Numerical obstructions to abelian surfaces in toric Fano 4-folds

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On the list of smooth toric Fano 4-folds ([1, 12]), of which there are 124, there remain 19 for which it is not known whether there are embedded abelian surfaces. In order to avoid trivialities, we consider only totally nondegenerate embedded abelian surfaces, in the sense of [11, 12].

Definition 1 *An embedding $\phi\colon Y \hookrightarrow X$ from a smooth variety Y to a smooth toric 4-fold X, or its image $\phi(Y) \subset X$, is totally nondegenerate if $\phi(Y) \cap D \neq \emptyset$ for any torus-invariant prime divisor D.*

Results from [3, 4, 5, 6, 7, 11, 13] about totally nondegenerate embeddings of abelian surfaces in toric Fano 4-folds may be summarised as follows.

Theorem 1 *Suppose X is a smooth toric Fano 4-fold. Then X admits a totally nondegenerate abelian surface if $X = \mathbb{P}^4$, if $X = \mathbb{P}^1 \times \mathbb{P}^3$ (type $\mathcal{B}_4$), or if X is a product of two smooth toric Del Pezzo surfaces (i.e. of type $\mathcal{C}_4$, $\mathcal{D}_{13}$, $\mathcal{H}_8$, $\mathcal{L}_7$, $\mathcal{L}_8$, $\mathcal{L}_9$, $\mathcal{Q}_{10}$, $\mathcal{Q}_{11}$, $\mathcal{K}_4$, $\mathcal{U}_5$, $\mathcal{S}_2 \times \mathcal{S}_2$, $\mathcal{S}_2 \times \mathcal{S}_3$ or $\mathcal{S}_3 \times \mathcal{S}_3$). Otherwise there is no such embedding, unless possibly X is of type $\mathcal{C}_3$, $\mathcal{D}_7$, $\mathcal{D}_{10}$, $\mathcal{D}_{11}$, $\mathcal{D}_{14}$, $\mathcal{D}_{17}$, $\mathcal{D}_{18}$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{L}_{11}$, $\mathcal{L}_{13}$, $\mathcal{I}_9$, $\mathcal{Q}_{16}$, $\mathcal{U}_8$, $\mathcal{V}_4$, $\mathcal{W}$, $\mathcal{Z}_1$ or $\mathcal{Z}_2$.*

The notation for the types is taken from [12], which apart from the addition of the missing case $\mathcal{W}$ is the same as in [1].

It is claimed in [11] that there are no totally nondegenerately embedded abelian surfaces in toric Fano 4-folds of types $\mathcal{C}_1$ or $\mathcal{C}_2$ but that there are in the case of type $\mathcal{C}_3$. However, it was discovered by T. Kajiwara that there is an error in the calculations in [11]: in the case of $\mathcal{C}_3$ it is claimed, wrongly, that a certain class satisfies the condition given by the double-point formula (Lemma 1.2 in this paper). I take this opportunity to thank Professor Kajiwara for pointing this out to me.

In view of this quantity of previous work it is perhaps surprising that more can be achieved by completely elementary methods. Nevertheless, in this paper we eliminate or restrict the remaining cases by examining the numerical conditions imposed on a class α in $A^2(X)$ by the condition that it should contain a minimal abelian surface. The methods are entirely elementary but they do rely on some small computer calculations.

There are also some geometric conditions. A simple example is that if $f\colon X \to S$ is a map to a rational surface then $f \circ \phi\colon A \to S$ cannot have degree 1. We make some remarks on such conditions but we do not attempt to exploit them systematically here.

The most conclusive results of this paper are as follows.

Theorem 2 *Suppose X is a smooth toric Fano 4-fold. Then X does not contain a totally nondegenerate abelian surface if it is of type $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{D}_{10}$ or $\mathcal{D}_{18}$.*

In many other cases we can prove that if a class $\alpha \in A^2(X)$ contains a totally nodegenerate abelian surface then it satisfies certain restrictive conditions.

- If X is of type $\mathcal{D}_7$, $\mathcal{G}_4$, $\mathcal{I}_9$, $\mathcal{L}_{13}$, $\mathcal{Q}_{16}$, $\mathcal{U}_8$ or $\mathcal{Z}_1$, then the class α of such a totally nondegenerate abelian surface in X must belong to a short finite list.
- If X is of type $\mathcal{D}_{14}$ or $\mathcal{D}_{17}$ then α belongs to a short finite list or satisfies other strong conditions.
- If X is of type $\mathcal{D}_{11}$, $\mathcal{G}_3$, $\mathcal{G}_5$, $\mathcal{W}$ or $\mathcal{Z}_2$ then the coefficients of the class α satisfy a condition modulo 3 and some other weak conditions.
- If X is of type $\mathcal{L}_{11}$ or $\mathcal{V}_4$ then we obtain only weak conditions on the class α.

The precise statements may be found in sections 3–6.

1 Preliminaries

In this section, unless otherwise stated, X is any smooth projective toric 4-fold.

Suppose that A is a (minimal) abelian surface, and that there is a totally nondegenerate embedding $\phi\colon A \to X$. Then ϕ determines a class $\alpha = [\phi(A)] \in A^2(X) \cong H^4(X, \mathbb{Z})$. There are strong restrictions on α. If X is one of the 124 smooth toric Fano 4-folds listed in [12], then in most cases the restrictions are so strong that no such ϕ can exist.

Let $D_1, \ldots, D_r$ be the torus-invariant prime divisors on X, and let Γ_i be the corresponding curve on A, so $\mathcal{O}_A(\Gamma_i) = \phi^*\big(\mathcal{O}_X(D_i)\big)$. We denote by L the quadratic form given by $L_{ij} = (\Gamma_i \cdot \Gamma_j)_A$: note that $L_{ij} = D_i D_j \alpha \in A^4(X) \cong \mathbb{Z}$. Since X is assumed to be smooth and projective, it follows easily (see [10, 2.1]) that there exist $\lambda_i \geq 0$ such that $\sum \lambda_i D_i$ is ample.

Lemma 1.1 *The quadratic form L has the following properties.*

(i) The rank of L is $e \leq 4$ and the signature is $(1, e-1)$.

(ii) L is an even form (in particular, L_{ii} is even), and $L_{ij} \geq 0$ for all i, j.

(iii) If $L_{ij} = 0$ then $L_{ii} = L_{jj} = 0$.

(iv) For any i, there exists j such that $L_{ij} > 0$.

(v) If $L_{ij} = 0$ then $(L_{i1} : \ \ldots : L_{ir}) = (L_{j1} : \ \ldots : L_{jr}) \in \mathbb{P}^{r-1}_{\mathbb{Q}}$. In particular $L_{ik}L_{jl} = L_{il}L_{jk}$ for all k and l.

(vi) If $L_{ii} = 0$ and $L_{ij} = 1$ then $2L_{jk} \geq L_{jj}L_{ik}$ for all k.

Proof.

(i) L is isomorphic to a sublattice of $H^2(A, \mathbb{Z})$ and is generated by algebraic cycles. But $\rho = \operatorname{rk} \operatorname{Pic} A \leq 4$ (for any abelian surface A), see [8], and the signature of the intersection form on $\operatorname{Pic} A$ is $(1, \rho - 1)$ by the Hodge index theorem. Moreover, $\sum \lambda_i D_i$ is ample on X and hence L is not negative semidefinite.

(ii) These are standard properties of the intersection form on an abelian surface.

(iii) See for instance [13], or note that by the Nakai criterion [8, Corollary 4.3.3] an effective curve with positive square on an abelian surface is ample.

(iv) Since $\sum \lambda_i D_i$ is ample on X and ϕ is an embedding, $\sum \lambda_i \Gamma_i$ is ample on A. Therefore $0 < \Gamma_i \cdot (\sum \lambda_j \Gamma_j) = \sum_j \lambda_j L_{ij}$ so some L_{ij} is positive.

(v) Since $L_{ij} = 0$ we have by Lemma 1.1(iii) that $\Gamma_i^2 = \Gamma_j^2 = 0$. Let Γ be a reduced irreducible component of the effective (possibly nonreduced) curve Γ_i. If Γ' is any reduced irreducible component of Γ_i (or Γ_j) we have $0 \leq \Gamma \cdot \Gamma' \leq \Gamma \cdot \Gamma_i \leq \Gamma_i^2 = 0$, so in particular $\Gamma^2 = 0$. Since A contains no rational curves, Γ must be an elliptic curve and we consider the exact sequence

$$0 \longrightarrow \Gamma \longrightarrow A \longrightarrow E \longrightarrow 0.$$

Every component Γ' as above is contained in a fibre of $A \to E$, since $\Gamma' \cdot \Gamma = 0$, so the numerical classes $[\Gamma_i]$ and $[\Gamma_j]$ are both integer multiples of $[\Gamma]$ and therefore rational multiples of each other.

(vi) As in the proof of Lemma 1.1(v), $[\Gamma_i] = n[\Gamma]$ for some elliptic curve Γ and some $n \in \mathbb{N}$. But $n|L_{ij}$ so $n = 1$ and Γ_i is a reduced elliptic curve. So we have an exact sequence

$$0 \longrightarrow \Gamma_i \longrightarrow A \longrightarrow E \longrightarrow 0$$

where E is an elliptic curve: in fact $A \cong \Gamma_i \times E$. But $\Gamma_j \to E$ is generically finite of degree 1, so $[\Gamma_j] = [E + m\Gamma_i]$ for some $m \in \mathbb{Z}$ (identifying E with the zero section). Since $E \cdot \Gamma_i = 1$ and $E^2 = \Gamma_i^2 = 0$, we have $L_{jj} = \Gamma_j^2 = 2m$, and hence

$$0 \leq 2\Gamma_k \cdot E = 2L_{jk} - 2mL_{ik} = 2L_{jk} - L_{jj}L_{ik}.$$

as claimed.

□

In particular, if $L_{ii} = 0$ then Γ_i is a union of disjoint smooth genus 1 curves (with multiplicity), all translates of one another.

Another constraint comes from the double-point formula.

Lemma 1.2 *If $\alpha \in A^2(X)$ is the class of a smooth minimal abelian surface embedded in X by $\phi\colon A \hookrightarrow X$ then $\alpha^2 - \alpha \cdot c_2(X) = 0$.*

Proof. The double-point scheme is given ([2, Theorem 9.3]) by

$$\begin{aligned} \mathbb{D}(\phi) &= \phi^*\phi_*[A] - \big(c(\phi^*T_X)c(T_A)^{-1}\big)_2 \cap [A] \\ &= \phi^*\phi_*[A] - c_2(\phi^*T_X) \cap [A] \\ &= \big([\phi(A)] - c_2(T_X)\big) \cap [\phi(A)]. \end{aligned}$$

Since $\phi(A)$ is smooth $\mathbb{D}(\phi)$ has length zero, i.e. $\big([\phi(A)] - c_2(T_X)\big) \cdot [\phi(A)] = 0$ in $A^4(X) \cong \mathbb{Z}$, which is what is claimed. □

If there is a toric map from X to a curve or a surface it is sometimes possible to use it to obtain further restrictions on L_{ij}. This happens for a few of the smooth toric Fano 4-folds.

