HYPERELLIPTIC GENUS 4 CURVES ON ABELIAN SURFACES

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Abstract. We study smooth curves on abelian surfaces, especially for genus 4, when the complementary subvariety in the Jacobian is also a surface. We show that up to translation there is exactly one genus 4 hyperelliptic curve on a general $(1,3)$-polarised abelian surface. We investigate these curves and show that their Jacobians contain a surface and its dual as complementary abelian subvarieties.

1. Introduction

A smooth curve C in an abelian surface A over the complex numbers cannot be of genus 0, and if it is of genus 1 then A is isogenous to a product $C \times C'$ of genus 1 curves. A smooth genus 2 curve can be embedded only in one abelian surface, namely as the theta divisor in its Jacobian. For curves of higher genus, we do not know much. There is one well known example: étale degree n cyclic covers of genus 2 curves, which provide curves embedded in $(1,n)$-polarised surfaces. On the other hand, a general section of a polarising line bundle on a (d_1,d_2)-polarised surface defines a smooth curve of genus $g = 1 + d_1d_2$. In general we know little about the geometry of these curves. For $(1,2)$ polarisation, W. Barth [B] shows that all curves in the linear series of a polarising line bundle are double covers of elliptic curves branched in 4 points. For $E \times E$ with a $(1,3)$ polarisation, Ch. Birkenhake and H. Lange [BL2] listed the types of curves in the linear series.

In this paper, we describe a distinguished family of curves in $(1,3)$-polarised abelian surfaces. As the construction is similar to the construction of the theta divisor on a principally polarised surface and their properties are similar, we call them $(1,3)$ theta divisors.

The paper is organised in the following way. Section 2 first provides some notation and background. Then, assuming that $f_C \colon C \to A$ is an embedding of a smooth complex curve of genus $g > 2$ in an abelian surface, we show (Lemma 2.6) that f_C induces an exact sequence with connected kernel

$$0 \longrightarrow K \longrightarrow JC \longrightarrow A \longrightarrow 0.$$

Thus JC is non-simple and contains K and $\hat{A}$ as complementary abelian subvarieties. Moreover, the restriction to $\hat{A}$ of the natural principal polarisation on JC is dual to the polarisation $c_1(\mathcal{O}_A(f_C(C)))$ on A.

Section 3 focuses on the case $g = 4$, where there is additional symmetry. Firstly, $g = 1 + d_1d_2$, so the type of polarisation has to be $(1,3)$, and secondly $\dim(JC) = 4$, so K is also a $(1,3)$-polarised surface. One main result of the paper (Theorem 3.7) is that a general $(1,3)$-polarised abelian surface contains, up to translation, exactly one smooth hyperelliptic curve of genus 4. This result is consistent with the number 9 of hyperelliptic genus 4 curves on a $(1,3)$-polarised surface given by the counting function $h^{A,FLS}_{g,\beta}$ defined in [BOPY]: see Remark 2.

Moreover, in this case of $g = 4$ we show that $K = A$, and indeed we characterise Jacobians of such curves as hyperelliptic Jacobians that contain a surface A and $\hat{A}$ as complementary abelian subvarieties: see Theorem 3.8.

Some results of this paper are contained in the PhD thesis of the first author.

2. Preliminaries

2.1. **Theta functions.** We recall very briefly some basic facts about theta functions, characteristics and theta constants. For simplicity of notation we shall immediately restrict our attention to $(1,3)$-polarised surfaces (A,H). A line bundle denoted L will always be a *polarising line bundle*, i.e. $c_1(L) = H$. For generalisations (to higher degree and higher dimension), proofs and details, we refer to [BL]. For $Z \in \mathfrak{h}_2$ (the Siegel upper half-plane), we denote by A_Z the complex torus $\mathbb{C}^2/(Z\mathbb{Z}^2 + \operatorname{diag}(1,3)\mathbb{Z}^2)$, always with the polarisation $H \in \operatorname{NS}(A_Z)$ whose Riemann form, also denoted by H, is given by $H = \operatorname{Im}(Z)^{-1}$. We take the standard decomposition $\mathbb{C}^2 = Z\mathbb{R}^2 \oplus \operatorname{diag}(1,3)\mathbb{R}^2$.

