Weighted Castelnuovo-Mumford Regularity and Weighted Global Generation

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April 15, 2016

Abstract

We introduce and study a notion of Castelnuovo-Mumford regularity suitable for weighted projective spaces.

0 Introduction

In chapter 14 of [12] Mumford introduced the concept of regularity for a coherent sheaf on a projective space $\mathbb{P}^n$. It was soon clear that it was a key notion and a fundamental tool in many areas of algebraic geometry and commutative algebra.

From the algebraic geometry point of view, regularity measures the complexity of a sheaf: the regularity of a coherent sheaf is an integer that estimates the smallest twist for which the sheaf is generated by its global sections. In Castelnuovo's much earlier version, if X is a closed subvariety of projective space and H is a general hyperplane, one uses linear systems (seen now as a precursor of sheaf cohomology) to get information about X from information about the intersection of X with H plus other geometrical or numerical assumptions on X.

From the computational and commutative algebra point of view, the regularity is one of the most important invariants of a finitely generated

Mathematics Subject Classification 2010: 14F05, 14J60.
Keywords: Castelnuovo-Mumford regularity, weighted projective spaces.

graded module over a polynomial ring. Roughly, it measures the amount of computational resources that working with that module requires. More precisely the regularity of a module bounds the largest degree of the minimal generators and the degree of syzygies.

Extensions of this notion have been proposed over the years to handle other ambient varieties instead of projective space: Grassmannians [1], quadrics [2], multiprojective spaces [3, 6], n-dimensional smooth projective varieties with an n-block collection [6], and abelian varieties [13].

In all these cases the ambient variety is smooth. Maclagan and Smith [11] gave a variant of multigraded Castelnuovo-Mumford regularity, motivated by toric geometry, which applies to some singular varieties: for a different approach to multigraded regularity, see [14].

Since it often happens that a variety can be conveniently embedded in a weighted projective space but embedding it in projective space requires some arbitrary choices, or the use of many variables or high degree equations, it is worthwhile to be able to import these ideas into weighted projective spaces.

The first aim of this project is to introduce and study a notion of regularity, and a related notion of globally generated sheaf, using Koszul complexes, for weighted projective spaces. The theory of [11] applies to weighted projective spaces, but as theirs is a general theory for all toric varieties, we believe that it should be possible to do better in this narrower context. In particular we want the structure sheaf to be regular, which in general does not happen in [11]. Specifically, the definitions in [11], applied to weighted projective spaces, take no account of the individual weights, and the results are therefore only those that hold for all weighted projective spaces (and more), irrespective of the weights.

1 Generalities

Fix a weighted projective space $\mathbb{P} = \mathbb{P}(w_0, \dots, w_n)$, which we always write with the weights in decreasing order, $w_0 \geq \cdots \geq w_n$. There is a natural quotient map $\pi\colon \mathbb{P}^n \to \mathbb{P}$ (see [4, Thm. 3A.1]).

We want to regard $\mathbb{P}$ as a stack, as in [8, Example 7.27]: we could instead follow [5] and regard $\mathbb{P}$ as a graded scheme. If we are interested only in schemes (or varieties), we may assume that the weights $(w_0, \dots, w_n)$ are reduced, i.e. no prime divides n of them, because every weighted projective space is isomorphic as a scheme to a weighted projective space with reduced weights (see [4, Prop. 3C.5]). If we are interested in orbifolds we may similarly assume that $\mathrm{hcf}(w_0, \dots, w_n) = 1$. However, the coordinate hyperplanes $\mathbb{H}_j$ (see Lemma 2.9 below) do not inherit these conditions, so we must continue to allow arbitrary weights.

For a subset $I = \{\nu_1, \dots, \nu_s\} \subset \{0, \dots, n\}$ with $0 \leq \nu_1 < \cdots < \nu_s \leq n$, we set $|w_I| = \sum_{\nu \in I} w_\nu$. For convenience, we also write $\mathbf{w}_i$ for the sum of the $i+1$ largest weights, $\mathbf{w}_i = w_0 + \ldots + w_i = |w_{0,\ldots,i}|$, and we write $\mathbf{w}$ for the total weight, $\mathbf{w} = \mathbf{w}_n$.

For $\mathcal{E}$ a coherent sheaf on $\mathbb{P}$, we define the modules

$$H^i(\mathcal{E}(\geq l)) = \bigoplus_{t \geq l} H^i(\mathbb{P}, \mathcal{E}(t)) \tag{1}$$

and similarly $H^i(\mathcal{E}(\leq l))$, for $l \in \mathbb{Z}$. We also use the notation $H^i_*(\mathbb{P}, \mathcal{E}) = \bigoplus_{t \in \mathbb{Z}} H^i(\mathbb{P}, \mathcal{E}(t)) = H^i(\mathcal{E}(> -\infty))$.

Remark 1.1. *$\pi^*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}_{\mathbb{P}^n}(1)$ and $\omega_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-\mathbf{w})$.*

Observe that $\pi_*\mathcal{O}_{\mathbb{P}^n}$ is a split vector bundle on $\mathbb{P}$ by [4, Cor. 3A.2], say

$$\pi_*\mathcal{O}_{\mathbb{P}^n} \cong \bigoplus_{j=1}^{m} \mathcal{O}_{\mathbb{P}}(-\textstyle\sum_{j=0}^{n} r_j). \tag{2}$$

where $0 \leq r_j < w_j$ for all $j = 0, \ldots, n$.

