_t/\{\pm\mathbf{1}\}$ is isomorphic to the stable special orthogonal group $\widetilde{\mathrm{SO}}^+(\Lambda_{2t})$ (see (2) below), associated with the lattice $\Lambda_{2t} = 2U \oplus \langle -2t \rangle$ where $2U = U \oplus U$ and U is the integral hyperbolic plane (see (3) below). Therefore (1) is a result about orthogonal groups of signature $(2, 3)$.

S. Kondo ([Ko1, Main theorem]) considered the lattice L_{2d} of signature $(2, 19)$ associated with moduli of polarised K3 surfaces of degree $2d$. He proved that the abelianisation of the modular group $\widetilde{\mathrm{O}}^+(L_{2d})$ is an elementary abelian 2-group whose order divides 8. We show in Theorem 1.7 that this group is in fact of order 2. Moreover a similar result is true for a large class of orthogonal groups that appear in the theory of moduli spaces.

The paper is organised as follows. In Section 1 we make some basic definitions, state our main results and give some examples. We prove some of the results straight away as corollaries of a theorem of Kneser. Section 2 gives an application to the theory of modular forms, showing that in many cases of interest the order of vanishing of a modular form at a cusp is necessarily an integer. In Section 3 we describe the Eichler transvections, which are special unipotent elements of the orthogonal groups we are interested in, and the Jacobi group, and use them to obtain suitable generators for the modular groups. Section 4 is mainly devoted to the proof of Theorem 1.7 but also includes some remarks about the number field case. We conclude in Section 5 with some applications to fundamental groups of moduli spaces. In particular we show that the compactified moduli spaces of polarised K3 surfaces and of polarised abelian surfaces are simply-connected.

1 Commutator subgroups

In this section L is always an integral even lattice, i.e. a free $\mathbb{Z}$-module with a non-degenerate symmetric bilinear form $(\cdot, \cdot) : L \times L \to \mathbb{Z}$ such that

$(u, u) = u^2 \in 2\mathbb{Z}$ for any $u \in L$. The dual lattice

$$L^\vee = \{v \in L \otimes \mathbb{Q} \mid (v, l) \in \mathbb{Z} \ \forall l \in L\}$$

contains L. We denote the *discriminant group* of L by $D(L) = L^\vee/L$. It carries a quadratic form with values in $\mathbb{Q}/2\mathbb{Z}$. The *stable orthogonal group* $\widetilde{O}(L)$ is defined as the kernel

$$\widetilde{O}(L) = \ker(O(L) \to O(D(L)))$$

of the natural projection to the finite orthogonal group $O(D(L))$.

For an indefinite lattice there are two ways to choose the real spinor norm because $O(L, b) = O(L, -b)$ where b is the bilinear form on L. We note that the different spinor norms agree on $SO(L \otimes \mathbb{Q})$. For any field K different from $\mathbb{F}_2$, any $g \in O(L \otimes K)$ can be represented as the product of reflections $g = \sigma_{v_1} \sigma_{v_2} \ldots \sigma_{v_m}$, where $v_i \in L \otimes K$. We define the *spinor norm over K* as follows (see [KnS]):

$$\mathrm{sn}_K(g) = \left(-\frac{(v_1, v_1)}{2}\right) \cdot \ldots \cdot \left(-\frac{(v_m, v_m)}{2}\right)(K^\times)^2.$$

Thus $\mathrm{sn}_K \colon O(L \otimes K) \to K^\times/(K^\times)^2$ is a group homomorphism. We have made this choice $-(\ ,\ )$ in the definition of sn because it is convenient for the geometric applications when the lattices have signature $(2, n)$. In that case, reflection with respect to a vector with negative norm fixes the connected component of the homogeneous domains.

We define three subgroups of $O(L)$:

$$
\begin{aligned}
O^+(L) &= O(L) \cap \ker \mathrm{sn}_\mathbb{R}, \\
\widetilde{O}^+(L) &= \widetilde{O}(L) \cap O^+(L), \\
O'(L) &= SO(L) \cap \ker \mathrm{sn}_\mathbb{Q}.
\end{aligned}
\tag{2}
$$

$O'(L)$ is sometimes called the *spinorial kernel*. We also use the notation $SO^+(L) = O^+(L) \cap SO(L)$ and $\widetilde{SO}^+(L) = \widetilde{O}^+(L) \cap SO(L)$; but $O'(L)$ is already a subgroup of $SO(L)$.

If $a \in L$ and $a^2 = -2$ then a is called a *(−2)-vector* or *root*. The reflection

$$\sigma_a \colon v \longrightarrow v - \frac{2(a, v)}{(a, a)} a$$

determined by a belongs to $\widetilde{O}^+(L)$.

The *Witt index* of L over a field K is the maximal dimension of a totally isotropic subspace of $L \otimes K$. For any prime p the *p-rank* of L, denoted by $\mathrm{rank}_p(L)$, is the maximal rank of the sublattices M in L such that $\det(M)$ is coprime to p. By the *integral hyperbolic plane* we mean the lattice

$$U := \mathbb{Z}e \oplus \mathbb{Z}f \qquad \text{where} \ \ (e, e) = (f, f) = 0, \ (e, f) = 1. \tag{3}$$

The following result of Kneser is very important for us. It allows us to use reflections to generate certain orthogonal groups over the integers, not just over a field.

Theorem 1.1 *([Kn2, Satz 4]) Let L be an even integral lattice of Witt index ≥ 2 over $\mathbb{R}$. We assume that L represents -2 and that $\mathrm{rank}_3(L) \geq 5$ and $\mathrm{rank}_2(L) \geq 6$. Then $\mathrm{O}'(L)$ is generated by the products of reflections $\sigma_a \sigma_b$ where $a, b \in L$ and $a^2 = b^2 = -2$.*

We shall say that a lattice *satisfies the Kneser conditions* if it satisfies the conditions of Theorem 1.1.

Corollary 1.2 *If L satisfies the Kneser conditions, then*

$$\mathrm{O}'(L) = \widetilde{\mathrm{SO}}^+(L).$$

Proof. According to Theorem 1.1, $\mathrm{O}'(L)$ is a subgroup of $\widetilde{\mathrm{SO}}^+(L)$. But by [Kn2, Satz 2], the local orthogonal group $\widetilde{\mathrm{O}}(L \otimes \mathbb{Z}_p)$ is generated by reflections with respect to (-2)-vectors for every finite prime p. Therefore for any $g \in \widetilde{\mathrm{SO}}(L)$ we have $\mathrm{sn}_{\mathbb{Q}_v}(g) = 1 \in \mathbb{Q}_v^\times / (\mathbb{Q}_v^\times)^2$ for every p-adic valuation v on $\mathbb{Q}$. Therefore for any $g \in \widetilde{\mathrm{SO}}^+(L)$ we obtain $\mathrm{sn}_{\mathbb{Q}}(g) = \mathrm{sn}_{\mathbb{R}}(g) \cdot \prod_p \mathrm{sn}_{\mathbb{Q}_p}(g) = 1$. $\qquad\qquad\square$

Note that Corollary 1.2 is also true with opposite signs, i.e. if L contains at least one 2-vector and we consider the reflections σ_a with $a^2 = 2$. To see this, simply multiply the quadratic form of the lattice L by -1.