Because we use both Riemann canonical and Riemann classical theta functions, we will have to abuse notation in the following way. For a canonical theta function we think of its characteristic as a point $c \in \mathbb{C}^2$ (a complex characteristic) and write $c = c_1 + c_2$, where $c_1 \in Z\mathbb{R}^2$ and $c_2 \in \mathbb{R}^2$. For a classical theta function its characteristic will be a pair (c_1, c_2) (a real characteristic), where c_1, $c_2 \in \mathbb{R}^2$, and we put $c = Zc_1 + \operatorname{diag}(1,3)c_2 \in \mathbb{C}^2$.

Any 2-torsion point $x \in A_Z[2]$ can be written as the image of $x = Zc_1 + \operatorname{diag}(1,3)c_2$ for some c_1, $c_2 \in \frac{1}{2}\mathbb{Z}^2$.

Definition 2.1. *Define* $e_*(c_1,c_2) = \exp(4\pi i\ {}^t c_1 c_2) \in \{\pm 1\}$. *A 2-torsion point* $x = Zc_1 + \operatorname{diag}(1,3)c_2$ *is called* even *or* odd *depending on the parity of* $e_*(c_1,c_2)$.

A simple computation shows that there are exactly ten even and six odd 2-torsion points.

On A_Z there exists a unique polarising line bundle of characteristic 0, which we denote L_0. Any other polarising line bundle is of the form $L = t_c^* L_0$ for some $c \in A_Z$ unique up to $K(L) = \{x \in A_Z \colon t_x^*(L) \cong L\} \cong \mathbb{Z}/3\mathbb{Z}$.

The *classical Riemann theta function* of real characteristic (c_1, c_2) is (see [BL, Section 3.2])

$$\theta\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}(v,Z) = \sum_{l\in\mathbb{Z}^2} \exp\Big(\pi i\ {}^t(l+c_1)Z(l+c_1) + 2\pi i\ {}^t(v+c_2)(l+c_1)\Big). \tag{1}$$

The *canonical Riemann theta function* of complex characteristic c is

$$\begin{aligned}\theta^c(v) = \exp\Big(&-\pi H(v,c) - \frac{\pi}{2}H(c,c) + \frac{\pi}{2}B(v+c,v+c)\Big)\\ &\cdot \sum_{\lambda\in Z\mathbb{Z}^2} \exp\Big(\pi(H-B)(v+c,\lambda) - \frac{\pi}{2}(H-B)(\lambda,\lambda)\Big).\end{aligned} \tag{2}$$

where H is a Riemann form and B is the bilinear extension of $H|_{\mathbb{Z}^2}$. For any $\eta \in K(L)_1 = K(L)|_{Z\mathbb{R}^2}$, we define

$$\theta^c_\eta = a_L(\eta,\cdot)^{-1}\theta^c(\cdot + \eta)$$

For details, again see [BL, Section 3.2].

In the case of $(1,3)$-polarised abelian surfaces, $K(L)_1$ is generated by $\omega = Z(0, \frac{1}{3})$ which, when we write classical theta functions, becomes $\omega = (0, \frac{1}{3})$ by the above convention. Then $H^0(L_0)$ can be identified with a space of classical theta functions and $\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\theta\begin{bmatrix} \omega \\ 0 \end{bmatrix}$, $\theta\begin{bmatrix} -\omega \\ 0 \end{bmatrix}$ form a basis. It can also be identified with a space of canonical theta functions and in this case θ^0_0, θ^0_ω, $\theta^0_{-\omega}$ form a basis.

Definition 2.2. *A polarising line bundle on $A = A_Z$ is said to be* symmetric *if* $(-1)^*L \cong L$.

For odd degree, the characteristic of a symmetric line bundle L can be chosen uniquely to be a 2-torsion point on A. If L is symmetric, then (-1) acts on $H^0(L)$, acting on (either kind of) theta function by $(-1)^*\theta(z) = \theta(-z)$. The Inverse Formula [BL, Inverse Formula 4.6.4] gives this action in terms of the basis of canonical theta functions.

