Cusp forms: a clarification

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It is necessary to check that the cusp forms constructed in [GHS1] are indeed cusp forms in the strong sense, i.e. that on the toroidal compactification they vanish to order at least 1 on every boundary component.

Lemma 1.1 *Let $L = 2U \oplus L_0$ be a lattice of signature $(2, n)$ containing two hyperbolic planes and let f be a modular form with character* det *or trivial character that vanishes at every cusp. Then f is a cusp form, vanishing to order at least 1 on every toroidal boundary component.*

Proof. It is clearly sufficient to show that the order of vanishing of f along any boundary component F is an integer. If f is of weight k then near the boundary component F we have

$$f(gZ) = j(g, Z)\chi(g)f(Z)$$

where $Z \in \mathcal{D}_L(F)$ and $g \in U(F)_{\mathbb{Z}}$, for some factor of automorphy j and χ the character of the modular form f. If the factor $j(g, Z)\chi(g)$ is equal to 1 for every $g \in U(F)_{\mathbb{Z}}$ then f is a section of a line bundle near F and its order of vanishing along F is therefore an integer.

Under the hypotheses of the lemma, we do indeed have $\chi(g) = 1$ because g is unipotent and therefore has trivial determinant. It therefore remains to check that the factor of automorphy $j(g, Z)$ is also trivial for $g \in U(F)_{\mathbb{Z}}$.

If F is of dimension 1 then according to [GHS1, Lemma 2.25] we have

$$U(F) = \left\{ \begin{pmatrix} I & 0 & \begin{pmatrix} 0 & ex \\ -x & 0 \end{pmatrix} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \mid x \in \mathbb{R} \right\}. \tag{1}$$

But the automorphy factor is given by the last ($(n + 2)$-th) coordinate of $g\big(p(Z)\big) \in \mathcal{D}_L$, where

$$\begin{aligned} p\colon \mathcal{H}_n &\longrightarrow \mathcal{D}_L \\ Z = (z_n, \ldots, z_1) &\longmapsto \big(-\frac{1}{2}(Z, Z)_{L_1} : z_n : \cdots : z_1 : 1\big) \end{aligned} \tag{2}$$

is the tube domain realisation of $\mathcal{D}_L$: see [GHS2, Section 3] or [G, Section 2]. From this description it is immediate that $j(g, Z) = 1$ for $g \in U(F)_{\mathbb{Z}}$.

If F is of dimension 0 then F corresponds to some isotropic vector $\mathbf{v} \in L$, and $U(F)$ is the centre of the unipotent radical of the stabiliser of $\mathbf{v}$. With respect to a basis of $L \otimes \mathbb{Q}$ in which $\mathbf{v}$ is the last ($(n + 2)$-th) element, the penultimate ($(n+1)$-th) element $\mathbf{w}$ is also isotropic and the remaining elements span the orthogonal complement L' of those two, we have

$$U(F) = \left\{ \begin{pmatrix} I_n & \mathbf{b} & 0 \\ 0 & 1 & 0 \\ \mathbf{c} & x & 1 \end{pmatrix} \mid L'\mathbf{b} + \alpha\mathbf{c} = 0,\ {}^t\mathbf{b}L'\mathbf{b} + 2\alpha\, x = 0 \right\}. \qquad (3)$$

Here $\mathbf{b}$ and $\mathbf{c}$ are column vectors, $x \in \mathbb{R}$ and $\alpha = (\mathbf{w}, \mathbf{v})_L$: compare [Ko, (2.7)]. In this case the tube domain is contained in $\mathbb{C}^n$ and is identified with a subset of the locus $z_{n+1} = 1 \subset \mathcal{D}_L^\bullet$. The automorphy factor $j(g, Z)$ is therefore equal to the $(n+1)$-th coordinate of $g(p(Z))$, where $p(Z)_{n+1} = 1$; but this is 1 as $p(Z)$ is a column vector. □

From the proof it follows that any cusp form f for an arithmetic subgroup $\Gamma < \mathrm{O}(L)$ vanishes to order at least 1 along a toroidal divisor unless the character χ associated with f is non-trivial (and not det) on $U(F)_\mathbb{Z} = U(F) \cap \Gamma$. The existence of such a character appears to be a strong condition on Γ: see [GHS3].

References

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