Our first result on the commutator is a corollary of Kneser's theorem. We consider the group $\widetilde{\mathrm{O}}^+(L)$, which is the main group in the geometric applications we shall give later.

Theorem 1.3 *Let L be a lattice which satisfies the Kneser conditions. Then $\widetilde{\mathrm{O}}^+(L)^{\mathrm{ab}}$ (resp. $\widetilde{\mathrm{SO}}^+(L)^{\mathrm{ab}}$) is an abelian 2-group. Its order divides 2^N (resp. 2^{N-1}), where N is the number of different $\widetilde{\mathrm{O}}^+(L)$-orbits (resp. $\widetilde{\mathrm{SO}}^+(L)$-orbits) of (-2)-vectors in L.*

Proof. For roots $a, b \in L$ we write $a \equiv b \mod \widetilde{\mathrm{O}}^+(L)$ if there exists $g \in \widetilde{\mathrm{O}}^+(L)$ such that $g(a) = b$. In this case $\sigma_{g(a)} = g \sigma_a g^{-1}$ and $\sigma_a \sigma_b \in [\widetilde{\mathrm{O}}^+(L), \widetilde{\mathrm{O}}^+(L)]$. By Theorem 1.1, any element of $\widetilde{\mathrm{SO}}^+(L)$ is a product of reflections by (-2)-vectors, and since L represents -2 the same is true for $\widetilde{\mathrm{O}}^+(L)$. Using this, and the evident property $\sigma_u \sigma_v = \sigma_{\sigma_u(v)} \sigma_u$, we can rewrite any class modulo commutator as the class of a product $\sigma_{a_1} \ldots \sigma_{a_n}$, where the (-2)-vectors a_i all belong to different $\widetilde{\mathrm{O}}^+(L)$-orbits. The square of such a class can be written as the class of a product of elements $\sigma_{b_i} \sigma_{c_i}$

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where $b_i \equiv c_i \mod \widetilde{O}^+(L)$, so it belongs to the commutator. Exactly the same argument works for $\widetilde{SO}^+(L)$. $\qquad\square$

Let us remark that if $L = 2U \oplus L_0$ and L_0 contains a sublattice isomorphic to A_2 then L satisfies the Kneser conditions.

We do not know exactly how far the conditions on $\mathrm{rank}_2(L)$ and $\mathrm{rank}_3(L)$ are necessary in Theorem 1.3. For $\widetilde{SO}^+(L)$ they cannot be weakened much, as the following examples show.

Example 1.4 *Take $t \not\equiv 0 \mod 3$ and take $L = \Lambda_{2t} = 2U \oplus \langle -2t \rangle$. Then Λ_{2t} satisfies all the Kneser conditions except that $\mathrm{rank}_2(\Lambda_{2t}) = 4$. In this case $\widetilde{SO}^+(\Lambda_{2t})^{\mathrm{ab}}$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z}$ if t is even.*

In this case $\widetilde{SO}^+(\Lambda_{2t})$ is isomorphic to the projective paramodular group $\Gamma_t/\{\pm \mathbf{1}_4\}$ (see [GH2]). Hence the 4-torsion element appears because of equation (1). If $3|t$, then there is also 3-torsion (and the Kneser condition on $\mathrm{rank}_3(\Lambda_{2t})$ fails also).

However, we do not know an example where the conclusion of Theorem 1.3 fails and the Kneser conditions fail only because $\mathrm{rank}_2(L) = 5$.

Example 1.5 *Take $L = 2U \oplus A_2(-3)$. Then L satisfies all the Kneser conditions except that $\mathrm{rank}_3(L) = 4$. In this case $\widetilde{SO}^+(L)^{\mathrm{ab}}$ contains a subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z}$.*

In fact, Desreumaux has constructed a modular form with respect to $\widetilde{SO}^+(L)$ with a character of order 3 (see [De]).

Proposition 1.6 *Let $L = 2U \oplus L_1$ be an even unimodular lattice of rank at least 6. Then $\widetilde{SO}^+(L)^{\mathrm{ab}}$ is trivial and $\widetilde{O}^+(L)^{\mathrm{ab}} \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. L satisfies the Kneser conditions so $\widetilde{SO}^+(L) = SO^+(L)$ is generated by products $\sigma_a \sigma_b$ with $a^2 = b^2 = -2$. The orbit of a (-2)-vector a is determined by its image in the discriminant group (this is a case of the Eichler criterion, from [Ei, §10]: see Proposition 3.3(i), below). But that group is trivial. Therefore there exists $g \in SO^+(L)$ such that $g(a) = b$. But then $\sigma_a \sigma_b = \sigma_a \sigma_{g(a)} = \sigma_a g \sigma_a g^{-1}$ is a commutator. $\qquad\square$

The lattices that appear in the theory of the moduli spaces of symplectic varieties frequently contain two integral hyperbolic planes even if they are not unimodular. In the case of polarised K3 surfaces of degree $2d$ the lattice

$$L_{2d} = 2U \oplus 2E_8(-1) \oplus \langle -2d \rangle \qquad (4)$$

occurs.

In the main theorem of [Ko1] it was proved that the order of $\widetilde{\mathrm{O}}(L_{2d})^{\mathrm{ab}}$ divides 16 or equivalently that the order of $\widetilde{\mathrm{O}}^+(L_{2d})^{\mathrm{ab}}$ divides 8. But there are at most two $\widetilde{\mathrm{O}}^+(L_{2d})$-orbits of (-2)-vectors in L_{2d} [GHS2, Proposition 2.4(ii)]. Hence by Theorem 1.3 the order of $\widetilde{\mathrm{O}}^+(L_{2d})^{\mathrm{ab}}$ divides 4. (There are two orbits if and only if $d \not\equiv 1 \bmod 4$.)

But in fact the order is 2, for any d. This is a special case of the following, which is our main theorem in this paper.

Theorem 1.7 *Let L be an even integral lattice containing at least two hyperbolic planes, such that $\mathrm{rank}_2(L) \geq 6$ and $\mathrm{rank}_3(L) \geq 5$. Then $\widetilde{\mathrm{SO}}^+(L)^{\mathrm{ab}}$ is trivial and $\widetilde{\mathrm{O}}^+(L)^{\mathrm{ab}} \cong \mathbb{Z}/2\mathbb{Z}$.*

The proof of Theorem 1.7 will be given in Section 4.1 below. The main tools are the Siegel–Eichler orthogonal transvections introduced in [Ei, Ch. 1–2].

For L of signature $(2, n)$ Borcherds proposed in [Bo] a very powerful construction of automorphic forms with respect to subgroups of $\mathrm{O}^+(L)$. We can use Theorem 1.7 to give an answer to the question discussed in the remark on page 546 of [Bo].