Proposition 2.3 (BL, Inverse Formula 4.6.4)**.** *Let L be a symmetric line bundle of characteristic $c = c_1 + c_2$. Then for $\eta \in K(L)_1$ we have*

$$(-1)^*\theta^c_\eta = \exp\big(4\pi i \operatorname{Im} H(\eta + c_1, c_2)\big)\theta^c_{-\eta-2c_1}.$$

We denote the (± 1)-eigenspaces by $H^0(L)_\pm$ and call the corresponding theta functions even and odd theta functions.

2.1.1. *Divisors.* Let $\mathfrak{D}$ be an effective ample divisor on an abelian surface A. We say that $\mathfrak{D}$ is symmetric if $(-1)^*\mathfrak{D} = \mathfrak{D}$. By $\operatorname{mult}_x \mathfrak{D}$ we denote the multiplicity of $\mathfrak{D}$ in $x \in A$. Then $\mathfrak{D}$ is called even or odd depending on the parity of $\operatorname{mult}_0 \mathfrak{D}$.

We need to understand the behaviour of a symmetric divisor in $A[2]$.

Definition 2.4. *Define*

$$A[2]^+_{\mathfrak{D}} = \{c \in A[2] \colon \operatorname{mult}_c(\mathfrak{D}) \equiv 0 \mod 2\},$$
$$A[2]^-_{\mathfrak{D}} = \{c \in A[2] \colon \operatorname{mult}_c(\mathfrak{D}) \equiv 1 \mod 2\}.$$

[BL, Exercise 4.12.14] gives us the following.

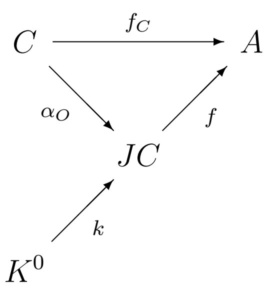
Proposition 2.5. *If $\mathfrak{D}$ is an effective symmetric divisor on (A, H) such that $\mathcal{O}(\mathfrak{D}) = L_0$, of type $(1,3)$, then*

$$\#A[2]^\pm_{\mathfrak{D}} = \begin{cases} 2(4 \pm 1) & \text{if } \mathfrak{D} \text{ is even} \\ 2(4 \mp 1) & \text{if } \mathfrak{D} \text{ is odd.} \end{cases}$$

Moreover, there are ten (respectively six) symmetric bundles L such that $\#A[2]^-_{\mathfrak{D}} = 6$ (respectively $\#A[2]^-_{\mathfrak{D}} = 10$) for all even symmetric divisors with $\mathcal{O}(\mathfrak{D}) = L$.

2.2. **Embedded curves.** Suppose that $f_C \colon C \to A$ is an embedding of a smooth curve of genus $g > 1$ in an abelian surface. Let (JC, Θ) denote the polarised Jacobian of C. Without loss of generality, we can choose a point $O \in C$ such that $f_C(O) = 0$. Then, by the universal property of Jacobians, we have the following diagram:

(3)



where α_O is the Abel-Jacobi map, f is the canonical homomorphism defined by the universal property and K^0 is the identity component of the kernel of f.

The image $f_C(C)$ generates A, so f must be surjective, and hence K^0 is an abelian subvariety of dimension $g-2$. The following lemma tells us that in fact $K^0 = \ker(f)$.

Lemma 2.6. *The kernel of f is connected. Hence, dualising the exact sequence*

$$0 \longrightarrow \ker(f) = K^0 \xrightarrow{k} JC \xrightarrow{f} A \longrightarrow 0$$

we get an embedding $\hat{f}\colon \hat{A} \hookrightarrow JC$.

Proof. The kernel of f is a reduced effective 2-cycle in JC consisting of a finitely many connected components $K^0, \dots, K^t$, each K^i being a copy (a translate) of the identity component K^0, and in particular numerically equivalent to K^0.