Lemma 1.2. *For any $i \in \mathbb{N}$, if $\mathcal{E}$ is a vector bundle on $\mathbb{P}$ then $H^i_*(\mathbb{P}, \mathcal{E}) = 0$ if and only if $H^i_*(\mathbb{P}^n, \pi^*\mathcal{E}) = 0$.*

Proof. Since π is a finite morphism, we have $H^i_*(\mathbb{P}^n, \pi^*\mathcal{E}) \cong H^i_*(\mathbb{P}, \pi_*\pi^*\mathcal{E})$, and it is enough to observe that (using the projection formula and (2))

$$\pi_*\pi^*\mathcal{E} \cong \mathcal{E} \otimes \pi_*\mathcal{O}_{\mathbb{P}^n} \cong \mathcal{E} \otimes \left(\bigoplus_{j=1}^{m} \mathcal{O}_{\mathbb{P}}(r_j) \right) \cong \bigoplus_{j=1}^{m} \mathcal{E}(r_j).$$

□

2 Weighted Castelnuovo-Mumford Regularity

We begin by recalling the usual definition of Castelnuovo-Mumford regularity on projective space: see, for example, [10, Chapter 1.8].

Definition 2.1. *A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is said to be m-*regular*, for $m \in \mathbb{Z}$, if*

$$H^i(\mathcal{F}(m-i)) = 0$$

for $i = 1, \ldots, n$.

It is well known (see [10, Theorem 1.8.3]) that being m-regular implies, in particular, that $H^0(\mathcal{F}(m+1)) \neq 0$, and in fact much more than that: it is globally generated (and this even holds for $\mathcal{F}(m)$).

Maclagan and Smith in [11] gave a definition of regularity for simplicial toric varieties. We refer to it as toric regularity. On $\mathbb{P}$ it reduces to the following (compare

Definition 2.2. *Let $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_n)$ and $k = \mathrm{lcm}(w_0, \ldots, w_n)$. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}$ is said to be m-*toric regular *if, for $i = 1, \ldots, n$*

$$H^i(\mathcal{F}(m - ik)) = 0.$$

Here we have taken $\mathcal{C} = \{\mathcal{O}(k)\}$ in [11, Definition 6.2]): according to the definition of $\mathcal{C}[i]$ given in [11, Section 4], $\mathcal{F}$ is toric m-regular if $H^i(\mathcal{F}(p)) = 0$ for every $p \in m + (-i\mathcal{O}(k) + \mathcal{C})$. That is, $H^i(\mathcal{F}(m - ik + t) = 0$ for every $t \in k\mathbb{N}$, but it is enough to consider $t = 0$.

In our more restricted context, we want a definition that takes account of the individual weights, which toric regularity does not.

Our motivation for the definition we make comes from the Koszul complex. On $\mathbb{P}$ (with, as usual, $w_0 \geq \cdots \geq w_n$) we define $\mathcal{A}_j = \bigoplus\limits_{|I|=j+1} \mathcal{O}(-|w_I|)$. Then (see [5]) there is a Koszul complex on $\mathbb{P}$ given by

$$0 \longrightarrow \mathcal{A}_n \longrightarrow \mathcal{A}_{n-1} \longrightarrow \ldots \longrightarrow \mathcal{A}_0 \longrightarrow \mathcal{O} \longrightarrow 0. \qquad (3)$$

For example, if $\mathbb{P} = \mathbb{P}(5, 3, 2)$ then the Koszul complex is

$$0 \to \mathcal{O}(-10) \to \begin{matrix} \mathcal{O}(-8) \\ \oplus \\ \mathcal{O}(-7) \\ \oplus \\ \mathcal{O}(-5) \end{matrix} \to \begin{matrix} \mathcal{O}(-5) \\ \oplus \\ \mathcal{O}(-3) \\ \oplus \\ \mathcal{O}(-2) \end{matrix} \to \mathcal{O} \to 0.$$

We give the following definition of weighted Castelnuovo-Mumford regularity.

Definition 2.3. *Let $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_n)$ with $w_0 \geq \cdots \geq w_n$ and $k = \mathrm{lcm}(w_0, \ldots, w_n)$. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}$ is said to be m-*weighted regular*, which we abbreviate to m-*wregular*, if for $i = 1, \ldots, n$*

$$H^i(\mathcal{F}(t + (m+1)k - \mathbf{w}_i) = 0$$

for every $t \geq 0$, and also

$$H^0(\mathcal{F}((m+1)k)) \neq 0.$$

We often write wregular *to mean 0-wregular.*
We define the wregularity *of $\mathcal{F}$,* $\mathrm{Wreg}(\mathcal{F})$*, as the smallest integer m such that $\mathcal{F}$ is m-wregular.*

Remark 2.4. *For $\mathbb{P} = \mathbb{P}^n$, wregularity and toric regularity both coincide with the usual notion of Castelnuovo-Mumford regularity.*

Indeed, in this case we have $m = 0$ and $w_0 = \cdots = w_n = 1$, so $k = 1$ and $w_i = i + 1$, so taking $t = 0$ in Definition 2.3 we get

$$H^i(\mathcal{F}(k - \mathbf{w}_i))) = H^i(\mathcal{F}(-i)).$$

Lemma 2.5. *For $\mathbb{P}$ any weighted projective space,* $\mathrm{Wreg}(\mathcal{O}_{\mathbb{P}}) = 0$.

Proof. In fact for any $t \in \mathbb{Z}$ we have $H^i(\mathcal{O}_{\mathbb{P}}(t)) = 0$ for $0 < i < n$, and $H^0(\mathcal{O}_{\mathbb{P}}(k)) \neq 0$. For 0-wregularity we also need $H^n(\mathcal{O}(k-\mathbf{w})) = 0$, but this holds because $H^n(\mathcal{O}(k-\mathbf{w}))$ is Serre dual to $H^0(\mathcal{O}(-k))$, which is zero. (See [4, Section 6B] and [5, Proposition 2.1.4] for Serre duality in this context.)