2 Notation and methods

For the rest of the paper, we let $X = T_N\mathrm{emb}(\Sigma)$ be a smooth toric Fano 4-fold given by a fan Σ. Following [1] we put $\Sigma^{(1)} = \{\tau_i \in \Sigma \mid \dim \tau_i = 1\}$ and $G(\Sigma) = \{x_i \mid \tau_i \in \Sigma^{(1)}\}$, where x_i is the unique element of τ_i that generates the lattice $\tau_i \cap N$. The fan Σ is determined up to an integral change of basis by the primitive relations (see [1]), which are tabulated in [14, 1.9]. We adopt the convention that $D_i = \overline{\mathrm{orb}(\tau_i)}$ denotes the divisor corresponding to $x_i \in G(\Sigma)$ as listed in [14].

For each type of Fano 4-fold we use *Macaulay2* to calculate the Stanley-Reisner ring, i.e. $H^{2*}(X,\mathbb{Z}) = A^*(X)$ with the intersection product. It is generated in degree 1, by $D_1,\dots,D_r$. Note that $\operatorname{rk}\operatorname{Pic}(X) = r-4$. There are linear relations $\sum_i m\big(\mathbf{e}(\tau_i)\big)D_i = 0$ for $m \in N^\vee$ (of course it is enough to take m in a $\mathbb{Z}$-basis of $N^\vee$) and multiplicative relations $\prod_{i\in I} D_i = 0$ if $\sum_{i\in I}\tau_i \notin \Sigma$. All these relations are easily computed from the primitive relations.

We also calculate a $\mathbb{Z}$-basis $\beta_1,\dots,\beta_s$ for $A^2(X)$, in which each β_k is of the form $[D_iD_j]$ for some $1 \le i \le j \le r$. This is simply a basis for the degree 2 part of the Stanley-Reisner ring. For a class $\alpha = \sum a_k\beta_k \in A^2(X)$ we calculate the intersection matrix $L_{ij} = D_iD_j\alpha \in A^4(X) \cong \mathbb{Z}$.

For ease of reading we write a, b, c, etc. instead of a_1, a_2, a_3 etc.

We do not always need the whole of L. If $i<j$ and $D_i \equiv D_j$ (numerical equivalence) then $L_{ik} = L_{jk}$ for all k and we may omit the redundant ith row and column altogether. We denote by Λ the submatrix of L thus obtained. Thus Λ is the restriction of L to the span of all D_i that are not numerically equivalent to any D_j with $j>i$, and $\Lambda_{\mu\nu} = L_{ij}$ if D_i and D_j are in the μth and νth (in the revlex order) numerical equivalence class respectively.

To apply this to particular cases we assume that α is the class of a smooth minimal abelian subvariety of X, and deduce conditions on $\mathsf{a},\mathsf{b},\dots$ which in many cases lead to a contradiction. There is some further computation involved, which was done using *Maple*.

3 Types $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$

In this section we re-examine the cases $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. In particular, we correct the error in [11] described in the introduction above, by giving a new analysis of the case $\mathcal{C}_3$. For $\mathcal{C}_1$ and $\mathcal{C}_2$ the results are as in [11], but we reprove them here more concisely and in the notation used in the rest of this paper.

In these cases $\operatorname{rk}\operatorname{Pic}(X) = 2$. The fans of the Fano 4-folds of these types are given by the primitive relations shown in the table.

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
$x_1+x_2+x_3 =$	0	0	0
$x_4+x_5+x_6 =$	$2x_1$	x_1	x_1+x_2

For X of type $\mathcal{C}_1$ or $\mathcal{C}_2$ we have $D_2 \equiv D_3$ and $D_4 \equiv D_5$, and a basis for $A^2(X)$ is $\beta_1 = D_3^2$, $\beta_2 = D_3D_6$, $\beta_3 = D_6^2$.

Proposition 3.1 *There are no totally nondegenerate abelian surfaces in toric Fano 4-folds of types $\mathcal{C}_1$ or $\mathcal{C}_2$.*

Proof. We need the observation from [11] that (in our present notation) $\mathsf{a} \ge 6$ (and is even). In fact $\mathsf{a} = \Lambda_{33} = D_6^2\alpha$ in both cases, and D_4, D_5

and D_6 are the pull-backs of lines in $\mathbb{P}^2$ under a projection $p\colon X \to \mathbb{P}^2$. These Fano 4-folds are the projectivisations of toric (hence decomposible) rank 3 vector bundles on $\mathbb{P}^2$, namely $\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)\big)$ for $\mathcal{C}_1$ and $\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\big)$ for $\mathcal{C}_2$, and p is the projection. Hence if $A \subset X$ is an abelian surface with $[A] = \alpha$, and $p|_A\colon A \to \mathbb{P}^2$ is surjective, then by Riemann-Roch

$$3 = h^0\big(\mathcal{O}_{\mathbb{P}^2}(1)\big) \le h^0\big(p|_A^*\mathcal{O}_{\mathbb{P}^2}(1)\big) = \frac{1}{2}D_6^2\alpha = \frac{1}{2}\mathsf{a}.$$

On the other hand, if $h^0\big(\mathcal{O}_A(D_6)\big) \le 2$ then $p|_A$ is given by a subsystem of the linear system $|D_6|$ and hence it maps A onto a line in $\mathbb{P}^2$. But the inverse image of this line is a smooth toric 3-fold and hence contains no abelian surface (see for instance [6]).

For $\mathcal{C}_1$ we find that $\Lambda_{12} = \mathsf{c}$ and $\Lambda_{13} = \mathsf{b}$, so $\mathsf{b} \ge 0$ and $\mathsf{c} \ge 0$ by Lemma 1.1(ii). But the double-point formula (Lemma 1.2) gives

$$\begin{aligned} 0 &= 4\mathsf{a}^2 + 4\mathsf{ab} + \mathsf{b}^2 + 2\mathsf{ac} - 19\mathsf{a} - 11\mathsf{b} - 3\mathsf{c} \\ &= (\mathsf{a}+\mathsf{b})^2 + (2\mathsf{a}-11)\mathsf{b} + (\mathsf{a}-3)\mathsf{c} + (3\mathsf{a} - 19 + \mathsf{c})\mathsf{a} \end{aligned}$$

which is obviously positive if $\mathsf{a} \ge 6$.

For $\mathcal{C}_2$ we again have $0 \le \Lambda_{12} = \mathsf{c}$ and $0 \le \Lambda_{13} = \mathsf{b}$. The double-point formula gives

$$\begin{aligned} 0 &= \mathsf{a}^2 + 2\mathsf{ab} + \mathsf{b}^2 + 2\mathsf{ac} - 10\mathsf{a} - 10\mathsf{b} - 3\mathsf{c} \\ &= (\mathsf{a}-10)\mathsf{a} + \mathsf{b}^2 + (2\mathsf{a}-10)\mathsf{b} + (2\mathsf{a}-3)\mathsf{c} \end{aligned}$$

which is positive if $\mathsf{a} \ge 10$. But $0 \le \Lambda_{11} = \mathsf{c} - \mathsf{b}$, so we get

$$0 = -16 + 6\mathsf{b} + \mathsf{b}^2 + 13\mathsf{c} \ge \mathsf{b}^2 + 19\mathsf{b} - 16$$

for $\mathsf{a} = 8$, and for $\mathsf{a} = 6$

$$0 = -24 + 2\mathsf{b} + \mathsf{b}^2 + 9\mathsf{c} \ge \mathsf{b}^2 + 11\mathsf{b} - 24.$$

Neither of these has a solution with b, c non-negative integers. $\square$

For X of type $\mathcal{C}_3$ we have $D_1 \equiv D_2$ and $D_4 \equiv D_5 \equiv D_6$. A basis for $A^2(X)$ is given by $\beta_1 = D_3^2$, $\beta_2 = D_3D_6$ and $\beta_3 = D_6^2$.

Proposition 3.2 *There are no totally nondegenerate abelian surfaces in a toric Fano 4-fold of type $\mathcal{C}_3$.*

Proof. The computation gives

$$\Lambda = \begin{pmatrix} \mathsf{c} & \mathsf{x}+\mathsf{c} & \mathsf{x} \\ \mathsf{x}+\mathsf{c} & \mathsf{a}+2\mathsf{x}+\mathsf{c} & \mathsf{a}+\mathsf{x} \\ \mathsf{x} & \mathsf{a}+\mathsf{x} & \mathsf{a} \end{pmatrix}$$

where $\mathsf{x} = \mathsf{a} + \mathsf{b}$. As Λ has rank at most 2, the Hodge index theorem gives $\mathsf{x}^2 \geq \mathsf{ac}$, and $\mathsf{x} > 0$ since otherwise a row of Λ vanishes. We may assume $\mathsf{a} \geq 6$, as in Proposition 3.1.

The double-point formula gives

$$\begin{aligned} 0 &= 3\mathsf{a}^2 + 4\mathsf{ab} + \mathsf{b}^2 + 2\mathsf{ac} - 17\mathsf{a} - 11\mathsf{b} - 3\mathsf{c} \\ &= 2\mathsf{ax} + \mathsf{x}^2 + 2\mathsf{ac} - 6\mathsf{a} - 11\mathsf{x} - 3\mathsf{c} \qquad (1) \\ &\geq \mathsf{x} + 3\mathsf{ac} - 6\mathsf{a} - 3\mathsf{c} \\ &> 3((\mathsf{a}-1)(\mathsf{c}-2) - 2). \end{aligned}$$

Hence $\mathsf{c} = 0$ or $\mathsf{c} = 2$. Since $\mathsf{x} \leq 11$ immediately from equation (1), it is simple to check that the only possibility is $\mathsf{c} = 0$, $\mathsf{x} = 4$, $\mathsf{a} = 14$.

Suppose this occurs. Then on $\phi(A)$ we have $D_1^2 = 0$, so D_1 is a union of elliptic curves in A. It cannot be a single reduced elliptic curve because it moves in the linear system $|D_1| \ni D_2$, so A has a $(1,7)$-polarisation given by D_6, though which $p\colon A \to \mathbb{P}^2$ factors, and an elliptic curve C on which D_6 has degree 1 or 2. The former case is evidently impossible since then we do not even get a morphism to $\mathbb{P}^2$ defined on C.