Corollary 1.8 *For L a lattice as in Theorem 1.7, the orthogonal group $\widetilde{\mathrm{O}}^+(L)$ has only one non-trivial character, namely $\det$, and $\widetilde{\mathrm{SO}}^+(L)$ has no non-trivial characters.*

Remark 1.9 *In many cases the quotient $\widetilde{\mathrm{O}}^+(L) \backslash \mathcal{D}_L$ of the homogeneous domain $\mathcal{D}_L$ associated to L (see Section 2 below) represents a moduli functor. In these cases, Corollary 1.8 also means that the torsion group of the Picard group of the associated moduli stack is $\mathbb{Z}/2\mathbb{Z}$. See also [GH1, Proposition 2.3] for the case of abelian surfaces.*

Returning to the polarised K3 lattice L_{2d} we note that $\mathrm{O}(L^\vee_{2d}/L_{2d}) \cong (\mathbb{Z}/2\mathbb{Z})^{\rho(d)}$ where $\rho(d)$ is the number of divisors of d (see [GH2]). Then according to Theorem 1.7, $[\mathrm{O}(L_{2d}), \mathrm{O}(L_{2d})] = \widetilde{\mathrm{SO}}^+(L_{2d})$ and

$$\mathrm{O}(L_{2d})^{\mathrm{ab}} \cong (\mathbb{Z}/2\mathbb{Z})^{\rho(d)+2}.$$

2 Vanishing order of cusp forms

The modular form η^2 is a cusp form for $\mathrm{SL}_2(\mathbb{Z})$, but it has highly non-trivial character and its order of vanishing at a cusp is not an integer (it is not a section of a line bundle, only of a $\mathbb{Q}$-line bundle). In [GH1] there are also many examples of modular forms with more complicated characters for orthogonal groups of lattices of signature $(2, 3)$. On the other hand, Corollary 1.8 shows that for lattices satisfying the conditions of Theorem 1.7 there are no

modular forms with complicated characters (indeed no complicated characters). In this section we consider lattices of signature $(2, n)$ and analyse the relation between the character of a modular form and its possible orders of vanishing at cusps. We use the following notation: $\mathcal{D}_L$ is the symmetric domain associated with the lattice L; $\mathcal{D}^\bullet$ is the affine cone on $\mathcal{D}$; F is a cusp, corresponding to an isotropic subspace defined over $\mathbb{Q}$, and $\mathcal{D}(F)$ a suitable neighbourhood of it; $U(F)$ is the centre of the unipotent radical of the stabiliser of F in $\mathrm{Aut}(\mathcal{D})$ and $U(F)_\mathbb{Z}$ is the intersection of $U(F)$ with the modular group; $\mathcal{H}_n$ is a tube domain. For more details we refer to [GHS2] and for the general theory of toroidal compactification to [AMRT].

Proposition 2.1 *Let $L = 2U \oplus L_0$ be a lattice of signature $(2, n)$ containing two hyperbolic planes and let ψ be a modular form with character $\det$ or trivial character for an arithmetic subgroup of $\widetilde{\mathrm{O}}^+(L)$. Then the order of vanishing of ψ along any boundary component F of $\mathcal{D}_L$ is an integer.*

Proof. If ψ is of weight k then near the boundary component F we have

$$\psi(gZ) = j(g, Z)\chi(g)\psi(Z),$$

where $Z \in \mathcal{D}_L(F)$ and $g \in U(F)_\mathbb{Z}$, for some factor of automorphy j and χ the character of the modular form ψ. If the factor $j(g, Z)\chi(g)$ is equal to 1 for every $g \in U(F)_\mathbb{Z}$ then ψ is a section of a line bundle near F and its order of vanishing along F is therefore an integer.

Under the hypotheses of the proposition, we do indeed have $\chi(g) = 1$ because g is unipotent and therefore has trivial determinant. It therefore remains to check that the factor of automorphy $j(g, Z)$ is also trivial for $g \in U(F)_\mathbb{Z}$.

If F is of dimension 1 then according to [GHS1, Lemma 2.25] we have

$$U(F) = \left\{ \begin{pmatrix} I & 0 & \begin{pmatrix} 0 & ex \\ -x & 0 \end{pmatrix} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \mid x \in \mathbb{R} \right\}.$$

But the automorphy factor is given by the last $((n+2)\mathrm{nd})$ coordinate of $g\big(p(Z)\big) \in \mathcal{D}_L$, where

$$
\begin{aligned}
p \colon \mathcal{H}_n &\longrightarrow \mathcal{D}_L \\
Z = (z_n, \ldots, z_1) &\longmapsto \big(-\tfrac{1}{2}(Z, Z)_{L_1} : z_n : \cdots : z_1 : 1 \big)
\end{aligned}
$$

is the tube domain realisation of $\mathcal{D}_L$: see [GHS2, Section 3] or [Gr2, Section 2] for the notation and more detail. From this description it is immediate that $j(g, Z) = 1$ for $g \in U(F)_\mathbb{Z}$.

If F is of dimension 0 then F corresponds to some isotropic vector $v \in L$, and $U(F)$ is the centre of the unipotent radical of the stabiliser of v. In this case the unipotent radical is abelian. With respect to a basis of $L \otimes \mathbb{Q}$ in which v is the last $((n+2)\text{nd})$ element, the penultimate $((n+1)\text{st})$ element w is also isotropic and the remaining elements span the orthogonal complement L' of those two, we have

$$
U(F) = \left\{ \begin{pmatrix} I_n & b & 0 \\ 0 & 1 & 0 \\ {}^t c & x & 1 \end{pmatrix} \mid L'b + \alpha c = 0, \ {}^t b L' b + 2\alpha x = 0 \right\}.
$$

Here b and c are column vectors, $x \in \mathbb{R}$ and $\alpha = (w, v)_L$: compare [Ko2, (2.7)]. In this case the tube domain is contained in $\mathbb{C}^n$ and is identified with a subset of the locus $(z_{n+1} = 1) \subset \mathcal{D}_L^\bullet$. The automorphy factor $j(g, Z)$ is therefore equal to the $n + 1$-st coordinate of $g(p(Z))$, where $p(Z)_{n+1} = 1$; but this is 1 as $p(Z)$ is a column vector. $\qquad\square$

From Proposition 2.1 and Corollary 1.8 we have immediately the following result.

Corollary 2.2 *If L is a lattice of signature $(2, n)$ satisfying the conditions of Theorem 1.7, then any cusp form for $\widetilde{\mathrm{O}}^+(L)$ or $\widetilde{\mathrm{SO}}^+(L)$ vanishes to integral order along any toroidal boundary divisor. In particular the order of vanishing of a cusp form along a boundary divisor is always at least 1.*

3 Eichler transvections and the Jacobi group

In this section we analyse the modular groups and construct useful sets of generators for them.

3.1 Eichler transvections

Let $V = L \otimes \mathbb{Q}$ be a quadratic space over $\mathbb{Q}$ and let e be an isotropic vector in V (i.e. $e^2 = 0$) and $a \in e_V^\perp$. The map

$$
t'(e, a) \colon v \longmapsto v - (a, v)e \qquad (v \in e_V^\perp)
$$

belongs to the orthogonal group $\mathrm{O}(e_V^\perp)$.