However, $\mathcal{O}$ is not -1-wregular because $H^n(\mathcal{O}(-\mathbf{w})) \cong H^0(\mathcal{O}) \neq 0$. □

On the other hand we cannot expect $\mathcal{O}_{\mathbb{P}}$ to be toric regular for arbitrary weights. In fact $H^n(\mathcal{O}(-nk)) \cong H^0(\mathcal{O}(nk-\mathbf{w}))$ which is non-zero in general. However, $\mathcal{O}(nk)$ is always toric regular.

A significant difference between Definition 2.3 and Definitions 2.1 and 2.2 is that we have imposed a non-vanishing condition, because we lack a counterpart to Mumford's theorem [10, Theorem 1.8.3]: see Example 2.7 below. With this in mind, we make the following definition.

Definition 2.6. *A coherent sheaf $\mathcal{F}$ on $\mathbb{P}$ is said to be m-*semiwregular *if for $i = 1, \ldots, n$*

$$H^i(\mathcal{F}(t + (m+1)k - \mathbf{w}_i) = 0$$

for every $t \geq 0$.

Example 2.7. *If $\mathbb{P} = \mathbb{P}(3,2)$ then $\mathcal{O}_{\mathbb{P}}(-5)$ is 0-semiwregular but not 0-wregular, whereas $\mathcal{O}_{\mathbb{P}}(-4)$ is 0-wregular.*

In fact, $m = 0$ and $k = 6$, so $H^1(\mathcal{O}(-5+6-5)) \cong H^0(\mathcal{O}(-1)) = 0$, and thus if we take $\mathcal{F} = \mathcal{O}(-5)$ then the condition $H^1(\mathcal{O}(-5) \otimes \mathcal{O}(t+6-(3+2))) = 0$ is satisfied for every $t \geq 0$, but $H^0(\mathcal{O}(-5) \otimes \mathcal{O}(k)) = H^0(\mathcal{O}(1)) = 0$.

On the other hand, for $\mathcal{O}(-4)$ the condition $H^1(\mathcal{O}(-4) \otimes \mathcal{O}(t+6-(3+2))) = 0$ is satisfied for every $t \geq 0$, but $H^0(\mathcal{O}(-4) \otimes \mathcal{O}(k)) = H^0(\mathcal{O}(2)) \neq 0$. So $\mathcal{F}(1) = \mathcal{O}(-4)$ is wregular.

Weighted regularity and weighted semiregularity behave well under pullback along the natural covering map from $\mathbb{P}^n$.

Lemma 2.8. *Let $\mathcal{F}$ be an m-semiwregular (or m-wregular) coherent sheaf on $\mathbb{P}$. Then $\pi^*\mathcal{F}$ is $((m+1)k - n + \mathbf{w} - \mathbf{w}_1)$-regular on $\mathbb{P}^n$.*

Proof. We want to show that, for $q = (m+1)k - n + \mathbf{w} - \mathbf{w}_1$ and for any $i = 1, \ldots, n$, we have

$$h^i(\mathbb{P}^n, \pi^*\mathcal{F}(q-i)) = 0.$$

By (2), $\pi_*\pi^*\mathcal{F}$ decomposes as direct sum of different twists of $\mathcal{F}$. The smallest twist that occurs is $-\mathbf{w} + n + 1$. Since

$$H^i(\mathbb{P}^n, \pi^*\mathcal{F}(q-i)) \cong H^i(\mathbb{P}, \pi_*\pi^*\mathcal{F}(q-i))$$

it is enough to show that $H^i(\mathbb{P}, \mathcal{F}(q - i - \mathbf{w} + n + 1))$ vanishes, for each i, with q as above.

If $i = 1$ and $q = (m+1)k - n + \mathbf{w} - \mathbf{w}_1$ we get

$$H^1(\mathbb{P}, \mathcal{F}(q - 1 - \mathbf{w} + n + 1)) = H^1(\mathcal{F}((m+1)k - \mathbf{w}_1))$$

which is zero because $\mathcal{F}$ is m-wregular. Hence

$$H^1(\mathbb{P}, \pi_*\pi^*\mathcal{F}((m+1)k - n + \mathbf{w} - \mathbf{w}_1 - 1)) = 0.$$

For $2 \leq i \leq n$ we have

$$H^i_*(\mathbb{P}, \mathcal{F}(t + (m+1)k - \mathbf{w}_i)) = 0$$

for every $t \geq 0$, by m-wregularity. However, $\mathbf{w}_i \geq \mathbf{w}_1 + i - 1$, since the weights are positive integers, so

$$\begin{aligned} q - i - \mathbf{w} + n + 1 = & \quad (m+1)k - n + \mathbf{w} - \mathbf{w}_1 - i - \mathbf{w} + n + 1 \\ = & \quad (m+1)k - \mathbf{w}_1 - i + 1 \\ \leq & \quad m(k+1) - \mathbf{w}_i \end{aligned}$$

and hence $H^i(\mathbb{P}, \mathcal{F}(q - i - \mathbf{w} + n + 1)) = 0$, as required. □

Weighted semiregularity also behaves well under restriction to coordinate hyperplanes: however, weighted regularity does not, in general.