Each $P \in A$ defines Abel-Jacobi map $\alpha_P\colon C \to JC$, given by $\alpha_P(Q) = \mathcal{O}_C(Q-P)$. We define the difference map $\delta\colon C \times C \to JC$ by

$$\delta(P,Q) \mapsto \mathcal{O}(P-Q) = \alpha_Q(P) \in JC.$$

We claim that the image $\delta(C \times C)$ is an effective 2-cycle. Certainly the image is effective and irreducible, and has dimension at least 1 because it contains $\delta(C \times \{O\}) = \alpha_O(C) \cong C$. Indeed it contains $\alpha_P(C)$ for every $P \in C$, so if it is of dimension 1 then $\alpha_P(C) = \alpha_O(C)$ for every $P \in C$. But then the isomorphism

$$\otimes \mathcal{O}_C(O-P)\colon \alpha_O(C) \longrightarrow \alpha_P(C),$$

which is the restriction of $t_{[\mathcal{O}(O-P)]}$, becomes an automorphism of $\alpha_O(C)$; moreover, the group of automorphisms acts transitively on $\alpha_O(C)$ because $\alpha_O(P) = -\alpha_P(O)$ is sent to $\alpha_O(Q)$ by $t_{[\mathcal{O}(Q-P)]}$. This is impossible, because a curve of genus $g > 1$ does not have transitive automorphism group.

Next, we show that $\delta(C \times C) \cap \ker f = \{0\}$, and in particular is connected. One inclusion is obvious. For the other, choose a point $\mathcal{O}(P-Q) \in \delta(C \times C) \cap \ker(f)$. It is obviously in the image of α_Q. Now, from the universal property of the Jacobian, f makes the following diagram commutative

$$\begin{array}{ccc} C & \xrightarrow{f_C} & A \\ \downarrow{\scriptstyle \alpha_Q} & & \uparrow{\scriptstyle t_{f(Q)}} \\ JC & \xrightarrow{f} & A \end{array}$$

But C is embedded in A and in JC so $f|_{\alpha_Q(C)}$ has to be injective. Hence $\ker(f) \cap \alpha_Q(C) = \{0\}$ and therefore $\mathcal{O}(P-Q) = 0$.

That means that $\delta(C \times C)$ has non-empty intersection with exactly one of the connected components of $\ker(f)$; but the connected components are all numerically equivalent, so there can only be one of them. $\square$

2.2.1. *Polarisations.* Because $g > 1$, the bundle $\mathcal{O}_A(f_C(C))$ is ample, of some type (d_1, d_2) with $g = 1 + d_1 d_2$. Recall that the dual abelian variety $\hat{A}$ of a polarised abelian variety (A, H) carries a uniquely defined dual polarisation, denoted $\hat{H}$ [BL, Prop. 14.1.1].

Proposition 2.7. *In Lemma 2.6, we have $\hat{f}^*\Theta \equiv \widehat{\mathcal{O}(f(C))}$ as classes in* $\mathrm{NS}(\hat{A})$.

Proof. This is a direct application of the following proposition. ☐

Proposition 2.8. [BL2, Proposition 4.3] *Let C be a smooth curve and (JC, Θ) its Jacobian. Let (A, H) be a polarised abelian surface and suppose $f_C\colon C \to A$ is a morphism and $f\colon JC \longrightarrow A$ is the canonical homomorphism defined by the universal property. Then the following are equivalent:*

(1) $\hat{f}^*\Theta \equiv \hat{H}$;
(2) $(f_C)_*[C] = H$ *in* $H^2(A, \mathbb{Z})$.

Corollary 2.9. *If $g(C) = 2$, then f is an isomorphism, so $(A, \mathcal{O}(f(C))) = (JC, \Theta)$ is the Jacobian of C.*

Proposition 2.8 allows us to invert the construction in the following way.

Proposition 2.10. *Let C be a smooth curve of genus g whose Jacobian contains an abelian surface, denoted $\hat{A}$. If $\Theta|_{\hat{A}}$ is of type (d_1, d_2) and $d_1 d_2 = g - 1$, then C can be embedded in the dual surface $A = \hat{\hat{A}}$ and we recover Diagram* (3).

Proof. Let $\hat{f}\colon \hat{A} \longrightarrow JC$ be the inclusion. We can dualise it to get a map $f\colon JC \longrightarrow A$. Choose a point $O \in C$ and consider the Abel-Jacobi map α_O. Then $f_C = f|_{\alpha_O(C)}\colon C \to A$ is a morphism and by construction and Proposition 2.8, $c_1(\mathcal{O}_A(f_C(C)))$ has to be of type (d_1, d_2). As the arithmetic genus of the image equals $1 + d_1 d_2 = g$ we get that f_C has to be an isomorphism onto its image, and therefore an embedding. ☐

As in [Bo] we denote by Is^g_D the locus in $\mathcal{A}_g$ of principally polarised abelian varieties of dimension g containing an abelian subvariety of dimension k on which the restricted polarisation has type $D = (d_1, \ldots, d_k)$. The restriction $k \leq \frac{g}{2}$, which is imposed in [Bo] to exclude empty cases, is not needed here, but note that $\mathrm{Is}^3_{(1,2)} = \mathrm{Is}^3_{(2)}$ because the complementary abelian subvariety to a $(1, 2)$-polarised surface is an elliptic curve with restricted polarisation of type (2).