In the latter case, the linear system $|D_6|$ is not very ample: $(A, \mathcal{O}_A(D_6))$ is bielliptic and $\phi_{|D_6|}$ maps A 2-to-1 onto its image S, which is an elliptic ruled surface in $\mathbb{P}^6$. The fibres of this ruled surface are the Kummer curves of the translates of C. Hence pairs of points in each such translate are identified both by the subsystem $|p^*(\mathcal{O}_{\mathbb{P}^2}(1))| \subset |D_6|$ and by the linear system $|D_1|$. In particular the induced map $A \to \mathbb{P}^2 \times \mathbb{P}^1$ factors through S. According to [11, Section 2], to specify a map to $\mathcal{C}_3$ we have to choose also a subsystem of $|D_1 + D_6|$ given by D_3, i.e. a section $\sigma \in H^0\big(\mathcal{O}_A(D_1 + D_6)\big)$.

The curve $\Gamma_3 = D_3|_A = \{\sigma = 0\} \subset A$ has genus given by $2p_a(\Gamma_3) - 2 = (D_1 + D_6)^2 = 22$ so $p_a(\Gamma_3) = 12$. Consider the image $\Delta_3 = \phi_{|D_6|}(\Gamma_3) \subset S$. It has $\Delta_3^2 = \Gamma_3^2 = 22$ and $\Delta_3.K_S \neq 0$ since $-K_S$ is effective and Δ_3 moves. So $p_a(\Delta_3) \neq p_a(\Gamma_3)$ so Δ_3 is not isomorphic to Γ_3; hence there are two (possibly infinitely near) points of Γ_3 that are identified by $A \to \mathbb{P}^2 \times \mathbb{P}^1$. But they are also not separated by σ, so the map $A \to \mathcal{C}_3$ is not an isomorphism on Γ_3. $\square$

4 Types $\mathcal{D}_7$, $\mathcal{D}_{10}$, $\mathcal{D}_{11}$, $\mathcal{D}_{14}$, $\mathcal{D}_{17}$ and $\mathcal{D}_{18}$

In these cases $\operatorname{rk}\operatorname{Pic}(X) = 3$. We obtain strong restrictions on α except in case $\mathcal{D}_{11}$. The fans of the Fano 4-folds of these types are given by the primitive relations shown in the table.

	$\mathcal{D}_7$	$\mathcal{D}_{10}$	$\mathcal{D}_{11}$	$\mathcal{D}_{14}$	$\mathcal{D}_{17}$	$\mathcal{D}_{18}$
$x_1 + x_2 + x_3 =$	0	x_6	0	0	0	$2x_7$
$x_4 + x_5 =$	x_1	x_1	x_1	0	x_1	x_6
$x_6 + x_7 =$	x_1	0	x_4	x_1	x_2	0

For X of type $\mathcal{D}_7$ we have $D_2 \equiv D_3$, $D_4 \equiv D_5$ and $D_6 \equiv D_7$. We choose the basis $\beta_1 = D_3^2$, $\beta_2 = D_3D_5$, $\beta_3 = D_3D_7$, $\beta_4 = D_5D_7$ for $A^2(X)$. Interchanging D_5 and D_7 if necessary, we may assume that $\mathsf{b} \geq \mathsf{c}$.

Proposition 4.1 *The class of any totally nondegenerate abelian surface in a Fano 4-fold of type $\mathcal{D}_7$ satisfies one of* $(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}) = (3, 4, 1, 7)$, $(3, 12, 0, 12)$, $(4, 4, 0, 4)$ *or* $(5, 2, 0, 2)$.

Proof. The 4-fold X of type $\mathcal{D}_7$ is a $\mathbb{P}^2$-bundle over $\mathbb{P} = \mathbb{P}^1 \times \mathbb{P}^1$. If $p\colon X \to \mathbb{P}$ is the projection then D_5 and D_7 are the pullbacks of lines in the two rulings. The computation gives $\Lambda_{34} = L_{57} = \mathsf{a}$, so if $\mathsf{a} \neq 0$ then

$$3 = h^0\big(\mathcal{O}_{\mathbb{P}}(1,1)\big) \leq h^0\big(\mathcal{O}_A(D_5 + D_7)\big) = \mathsf{a}$$

by Riemann-Roch. The double-point formula gives

$$(2\mathsf{a}^2 - 14\mathsf{a}) + (2\mathsf{a} - 7)(\mathsf{b} + \mathsf{c}) + (2\mathsf{a} - 3)\mathsf{d} = 0 \tag{2}$$

so evidently $0 < \mathsf{a} \leq 6$. Moreover, $\Lambda_{11} = \mathsf{d} - \mathsf{b} - \mathsf{c}$ and $\Lambda_{13} = \mathsf{c}$, so by Lemma 1.1(ii) $\mathsf{d} \geq \mathsf{b} + \mathsf{c}$ and $\mathsf{c} \geq 0$, and if $\mathsf{c} = 0$ then $\mathsf{b} = \mathsf{d}$ by Lemma 1.1(iii).

With these restrictions it is easy to see that the solutions to equation (2) are $(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}) = (3, 1, 1, 8)$, $(3, 4, 1, 7)$, $(3, 12, 0, 12)$, $(4, 1, 1, 4)$, $(4, 4, 0, 4)$ and $(5, 2, 0, 2)$. However the cases $(3, 1, 1, 8)$ and $(4, 1, 1, 4)$ are excluded by Lemma 1.1(vi). $\square$

For X of type $\mathcal{D}_{10}$ we have $D_2 \equiv D_3$ and $D_4 \equiv D_5$. We choose the basis $\beta_1 = D_5D_6$, $\beta_2 = D_5D_7$, $\beta_3 = D_6^2$, $\beta_4 = D_7^2$ for $A^2(X)$.

Proposition 4.2 *There are no totally nondegenerate abelian surfaces in a toric Fano 4-fold of type $\mathcal{D}_{10}$.*

Proof. The computation yields

$$\Lambda_{4*} = \begin{pmatrix} -\mathsf{a} & -\mathsf{a} + \mathsf{c} & \mathsf{c} & \mathsf{a} - \mathsf{c} & 0 \end{pmatrix}.$$

So $-\mathsf{a} + \mathsf{c} \geq 0$ and $\mathsf{a} - \mathsf{c} \geq 0$ by Lemma 1.1(ii), so $\mathsf{a} = \mathsf{c}$; but again by Lemma 1.1(ii), $-\mathsf{a} \geq 0$ and $\mathsf{c} \geq 0$ so $\mathsf{a} = \mathsf{c} = 0$. But now $\Lambda_{4j} = 0$ for all j, so by Lemma 1.1(iv) no totally nondegenerate embedding exists. $\square$

For $\mathcal{D}_{11}$ the numerical conditions are rather weak. We have $D_2 \equiv D_3$ and $D_6 = D_7$, and we choose the basis $\beta_1 = D_3^2$, $\beta_2 = D_3D_5$, $\beta_3 = D_3D_7$, $\beta_4 = D_5D_7$.

Proposition 4.3 *The class of any totally nondegenerate abelian surface in a toric Fano 4-fold of type $\mathcal{D}_{11}$ satisfies* $\mathsf{a} = 0$, $\mathsf{d} - \mathsf{b} - \mathsf{c} \geq 0$ *and* $3\mathsf{d} = \mathsf{b}^2 + 2\mathsf{b}\mathsf{c} - 10\mathsf{b} - 7\mathsf{c}$.

Proof. The computation gives $\Lambda_{55} = -\Lambda_{33} = \mathsf{a}$, so $\mathsf{a} = 0$, and $\Lambda_{11} = \mathsf{d}-\mathsf{b}-\mathsf{c}$, and the condition $3\mathsf{d} = \mathsf{b}^2 + 2\mathsf{bc} - 10\mathsf{b} - 7\mathsf{c}$ comes from the double-point formula. □

For $\mathcal{D}_{14}$ the numerical conditions are fairly restrictive and the existence of a map to $\mathbb{P}^1 \times \mathbb{P}^1$ reduces the possibilities still further. We have $D_2 \equiv D_3$, $D_4 \equiv D_5$ and $D_6 \equiv D_7$ and we choose the basis $\beta_1 = D_3^2$, $\beta_2 = D_3D_5$, $\beta_3 = D_3D_7$, $\beta_4 = D_5D_7$.

Proposition 4.4 *The class of any totally nondegenerate abelian surface in a toric Fano 4-fold of type $\mathcal{D}_{14}$ satisfies one of $\mathsf{a} \le 3$, $\mathsf{b} \le 2$ or $(\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}) = (4,4,4,4)$ or $(5,4,0,4)$.*

Proof. The computation yields $\Lambda_{11} = \mathsf{d} - \mathsf{b}$, $\Lambda_{13} = \mathsf{c}$, $\Lambda_{14} = \mathsf{b}$ and $\Lambda_{34} = \mathsf{a}$, and the double-point formula gives

$$\begin{aligned} 0 &= 2\mathsf{ab} + 2\mathsf{bc} + 2\mathsf{ad} - 8\mathsf{a} - 7\mathsf{b} - 6\mathsf{c} - 3\mathsf{d} \\ &= 4\mathsf{ab} + 2\mathsf{bc} + 2\mathsf{a}(\mathsf{d}-\mathsf{b}) - 8\mathsf{a} - 10\mathsf{b} - 6\mathsf{c} - 3(\mathsf{d}-\mathsf{b}) \\ &= \mathsf{a}(2\mathsf{b}-8) + \mathsf{b}(2\mathsf{a}-10) + \mathsf{c}(2\mathsf{b}-6) + \Lambda_{11}(2\mathsf{a}-3). \end{aligned}$$

If we assume that $\mathsf{a} \ge 4$ and $\mathsf{b} \ge 3$, this immediately gives $\mathsf{b} \le 4$ or $\mathsf{a} \le 5$. Moreover the Hodge index theorem gives $2\mathsf{bc} + \mathsf{a}(\mathsf{d}-\mathsf{b}) \le 0$. It is easy to check that the only solutions to this are those claimed. □

For $\mathcal{D}_{17}$ the position is similar to that for $\mathcal{D}_{14}$. For X of type $\mathcal{D}_{17}$ we have $D_4 \equiv D_5$ and $D_6 \equiv D_7$ and we choose the basis $\beta_1 = D_3^2$, $\beta_2 = D_3D_5$, $\beta_3 = D_3D_7$, $\beta_4 = D_5D_7$. Interchanging D_5 and D_7 if necessary, we may assume that $\mathsf{b} \ge \mathsf{c}$, and we write $\mathsf{x} = \mathsf{b} - \mathsf{c} \ge 0$.