Lemma 2.9. *Suppose that $\mathcal{F}$ is an m-semiwregular coherent sheaf on $\mathbb{P}$ and let $\mathbb{H}_j = \{x_j = 0\}$ be the j-th coordinate hyperplane. We put $k_j = \operatorname{lcm}(w_0, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n)$ and $z_j = k/k_j$. Then $\mathcal{F}_{\mathbb{H}_j} = \mathcal{F} \otimes \mathcal{I}_{\mathbb{H}_j}$ is $((m+1)z_j - 1)$-semiwregular on $\mathbb{H}_j$.*

Proof. We consider $\mathbb{H}_j \cong \mathbb{P}(w_0, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n)$: note that the sum of the first $i+1$ of these weights is $\mathbf{w}_i$ if $i < j$ and is $\mathbf{w}_{i+1} - w_j$ if $i \geq j$. Thus, writing $\mathcal{E} = \mathcal{F}_{\mathbb{H}_j}$, we want to show that, for $q = (m+1)z_j - 1$ and for any $i = 1, \ldots, n-1$

$$H^i(\mathbb{H}_j, \mathcal{E}(t + (q+1)k_j - \mathbf{w}_i)) = 0 \qquad \text{if } i < j \tag{4}$$
$$H^i(\mathbb{H}_j, \mathcal{E}(t + (q+1)k_j - (\mathbf{w}_{i+1} - w_j))) = 0 \qquad \text{if } i \geq j, \tag{5}$$

for every integer $t \geq 0$.

Let us consider the exact sequence

$$0 \longrightarrow \mathcal{F}(-w_j) \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0, \tag{6}$$

coming from tensoring $\mathcal{F}$ with the restriction sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-w_j) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{H}_j} \longrightarrow 0.$$

If $i < j$ we twist (6) by $t + (m+1)k - \mathbf{w}_i$: in cohomology, this gives (for $0 < i < n$ and $t \geq 0$)

$$H^i(\mathbb{P}, \mathcal{F}(t + (m+1)k - \mathbf{w}_i)) \longrightarrow H^i(\mathbb{H}_j, \mathcal{E}(t + (m+1)k - \mathbf{w}_i)) \\ \longrightarrow H^{i+1}(\mathbb{P}, \mathcal{F}(t + (m+1)k - \mathbf{w}_i - w_j)). \quad (7)$$

The first of these terms vanishes because $\mathcal{F}$ is m-wregular, and the m-wregularity also gives $H^{i+1}(\mathbb{P}, \mathcal{F}(t' + (m+1)k - \mathbf{w}_{i+1}) = 0$ for any $t' \geq 0$. In particular, since $i + 1 \leq j$ we have $w_{i+1} \geq w_j$ and we may take $t' = t + w_{i+1} - w_j$, giving us vanishing of the third term in (7). Thus the middle term also vanishes, and since $(q+1)k_j = (m+1)k$ that proves (4).

The proof for the second case, $i \geq j$, is similar. This time we twist (6) by $t + (m+1)k - (\mathbf{w}_{i+1} - w_j)$. In cohomology this gives

$$H^i(\mathbb{P}, \mathcal{F}(t+(m+1)k-(\mathbf{w}_{i+1}-w_j)) \longrightarrow H^i(\mathbb{H}_j, \mathcal{E}(t+(m+1)k-(\mathbf{w}_{i+1}-w_j))) \\ \longrightarrow H^{i+1}(\mathbb{P}, \mathcal{F}(t + (m+1)k - \mathbf{w}_{i+1})). \quad (8)$$

The third of these terms vanishes because $\mathcal{F}$ is m-wregular, and the m-wregularity also gives $H^i(\mathbb{P}, \mathcal{F}(t' + (m+1)k - \mathbf{w}_i) = 0$ for any $t' > 0$. Now since $i+1 \geq j$ we have $w_{i+1} \leq w_j$ and we may take $t' = t - w_{i+1} + w_j$, giving us vanishing of the first term in (8). Again, the middle term also vanishes and this proves (5). $\square$

Example 2.10. *If $\mathbb{P} = \mathbb{P}(5, 3, 2)$ then it is easy to check that $\mathcal{F} = \mathcal{O}_{\mathbb{P}}(-5)$ is 0-wregular but $\mathcal{F}_{\mathbb{H}_0}$ is not 0-wregular by Example 2.7.*

Remark 2.11. *Let $\mathcal{F}$ be an m-wregular coherent sheaf on $\mathbb{P}$. If $w_0 = \cdots = w_n = 1$, then $(m+1)k - n + \mathbf{w}_{n-2} = m$, and $\pi^*\mathcal{F}$ is m-regular. More generally, if $w_j = 1$, then $z_j(m+1) - 1 = m$ and $\mathcal{F}_{\mathbb{H}_j}$ is m-wregular on $\mathbb{H}_j$.*

We cannot expect the above properties for toric regularity.

We can give a notion of global generation adapted to this weighted situation.

Definition 2.12. *A coherent sheaf $\mathcal{F}$ on $\mathbb{P}$ is said to be* weighted globally generated *(abbreviated to* wgg*) if, for any $x \in \mathbb{P}$, the map*

$$\mu\colon \bigoplus_{i=0}^{n} H^0(\mathcal{F}(k - w_i)) \otimes \mathcal{O}_x \to \mathcal{F}_x(k),$$

where $\mu(\sum_{i=0}^n f_i \otimes e_x) = \sum_{i=0}^n f_i x_i e_x$, is surjective.