Theorem 2.11. *For every $g \geq 3$ and any $D = (d_1, d_2)$ such that $g = 1 + d_1 d_2$, there is a $(g+1)$-dimensional family of non-isomorphic smooth curves of genus g that have non-simple Jacobians belonging to $\mathrm{Is}^g_{(d_1,d_2)}$.*

Proof. For a general (d_1, d_2)-polarised surface (A, H), we have $h^0(A, L) = d_1 d_2$ and the zero set of a general section is a smooth curve of genus $g = 1 + d_1 d_2$. The moduli space $\mathcal{A}_D$ of (d_1, d_2)-polarised abelian surfaces is of dimension 3. We claim that a given genus g curve has only finitely many embeddings of this kind, up to translation. More precisely, if $O \in C$ is a base point, the set

$$\Phi_D = \{f_C\colon (C, O) \to (B, 0) \mid f_c \,\text{an embedding},\ (B, H_B) = (B, [\mathcal{O}(f(C))]) \in \mathcal{A}_D\}$$

is finite. This follows because Propositions 2.8 and 2.10 give a bijection between Φ_D and

$$\Psi_D = \left\{\hat{f} \in \mathrm{Hom}(\hat{B}, JC) \mid \hat{f}\,\text{injective},\ \hat{f}^*\Theta = \hat{H}_B\right\}.$$

But JC contains only finitely many abelian surfaces such that Θ restricts to a polarisation of type (d_1, d_2), and $\mathrm{Aut}(\hat{B})$ is also finite, so Ψ_D is finite.

Hence the family of curves that arises in this way is of dimension $d_1 d_2 - 1 + 3 = g + 1$. ☐

In genus 3, we have recovered a result by W. Barth [B, Prop 1.8].

Corollary 2.12. *The following conditions on a smooth genus* 3 *curve* C *are equivalent:*

- C *can be embedded in a* $(1,2)$*-polarised abelian surface*
- $JC \in \mathrm{Is}^3_{(2)} = \mathrm{Is}^3_{(1,2)}$
- C *is bielliptic, i.e. a double cover of an elliptic curve branched in four points.*

The family of such curves is irreducible of dimension 4.

Remark 1. If a genus 3 curve C is both bielliptic and hyperelliptic then C has to be an étale double cover of a genus 2 curve, say T: see [Bo, Proposition 5]. Then there exists a polarised isogeny $\pi\colon A \to JT$ such that $C = \pi^{-1}(T)$. The converse also holds: any étale double cover of a genus 2 curve is hyperelliptic and bielliptic [M, page 346]. In particular, as the number of non-zero 2-torsion points in the kernel of the polarisation on A is three, there are exactly three hyperelliptic curves in the linear system of a $(1,2)$-polarising line bundle on a very general abelian surface [Bo, Proposition 6].

3. Genus four curves and $(1,3)$ theta divisors

Now we focus on genus 4 curves. They have special properties because, firstly, they can be embedded only in $(1,3)$ polarised surfaces and, secondly, the complementary abelian subvariety is also a $(1,3)$-polarised surface. Using the additional symmetry, we have the following theorem.

Theorem 3.1. *Let* C *be a smooth genus* 4 *curve. The following conditions are equivalent.*

(i) $JC \in \mathrm{Is}^4_{(1,3)}$.
(ii) C *can be embedded in an abelian surface* A.
(iii) C *has two non-isomorphic embeddings into abelian surfaces* A_1 *and* A_2.
(iv) JC *contains two complementary abelian surfaces* $\hat{A}$ *and* $\hat{B}$ *with restricted polarisation of type* $(1,3)$. □

We say that two embeddings $\varphi_i\colon C \to A_i$ are isomorphic if there is a polarised isomorphism $\psi\colon A_1 \to A_2$ such that $\varphi_2 = \psi\varphi_1$.