Proposition 4.5 *The class of any totally nondegenerate abelian surface in a toric Fano 4-fold of type $\mathcal{D}_{17}$ satisfies one of $\mathsf{a} = 0$ or $(\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}) = (3,3,3,5)$, $(3,4,1,8)$, $(3,9,0,12)$, $(4,2,0,6)$, $(4,2,1,5)$ or $(4,2,2,4)$.*

Proof. Again $\mathsf{a} = \Lambda_{45}$ is the degree of a map $A \to \mathbb{P}^1 \times \mathbb{P}^1$ and hence $\mathsf{a} = 0$ or $\mathsf{a} \ge 3$. We also have $\Lambda_{12} = \mathsf{d}$, $\Lambda_{15} = \mathsf{b}$ and $\Lambda_{24} = \mathsf{c}$. Moreover $\Lambda_{11} + \Lambda_{22} = 2(\mathsf{d}-\mathsf{a})$ so $\mathsf{d} \ge \mathsf{a}$, and we write $\mathsf{y} = \mathsf{d} - \mathsf{a}$. Now the double-point formula gives

$$\begin{aligned} 0 &= \mathsf{a}^2 + 2\mathsf{ab} + 2\mathsf{ac} + 2\mathsf{bc} + 2\mathsf{ad} - 12\mathsf{a} - 7\mathsf{b} - 7\mathsf{c} - 3\mathsf{d} \\ &= (3\mathsf{a}-15)\mathsf{a} + (4\mathsf{a}-14)\mathsf{c} + (2\mathsf{a}-7)\mathsf{x} + (2\mathsf{a}-3)\mathsf{y} + 2\mathsf{c}^2 + 2\mathsf{cx} \end{aligned}$$

which immediately implies that $\mathsf{a} \le 4$. Together with the Hodge index theorem, which gives $\mathsf{a}^2 + \mathsf{ac} - \mathsf{ad} + \mathsf{ab} + 2\mathsf{bc} \ge 0$ for $\mathsf{a} \ne 0$, this is enough to yield the result by a simple calculation. □

Finally, the case $\mathcal{D}_{18}$ may also be excluded altogether. In this case we have $D_1 \equiv D_2 \equiv D_3$ and $D_4 \equiv D_5$, and we choose the basis $\beta_1 = D_3D_5$, $\beta_2 = D_3D_7$, $\beta_3 = D_5D_7$, $\beta_4 = D_3^2$.

Proposition 4.6 *There are no totally nondegenerate abelian surfaces in a toric Fano 4-fold of type $\mathcal{D}_{18}$.*

Proof. We find

$$\Lambda = \begin{pmatrix} \mathsf{c} & \mathsf{b} & \mathsf{a} & \mathsf{a}+\mathsf{b}-2\mathsf{c} \\ \mathsf{b} & 0 & \mathsf{d} & -2\mathsf{b}+\mathsf{d} \\ \mathsf{a} & \mathsf{d} & 2\mathsf{a}-\mathsf{d} & 0 \\ \mathsf{a}+\mathsf{b}-2\mathsf{c} & -2\mathsf{b}+\mathsf{d} & 0 & -2\mathsf{a}-4\mathsf{b}+4\mathsf{c}+\mathsf{d} \end{pmatrix},$$

so by Lemma 1.1(iii) we have $\mathsf{d} = 2\mathsf{a}$ and $\mathsf{b} = \mathsf{c}$. The double-point formula gives

$$\begin{aligned} 0 &= 2\mathsf{ab} + \mathsf{b}^2 - 4\mathsf{bc} + 2\mathsf{cd} - 6\mathsf{a} - 5\mathsf{b} + 3\mathsf{c} - 4\mathsf{d} \\ &= 6\mathsf{ab} - 3\mathsf{b}^2 - 14\mathsf{a} - 2\mathsf{b} \end{aligned} \tag{3}$$

and by Lemma 1.1(ii) $\mathsf{a} \geq 0$, $\mathsf{b} \geq 0$ and $0 \leq \mathsf{a}+\mathsf{b}-2\mathsf{c} = \mathsf{a}-\mathsf{b}$. If $\mathsf{b} = 0$ then $\mathsf{a} = 0$ by equation (3), which is impossible by Lemma 1.1(iv) because $\Lambda_{3*} = 0$, and if $\mathsf{a} = \mathsf{b}$ then $\Lambda_{14} = 0$ so $\Lambda_{11} = 0$ by Lemma 1.1(iii); but then $\mathsf{b} = 0$. Hence $0 < \mathsf{b} < \mathsf{a}$, and $\mathsf{b} = \Lambda_{11}$ is even by Lemma 1.1(ii). So

$$2 \leq \mathsf{b} < \mathsf{a} = (3\mathsf{b}^2 + 2\mathsf{b})/(6\mathsf{b} - 14)$$

from equation (3), since $6\mathsf{b} - 14 \neq 0$. So either $6\mathsf{b} - 14 < 0$, i.e. $\mathsf{b} = 2$, or $\mathsf{b}(6\mathsf{b} - 14) < 3\mathsf{b}^2 + 2\mathsf{b}$, which gives $\mathsf{b} \leq 16/3 < 6$. So $\mathsf{b} = 2$ or $\mathsf{b} = 4$; but if $\mathsf{b} = 2$ then $\mathsf{a} = -8 < 0$, and if $\mathsf{b} = 4$ then $\mathsf{a} = 28/5 \notin \mathbb{Z}$. $\square$

In Proposition 4.4 the conditions $\mathsf{a} \geq 4$ and $\mathsf{b} \geq 3$ are not arbitrary: they correspond to geometric conditions on A. Exactly as in Proposition 4.1, there is a map to $\mathbb{P}^1 \times \mathbb{P}^1$ whose restriction is of degree $\mathsf{a} = L_{57}$. Therefore in any case $\mathsf{a} = 0$ or $\mathsf{a} \geq 3$. There is also a map from a blow-up $\tilde{X}$ of X to the Hirzebruch surface $\mathbb{F}_1$ which is of degree b on the proper transform $\tilde{A}$ of A. We obtain the blow-up by introducing the ray $\mathbb{R}_+(-x_2)$ into $\Sigma^{(1)}$: the map to $\mathbb{F}_1$ is induced by projection to the plane spanned by x_1 and x_7. Since $\mathbb{F}_1$ is rational and $\tilde{A}$ is not, $\mathsf{b} \neq 1$.

However, if $\mathsf{a} = 0$ or $\mathsf{b} = 0$ then A or $\tilde{A}$ maps to a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$; if $\mathsf{a} = 3$ then A is a triple cover of $\mathbb{P}^1 \times \mathbb{P}^1$ and if $\mathsf{b} = 2$ then $\tilde{A}$ is a double cover of $\mathbb{F}_1$. All these are strong geometric constraints. For instance, it is easy to see, using the results of Miranda [9] on triple covers, that $A \to \mathbb{P}^1 \times \mathbb{P}^1$ cannot have general branching behaviour. A similar remark applies to the cases in Proposition 4.1 and Proposition 4.5 for which $\mathsf{a} = 3$.

5 Types $\mathcal{G}_3$, $\mathcal{G}_4$ and $\mathcal{G}_5$

In these cases $\operatorname{rk}\operatorname{Pic} X = 3$. For $\mathcal{G}_3$ and $\mathcal{G}_5$ the numerical conditions are quite weak, but for $\mathcal{G}_4$ they are very restrictive. The fans of the Fano 4-folds of these types are given by the primitive relations shown in the table.

	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
$x_1+x_7=$	0	x_4	x_4
$x_2+x_3+x_4=$	0	x_7	x_7
$x_4+x_5+x_6=$	x_1	x_1+x_2	0
$x_5+x_6+x_7=$	x_2+x_3	x_2	x_2+x_3
$x_1+x_2+x_3=$	x_5+x_6	0	0

For X of type $\mathcal{G}_3$ we have $D_2 \equiv D_3$ and $D_5 \equiv D_6$ and we choose the basis $\beta_1 = D_4^2$, $\beta_2 = D_4D_6$, $\beta_3 = D_4D_7$, $\beta_4 = D_6^2$, $\beta_5 = D_7^2$.

Proposition 5.1 *The class of any totally nondegenerate abelian surface in a toric Fano 4-fold of type $\mathcal{G}_3$ satisfies $\mathsf{a} = \mathsf{d} = 0$, $\mathsf{b} > 0$, $\mathsf{c} > 0$ and $3\mathsf{e} = 2\mathsf{bc} + \mathsf{c}^2 - 6\mathsf{b} - 11\mathsf{c} \geq 0$.*

Proof. We have $\Lambda_{15} = 0$ and $\Lambda_{15} = \mathsf{d} - \mathsf{a}$, $\Lambda_{55} = \mathsf{a}$. The formula for e in terms of b and c comes from the double-point formula. Each of b, c and e occurs as an entry in Λ, so they are non-negative by Lemma 1.1(ii) and b and c are the only non-zero values in some row, and hence positive by Lemma 1.1(iv). $\square$

In the case of $\mathcal{G}_4$ we have $D_5 \equiv D_6$ and we choose the basis $\beta_1 = D_3^2$, $\beta_2 = D_3D_6$, $\beta_3 = D_6^2$, $\beta_4 = D_6D_7$, $\beta_5 = D_7^2$.

Proposition 5.2 *If $\mathsf{a}\beta_1 + \cdots + \mathsf{e}\beta_5$ is the class of a totally nondegenerate abelian surface in a toric Fano 4-fold of type $\mathcal{G}_4$ then $(\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d},\mathsf{e})$ is one of*

$$(2,6,0,8,-1), (12,-8,0,12,-10),$$

$$(12,-8,0,14,-11) \text{ or } (14,-10,0,12,-11).$$

Proof. We have $\Lambda_{11} = \mathsf{c}$ and $\Lambda_{16} = 0$ so $\mathsf{c} = 0$: also $\Lambda_{66} = \mathsf{d} + 2\mathsf{e} - \mathsf{b}$ so $\mathsf{b} = \mathsf{d} + 2\mathsf{e}$. Moreover $\mathsf{a} = \Lambda_{55} \geq 0$ and $\mathsf{e} = -\Lambda_{45} \leq 0$. Putting $\mathsf{x} = \mathsf{a} + \mathsf{b}$ and $\mathsf{y} = \mathsf{a} + \mathsf{e}$ we find $\Lambda_{23} = \mathsf{x} - \mathsf{y}$ and $\Lambda_{63} = \mathsf{y}$, and both are positive by Lemma 1.1(ii), so $\mathsf{x} > \mathsf{y} > 0 \geq \mathsf{e}$. But the double-point formula gives

$$\begin{aligned} 0 &= -2\mathsf{y}^2 + 2\mathsf{yx} - 2\mathsf{ex} + \mathsf{x}^2 + 6\mathsf{e} - 11\mathsf{x} \\ &= \mathsf{x}(\mathsf{x} - 11) - \mathsf{e}(2\mathsf{x} - 6) + 2\mathsf{y}(\mathsf{x} - \mathsf{y}) \end{aligned} \tag{4}$$

which implies $\mathsf{x} < 11$. Since equation (4) has no integer solutions with $\mathsf{x} = 3$, e is also bounded and a simple search gives the solutions claimed, along with the solution $(3,4,0,8,-2)$ which is excluded because Λ_{33} is odd. $\square$

For $\mathcal{G}_5$ we have $D_2 \equiv D_3$ and $D_5 \equiv D_6$, and we choose the basis $\beta_1 = D_3^2$, $\beta_2 = D_3D_6$, $\beta_3 = D_6^2$, $\beta_4 = D_6D_7$, $\beta_5 = D_7^2$.