This reduces to the usual definition of globally generated in the case of $\mathbb{P}^n$, when $w_0 = \cdots = w_n = 1$ and $k = 1$, so $\bigoplus_{i=0}^n H^0(\mathcal{F}(k - w_i)) \cong H^0(\mathcal{F}) \otimes H^0(\mathcal{O}(1))$. In fact we have a surjection

$$\mu\colon H^0(\mathcal{F}) \otimes H^0(\mathcal{O}(1)) \otimes \mathcal{O}_x \to \mathcal{F}_x(1)$$

and a surjection

$$H^0(\mathcal{F}) \otimes H^0(\mathcal{O}(1)) \otimes \mathcal{O}_x \to H^0(\mathcal{F}) \otimes \mathcal{O}_x(1).$$

So we may construct a surjection

$$H^0(\mathcal{F}) \otimes \mathcal{O}_x(1) \to \mathcal{F}_x(1).$$

Proposition 2.13. *$\mathcal{O}_{\mathbb{P}}$ is wgg.*

Proof. We want to show that for any $i = 0, \dots, n$ the map

$$\mu \colon \bigoplus_{i=0}^{n} H^0(\mathcal{O}(k - w_i)) \otimes \Gamma(D_+(x_i), \mathcal{O}) \to \Gamma(D_+(x_i), \mathcal{O}(k))$$

is surjective, where $D_+(x_i)$ denotes as usual the locus $(x_i \neq 0)$.

Let $u \in \Gamma(D_+(x_i), \mathcal{O}(k))$; then $u = a x_i^{-s}$ with $s > 0$ and a a monomial of degree $sw_i + k$. Therefore $u = a x_i^{sk-s}/x_i^{sk}$ and

$$\deg(a x_i^{sk-s}) = sw_i + k + w_i(sk - s) = w_i sk + k = k(w_i s + 1).$$

So $a x_i^{sk-s}$ is a monomial containing x_i^{sk-s} and its degree is a multiple of k. This means that we can write $a x_i^{sk-s} = a'b$ where $a' = x_i^{k/w_i - 1}$ and has degree $k - w_i$; then b/x_i^{sk} has degree $k(sw_i + 1) - k + w_i - skw_i = w_i$ so the map

$$H^0(\mathcal{O}(k - w_i)) \otimes \Gamma(D_+(x_i), \mathcal{O}(w_i)) \longrightarrow \Gamma(D_+(x_i), \mathcal{O}(k))$$

is surjective.

Finally let us notice that $\Gamma(D_+(x_i), \mathcal{O}(w_i)) \cong \Gamma(D_+(x_i), \mathcal{O})$. □

In general, $\mathcal{O}_{\mathbb{P}}(m)$ is not globally generated in the usual sense: see for example [4, Theorem 4B.7]. On the other hand we have the following proposition.

Proposition 2.14. *If $m > 0$ and $\mathcal{O}_{\mathbb{P}}(m)$ is globally generated then $\mathcal{O}_{\mathbb{P}}(m)$ is wgg.*

Proof. We want to show that for any $i = 0, \dots, n$ the map

$$\mu \colon \bigoplus_{i=0}^{n} H^0(\mathcal{O}(m - w_i)) \otimes \Gamma(D_+(x_i), \mathcal{O}) \longrightarrow \Gamma(D_+(x_i), \mathcal{O}(m))$$

is surjective.

Let $u \in \Gamma(D_+(x_i), \mathcal{O}(m))$: then $u = a x_i^{-s}$ with $s > 0$ and a a monomial of degree $sw_i + m$. Therefore $u = a x_i^{sk-s}/x_i^{sk}$ and

$$\deg(a x_i^{sk-s}) = sw_i + m + w_i(sk - s) = w_i sk + m$$

Now since $\mathcal{O}(m)$ is globally generated we can write $ax_i^{sk-s} = a'b$, where a' has degree m and b/x_i^{sk} has degree 0. This means that we can write $a'b = a''b'$ where $a'' = a'x_i^{-1}$ and has degree $m - w_i$, and $b' = bx_i$ so that b'/x_i^{sk} has degree w_i. In this way we have that the map

$$H^0(\mathcal{O}(m - w_i)) \otimes \Gamma(D_+(x_i), \mathcal{O}(w_i)) \longrightarrow \Gamma(D_+(x_i), \mathcal{O}(m))$$

is surjective. $\square$

Now we prove the analogues for weighted regularity of the main properties of Castelnuovo-Mumford regularity.

Theorem 2.15. *Let $\mathcal{F}$ be a wregular coherent sheaf on $\mathbb{P}$.*

(i) $H^0(\mathcal{F}(k))$ is spanned by $H^0(\mathcal{F}(k - w_0)) \oplus \cdots \oplus H^0(\mathcal{F}(k - w_n))$.

(ii) $\mathcal{F}$ is m-wregular for all $m \geq 0$.

(iii) $\mathcal{F}$ is wgg.

Proof. (i) is clear from the Koszul sequence (3) twisted by k. Moreover since $H^0(\mathcal{F}(k)) \neq 0$ the surjection is non-trivial.
(ii) is clear by the definition of wregularity.
(iii) we prove as follows. Choose $l \in k\mathbb{Z}$ so that $\mathcal{F}(k+l)$ and $\mathcal{O}(l)$ are globally generated, which holds for $l \gg 0$, and consider the (not exact!) sequence

$$\bigoplus_i H^0(\mathcal{F}(k - w_i)) \otimes H^0(\mathcal{O}(l)) \otimes \mathcal{O} \xrightarrow{\mu} H^0(\mathcal{F}(k)) \otimes H^0(\mathcal{O}(l)) \otimes \mathcal{O}$$
$$\xrightarrow{\mu'} H^0(\mathcal{F}(k + l)) \otimes \mathcal{O} \xrightarrow{\mu''} \mathcal{F}(k + l).$$

Notice that μ is non-trivial and surjective by (i), and μ' and μ'' are both surjective because $\mathcal{F}(k + l)$ and $\mathcal{O}(l)$ are globally generated. Near a point x, fix an isomorphism between $\mathcal{O}(l)$ and $\mathcal{O}$: this identifies $\mathcal{O}(l)$ with $\mathcal{O}$ and $\mathcal{F}(k)_x$ with $\mathcal{F}(k + l)_x$. Then $H^0(\mathcal{O}(l))$ becomes just a vector space of elements of the local ring $\mathcal{O}_x$, so we have that $\mathcal{F}(k)$ is wgg. $\square$

3 Monads on weighted projective spaces

In this section we assume that $n = \dim \mathbb{P} \geq 3$. We begin with a preliminary definition.