Lemma 3.1 *$t'(e, a)$ extends to a unique element $t(e, a) \in \mathrm{O}(V)$.*

Proof. We first complete e to a rational hyperbolic plane $\mathbb{Q}e \oplus \mathbb{Q}f \subseteq V$. If there exist γ_1, $\gamma_2 \in \mathrm{O}(V)$ such that $\gamma_1(e) = \gamma_2(e) = e$ and $\gamma_1|_{e_V^\perp} = \gamma_2|_{e_V^\perp}$, then they take the same value on f. The unique orthogonal extension of $t'(e, a)$ on V is given by the map

$$
t(e, a) \colon v \longmapsto v - (a, v)e + (e, v)a - \frac{1}{2}(a, a)(e, v)e. \qquad (5)
$$

This element is called an *Eichler transvection* (see [Ei, §3]). □

We note that $t(e,a)$ acts as the identity on $e_V^\perp \cap a_V^\perp \subset V$. In particular $t(e,a)(e) = e$. Using Lemma 3.1 it is easy to see that

$$t(e,a)t(e,b) = t(e,a+b) \quad \text{and} \quad t(e,a)^{-1} = t(e,-a), \tag{6}$$

$$\gamma t(e,a)\gamma^{-1} = t(\gamma(e),\gamma(a)) \qquad \forall\, \gamma \in \mathrm{O}(V), \tag{7}$$

$$t(xe,a) = t(e,xa), \quad t(e,xe) = \mathrm{id} \qquad \forall\, x \in \mathbb{Q}^*, \tag{8}$$

$$t(e,a) = \sigma_a \sigma_{a+\frac{1}{2}(a,a)e} \qquad \text{if } (a,a) \neq 0. \tag{9}$$

Using equation (9) one can prove (see [Ei, (3.12)]) that for any non-isotropic a orthogonal to the rational hyperbolic plane $\mathbb{Q}e \oplus \mathbb{Q}f$

$$t(f,a)\, t(e, \tfrac{2}{(a,a)}a)\, t(f,a) = \sigma_a \sigma_{f+(2/(a,a))e}. \tag{10}$$

From the definition (5) we see that any transvection $t(e,a)$ is unipotent. From equation (9) we have that $t(e,a) \in \mathrm{SO}^+(L \otimes \mathbb{Q})$. According to equation (5)

$$t(e,a) \in \widetilde{\mathrm{SO}}^+(L) \quad \text{for any } e \in L,\ a \in L \text{ with } (e,e) = (e,a) = 0. \tag{11}$$

Moreover for any primitive isotropic e in L

$$t(e,*)\colon e_L^\perp \longrightarrow \widetilde{\mathrm{SO}}^+(L)$$

is a homomorphism of groups with kernel $\mathbb{Z}e$.

One can also give a description of the transvections in the terms of the Clifford algebra of L. For any isotropic $e \in L \otimes \mathbb{Q}$ and any a such that $(a,e) = 0$ we have that $1 - ea \in \mathrm{Spin}(L \otimes \mathbb{Q})$ and $\pi(1 - ea) = t(e,a)$, where $\pi(\gamma)(v) = \gamma v \gamma^{-1}$ for any γ in the Clifford group (see, e.g., [HO'M]).

3.2 The Jacobi group

Suppose $L = U \oplus U_1 \oplus L_0$, where $U = \mathbb{Z}e \oplus \mathbb{Z}f$ and $U_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}f_1$ are two integral hyperbolic planes. Let F be the totally isotropic plane spanned by f and f_1 and let P_F be the parabolic subgroup of $\mathrm{SO}^+(L)$ that preserves F. This corresponds to a 1-dimensional cusp of the modular variety $\mathrm{SO}^+(L)\backslash \mathcal{D}_L$. We choose a basis of L of the form $(e, e_1, \ldots, f_1, f)$. The subgroup $\Gamma^J(L_0)$ of P_F of elements acting trivially on the sublattice L_0 is called the *Jacobi group*.

The Jacobi group is isomorphic to the semidirect product of $\mathrm{SL}_2(\mathbb{Z})$ with the Heisenberg group $H(L_0)$, the central extension $\mathbb{Z} \rtimes (L_0 \times L_0)$. More

precisely (see [Gr2] for more information) we define elements $[A] \in \Gamma^J(L_0)$ for $A \in \mathrm{SL}_2(\mathbb{Z})$ and $[u, v; z] \in \Gamma^J(L_0)$ for $u, v \in L_0$, $z \in \mathbb{Z}$ by

$$[A] := \begin{pmatrix} A^* & 0 & 0 \\ 0 & \mathbf{1}_{n_0} & 0 \\ 0 & 0 & A \end{pmatrix},$$

$$[u, v; z] := \begin{pmatrix} 1 & 0 & -{}^tvS_0 & -(u,v) - z & -(v,v)/2 \\ 0 & 1 & -{}^tuS_0 & -(u,u)/2 & z \\ 0 & 0 & \mathbf{1}_{n_0} & u & v \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where S_0 is the matrix of the quadratic form L_0 of rank n_0, we consider u and v as column vectors, and $A^* = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) A^{-1} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus any element of $\Gamma^J(L_0)$ may be written in the form $[A] \cdot [u, v; z]$ for suitable A, u, v and z.

The Jacobi group is generated by the transvections

$$t(e, f_1) = \left[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right], \qquad t(f, e_1) = \left[\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)\right], \tag{12}$$

$$t(e, v) = [0, v; 0], \quad t(e_1, u) = [u, 0; 0], \quad t(e, e_1) = [0, 0; 1]. \tag{13}$$

Note that $t(e, e_1)$ generates the centre of the Heisenberg group. It is easy to see, using only the elementary divisor theorem, that

$$\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})/\{\pm(\mathbf{1}_2, \mathbf{1}_2)\} \cong \mathrm{SO}^+(U \oplus U_1).$$

If we identify $xe + x_1e_1 + y_1f_1 + yf \in U \oplus U_1$ with $X = \left(\begin{smallmatrix} x_1 & x \\ y & -y_1 \end{smallmatrix}\right)$, the isomorphism is given by

$$(B, A) \longmapsto \left(X \mapsto BXA^{-1}\right). \tag{14}$$

The map $X \to BXA^{-1}$ certainly preserves the quadratic form $-\det X$. Its kernel is the centre $\pm(\mathbf{1}_2, \mathbf{1}_2)$ of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$. The first copy of $\mathrm{SL}_2(\mathbb{Z})$ (parametrised by B in (14)) is generated by the transvections $t(e, e_1)$ and $t(f, f_1)$, the second by $t(e, f_1)$ and $t(f, e_1)$. From the representation above and from the elementary divisor theorem for 2×2 matrices there follows the next lemma, which is well known.

Lemma 3.2 $\mathrm{SO}^+(U \oplus U_1)$ is generated by the four transvections $t(e, e_1)$, $t(e, f_1)$, $t(f, e_1)$ and $t(f, f_1)$. For any $v \in U \oplus U_1$ there exists $g \in \mathrm{SO}^+(U \oplus U_1)$ such that $g(v) \in U_1$.