Proposition 5.3 *The class of any totally nondegenerate abelian surface in a toric Fano 4-fold of type $\mathcal{G}_5$ satisfies $\mathsf{c} = 0$, $\mathsf{d} > 0$, $\mathsf{b} = \mathsf{d} + \mathsf{x}$, $\mathsf{a} \geq 2\mathsf{x} > 0$ and $3\mathsf{a} = 2\mathsf{dx} + \mathsf{d}^2 - 4\mathsf{x} - 9\mathsf{d}$, where $\mathsf{x} = \mathsf{a} + \mathsf{e}$.*

Proof. $\Lambda_{15} = 0$ so Λ_{11} and Λ_{55} both vanish: they are equal to c and $\mathsf{d}+\mathsf{x}-\mathsf{b}$ respectively. $\mathsf{d} = \Lambda_{23}$ and $\mathsf{x} = \Lambda_{35}$ and both are the only non-zero values in some row of Λ. The double-point formula gives the equation for a in terms of d and x, and the remaining inequality comes from the fact that $\Lambda_{33} = \mathsf{a} - 2\mathsf{x}$. □

Again it might be possible to analyse these cases further by considering maps to toric surfaces.

6 Types $\mathcal{I}_9$, $\mathcal{L}_{11}$, $\mathcal{L}_{13}$, $\mathcal{Q}_{16}$, $\mathcal{U}_8$, $\mathcal{V}_4$, $\mathcal{W}$, $\mathcal{Z}_1$ and $\mathcal{Z}_2$

In these miscellaneous cases the calculations do not usually have much in common: even when they do, as in the cases of $\mathcal{Z}_1$ and $\mathcal{Z}_2$, the outcomes can be very different.

We start with the case of $\mathcal{I}_9$, where the restrictions are strong. In this case $\operatorname{rk}\operatorname{Pic} X = 4$. The primitive relations are

$$\begin{array}{lll} x_7 + x_8 = x_3, & x_3 + x_6 = x_7, & x_6 + x_8 = 0, \\ x_1 + x_2 = x_7, & x_3 + x_4 + x_5 = x_8, & x_4 + x_5 + x_7 = 0. \end{array}$$

We have $D_1 \equiv D_2$ and $D_4 \equiv D_5$. We choose the basis $\beta_1 = D_4D_6$, $\beta_2 = D_4D_7$, $\beta_3 = D_4D_8$, $\beta_4 = D_6D_7$, $\beta_5 = D_7^2$, $\beta_6 = D_8^2$.

Proposition 6.1 *If $\mathsf{a}\beta_1 + \cdots + \mathsf{f}\beta_6$ is the class of a totally nondegenerate abelian surface in a toric Fano 4-fold of type $\mathcal{I}_9$ then $(\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d},\mathsf{e},\mathsf{f})$ is one of*

$$(18, 14, -12, -16, -7, -6), (16, 8, -6, -12, -4, -3), (26, 6, -4, -16, -3, -2),$$

$$(14, 10, -6, -12, -5, -3) \text{ or } (26, 6, -2, -16, -3, -1).$$

Proof. $\Lambda_{26} = 0$ so $\Lambda_{66} = 2\mathsf{f} - \mathsf{c}$ vanishes by Lemma 1.1(iii) and the rows Λ_{2*} and Λ_{6*} are proportional by Lemma 1.1(v). Since $\Lambda_{25} = -\mathsf{b} - 2\mathsf{e}$ and $\Lambda_{65} = 0$ this gives $\mathsf{b} = -2\mathsf{e}$. Moreover $\Lambda_{55} = \mathsf{e} - \mathsf{a} - 2\mathsf{d}$ vanishes.

Now it is convenient to use the variables $\mathsf{f} = -\Lambda_{61}$, $\mathsf{x} = 2\mathsf{e} - \mathsf{d} = \Lambda_{51}$ and $\mathsf{y} = \mathsf{f} - \mathsf{e} = \Lambda_{21}$: as these are the only non-zero values in those rows, we have $\mathsf{x} > 0$, $\mathsf{y} > 0$ and $\mathsf{f} < 0$. The double-point formula gives

$$\begin{aligned} 0 &= -2\mathsf{f}\mathsf{y} + 2\mathsf{y}\mathsf{x} + \mathsf{y}^2 - 2\mathsf{f}\mathsf{x} - 5\mathsf{x} + 4\mathsf{f} - 7\mathsf{y} \\ &= \mathsf{y}(\mathsf{y}-7) + \mathsf{x}(2\mathsf{y}-5) + (-\mathsf{f})(2\mathsf{y}-4) + 2(-\mathsf{f})\mathsf{x} \qquad (5) \end{aligned}$$

so $\mathsf{y} \leq 6$. From this, by solving equation (5) for f, it is trivial to obtain the result claimed. □

For $\mathcal{L}_{11}$ the restrictions are rather weak. In this case $\operatorname{rk}\operatorname{Pic} X = 3$. The primitive relations are

$$x_1 + x_8 = 0, \; x_2 + x_3 = 0, \; x_4 + x_5 = x_3, \; x_6 + x_7 = x_2.$$

We have $D_1 \equiv D_8$, $D_4 \equiv D_5$ and $D_6 \equiv D_7$. We choose the basis $\beta_1 = D_3D_5$, $\beta_2 = D_3D_7$, $\beta_3 = D_3D_8$, $\beta_4 = D_5D_7$, $\beta_5 = D_5D_8$, $\beta_6 = D_7D_8$.

Proposition 6.2 *If $\mathsf{a}\beta_1 + \cdots + \mathsf{f}\beta_6$ is the class of a totally nondegenerate abelian surface in a toric Fano 4-fold of type $\mathcal{L}_{11}$ then $\mathsf{a} = \mathsf{b} > 0$, $\mathsf{c} = 0$, $\mathsf{e} = \mathsf{f} > 0$ and $\mathsf{d} = \mathsf{bf} - 2\mathsf{b} - 2\mathsf{f} \geq 0$.*

Proof. Since $\Lambda_{12} = 0$, we have $0 = \Lambda_{11} = \mathsf{f} - \mathsf{e}$ and $0 = \Lambda_{22} = \mathsf{e} - \mathsf{f} - 2\mathsf{c}$, so $\mathsf{e} = \mathsf{f}$ and $\mathsf{c} = 0$. The rows Λ_{1*} and Λ_{2*} are proportional by Lemma 1.1(v), and since $\Lambda_{13} = \Lambda_{23} = \mathsf{f}$ we have either $\mathsf{f} = 0$ or $\mathsf{f} > 0$ and $\Lambda_{15} = \Lambda_{25}$. But $\Lambda_{15} = \mathsf{d}$ and $\Lambda_{25} = \mathsf{a} + \mathsf{b} - \mathsf{d}$, so if $\mathsf{f} \neq 0$ we have $\mathsf{a} = \mathsf{b}$.

The double-point formula gives $\mathsf{af} + \mathsf{bf} - 3\mathsf{a} - \mathsf{b} - 2\mathsf{d} - 4\mathsf{f} = 0$. This immediately excludes $\mathsf{f} = 0$ and hence gives the formula for d. Since $\mathsf{b} = \Lambda_{35}$ it is non-negative, and $\mathsf{b} \neq 0$ because otherwise $\mathsf{d} < 0$. $\square$

For $\mathcal{L}_{13}$ the restrictions are much stronger. In this case $\operatorname{rk}\operatorname{Pic} X = 3$. The primitive relations are

$$x_1 + x_8 = 0,\ x_2 + x_3 = x_1,\ x_4 + x_5 = x_1,\ x_6 + x_7 = x_8.$$

We have $D_2 \equiv D_3$, $D_4 \equiv D_5$ and $D_6 \equiv D_7$ and we choose the basis $\beta_1 = D_3D_5$, $\beta_2 = D_3D_7$, $\beta_3 = D_5D_7$, $\beta_4 = D_5D_8$, $\beta_5 = D_7D_8$, $\beta_6 = D_8^2$.

Proposition 6.3 *If $\mathsf{a}\beta_1 + \cdots + \mathsf{f}\beta_6$ is the class of a totally nondegenerate abelian surface in a toric Fano 4-fold of type $\mathcal{L}_{13}$ then $(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}) = (2, 2, 0, 4, 4, 0)$, $(6, 3, 3, 0, 6, 2)$ or $(2, 0, 2, -4, 8, 4)$.*

Proof. Since $\Lambda_{15} = 0$ we have $0 = \Lambda_{11} = \mathsf{a} - \mathsf{b} - \mathsf{c}$ and also $\Lambda_{55} = 0$, which gives $\mathsf{e} = \mathsf{d} + 3\mathsf{f}$. The double-point formula gives

$$\begin{aligned} 0 &= 2\mathsf{bd} + 3\mathsf{bf} + \mathsf{cd} + 3\mathsf{cf} + \mathsf{d}^2 + 3\mathsf{df} + 3\mathsf{f}^2 - 4\mathsf{b} - 4\mathsf{c} - 6\mathsf{d} - 12\mathsf{f} \\ &= 2\mathsf{bx} + (\mathsf{f} - 4)\mathsf{b} + \mathsf{cx} + (2\mathsf{f} - 4)\mathsf{c} + \mathsf{x}^2 + (\mathsf{f} - 6)\mathsf{x} + (\mathsf{f} - 6)\mathsf{f} \qquad (6) \end{aligned}$$

where $\mathsf{x} = \mathsf{d} + \mathsf{f} = \Lambda_{24}$. Since $\mathsf{c} = \Lambda_{12}$, $\mathsf{b} = \Lambda_{13}$ and $\mathsf{f} = \Lambda_{34}$ and they are thus nonnegative, we deduce that $\mathsf{f} \leq 6$. Moreover b and c are not both zero, since otherwise $\Lambda_{1j} = 0$ for all j because $\Lambda_{14} = \mathsf{b} + \mathsf{c}$.