Definition 3.1. *Suppose that $\mathcal{E}$ and $\mathcal{E}'$ are vector bundles on a projective variety X. A surjective map $\eta\colon \mathcal{E} \to \mathcal{E}'$ is said to be* minimal *if no rank* 1 *direct summand of $\mathcal{E}'$ is the image of a line subbundle of $\mathcal{E}$.*

Next we recall the basic definitions about monads, due to Horrocks [9].

Definition 3.2. *A sequence of bundles on a projective variety X*

$$\mathcal{A} \xhookrightarrow{\alpha} \mathcal{B} \overset{\beta}{\twoheadrightarrow} \mathcal{C}$$

such that $\mathcal{A}$ and $\mathcal{C}$ are sums of line bundles, α is injective, β is surjective and $\beta\alpha = 0$ is called a monad *on X.*
The vector bundle $\mathcal{E} = \frac{\ker \beta}{\operatorname{im} \alpha}$ is called the homology *of the monad.*
A monad is said to be minimal *if the maps $\alpha^\vee \colon \mathcal{B}^\vee \to \mathcal{A}^\vee$ and $\beta \colon \mathcal{B} \to \mathcal{C}$ are minimal.*

In particular if $\mathcal{B}$ is a sum of line bundles, the maps α and β are just matrices and then minimal means that no matrix entry is a non-zero scalar both in α and in β.

Horrocks showed in [9] that every bundle $\mathcal{E}$ on $\mathbb{P}^n$ with $n \geq 3$ is the homology of a minimal monad. Now we extend this correspondence to $\mathbb{P}$. First we need a definition (see equation (1) in Section 1 for the notation).

Definition 3.3. *For $l \in \mathbb{Z}$, a* minimal l-resolution *of a bundle $\mathcal{E}$ is an exact sequence*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{C} \longrightarrow 0$$

in which $\mathcal{C}$ splits, π is minimal and $H^1(\mathcal{P}(\geq l)) = 0$.

Theorem 3.4. *Every bundle $\mathcal{E}$ on $\mathbb{P}$ is the homology of a minimal monad with $\mathcal{B}$ satisfying*

(i) $H^1_*(\mathcal{B}) = H^{n-1}_*(\mathcal{B}) = 0$

(ii) $H^i_*(\mathcal{B}) = H^i_*(\mathcal{E})$ *if* $1 < i < n-1$.

Proof. The module $H^i_*(\mathcal{E})$ has finite length for $0 < i < n$, because $\mathbb{P}$ is arithmetically Cohen-Macaulay and subcanonical, and for any $t \in \mathbb{Z}$ we have

$$H^i(X, \mathcal{E}(t)) \cong H^{n-i}(X, \mathcal{E}^\vee(-t+e))^\vee.$$

We start by proving that every bundle $\mathcal{E}$ has a minimal l-resolution for each $l \in \mathbb{Z}$. Consider the module $H^1(\mathcal{E}(\geq l))$ and a minimal system of generators $g_1, \ldots, g_r$. For each $i \geq l$, write q_i for the number of generators in degree i: since the module has finite length there is an l_0 such that $q_i = 0$ if $i > l_0$. So our system $g_1, \ldots, g_r$ is an element of

$$q_l H^1(\mathcal{E}(l)) \oplus \cdots \oplus q_{l_0} H^1(\mathcal{E}(l_0)) \cong H^1(\mathcal{E} \otimes \mathcal{C}^\vee)$$

where $\mathcal{C} = q_l\mathcal{O}(-l) \oplus \cdots \oplus q_{l_0}\mathcal{O}(-l_0)$. But, since $H^1(\mathcal{E} \otimes \mathcal{C}^\vee) \cong \operatorname{Ext}^1(\mathcal{C}, \mathcal{E})$ we have $\{g_i\} \in \operatorname{Ext}^1(\mathcal{C}, \mathcal{E})$, so we can associate an extension

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{P} \longrightarrow \mathcal{C} \longrightarrow 0 \qquad (9)$$

to our system of generators.

Now, looking at the sequence in cohomology we see that the map

$$f\colon H^0(\mathcal{C}(\geq l)) \longrightarrow H^1((\mathcal{E}(\geq l))$$

is surjective by construction. Moreover, since $\mathbb{P}$ is ACM, all the intermediate cohomology of $\mathcal{C}$ vanishes and we can conclude that $H^1(\mathcal{P}(\geq l)) = 0$. Thus the sequence (9) is an l-resolution, and it is minimal because the system $\{g_i\}$ was chosen minimal.

This l-resolution will be the last column of our display.