3.3 The group $E(L)$ of unimodular transvections

The *divisor* $\mathrm{div}(l)$ of $l \in L$ is the positive generator of the ideal $(l, L) \subset \mathbb{Z}$, so $l^* = l/\mathrm{div}(l)$ is a primitive element of the dual lattice $L^\vee$. Therefore l^* (mod L) is an element of order $\mathrm{div}(l)$ of the discriminant group $D(L)$ and $\mathrm{div}(l)$ is a divisor of $\mathrm{ord}(D(L)) = |\det(L)|$. One can complete an isotropic element $e \in L$ to an integral isotropic plane $U = \mathbb{Z}e \oplus \mathbb{Z}f \subset L$ if and only if $\mathrm{div}(e) = 1$. We call such an isotropic vector *unimodular*. For a unimodular isotropic vector e we have $L = U \oplus L_1$.

We define $E(L)$ to be the group generated by all transvections by unimodular isotropic vectors:

$$E(L) := \langle \{t(e, a) \mid e, a \in L, \ (e, e) = (e, a) = 0, \ \mathrm{div}(e) = 1\} \rangle.$$

We have seen that $E(L)$ is a subgroup of $\widetilde{\mathrm{SO}}^+(L)$. Now let us fix a unimodular isotropic vector $e \in L$ and the decomposition $L = U \oplus L_1$ where $U = \mathbb{Z}e \oplus \mathbb{Z}f$. Then we set

$$E_U(L_1) := \langle \{t(e, a), \ t(f, a) \mid a \in L_1\} \rangle.$$

Proposition 3.3 *Let $L = U \oplus U_1 \oplus L_0$, where $U = \mathbb{Z}e \oplus \mathbb{Z}f$, U_1 is the second copy of the integral hyperbolic plane in L and $L_1 = U_1 \oplus L_0$.*

(i) *If $u, v \in L$ are primitive, $(u, u) = (v, v)$ and $u^* \equiv v^* \mod L$, then there exists $\tau \in E_U(L_1)$ such that $\tau(u) = v$.*

(ii) *$E(L) = E_U(L_1)$.*

(iii) *$\mathrm{O}(L) = \langle E_U(L_1), \ \mathrm{O}(L_1) \rangle$.*

(iv) *For any (-2)-vector $r \in L$ there exists $\rho \in E_U(L_1)$ such that $\sigma_r = \rho \cdot \sigma_{e-f}$.*

Proof. (i) First we note that $\mathrm{div}(u) = \mathrm{ord}_{D(L)}(u^*)$. Therefore $\mathrm{div}(u) = \mathrm{div}(v) = d$. According to Lemma 3.2 there exists $\tau_1 \in E_U(U_1)$ such that $\tau_1(u) \in L_1$. Thus we may assume that u and v are in L_1. Then we can realise the translation by $w = (u - v)/d$ in the sublattice L_1 orthogonal to U as a composition of Eichler transvections:

$$u \ \overset{t(e, u')}{\longmapsto} \ (u - de) \ \overset{t(f, w)}{\longmapsto} \ (v - de) \ \overset{t(e, -v')}{\longmapsto} \ v,$$

where $u', v' \in L_1$ are such that $(u, u') = (v, v') = d$.

(ii) Let $t(u, a)$ be an arbitrary unimodular transvection in $E(L)$ with $(u, u) = 0$ and $\mathrm{div}(u) = 1$. According to (i) there exists $\tau \in E_U(L_1)$ such that $\tau(u) = e$. By equation (7) we obtain that $\tau\, t(u, a)\tau^{-1} = t(\tau(u), \tau(a)) = t(e, \tau(a))$ is in $E_U(L_1)$.

(iii) Let $g \in O(L)$. According to (i) and (ii) there exists $\tau \in E_U(L_1)$ such that $\tau(g(e)) = e$. We have $\bigl(\tau g(e), \tau g(f)\bigr) = (e, \tau g(f)) = (e, f) = 1$. Therefore

$$(\tau g)(f) = f + b - \frac{1}{2}(b, b)e = t(e, b)(f),$$

where $b \in L_1$. Now we see that $h = t(e, -b)\tau g$ acts trivially on U. Therefore $h \in O(L_1)$.

(iv) There exists $\tau \in E_U(L_1)$ such that $\tau(r) = a \in L_1$. According to equation (10)

$$\tau \sigma_r \tau^{-1} = \sigma_a = t(f, a)\, t(e, -a)\, t(f, a)\, \sigma_{e-f}$$

(σ_a and σ_{e-f} commute). To finish we use that $\sigma_{e-f}\tau\sigma_{e-f} \in E_U(L_1)$ for any $\tau \in E_U(L_1)$. $\qquad\square$

Notice that (iii) is true for all the groups we have considered: for instance, $\widetilde{O}^+(L) = \langle E_U(L_1),\ \widetilde{O}^+(L_1) \rangle$ and similarly for SO, $\widetilde{\text{SO}}$, etc. This is because in the proof of (iii) the product $t(e, -b)\tau \in \widetilde{\text{SO}}^+(L)$, which is a subgroup of all of these groups.

All the results of Proposition 3.3 are essentially to be found in [Ei]. (i), which is sometimes called the *Eichler criterion*, is proved in [Ei, Satz 10.4] for lattices over local rings. See also the second proof given in "Anmerkungen zum zweiten Kapitel" [Ei, p. 231]. There is a global variant in [Br, p.85]. (iii) was proved in [Wa, 5.2] for unimodular lattices (see also [P-SS], [Eb], [Gr2]). One can prove (ii), under an additional condition on $\text{rank}_\pi(L)$ for all prime ideals π, over any commutative ring, but the proof is much longer: see [Va2, Theorem 3.3(a)]).

Proposition 3.3 gives us the following result about generators of the orthogonal group which was briefly indicated in [Gr2, p.1194].

Proposition 3.4 *Let $L = U \oplus U_1 \oplus L_0$ be an even lattice with two hyperbolic planes, such that $\text{rank}_3(L) \geq 5$ and $\text{rank}_2(L) \geq 6$. Then*

$$\widetilde{\text{SO}}^+(L) = O'(L) = E(L) = E_U(L_1) \tag{15}$$

and

$$\widetilde{O}^+(L) = \langle \Gamma^J(L_0),\ \sigma_1 \rangle, \tag{16}$$

where $L_1 = U_1 \oplus L_0$, $\Gamma^J(L_0)$ is the Jacobi group, $U_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}f_1$ and $\sigma_1 = \sigma_{e_1 - f_1}$.