By Lemma 1.1(v) we have $\Lambda_{12}\Lambda_{53} = \Lambda_{13}\Lambda_{52}$, and as $\Lambda_{25} = \mathsf{c} + \mathsf{f}$ and $\Lambda_{53} = \mathsf{b} + \mathsf{x}$ we have $\mathsf{bf} = \mathsf{cx}$. This, equation (6) and the bound $0 \leq \mathsf{f} \leq 6$ is sufficient to give the result by a simple calculation. $\square$

For $\mathcal{Q}_{16}$ we get strong restrictions. In this case $\operatorname{rk}\operatorname{Pic} X = 5$. The primitive relations are

$$\begin{aligned} &x_8 + x_9 = 0, \quad x_7 + x_9 = x_1, \quad x_1 + x_2 = x_9, \quad x_1 + x_8 = x_7, \\ &x_2 + x_7 = 0, \quad x_3 + x_5 = x_9, \quad x_4 + x_6 = x_8. \end{aligned}$$

We have $D_3 \equiv D_5$, $D_4 \equiv D_6$. We choose the basis $\beta_1 = D_5D_6$, $\beta_2 = D_5D_7$, $\beta_3 = D_5D_8$, $\beta_4 = D_5D_9$, $\beta_5 = D_6D_7$, $\beta_6 = D_6D_8$, $\beta_7 = D_6D_9$, $\beta_8 = D_7^2$.

Proposition 6.4 *If* $\mathsf{a}\beta_1 + \cdots + \mathsf{h}\beta_8$ *is the class of a totally nondegenerate abelian surface in a toric Fano 4-fold of type* $\mathcal{Q}_{16}$ *then* $(\mathsf{a}, \ldots \mathsf{h})$ *is one of* $(0, 6, 7, -4, 6, 7, -4, 0)$, $(0, 4, 6, -2, 4, 6, -2, 0)$ *or* $(0, 3, 7, -1, 3, 7, -1, 0)$.

Proof. The computation shows that $\Lambda_{57} = \Lambda_{67} = 0$ so by Lemma 1.1(iii) we have $\Lambda_{55} = \Lambda_{66} = \Lambda_{77} = 0$. These give $-\mathsf{a} + \mathsf{b} - \mathsf{e} + 2\mathsf{h} = -\mathsf{b} + \mathsf{e} = -\mathsf{a} - 2\mathsf{d} + 2\mathsf{g} = 0$. Using these equations to eliminate a, e and g we are left with $\Lambda_{17} = -\Lambda_{34} = \mathsf{h}$, so $\mathsf{h} = 0$. Also $\Lambda_{12} = 0$, and therefore Λ_{1*} and Λ_{2*} are proportional; but $\Lambda_{16} = 0$ and $\Lambda_{26} = \mathsf{f} - \mathsf{c}$, so $\mathsf{f} = \mathsf{c}$. Now it is convenient to put $\mathsf{x} = \mathsf{b} + \mathsf{d} = \Lambda_{13} > 0$ (it is not zero, by Lemma 1.1(iv)) and $\mathsf{y} = \mathsf{c} - \mathsf{b} = \Lambda_{53} > 0$ similarly. We also have $0 < \Lambda_{73} = -\mathsf{d}$. The double-point formula gives

$$\begin{aligned} 0 &= \mathsf{x}^2 - 2\mathsf{xz} + 2\mathsf{yx} - 2\mathsf{yd} - 6\mathsf{x} + 4\mathsf{d} - 4\mathsf{y} \\ &= (\mathsf{x} - 6)\mathsf{x} + (2\mathsf{x} - 4)(-\mathsf{d}) + (2sfx - 4)\mathsf{y} + 2\mathsf{y}(-\mathsf{d}) \end{aligned} \tag{7}$$

so $1 \leq \mathsf{x} \leq 5$. By solving equation (7) for y or d it is easy to see that the only solutions are as claimed. $\square$

For $\mathcal{U}_8$ we again get strong restrictions. In this case $\operatorname{rk} \operatorname{Pic} X = 6$. The primitive relations are

$$\begin{array}{llll} x_1 + x_3 = x_2, & x_2 + x_4 = x_3, & x_1 + x_4 = 0, & x_3 + x_5 = x_4, \\ x_4 + x_6 = x_5, & x_2 + x_5 = 0, & x_1 + x_5 = x_6, & x_2 + x_6 = x_1, \\ x_3 + x_6 = 0, & x_7 + x_8 = x_1, & x_9 + x_{10} = x_4. & \end{array}$$

We have $D_7 \equiv D_8$, $D_9 \equiv D_{10}$. We choose the basis $\beta_1 = D_3D_8$, $\beta_2 = D_3D_{10}$, $\beta_3 = D_4D_8$, $\beta_4 = D_4D_{10}$, $\beta_5 = D_5D_8$, $\beta_6 = D_5D_{10}$, $\beta_7 = D_6^2$, $\beta_8 = D_6D_8$, $\beta_9 = D_6D_{10}$, $\beta_{10} = D_8D_{10}$.

Proposition 6.5 *If* $\mathsf{a}\beta_1 + \cdots + \mathsf{j}\beta_{10}$ *is the class of a totally nondegenerate abelian surface in a toric Fano 4-fold of type* $\mathcal{U}_8$ *then* $(\mathsf{a}, \mathsf{x}, \mathsf{y}) = (4, 2, 1)$ *or* $(2, 2, 2)$ *up to permutation, and* $(\mathsf{a}, \ldots, \mathsf{j}) = (\mathsf{a}, \mathsf{a}, \mathsf{a} + \mathsf{x}, \mathsf{a} + \mathsf{x}, \mathsf{y} + \mathsf{x}, \mathsf{y} + \mathsf{x}, 0, \mathsf{y}, \mathsf{y}, 0)$

Proof. The computation shows that $\Lambda_{13} = 0$ so by Lemma 1.1(iii) we have $\Lambda_{11} = \Lambda_{33} = 0$. These give $\mathsf{i} = \mathsf{h} - \mathsf{j}$ and $\mathsf{b} = \mathsf{a} - \mathsf{j}$. The proportionality between Λ_{1*} and Λ_3* together with $\Lambda_{16} = 0$ and $\Lambda_{36} = \mathsf{j}$ gives $\mathsf{j} = 0$. Also $\Lambda_{14} = 0$ so $\Lambda_{44} = 0$ and by the proportionality between Λ_{1*} and Λ_{4*} we have $\Lambda_{43} = 0$. These give $\mathsf{d} = \mathsf{c}$ and $\mathsf{e} = \mathsf{f}$. Finally, $\Lambda_{55} = 2\mathsf{g} = -\Lambda_{56}$ so $\mathsf{g} = 0$.

Now we write the double-point formula, in terms of $\mathsf{a} = \Lambda_{72}$, $\mathsf{x} = \mathsf{f} - \mathsf{h} = \Lambda_{76}$, $\mathsf{y} = \mathsf{a} - \mathsf{c} + \mathsf{f} = \Lambda_{74}$ and $\mathsf{z} = \mathsf{c} - \mathsf{f}$. By Lemma 1.1(iv) we get $\mathsf{a}, \mathsf{x}, \mathsf{y} > 0$. The double-point formula gives

$$\mathsf{z}^2 + (2\mathsf{a} + 2\mathsf{x} + 2\mathsf{y} - 6)\mathsf{z} + (2\mathsf{ax} + 2\mathsf{xy} + 2\mathsf{ya} - 4(\mathsf{a} + \mathsf{x} + \mathsf{y})) = 0 \tag{8}$$

which is symmetric in a, x and y. So to solve it we may assume that $\mathsf{a} \geq \mathsf{x} \geq \mathsf{y} \geq -\mathsf{z}$ (since $\mathsf{y}+\mathsf{z} = \Lambda_{81} > 0$) and $\mathsf{y} > 0$. It is straightforward to check that the only solutions of equation (8) satisfying these conditions are those given. $\square$

For $\mathcal{V}_4$ we get some interesting restrictions. In this case $\operatorname{rk}\operatorname{Pic} X = 6$. The primitive relations are

$$x_4 + x_{10} = 0,\ x_1 + x_5 = 0,\ x_2 + x_6 = 0,\ x_3 + x_7 = 0,\ x_8 + x_9 = 0,$$
$$\begin{aligned} x_1 + x_2 + x_{10} &= x_7 + x_8, & x_1 + x_3 + x_{10} &= x_6 + x_8,\\ x_2 + x_3 + x_{10} &= x_5 + x_8, & x_1 + x_2 + x_3 &= x_4 + x_8,\\ x_1 + x_9 + x_{10} &= x_6 + x_7, & x_2 + x_9 + x_{10} &= x_5 + x_7,\\ x_3 + x_9 + x_{10} &= x_5 + x_6, & x_1 + x_2 + x_9 &= x_4 + x_7,\\ x_1 + x_3 + x_9 &= x_4 + x_6, & x_2 + x_3 + x_9 &= x_4 + x_5,\\ x_4 + x_5 + x_6 &= x_3 + x_9, & x_4 + x_5 + x_7 &= x_2 + x_9,\\ x_4 + x_6 + x_7 &= x_1 + x_9, & x_5 + x_6 + x_7 &= x_9 + x_{10},\\ x_4 + x_5 + x_8 &= x_2 + x_3, & x_4 + x_6 + x_8 &= x_1 + x_3,\\ x_4 + x_7 + x_8 &= x_1 + x_2, & x_5 + x_6 + x_8 &= x_3 + x_{10},\\ x_5 + x_7 + x_8 &= x_2 + x_{10}, & x_6 + x_7 + x_8 &= x_1 + x_{10}. \end{aligned}$$

No two D_i are numerically equivalent, so $\Lambda = L$. We choose the basis $\beta_1 = D_5D_6$, $\beta_2 = D_5D_7$, $\beta_3 = D_5D_8$, $\beta_4 = D_5D_9$, $\beta_5 = D_5D_{10}$, $\beta_6 = D_6D_7$, $\beta_7 = D_6D_8$, $\beta_8 = D_6D_9$, $\beta_9 = D_6D_{10}$, $\beta_{10} = D_7D_8$, $\beta_{11} = D_7D_9$, $\beta_{12} = D_7D_{10}$, $\beta_{13} = D_8^2$, $\beta_{14} = D_9^2$, $\beta_{15} = D_9D_{10}$, $\beta_{16} = D_{10}^2$. Thus L is a 10×10 matrix of linear forms in 16 variables $\mathsf{a}, \ldots, \mathsf{p}$.