In the same way, for every $l' \in \mathbb{Z}$, we can find an l'-resolution

$$0 \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{Q}^\vee \longrightarrow \mathcal{A}^\vee \longrightarrow 0 \tag{10}$$

for $\mathcal{E}^\vee$ and the dual of (10) will be the first row, so we have

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \rightarrow & \mathcal{A} & \rightarrow & \mathcal{Q} & \rightarrow & \mathcal{E} & \rightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & \mathcal{P} & & \\ & & & & & & \downarrow & & \\ & & & & & & \mathcal{C} & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

where $\mathcal{A} = \bigoplus_i \mathcal{O}(a_i)$ is a bundle without intermediate cohomology, with $a_i \geq l'$ for all i. Moreover

$$H^{n-1}(\mathcal{Q}(\leq -l' + e)) \cong H^1(\mathcal{Q}^\vee(\geq l')) = 0.$$

We observe that if $n \geq 3$

$$\operatorname{Ext}^i(\mathcal{C}, \mathcal{A}) = H^i(\mathcal{C}^\vee \otimes \mathcal{A}) = 0$$

for $i = 1$, 2. Then applying the functor $\operatorname{Hom}(\bullet, \mathcal{A})$ to (9) we have

$$0 = \operatorname{Ext}^1(\mathcal{C}, \mathcal{A}) \to \operatorname{Ext}^1(\mathcal{P}, \mathcal{A}) \to \operatorname{Ext}^1(\mathcal{E}, \mathcal{A}) \to \operatorname{Ext}^2(\mathcal{C}, \mathcal{A}) = 0,$$

so

$$\operatorname{Ext}^1(\mathcal{P}, \mathcal{A}) \cong \operatorname{Ext}^1(\mathcal{E}, \mathcal{A}).$$

This means that the extension in our row (the dual of (10)) comes from the unique extension

$$0 \longrightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \longrightarrow \mathcal{P} \longrightarrow 0$$

and we have

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{A} & \to & \mathcal{Q} & \to & \mathcal{E} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{A} & \xrightarrow{\alpha} & \mathcal{B} & \to & \mathcal{P} & \to & 0\ . \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{C} & = & \mathcal{C} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

This is the display of the monad

$$\mathcal{A} \xhookrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C}.$$

The minimality comes from the minimality of the two resolutions. From the first row we see that $H^i_*(\mathcal{E}) \cong H^i_*(\mathcal{Q})$ for $0 < i < n-1$. Looking at the first column in cohomology,

$$0 = H^{n-1}(\mathcal{Q}(\leq -l'+e)) \to H^{n-1}(\mathcal{B}(\leq -l'+e)) \to H^{n-1}(\mathcal{C}(\leq -l'+e)) = 0,$$

we have that $H^{n-1}(\mathcal{B}(\leq -l'+e)) = 0$, and $H^i_*(\mathcal{E}) \cong H^i_*(\mathcal{Q}) \cong H^i_*(\mathcal{B})$ for $1 < i < n-1$.

Looking at the second row in cohomology,

$$0 = H^1(\mathcal{A}(\geq l)) \to H^{n-1}(\mathcal{B}(\geq l)) \to H^{n-1}(\mathcal{C}(\geq l)) = 0,$$

we see that $H^1(\mathcal{B}(\geq l)) = 0$. If we choose l and l' small enough we get the claimed conditions (i) and (ii) above:

(i) $H^1_*(\mathcal{B}) = H^{n-1}_*(\mathcal{B}) = 0$

(ii) $H^i_*(\mathcal{B}) = H^i_*(\mathcal{E})$ for $1 < i < n-1$.

□

If the bundles in the monad all split, we can get some results about wregularity for $\mathcal{E}$.

Definition 3.5. *A monad on $\mathbb{P}$ is called* quasi-linear *if it has the form*

$$\bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}}(a_i) \xhookrightarrow{\alpha} \bigoplus_{l=1}^{r+s+t} \mathcal{O}_{\mathbb{P}}(b_l) \xrightarrow{\beta} \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}}(c_j). \tag{11}$$

By convention, we shall write the twists in increasing order, $a_i \leq a_{i+1}$, etc.: note that this is the opposite of our convention for the weights.

We prove an analogue of [7, Theorem 3.2].

Theorem 3.6. *Let $\mathcal{E}$ be a rank r vector bundle on $\mathbb{P}$ which is the homology of a quasi-linear monad* (11). *Put $c = \sum_{j=1}^{t} c_t$. Then $\mathcal{E}$ is m-wregular for any integer m such that $H^0(\mathcal{E}((m+1)k)) \neq 0$ and*

$$(m+1)k \geq \max\{(n-1)c_t - (b_1 + \cdots + b_{t+n}) - (\mathbf{w} - \mathbf{w}_1) + 1 + c, -b_1 + 1, -a_1 + 1\}. \tag{12}$$

Proof. Let us consider the short exact sequences from the display of the monad:

$$0 \to \mathcal{K} \to \bigoplus_{l=1}^{r+s+t} \mathcal{O}_{\mathbb{P}}(b_l) \xrightarrow{\beta} \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}}(c_j) \to 0$$

and

$$0 \to \bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}}(a_i) \to \mathcal{K} \to \mathcal{E} \to 0.$$

We get $H^i(\mathcal{K}(p)) = H^i(\mathcal{E}(p)) = 0$ for any integer p and any $i = 2, \ldots, n-2$. Moreover if $p \geq \max\{-b_1 - \mathbf{w} + 1, -a_1 - \mathbf{w} + 1\}$ we have also $H^i(\mathcal{K}(p)) = H^i(\mathcal{E}(p)) = 0$ for $i \geq n-1$. So if $(m+1)k \geq \max\{-b_1 + 1, -a_1 + 1\}$ we may conclude that

$$H^n(\mathcal{E}((m+1)k - \mathbf{w}) = 0.$$

To see which are the p for which $H^1(\mathcal{K}(p)) \cong H^1(\mathcal{E}(p)) = 0$, we consider the Buchsbaum-Rim complex associated to

$$\mathcal{F} = \bigoplus_{l=1}^{r+s+t} \mathcal{O}_{\mathbb{P}}(b_l) \xrightarrow{\beta} \mathcal{G} = \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}}(c_j),$$

which is the complex

$$\begin{aligned} S^{r+s-1}\mathcal{G}^\vee \otimes \wedge^{r+s+t}\mathcal{F} \to S^{r+s-2}\mathcal{G}^\vee \otimes \wedge^{r+s+t-1}\mathcal{F} \to \cdots \to S^2\mathcal{G}^\vee \otimes \wedge^{3+t}\mathcal{F} \\ \to \mathcal{G}^\vee \otimes \wedge^{2+t}\mathcal{F} \to \wedge^{1+t}\mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}(c) \to \mathcal{G} \otimes \mathcal{O}_{\mathbb{P}}(c) \to 0. \end{aligned} \tag{13}$$