Proof. According to Proposition 3.3(iv) the product $\sigma_a \sigma_b$ of any two reflections with $(a, a) = (b, b) = -2$ belongs to $E(L)$. Therefore from Theorem 1.1 and Proposition 3.3(ii) it follows that

$$\widetilde{\text{SO}}^+(L) = O'(L) = E(L) = E_U(L_1) = \langle \{t(c, a) \mid a \in L_1,\ c = e \text{ or } f\} \rangle. \tag{17}$$

The Jacobi group $\Gamma^J(L_0)$ contains the transvections $t(e, v)$ $(v \in L_1)$ and $t(f, e_1)$ (see (12)–(13)). To have the whole group $\widetilde{\mathrm{SO}}^+(L) = E(L)$ we have to add $t(f, u + xf_1)$ with $u \in L_0$ and $x \in \mathbb{Z}$. The $\mathrm{SL}_2(\mathbb{Z})$-subgroup of the Jacobi group is generated by $t(e, f_1)$ and $t(f, e_1)$. Consider the element $S = \left[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \right] \in \Gamma^J(L_0)$. We have

$$S(e) = -e_1, \quad S(f) = -f_1, \quad S^2 = -\mathrm{id}.$$

Using equation (7) we deduce

$$
\begin{aligned}
\sigma_1 t(f, e_1) \sigma_1 &= t(f, f_1), \\
(S\sigma_1 S\sigma_1) t(e, u) (S\sigma_1 S\sigma_1)^{-1} &= t(f, u) \quad \text{for all } u \in L_0.
\end{aligned}
$$

Therefore $\langle \Gamma^J(L_0), \sigma_1 \rangle$ contains all the generators of $E_U(L_1)$. The proposition follows from equation (17). $\qquad \square$

4 Strong Approximation

In this section we prove Theorem 1.7, and make some remarks about similar results over number fields.

It is enough to prove Theorem 1.7 for $\widetilde{\mathrm{SO}}^+(L)$ (or, equivalently by equation (17), for $E(L)$), because $\widetilde{\mathrm{O}}^+(L) = \langle \widetilde{\mathrm{SO}}^+(L), \sigma_{e-f} \rangle$. Vaserstein [Va2, Theorem 3(c)] did this under the extra assumption that $\mathrm{rank}_p(L) \geq 5$ for any odd prime p.

Our method is different and Theorem 1.7 does not have the infinite set of conditions $\mathrm{rank}_p(L) \geq 5$ for $p > 3$. We use the strong approximation theorem (L is indefinite) and the positive solution of the principal congruence problem for the spinorial kernel $\mathrm{O}'(L) = \widetilde{\mathrm{SO}}^+(L)$ for a lattice L with real Witt index ≥ 2 (see [Kn1, 11.4]).

4.1 Proof of Theorem 1.7

First we note that $[E(L), E(L)]$ is an infinite normal subgroup of $\mathrm{O}'(L)$ which is not a subgroup of its centre. Therefore $[E(L), E(L)]$ contains a congruence subgroup of $\mathrm{O}'(L)$ of some level m (by [Kn1, 11.4]) . We may assume that 6 divides m. According to Proposition 3.3(ii), the group $E(L)$ is generated by all $t(e, u)$ and $t(f, v)$ where $u, v \in L_1$. We prove that these generators are the products of commutators in $E(L_p)$, where $L_p = L \otimes \mathbb{Z}_p$, for any prime divisor p of m. For this purpose we introduce the Eichler orthogonal transformation $P(s) \in \mathrm{SO}(L \otimes \mathbb{Q}_p)$ for $s \in \mathbb{Q}_p^\times$:

$$P(s) \colon e \longmapsto s^{-1} e, \ f \longmapsto sf, \ u \longmapsto u \ \forall \, u \in L_1.$$

We have $P(s)^{-1} = P(s^{-1})$. We can describe $P(s)$ in terms of reflections because $\sigma_{e-sf} = P(s^{-1}) \psi$, where $\psi \in \mathrm{O}(L)$ is the permutation of e and

f. Thus $P(s) = \sigma_{e-f}\sigma_{e-sf}$. The following formula (see [Ei, (3.16)]) can be obtained as a corollary of (10):

$$t(f, sw)t(e, w) = t(e, (1 - s\tfrac{w^2}{2})^{-1}w)\, t(f, s(1 - s\tfrac{w^2}{2})w)P((1 - s\tfrac{w^2}{2})^2) \quad (18)$$

for any $w \in L_1 \otimes \mathbb{Q}_p$ and $s \in \mathbb{Q}_p$ such that $1 - s\tfrac{w^2}{2} \neq 0$. In particular for any $v_6 \in L_1 \otimes \mathbb{Z}_p$ such that $(v_6, v_6) = 6$ and $s = 1$ we obtain that $P(4)$ is an element of $E(L_p)$ if $p \neq 2$:

$$P(4) = t(f, 2v_6)t(e, 2^{-1}v_6)t(f, v_6)t(e, v_6). \quad (19)$$

It follows that $t(e, u)$ and $t(f, v)$ are commutators in $E(L_p)$ if $p \neq 2$ or 3:

$$t(e, u) = P(4)^{-1}t(e, 3^{-1}u)P(4)t(e, -3^{-1}u). \quad (20)$$

Now we consider $p = 2$. Let $L = U \oplus U_1 \oplus L_0$, with $U = \mathbb{Z}e \oplus \mathbb{Z}f$ and $U_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}f_1$. For any u orthogonal to e and f_1 we have

$$t(e_1, u)(e) = e, \qquad t(e_1, u)(f_1) = f_1 + u - \frac{1}{2}(u, u)e_1.$$

Therefore for any $s \in \mathbb{Q}_2^\times$ we have

$$[t(e, -sf_1), t(e_1, u)] = t(e, su - s\tfrac{u^2}{2}e_1).$$

Using the same formula for (s, v) and $(-s, u + v)$ we obtain the following representation

$$t(e, s(u, v)e_1) =$$
$$[t(e, -sf_1), t(e_1, u)] \cdot [t(e, -sf_1), t(e_1, v)] \cdot [t(e, sf_1), t(e_1, u + v)].$$

Since $\mathrm{rank}_2(L) \geq 6$, we can find $u, v \in L_0 \otimes \mathbb{Z}_2$ such that $(u, v) \in \mathbb{Z}_2^\times$. Therefore taking $s = (u, v)^{-1}$ we obtain $t(e, e_1)$ as the product of three commutators in $E(L \otimes \mathbb{Z}_2)$. The same argument works for $t(e, f_1)$. Then we can replace e_1 by any unimodular isotropic vector of the form $e_1' = e_1 + w - \tfrac{w^2}{2}f_1$ where $w \in L_0$. We note that $(e_1', f_1) = 1$. We can repeat the arguments above for this new hyperbolic plane $U_1' = \langle e_1', f_1 \rangle$ and we obtain that $t(e, e_1 + w - \tfrac{w^2}{2}f_1)$ belongs to the commutator subgroup of $E(L \otimes \mathbb{Z}_2)$. Using $t(e, e_1)$, $t(e, f_1)$ and $t(e, e_1 + w - \tfrac{w^2}{2}f_1)$, we see that $t(e, l)$ for any $l \in L_1$ is a commutator in $E(L \otimes \mathbb{Z}_2)$.

For $p = 3$ we can use the same calculation with a vector $u \in L_0$ such that $(u, u) \in \mathbb{Z}_3^\times$ ($\mathrm{rank}_3(L) \geq 5$).

We have proved that the generators $t(e, u)$ and $t(f, v)$ $(u, v \in L_1)$ are elements of the commutator subgroup of $E(L_p)$ for any prime divisor p of the level m. So we can write

$$t(e, u) = [t_1^{(p)}, t_2^{(p)}] \cdot \ldots \cdot [t_{2n-1}^{(p)}, t_{2n}^{(p)}],$$

14

where the index n does not depend on p (some of the factors may be trivial). We denote this product of commutators by $[\![t_i]\!]$.