Proposition 6.6 *If $\mathsf{a}\beta_1 + \cdots + \mathsf{p}\beta_{16}$ is the class of a totally nondegenerate abelian surface in a toric Fano 4-fold of type $\mathcal{V}_4$ then there is a symmetric matrix*

$$M = \begin{pmatrix} 0 & \mathsf{q} & \mathsf{u} & \mathsf{x} & \mathsf{z} \\ \mathsf{q} & 0 & \mathsf{r} & \mathsf{v} & \mathsf{y} \\ \mathsf{u} & \mathsf{r} & 0 & \mathsf{s} & \mathsf{w} \\ \mathsf{x} & \mathsf{v} & \mathsf{s} & 0 & \mathsf{t} \\ \mathsf{z} & \mathsf{y} & \mathsf{w} & \mathsf{t} & 0 \end{pmatrix}$$

with non-negative off-diagonal entries that determines $\mathsf{a}, \ldots, \mathsf{p}$ by

$$\begin{aligned} (\mathsf{a}, \ldots, \mathsf{p}) \quad = \quad & (\mathsf{s}+\mathsf{t}+\mathsf{w}, \mathsf{t}+\mathsf{v}+\mathsf{y}, \mathsf{r}-\mathsf{t}, \mathsf{s}+\mathsf{t}+\mathsf{v}, \mathsf{r}+\mathsf{w}+\mathsf{y}, \mathsf{t}+\mathsf{x}+\mathsf{z},\\ & -\mathsf{t}+\mathsf{u}, \mathsf{s}+\mathsf{t}+\mathsf{x}, \mathsf{u}+\mathsf{w}+\mathsf{z}, \mathsf{q}-\mathsf{t}, \mathsf{t}+\mathsf{v}+\mathsf{x}, \mathsf{q}+\mathsf{y}+\mathsf{z}, \mathsf{t},\\ & \mathsf{s}+\mathsf{t}+\mathsf{v}+\mathsf{x}, \mathsf{q}+\mathsf{r}+\mathsf{u}, \mathsf{w}+\mathsf{y}+\mathsf{z}). \end{aligned}$$

The rank of M is at most 4, no row of M vanishes, and

$$\sum_{\{i,j\}\cap\{k,l\}=\emptyset} M_{ij}M_{kl} = 4 \sum_{1\leq i,j\leq 5} M_{ij}. \tag{9}$$

Proof. There are off-diagonal entries of L that vanish in each row, so every diagonal entry is zero by Lemma 1.1(iii). This gives a system of ten linear equations in the sixteen variables $\mathsf{a}, \dots, \mathsf{p}$, which is of rank six, so ten of the sixteen variables are independent. The matrix M is obtained by selecting the submatrix of L given by the first four rows and columns and the eighth (which is independent of the first four for general values of $(\mathsf{a}, \dots, \mathsf{p})$). Then we set $\mathsf{q} = M_{1,2}$, etc., and use these ten variables subsequently.

Equation (9) is the double-point formula and the condition that $0 < \operatorname{rk} M < 5$ comes from Lemma 1.1(i). If a row of M vanishes then (it is simple to check by computation) the corresponding row of L vanishes also, contrary to Lemma 1.1(iii). □

For $\mathcal{W}$ we get some restrictions. In this case $\operatorname{rk}\operatorname{Pic} X = 5$. The primitive relations are

$$\begin{array}{lll} x_1 + x_4 = x_7, & x_2 + x_5 = x_8, & x_3 + x_6 = x_9, \\ x_1 + x_2 + x_3 = 0, & x_4 + x_5 + x_6 = 0, & x_7 + x_8 + x_9 = 0, \\ x_1 + x_2 + x_9 = x_6, & x_4 + x_5 + x_9 = x_3, & x_1 + x_3 + x_8 = x_5, \\ x_4 + x_6 + x_8 = x_2, & x_2 + x_3 + x_7 = x_4, & x_5 + x_6 + x_7 = x_1, \end{array}$$
$$\begin{array}{ll} x_1 + x_8 + x_9 = x_5 + x_6, & x_4 + x_8 + x_9 = x_2 + x_3, \\ x_2 + x_7 + x_9 = x_4 + x_6, & x_5 + x_7 + x_9 = x_1 + x_3, \\ x_3 + x_7 + x_8 = x_4 + x_5, & x_6 + x_7 + x_8 = x_1 + x_2. \end{array}$$

No two D_i are equivalent, so $\Lambda = L$. We choose the basis $\beta_1 = D_3^2$, $\beta_2 = D_3D_9$, $\beta_3 = D_6^2$, $\beta_4 = D_6D_7$, $\beta_5 = D_6D_8$, $\beta_6 = D_6D_9$, $\beta_7 = D_7^2$, $\beta_8 = D_7D_8$, $\beta_9 = D_7D_9$, $\beta_{10} = D_8^2$, $\beta_{11} = D_8D_9$, $\beta_{12} = D_9^2$. Thus L is a 9×9 matrix of linear forms in 12 variables $\mathsf{a}, \dots, \mathsf{l}$.

Proposition 6.7 *If $\mathsf{a}\beta_1 + \dots + \mathsf{l}\beta_{12}$ is the class of a totally nondegenerate abelian surface in a toric Fano 4-fold of type $\mathcal{W}$ then there are positive integers w, x, y and an integer $\mathsf{t} > -\min(\mathsf{w}, \mathsf{x}, \mathsf{y})$ such that*

$$3\mathsf{h} = \mathsf{t}^2 + (2\mathsf{w} + 2\mathsf{y} + 2\mathsf{x} - 9)\mathsf{t} + 2\mathsf{y}(\mathsf{w} + \mathsf{x}) + 2\mathsf{w}\mathsf{x} - 7(\mathsf{w} + \mathsf{x}) - 4\mathsf{y}. \qquad (10)$$

Moreover $\mathsf{h} \geq 0$ and $\mathsf{h} \pm (\mathsf{w} - \mathsf{x})$, $\mathsf{h} \pm (\mathsf{x} - \mathsf{y})$, $\mathsf{h} \pm (\mathsf{y} - \mathsf{w})$ and $\mathsf{h} + \mathsf{w} + \mathsf{x} - 2\mathsf{y}$ are also all non-negative.

These numbers determine $(\mathsf{a}, \dots, \mathsf{l})$ by

$$\begin{aligned} (\mathsf{a}, \dots, \mathsf{l}) \;=\; & (\mathsf{h} + 2\mathsf{t} + \mathsf{x} + \mathsf{w}, 2\mathsf{t} + \mathsf{w} + \mathsf{x} + \mathsf{y}, \mathsf{h} + \mathsf{x} + \mathsf{w}, \mathsf{t}, \\ & \mathsf{t}, -\mathsf{t} + \mathsf{w} + \mathsf{x} + \mathsf{y}, -\mathsf{h} - \mathsf{t} - \mathsf{w}, \mathsf{h}, \\ & \mathsf{h} + \mathsf{t} + \mathsf{w} - \mathsf{y}, -\mathsf{h} - \mathsf{t} - \mathsf{x}, \mathsf{h} + \mathsf{t} + \mathsf{x} - \mathsf{y}, -\mathsf{h} + 2\mathsf{y}). \end{aligned}$$

Proof. The entries L_{14}, L_{25} and L_{36} vanish, so Lemma 1.1(iii) gives six linear relations $L_{11} = \cdots = L_{66} = 0$. Computing L_{ii} we find

$$\begin{aligned}
-\mathsf{f}+\mathsf{g}+\mathsf{j}+\mathsf{l} &= 0\\
\mathsf{a}-\mathsf{b}+2\mathsf{c}+\mathsf{e}-\mathsf{f}+\mathsf{g}+\mathsf{j}+\mathsf{l} &= 0\\
3\mathsf{a}-2\mathsf{b}+\mathsf{g}+\mathsf{j}+\mathsf{l} &= 0\\
2\mathsf{a}-\mathsf{b}+\mathsf{c}-\mathsf{d}-\mathsf{f}+\mathsf{g}+\mathsf{j}+\mathsf{l} &= 0\\
2\mathsf{a}-\mathsf{b}+\mathsf{c}-\mathsf{e}-\mathsf{f}+\mathsf{g}+\mathsf{j}+\mathsf{l} &= 0\\
3\mathsf{c}-2\mathsf{f}+\mathsf{g}+\mathsf{j}+\mathsf{l} &= 0.
\end{aligned}$$

This system has rank 4 and we use it to eliminate c, e, f and g.

Note that $\mathsf{d} \neq 0$, because if $\mathsf{d} = 0$ then $L_{58} = \mathsf{a}+\mathsf{j}$ and $L_{88} = -\mathsf{a}-2\mathsf{j}$ so $\mathsf{a} = \mathsf{j} = 0$, and then

$$L_{8*} = (-\mathsf{h}, 0, -\mathsf{k}, -\mathsf{h}, 0, -\mathsf{k}, \mathsf{h}, 0, \mathsf{k})$$

so $\mathsf{h} = \mathsf{k} = 0$ and L_{8*} vanishes.

Because $L_{14} = L_{25} = 0$ we have, from Lemma 1.1(v), that $L_{19}L_{48} = L_{18}L_{49}$, $L_{17}L_{48} = L_{18}L_{47}$ and $L_{29}L_{58} = L_{28}L_{59}$. After dividing by d these give three independent linear equations

$$\begin{aligned}
-\mathsf{h}-\mathsf{j}-\mathsf{b}+\mathsf{i}+\mathsf{l} &= 0\\
2\mathsf{a}-2\mathsf{b}+\mathsf{l}-\mathsf{h} &= 0\\
\mathsf{a}-\mathsf{b}+\mathsf{j}+\mathsf{k}+\mathsf{l} &= 0.
\end{aligned}$$

We use them to eliminate a, b and l. Now we introduce $\mathsf{w} = L_{17}$, $\mathsf{x} = L_{27}$ and $\mathsf{y} = L_{37}$, which are all positive (otherwise a row of L vanishes) and $\mathsf{t} = L_{47} - \mathsf{w} = L_{57} - \mathsf{x} = L_{67} - \mathsf{y}$. We have $L_{78} = \mathsf{h}$. The other non-zero values occurring are the ones listed in the theorem as being non-negative, and equation (10) is the double-point formula. □

For $\mathcal{Z}_1$ we get strong restrictions. In this case $\operatorname{rk}\operatorname{Pic} X = 4$. The primitive relations are

$$\begin{array}{lll}
x_1+x_2+x_5=0, & x_1+x_2+x_6=x_7, & x_2+x_4+x_5=x_8,\\
x_2+x_4+x_6=x_7+x_8, & x_3+x_7+x_8=0, & x_3+x_4+x_6=x_1+x_5,\\
x_3+x_4+x_7=x_1, & x_3+x_6+x_8=x_5. &
\end{array}$$

No two D_i are equivalent, so $\Lambda = L$. We choose the basis $\beta_1 = D_5^2$, $\beta_2 = D_5D_6$, $\beta_3 = D_6^2$, $\beta_4 = D_6D_7$, $\beta_5 = D_6D_8$, $\beta_6 = D_7^2$, $\beta_7 = D_7D_8$, $\beta_8 = D_8^2$. Thus L is an 8×8 matrix of linear forms in 8 variables $\mathsf{a}, \dots, \mathsf{h}$.