There is a well-known family of curves that fulfill the conditions of Theorem 3.1. We will follow results from [Ri]. Let $\pi\colon C \longrightarrow C'$ be an étale triple cyclic cover of a genus 2 curve, defined by a 3-torsion point $\eta \in JC'$. Choose σ, a lift to C of the hyperelliptic involution of C'. Then $E = C/\sigma$ is an elliptic curve. Moreover, $A = JC/\eta$ is embedded in JC, and by [Ri, Thm. 1], the Prym surface $P(C/C')$ is $E \times E$ with a polarisation given by $\mathcal{O}(E \times \{0\} \cup \{0\} \times E \cup \Delta)$, where Δ is the diagonal: see also [BL2]. Moreover, A and $P(C/C')$ are complementary to each other and of type $(1,3)$.

3.1. **Theta divisor.** Let $A = A_Z$ be an abelian surface with a $(1,3)$ polarisation H, as in Section 1. There is a unique odd section (up to scalar) of L_0, given in terms of classical theta functions by $\theta_A = \theta[{}^{-\omega}_{\ 0}] - \theta[{}^{\omega}_{0}]$ or, in terms of canonical theta functions, by $\theta_A = \theta^0_{-\omega} - \theta^0_{\omega}$.

Definition 3.2. *The* $(1,3)$ *theta divisor of* A *is the curve* $C_A = (\theta_A = 0)$ *in* A.

We made a few choices to define C_A, but it is well-defined up to translation, as the following (essentially [BL, Prop. 4.6.5]) shows.

Proposition 3.3. *Suppose L is a symmetric polarising line bundle of type $(1,3)$ and characteristic $c = c_1 + c_2$ with respect to some decomposition on an abelian surface A, so that $K(L)_1 = \{0, \eta, -\eta\}$, for some $\eta \in A[3]$. Let $h^0_\pm$ be the dimensions of the (± 1)-eigenspaces of the (-1) action on $H^0(L)$. Then*

- *if c is even then $h^0_+ = 2$, $h^0_- = 1$;*
- *if c is odd then $h^0_+ = 1$, $h^0_- = 2$.*

In both cases

$$\theta_{A,L} := \theta^c_\eta - \exp(4\pi i \operatorname{Im}(H)(\eta, c_2))\theta^c_{-\eta}$$

generates the 1-dimensional eigenspace, and so for every characteristic c we have

$$(\theta_A = 0) = t^*_c(\theta_{A,L} = 0).$$

The following lemma gives us a few basic properties of C_A.

Lemma 3.4. *Let A be a $(1,3)$-polarised surface and C_A be the $(1,3)$ theta divisor. Then:*

(i) *C_A is of arithmetic genus 4.*
(ii) *C_A passes through at least ten 2-torsion points on A.*
(iii) *If A is a general abelian surface, then C_A is smooth.*
(iv) *If C_A is smooth then it is hyperelliptic.*
(v) *For $A = E \times F$ with the product polarisation $\mathcal{O}_E(1) \boxtimes \mathcal{O}_F(3)$, where E, F are elliptic curves, the curve C_A is reducible and consists of one copy of F and three copies of E.*

Proof. (i) follows from adjunction and Riemann-Roch, (ii) is a consequence of Proposition 2.5, and (iii) follows from the work of Andreotti and Mayer: see [AM, Prop 6] or [ACGH, Ch. 6.4] for details.

For (iv), the involution (-1) on A restricts to C_A and the quotient $C' = C_A/\pm 1$ is a smooth curve. The quotient map $C_A \to C'$ is ramified at the 2-torsion points of A lying on C_A, of which there are $b \geq 10$ by (ii). The Hurwitz formula now gives

$$2g(C_A) - 2 = 2(2g(C') - 2) + b$$

and since $g(C_A) = 4$ and $b \geq 10$, the only possibility is $g(C') = 0$ and $b = 10$.