Using the strong approximation theorem for the spinorial kernel $\mathrm{O}'(L)$ (see [O'M, 104:4]) we find $h_i \in \mathrm{O}'(L \otimes \mathbb{Q})$ such that

$$\|t_i^{(p)} - h_i\|_p < \varepsilon \quad \forall\, p|m \quad \text{and} \quad \|h_i\|_p = 1 \quad \forall\, p \nmid m.$$

If ε is sufficiently small then $h_i \in \mathrm{O}'(L)$ and $\|t(e,u)[\![h_i]\!]^{-1} - 1\|_p$ will be small for any prime divisor of m. Then $t(e,u)[\![h_i]\!]^{-1} \equiv 1 \mod m$. It follows that $t(e,u)$ belongs to the commutator subgroup of $\mathrm{O}'(L) = E(L)$.

This completes the proof of Theorem 1.7.

4.2   Orthogonal groups over number fields.

A version of Theorem 1.1 holds over an algebraic number field. To formulate this, collecting the remarks in [Kn2, §5], we must give a suitable extended version of the Kneser conditions. We say that a lattice L over the ring of integers $\mathcal{O}_K$ of a number field K satisfies the Kneser conditions if L is even and represents -2; there exists a real place ν of K such that the Witt index of $L \otimes K_\nu$ is at least 2; and the π-rank $\mathrm{rank}_\pi(L)$ is at least 5 (respectively at least 6) if π is a place such that the residue field k_π is $\mathbb{F}_3$ (respectively $k_\pi = \mathbb{F}_2$).

Theorem 4.1 *([Kn2]) Suppose L is an integral lattice over $\mathcal{O}_K$ satisfying the Kneser conditions. Then $\mathrm{O}'(L) = \mathrm{SO}(L) \cap \ker \mathrm{sn}_K$ is generated by the products of reflections $\sigma_a \sigma_b$ where $a, b \in L$ and $a^2 = b^2 = -2$.*

In this context we have the following result, analogous to Theorems 1.3 and 1.7.

Theorem 4.2 *Let L be a lattice over the ring of integers $\mathcal{O}_K$ of an algebraic number field K that satisfies the Kneser conditions. Then $\mathrm{O}'(L)^{\mathrm{ab}}$ is an abelian 2-group. Its order divides 2^{N-1}, where N is the number of different $\mathrm{O}'(L)$-orbits of (-2)-vectors in L.*

If L contains two hyperbolic planes and $\mathcal{O}_K$ is a principal ideal ring then $\mathrm{O}'(L)^{\mathrm{ab}}$ is trivial.

Proof. The first part of the theorem is similar to Theorem 1.3. We show briefly how to generalise the proof of Theorem 1.7 to the case of algebraic number fields. According to [Va1] (see also [HO'M]) the group $\mathrm{SL}_2(\mathcal{O}_K)$ is generated by the unipotent matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ where $a, b \in \mathcal{O}_K$.

If $\mathcal{O}_K$ is a principal ideal domain then Lemma 3.2 is still true if we replace $\mathrm{SO}^+(U \oplus U_1)$ by $\mathrm{O}'(U \oplus U_1)$. The proof is the same: one uses the action (14) and the elementary divisor theorem, which is true for principal ideal domains in its classical matrix form (there exist $g, h \in \mathrm{SL}_2(\mathcal{O}_K)$ such that gMh is

diagonal). Moreover using (12) and Vaserstein's result from [Va1] we obtain $O'(U \oplus U_1) = E(U \oplus U_1)$. Using this version of Lemma 3.2, we see that Proposition 3.3 is still true over a principal ideal domain. (There are no changes in the proof.) Now we can repeat the proof of Theorem 1.7 using the strong approximation theorem and the positive solution of the congruence subgroup problem (see [Kn1]). □

5   Fundamental groups

In this section we use our results above to compute the fundamental groups of some locally symmetric varieties and their compactifications.

Let $\mathcal{D}$ be a bounded symmetric domain and let Γ be an arithmetic group acting on $\mathcal{D}$. Put $X = \Gamma \backslash \mathcal{D}$.

Lemma 5.1 *There is a surjective homomorphism* $\Gamma \twoheadrightarrow \pi_1(X)$, *which is an isomorphism if* Γ *acts freely on* $\mathcal{D}$.

Proof. The map $\phi \colon \Gamma \to \pi_1(\mathcal{D}/\Gamma)$ is defined as follows. Choose a base point $p_0 \in \mathcal{D}$, and suppose $\gamma \in \Gamma$. Since $\mathcal{D}$ is connected and simply-connected, we may join p_0 and $\gamma(p_0)$ by a path σ_γ and any two such paths are homotopic. The quotient map $\pi \colon \mathcal{D} \to \Gamma \backslash \mathcal{D}$ makes this into a loop $\pi \circ \sigma_\gamma$ based at $x_0 = \pi(p_0)$, and we define $\phi(\gamma)$ to be the homotopy class $[\pi \circ \sigma_\gamma] \in \pi_1(X, x_0)$.

However, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x)$ for any base point $x \in X$. It is easy to check that the map ϕ is well defined and has the required properties. □

Lemma 5.2 *If* γ *has fixed points in* $\mathcal{D}$ *then* $\gamma \in \ker \phi$.

Proof. In the proof of Lemma 5.1 we may choose p_0 and σ_γ freely, so we choose p_0 to be a fixed point of γ and σ_γ to be the constant path at p_0. Then $[\pi \circ \sigma_\gamma] = 1$. □

Now we pass to compactifications of X. Let $\overline{X}$ denote a normal compactification of X and let $\widetilde{X}$ denote a projective smooth model of $\overline{X}$.

Proposition 5.3 *There are surjections* $\Gamma \twoheadrightarrow \pi_1(\widetilde{X})$ *and* $\Gamma \twoheadrightarrow \pi_1(\overline{X})$, *both factoring through* $\phi \colon \Gamma \to \pi_1(X)$.

Proof. Note first of all that $\pi_1(\widetilde{X})$ does not depend on the choice of the model $\widetilde{X}$ (see for example [HK] or [Sa, Lemma 1.3]). So we may take a toroidal compactification X' of X with only finite quotient singularities and $\widetilde{X}$ a resolution of X'. Using the remark [Fu, p. 56] that the inclusion of an open subvariety in a normal variety induces a surjection on fundamental groups, there is a surjection $\pi_1(X) \to \pi_1(X')$. By [Kol, §7], resolving finite quotient singularities does not change the fundamental group, so we have

(by, for example, [Sa, Lemma 1.2]) a surjection $\Gamma \twoheadrightarrow \pi_1(\widetilde{X})$ factoring through π.