Proposition 6.8 *If $\mathsf{a}\beta_1 + \cdots + \mathsf{h}\beta_8$ is the class of a totally nondegenerate abelian surface in a toric Fano 4-fold of type $\mathcal{Z}_1$ then* $(\mathsf{w},\mathsf{x},\mathsf{y},\mathsf{z}) = (0,2,2,6)$, $(0,3,1,8)$, $(1,2,1,2)$, $(0,1,3,8)$ *or* $(1,1,2,2)$. *These integers determine* $(\mathsf{a},\dots,\mathsf{h})$ *by*

$$\begin{aligned}
(\mathsf{a},\dots,\mathsf{h}) = {} & (\mathsf{w}+\mathsf{x}+\mathsf{y}+\mathsf{z}, 3\mathsf{w}+2\mathsf{x}+2\mathsf{y}+\mathsf{z}, 3\mathsf{w}+\mathsf{x}+\mathsf{y},\\
& 3\mathsf{w}+2\mathsf{x}, 2\mathsf{y}+\mathsf{z}, \mathsf{w}+\mathsf{x}, 2\mathsf{y}+2\mathsf{z}, -\mathsf{y}-\mathsf{z}).
\end{aligned}$$

Proof. The entries L_{18} and L_{57} vanish, so Lemma 1.1(iii) gives four linear relations $L_{11} = L_{55} = L_{77} = L_{88} = 0$, namely

$$\begin{aligned} \mathsf{a} - \mathsf{b} + \mathsf{c} - \mathsf{d} + 2\mathsf{f} &= 0 \\ 3\mathsf{a} - 2\mathsf{b} + \mathsf{c} + \mathsf{e} + 2\mathsf{h} &= 0 \\ \mathsf{c} - 2\mathsf{d} - \mathsf{e} + 3\mathsf{f} + \mathsf{g} + \mathsf{h} &= 0 \\ 2\mathsf{a} - \mathsf{b} + \mathsf{c} - \mathsf{d} + \mathsf{f} + \mathsf{g} + 3\mathsf{h} &= 0. \end{aligned}$$

This system has rank 3 and we use it to eliminate a, b and c.

Since $L_{18} = 0$ we have $L_{14}L_{85} = L_{15}L_{84}$ by Lemma 1.1(v) and this gives $\mathsf{g} = -2\mathsf{h}$. Now we may use the variables $\mathsf{w} = L_{15}$, $\mathsf{x} = L_{16}$, $\mathsf{y} = L_{45}$ and $\mathsf{z} = L_{46}$. The double-point formula gives

$$(2(\mathsf{x}+\mathsf{y})+6\mathsf{w}-6)\mathsf{z}+(3\mathsf{w}-17)\mathsf{w}+(6\mathsf{w}-11)(\mathsf{x}+\mathsf{y})+(\mathsf{x}+\mathsf{y})^2+4\mathsf{xy} = 0 \quad (11)$$

from which it follows immediately that $\mathsf{w} \leq 5$. Then it is simple to list the integer solutions for $(\mathsf{w}, \mathsf{x}, \mathsf{y}, \mathsf{z})$. Note that L and equation (11), but not $\mathsf{a}, \ldots, \mathsf{h}$, are symmetric in x and y so we may assume that $\mathsf{x} \geq \mathsf{y}$ at this stage. The solutions $(0, 6, 0, 5)$, $(0, 11, 0, 0)$, $(1, 1, 0, 9)$ and $(1, 1, 1, 4)$ all give odd values for some L_{ii}, and $(0, 4, 0, 14)$ is excluded because L_{5*} vanishes. $(1, 7, 0, 0)$ gives $L_{44} < 0$ and $(1, 2, 0, 5)$ gives $L_{45} = 0$ but $L_{44} > 0$, so both are excluded.

The remaining solutions are $(0, 2, 2, 6)$, $(0, 3, 1, 8)$ and $(1, 2, 1, 2)$ which, after possibly interchanging x and y, give the result claimed. □

For $\mathcal{Z}_2$ we get only weak restrictions. In this case $\operatorname{rk}\operatorname{Pic} X = 4$. The primitive relations are

$$\begin{array}{lll} x_1 + x_2 + x_5 = 0, & x_1 + x_2 + x_6 = x_7, & x_2 + x_4 + x_5 = x_8, \\ x_2 + x_4 + x_6 = x_7 + x_8, & x_3 + x_7 + x_8 = x_2, & x_3 + x_4 + x_6 = 0, \\ x_3 + x_4 + x_7 = x_1 + x_2 & x_3 + x_6 + x_8 = x_2 + x_5 & \end{array}$$

No two D_i are equivalent, so $\Lambda = L$. We choose the basis $\beta_1 = D_5^2$, $\beta_2 = D_5D_6$, $\beta_3 = D_6^2$, $\beta_4 = D_6D_7$, $\beta_5 = D_6D_8$, $\beta_6 = D_7^2$, $\beta_7 = D_7D_8$, $\beta_8 = D_8^2$. Thus L is an 8×8 matrix of linear forms in 8 variables $\mathsf{a}, \ldots, \mathsf{h}$.

Proposition 6.9 *If $\mathsf{a}\beta_1 + \cdots + \mathsf{h}\beta_8$ is the class of a totally nondegenerate abelian surface in a toric Fano 4-fold of type $\mathcal{Z}_2$ then there are positive integers x, y, z and a non-negative integer w such that*

$$3\mathsf{w} = \mathsf{x}^2 + 2\mathsf{xy} + 2\mathsf{yz} + 2\mathsf{zx} - 9\mathsf{x} - 4\mathsf{y} - 10\mathsf{z}. \quad (12)$$

Moreover $\mathsf{w} + \mathsf{z} \geq \mathsf{y}$ and $\mathsf{w} + 2\mathsf{z} \geq 2\mathsf{y}$.

These numbers determine $(\mathsf{a}, \ldots, \mathsf{h})$ by

$$(\mathsf{a}, \ldots, \mathsf{h}) = (\mathsf{w} + 2\mathsf{z}, \mathsf{x} + \mathsf{y} + \mathsf{z}, 0, -\mathsf{y}, \mathsf{x} + \mathsf{y}, 0, \mathsf{w} + \mathsf{x} + \mathsf{z}, -\mathsf{w} - \mathsf{z})$$

Proof. $L_{18} = 0$ so $\mathsf{f} = L_{11} = 0$. But $L_{12} = \mathsf{f}$ so $L_{22} = 0$. Also $L_{57} = 0$ so $L_{55} = L_{77} = 0$. This gives two independent equations, which we use to eliminate a and d. Applying Lemma 1.1(v) to the first two rows gives $\mathsf{c} = 0$, since $L_{15} = \mathsf{c}$. Now we use the variables $\mathsf{x} = L_{32}$, $\mathsf{y} = L_{37}$ and $\mathsf{z} = L_{38}$, all of which are positive (otherwise a row vanishes), and $\mathsf{w} = L_{44}$. Equation (12) is the double-point formula and the inequalities are imposed by $L_{46} = \mathsf{w} - \mathsf{y} + \mathsf{z}$ and $L_{46} = \mathsf{w} - 2\mathsf{y} + 2\mathsf{z}$. □

7 Summary

We conclude with a table that shows what is known about the existence of totally nondegenerate abelian surfaces in smooth toric Fano 4-folds. The Fano 4-folds are listed in the order of the tables in [14]. In the second column, ✓ indicates that it is known that such surfaces do exist and × indicates that it is known that they do not. The symbols ♭ and ♯ indicate that it is not known whether such surfaces exist: ♯ is used when there are known to be only finitely many classes that could possibly accommodate such a surface. The third column in each block gives a reference to a paper or to a theorem in this paper, where more details may be found.

$\mathbb{P}^4$	✓	[3]	$\mathcal{G}_1$	×	[13]	$\mathcal{Q}_{10}, \mathcal{Q}_{11}$	✓	[13]
$\mathcal{B}_1$-$\mathcal{B}_3$	×	[13]	$\mathcal{G}_2$	×	[6]	$\mathcal{Q}_{12} - \mathcal{Q}_{15}$	×	[13]
$\mathcal{B}_4$	✓	[7]	$\mathcal{G}_3$	♭	5.1	$\mathcal{Q}_{16}$	♯	6.4
$\mathcal{B}_5$	×	[11]	$\mathcal{G}_4$	♯	5.2	$\mathcal{Q}_{17}$	×	[13]
$\mathcal{C}_1, \mathcal{C}_2$	×	[11]	$\mathcal{G}_5$	♭	5.3	$\mathcal{K}_1 - \mathcal{K}_3$	×	[13]
$\mathcal{C}_3$	×	3.2	$\mathcal{G}_6$	×	[6]	$\mathcal{K}_4$	✓	[13]
$\mathcal{C}_4$	✓	[13]	$\mathcal{H}_1 - \mathcal{H}_7$	×	[13]	$\mathcal{R}_1 - \mathcal{R}_3$	×	[13]
$\mathcal{E}_1 - \mathcal{E}_3$	×	[13]	$\mathcal{H}_8$	✓	[13]	$\mathcal{P}$	×	[13]
$\mathcal{D}_1 - \mathcal{D}_6$	×	[13]	$\mathcal{H}_9, \mathcal{H}_{10}$	×	[13]	$\mathcal{U}_1 - \mathcal{U}_4$	×	[13]
$\mathcal{D}_7$	♯	4.1	$\mathcal{L}_1 - \mathcal{L}_6$	×	[13]	$\mathcal{U}_5$	✓	[13]
$\mathcal{D}_8, \mathcal{D}_9$	×	[13]	$\mathcal{L}_7 - \mathcal{L}_9$	✓	[13]	$\mathcal{U}_6, \mathcal{U}_7$	×	[13]
$\mathcal{D}_{10}$	×	4.2	$\mathcal{L}_{10}$	×	[13]	$\mathcal{U}_8$	♯	6.5
$\mathcal{D}_{11}$	♭	4.3	$\mathcal{L}_{11}$	♭	6.2	$\widetilde{\mathcal{V}}_4$	×	[6]
$\mathcal{D}_{12}$	×	[13]	$\mathcal{L}_{12}$	×	[13]	$\mathcal{V}_4$	♭	6.6
$\mathcal{D}_{13}$	✓	[13]	$\mathcal{L}_{13}$	♯	6.3	$S_2 \times S_2$	✓	[13]
$\mathcal{D}_{14}$	♭	4.4	$\mathcal{I}_1 - \mathcal{I}_8$	×	[13]	$S_2 \times S_3$	✓	[13]
$\mathcal{D}_{15}$	✓	[13]	$\mathcal{I}_9$	♯	6.1	$S_3 \times S_3$	✓	[13]
$\mathcal{D}_{16}$	×	[13]	$\mathcal{I}_{10} - \mathcal{I}_{15}$	×	[13]	$\mathcal{Z}_1$	♯	6.8
$\mathcal{D}_{17}$	♭	4.5	$\mathcal{M}_1 - \mathcal{M}_5$	×	[13]	$\mathcal{Z}_2$	♭	6.9
$\mathcal{D}_{18}$	×	4.6	$\mathcal{J}_1 . \mathcal{J}_2$	×	[13]	$\mathcal{W}$	♭	6.7
$\mathcal{D}_{19}$	×	[6]	$\mathcal{Q}_1 - \mathcal{Q}_9$	×	[13]			

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