We cut (13) into short exact sequences

$$\begin{aligned} &0 \to \mathcal{K} \otimes \mathcal{O}_{\mathbb{P}}(c) \to \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}(c) \to \mathcal{G} \otimes \mathcal{O}_{\mathbb{P}}(c) \to 0, \\ &0 \to \mathcal{K}_2 \to \wedge^{1+t}\mathcal{F} \to \mathcal{K} \otimes \mathcal{O}_{\mathbb{P}}(c) \to 0, \\ &0 \to \mathcal{K}_3 \to \mathcal{G}^\vee \otimes \wedge^{2+t}\mathcal{F} \to \mathcal{K}_2 \to 0, \\ &\qquad\qquad \vdots \\ &0 \to \mathcal{K}_n \to S^{n-2}\mathcal{G}^\vee \otimes \wedge^{t+n-1}\mathcal{F} \to \mathcal{K}_{n-1} \to 0, \\ &0 \to \mathcal{K}_{n+1} \to S^{n-1}\mathcal{G}^\vee \otimes \wedge^{t+n}\mathcal{F} \to \mathcal{K}_n \to 0. \end{aligned}$$

Note that

$$S^{n-1}\mathcal{G}^\vee \otimes \wedge^{t+n}\mathcal{F} = \bigoplus_q \mathcal{O}_{\mathbb{P}}(d_q)$$

where $d_q = (b_{l_1} + \cdots + b_{l_{t+n}}) - (c_{j_1} + \cdots + c_{j_{n-1}})$ with $l_1 < \cdots < l_{t+n}$ and $j_1 \leq \cdots \leq j_{n-1}$. Now from the cohomological exact sequences associated to the above short exact sequences tensored by $\mathcal{O}_{\mathbb{P}}(p-c)$ we get

$$\begin{aligned} h^1(\mathcal{K}(p)) = h^2(\mathcal{K}_2(p-c)) = \cdots = h^n(\mathcal{K}_n(p-c)) \\ \leq h^n(S^{n-1}\mathcal{G}^\vee \otimes \wedge^{t+n}\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}(p-c)) \end{aligned} \quad (14)$$

which is zero if $p \geq (n-1)c_t - (b_1 + \cdots + b_{t+n}) - \mathbf{w} + 1 + c$. In fact, since

$$(n-1)c_t - (c_{j_1} + \cdots + c_{j_{n-1}}) \geq 0$$

and

$$(b_{l_1} + \cdots + b_{l_{t+n}}) - (b_1 + \cdots + b_{t+n}) \geq 0,$$

we have $d_q + p - c \geq -\mathbf{w}$. So we get

$$H^1(\mathcal{E}((m+1)k - (w_{n-1} + w_n))) = 0$$

if $(m+1)k \geq (n-1)c_t - (b_1 + \cdots + b_{t+n}) - (w_0 + \cdots + w_{n-2}) + 1 + c$. $\square$

Remark 3.7. *In the case of* $\mathbb{P} = \mathbb{P}^n$ *the bound* (12) *reduces to*

$$m + 1 \geq \max\{(n-1)c_t - (b_1 + \cdots + b_{t+n}) - (n-1) + 1 + c, -b_1 + 1, -a_1 + 1\}$$

which is precisely the bound of [7, Theorem 3.2]

Finally we want to discuss the sharpness of the bound in Theorem 3.6.

Example 3.8. *Take* $\mathbb{P} = \mathbb{P}(3, 2, 2, 1)$ *and consider the bundle* $\mathcal{E}$ *given by the monad*

$$\mathcal{O}_{\mathbb{P}}(-2) \xhookrightarrow{\alpha} \mathcal{O}_{\mathbb{P}}(-1) \oplus \mathcal{O}_{\mathbb{P}}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}}(1) \xrightarrow{\alpha^\vee} \mathcal{O}_{\mathbb{P}}(2),$$

where $\alpha^\vee = (x_0, x_1, x_2, x_3)$. *In this case the bound given by* (12) *is sharp.*

In fact, we have $k = 3$, $a_1 = -2, b_1 = -1, b_2 = b_3 = 0, b_4 = 1$ and $c_t = c = c_1 = 2$, so we get

$$(m+1)3 \geq \max\{(2)2 - (0) - (3) + 1 + 2, 1 + 1, 2 + 1\} = 4,$$

so $m = 1$. On the other hand we notice that $\mathcal{E}$ is not wregular (i.e. we cannot take $m = 0$), so the bound is sharp. In fact from the sequences

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}}(-1) \oplus \mathcal{O}_{\mathbb{P}}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}}(1) \to \mathcal{O}_{\mathbb{P}}(2) \to 0$$

and

$$0 \to \mathcal{O}_{\mathbb{P}}(-2) \to \mathcal{K} \to E \to 0,$$

we get $H^3(\mathcal{E}(3 - 8)) \neq 0$.

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