For the case of $\overline{X}$, in particular for the Satake compactification, one may again use the remark in [Fu], as in [Sa, p. 42]. $\qquad\square$

Corollary 5.4 *If Γ is generated by elements $\gamma \in \Gamma$ with fixed points in $\mathcal{D}$, then $\pi_1(X) = \pi_1(\widetilde{X}) = 1$.*

Proof. This follows immediately from Lemma 5.2 and Proposition 5.3. $\qquad\square$

For an integral lattice L of real signature $(2, n)$ one can determine the hermitian homogeneous domain of type IV

$$\mathcal{D}(L) = \{[Z] \in \mathbb{P}(L \otimes \mathbb{C}) \,|\, (Z, Z) = 0, \ (Z, \bar{Z}) > 0\}^+$$

where $+$ means a connected component. In [GHS1]–[GHS3] we studied the geometry of the modular varieties

$$\mathcal{F}(L) = \widetilde{\mathrm{O}}^+(L)\backslash\mathcal{D}(L) \quad \text{and} \quad \mathcal{SF}(L) = \widetilde{\mathrm{SO}}^+(L)\backslash\mathcal{D}(L).$$

For $L = L_{2d}$ (see (4)) the variety $\mathcal{F}_{2d} = \mathcal{F}(L_{2d})$ is the moduli space of K3 surfaces with a polarisation of degree $2d$. The variety $\mathcal{SF}_{2d}$ corresponds to the addition of a spin structure (see [GHS1, § 5]).

Theorem 5.5 *Let L be a lattice with $\mathrm{sign}_{\mathbb{R}}(L) = (2, n)$ satisfying the condition of Theorem 1.7. Then $\mathcal{F}(L)$ and $\mathcal{SF}(L)$, as well as any smooth complete model of $\mathcal{F}(L)$ or $\mathcal{SF}(L)$, are simply connected. In particular this is true for the moduli spaces $\mathcal{F}_{2d}$ and $\mathcal{SF}_{2d}$.*

Proof. In view of Corollary 5.4 it is enough to verify that $\widetilde{\mathrm{O}}^+(L_{2d})$ and $\widetilde{\mathrm{SO}}^+(L_{2d})$ are generated by elements having fixed points in $\mathcal{D}_{L_{2d}}$. It is easy to see that L_{2d} satisfies the Kneser conditions. So by Theorem 1.1 and Corollary 1.2, $\widetilde{\mathrm{SO}}^+(L_{2d}) = \mathrm{O}'(L_{2d})$ is generated by products of pairs of reflections, and $\widetilde{\mathrm{O}}^+(L_{2d})$ is generated by reflections. Both reflections and the products of two reflections have fixed points, so the result follows. $\qquad\square$

Proposition 5.6 *The moduli space $\mathcal{E}$ of Enriques surfaces, and any smooth compactification of it, are simply-connected.*

Proof. This follows from the hard fact that the moduli space of Enriques surfaces is rational [Ko3]. However, for simply-connectedness we can give a quick proof using the results above. The moduli space $\mathcal{E}$ is associated with the lattice $L = U(2) \oplus U \oplus E_8(-2)$, which has 2-rank 2 and therefore does not satisfy the Kneser conditions. But $\mathcal{E} = \mathrm{O}^+(L)\backslash\mathcal{D}_L$ is also equal to $\mathrm{O}^+(L')\backslash\mathcal{D}_{L'}$, where $L' = U \oplus U(2) \oplus E_8(-1)$, since L is obtained from L'

as the sublattice of $L'(2)$ of index 4 where the generators e, f of $U(4)$ are replaced by $e/2$ and $f/2$: see [Ko3].

Since L' does satisfy the Kneser conditions, Theorem 1.3 tells us that $\widetilde{O}^+(L')$ is generated by pairs of reflections, and these have fixed points. But $\widetilde{O}^+(L')$ is of index 2 in $O^+(L')$, and the reflection that interchanges the two generators of $U(2)$ is the extra generator that we need. It also has fixed points in $\mathcal{D}_{L'}$, so by Lemma 5.4 we are done. $\qquad\square$

Apart from elements with fixed points there are also other elements in the kernel of $\Gamma \twoheadrightarrow \pi_1(\overline{X})$, namely those coming from the unipotent radical of parabolic subgroups. By Lemma 3.1, a unimodular transvection $t(e,v)$ is determined by a unimodular isotropic vector, $e^2 = 0$, $\mathrm{div}(e) = 1$, and by $v \in e_L^\perp$. Thus e defines a zero-dimensional cusp of the modular variety $X = \widetilde{SO}^+(L) \setminus \mathcal{D}_L$. In other words $t(e,v)$ is an element of the corresponding parabolic subgroup P, and hence it belongs to the centre of the unipotent radical U_P of P. Different transvections correspond to different 0-dimensional cusps. According [Sa, Theorem 1.5 and Corollary 1.6], $E(L)$ is contained in the kernel of the surjection $\phi\colon \widetilde{SO}^+(L) \to \pi_1(\widetilde{X})$.

For the moduli space $\mathcal{A}_t$ of abelian surfaces with a polarisation of type $(1,t)$ the lattice that occurs is $\Lambda_{2t} = 2U \oplus \langle -2t \rangle$ and the group is the paramodular group Γ_{2t}. As we have seen, both the Kneser conditions and the conclusions of Theorem 1.3 fail in this case. However, the results of this paper together with those of [Sa] still give us results about the fundamental groups.

Theorem 5.7 *Any smooth model $\widetilde{\mathcal{A}}_t$ of a compactification of $\mathcal{A}_t$ is simply-connected.*

Proof. We cannot apply Proposition 3.4 to the lattice Λ_{2t} but the last identity (16) of this proposition is still true for Λ_{2t}. According to [GH2] there exists an isomorphism $\Phi\colon \Gamma_t/\{\pm\mathbf{1}\} \to \widetilde{SO}^+(\Lambda_{2t})$. For the paramodular group Γ_t we have

$$\Gamma_t = \langle \Gamma_t^J, J_t \rangle \quad \text{where} \quad J_t = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1/t \\ 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \end{pmatrix}$$

and Γ_t^J is the Jacobi subgroup of the paramodular group. This follows from the elementary divisor theorem for the symplectic group: see, for example, [Gr1].

We know (see [GH2]) that $\Phi(\Gamma_t^J) = \Gamma^J(\Lambda_{2t})$, which is generated by

transvections (see Section 3.2). Then

$$\Phi(J_t) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix} = \sigma_{e+f}\,\sigma_{e_1+f_1}$$

where we use notations of Section 3.2. As in the proof of Proposition 1.6 we see that $\Phi(J_t)$ is a transvection. Therefore

$$\widetilde{\mathrm{SO}}^+(\Lambda_{2t}) = E(\Lambda_{2t}), \qquad \widetilde{\mathrm{O}}^+(\Lambda_{2t}) = \langle \Gamma^J(\Lambda_{2t}), \sigma_{e_1-f_1} \rangle \tag{21}$$

$\square$

In [Sa, Theorem 3.4] it was proved that $\widetilde{\mathcal{A}}_p$ is simply-connected for any odd prime p. Also in [Sa] one may find examples of locally symmetric varieties that are not simply-connected. However, in all these cases one has, in particular, that the fundamental group is finite and therefore the irregularity is zero.

In a similar way, by combining the results of [Sa] and those of Section 4.2, one can prove that some Shimura varieties (considered as complex manifolds) are simply-connected.

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