In the situation of (v), the matrix Z can be chosen to be diagonal, so the theta function is of the form $\theta(v_1, v_2) = f(v_1)g(v_2)$, where $f \in H^0(\mathcal{O}_E(1))$ and $g \in H^0(\mathcal{O}_F(3))$. Therefore, f has exactly one zero and g has three zeros, which gives the assertion. ∎

3.2. **Product of elliptic curves.** If (A, H) is a product, we can compute more details. Let

$$E = \mathbb{C}/\tau_1\mathbb{Z} + \mathbb{Z},\ F = \mathbb{C}/\tau_2\mathbb{Z} + \mathbb{Z},\ \Lambda = \begin{bmatrix} \tau_1 & 0 & 1 & 0 \\ 0 & \tau_2 & 0 & 3 \end{bmatrix},\ A = \mathbb{C}^2/\Lambda.$$

Then $A = E \times F$, with the product polarisation. We can take a standard decomposition

$$\mathbb{C}^2 = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix} \mathbb{R}^2 + \mathbb{R}^2$$

and write theta functions explicitly: for $v = (v_1, v_2) \in \mathbb{C}^2$ we have

$$\theta\left[\begin{smallmatrix} \pm\omega \\ 0 \end{smallmatrix}\right](v, Z) = \sum_{l_1, l_2 \in \mathbb{Z}} \exp(\pi i l_1^2 \tau_1 + \pi i (l_2 \pm \frac{1}{3})^2 \tau_2 + 2\pi i l_1 v_1 + 2\pi i v_2 (l_2 + \frac{1}{3})).$$

For the computations, let us denote

$$a_l = \exp(\pi i l^2 \tau_1 + 2\pi i v_1 l),$$
$$b_l^{\pm} = \exp(\pi i (l \pm \frac{1}{3})^2 \tau_2 + 2\pi i v_2 (l \pm \frac{1}{3})).$$

Then, since the series converge absolutely

$$\theta_A = \theta[{}^{\omega}_{0}] - \theta[{}^{-\omega}_{0}] = \sum_{l_1,l_2} a_{l_1} b_{l_2}^+ - \sum_{l_1,l_2} a_{l_1} b_{l_2}^- = \sum_{l_1} a_{l_1} (\sum_{l_2} b_{l_2}^+ - \sum_{l_2} b_{l_2}^-).$$

For $v_1 = \frac{1}{2} + \frac{1}{2}\tau_1$ we have

$$\begin{aligned} a_l &= \exp(\pi i l^2 \tau_1 + \pi i l \tau_1 + \pi i l) \\ &= \exp(\pi i (l + \frac{1}{2})^2 \tau_1 - \frac{1}{4}\pi i \tau_1 + \pi i l) \\ &= (-1)^l \exp(\pi i (l + \frac{1}{2})^2 - \frac{1}{4}\pi i \tau_1). \end{aligned}$$

Now, $a_l = -a_{-l-1}$, so $\sum_l a_l = 0$ and therefore for any v_2 we find $\theta_A(\frac{1}{2} + \frac{1}{2}\tau_1, v_2) = 0$. The image of this component of $\{v \mid \theta_A(v) = 0\}$ in A is a curve isomorphic to F.

Similar computations can be carried out for $v_2 = 0$, $v_2 = \frac{3}{2}$ and $v_2 = \frac{1}{2}\tau_2$. For $v_2 = 0$ and $v_2 = \frac{3}{2}$, we have $b_l^+ = b_l^-$. For $v_2 = \frac{1}{2}\tau_2$ we have $b_l^+ = b_{-l-1}^-$. In all three cases we get $\sum_l b_l^+ = \sum_l b_l^-$. The images in A of those zeros are isomorphic to E, so by Lemma 3.4, we know we have found all zeros of θ_A.

This gives us some more information about C_A in general.

Proposition 3.5. *For a general (A, H) of type $(1,3)$, the $(1,3)$ theta divisor C_A is not a finite cover of a curve of lower positive genus.*

Proof. If not, then $3E \cup F$ would be specialisations and so they would be covers of possibly singular curves. The only nontrivial case is to show that $3E \cup F$ is not a triple cover of a genus 2 curve. Then the intersection points of F and the copies of E would be 3-torsion points on F, but they are in fact 2-torsion points. □

In [BL2], Ch. Birkenhake and H. Lange consider $E \times E$ with the polarisation given by $E \times \{0\} \cup \{0\} \times E \cup \pm\Delta$, where $\pm\Delta$ is the diagonal or antidiagonal respectively. They prove that both polarisations are of type $(1,3)$ and that they are dual to each other. As C_A is characterised as the only symmetric divisor that passes through ten 2-torsion points with odd multiplicity, it is easy to see that in this case both of $E \times \{0\} \cup \{0\} \times E \cup \pm\Delta$ are $(1,3)$ theta divisors (for different polarised surfaces).

3.3. **Jacobian of C_A.**

Proposition 3.6. *Let C be a smooth hyperelliptic genus 4 curve. Then JC contains a $(1,3)$-polarised surface $\hat{A}$ if and only if C can be embedded into A as the $(1,3)$ theta divisor.*

Proof. In view of Theorem 3.1 and Lemma 3.4, it remains to prove that if C is hyperelliptic with $JC \in \mathrm{Is}^4_{(1,3)}$ then $C = C_A$ is the $(1,3)$ theta divisor of A. For $i\colon \hat{A} \to JC$ the inclusion of a surface with restricted polarisation of type $(1,3)$, we write $\hat{\imath}\colon JC \to A$ for the dual map, ι for the hyperelliptic involution, and α for the Abel-Jacobi map, and identify C with $\alpha(C) \subset JC$. Then $(-1)^*C = C$ is a symmetric curve because $\alpha(P) = -\alpha(\iota(P))$. Therefore the image $\hat{\imath}(C)$ is also symmetric in A. Note that by Proposition 2.8, $\hat{\imath}(C)$ is isomorphic to C.

Let $\pi_{JC}\colon JC \longrightarrow JC/(-1)$ be the Kummer map. Then $\pi_{JC}(C)$ is isomorphic to $\mathbb{P}^1$, because $(-1)|_C = \iota$. Let π_A be the Kummer map of A. Because $\hat{\imath}$ is a homomorphism, it descends to a map j making the diagram

$$\begin{array}{ccc} JC & \xrightarrow{\hat{\imath}} & A \\ \downarrow{\scriptstyle \pi_{JC}} & & \downarrow{\scriptstyle \pi_A} \\ JC/(-1) & \xrightarrow{j} & A/(-1) \end{array}$$

commute, given by $j(\pm x) = \pm\hat{\imath}(x)$. Now, $j(\pi_{JC}(C))$ has to be a rational curve, equal to $\pi_A(\hat{\imath}(C))$. As π_A is a 2-to-1 map, by the Hurwitz formula it has to be branched in ten points. The only possible branch points are 2-torsion points, so $\hat{\imath}(C)$ has to go through ten 2-torsion points of A. By Proposition 2.5, $\hat{\imath}(C) \cong C$ is the zero locus of an odd global section, which finishes the proof. □

From Proposition 3.6 we get the following.

Theorem 3.7. *A general* $(1,3)$*-polarised surface contains exactly one hyperelliptic curve up to translation.*

Remark 2. The authors of [BOPY] use Gromov-Witten theory of K3 surfaces to count hyperelliptic curves of arithmetic genus g on a fixed linear system of a $(1,d)$-polarised abelian surface: see also [Ro]. These numbers, called $h_{g,\beta}^{A,FLS}$, are presented in [BOPY, Table 1]. The numbers on the diagonal of that table, where $d = g+1$, are of particular interest because then the curves have to be smooth. These numbers are non-zero only for $g = 2$, 3, 4 and 5, and we can explain them as follows.

For $g = 2$, the surfaces are principally polarised so the number $h_{g,\beta}^{A,FLS} = 1$ corresponds to the curve embedded in its Jacobian.

For $g = 3$, there are six hyperelliptic curves according to [BOPY], yet by Remark 1 there are only three such curves up to translation. The difference follows from the fact that we can translate any curve by an element of the kernel of the polarisation and still stay in the fixed linear system. In this case, although the kernel has order four, we get only two copies of each curve in this way, because both copies are invariant under translation by a 2-torsion point that defines the polarised isogeny $\pi\colon A \to JT$ described in Remark 1.

For $g = 4$, the order of the kernel of the polarisation is 9. By Proposition 3.5, we know we get exactly 9 translates of the $(1,3)$ theta divisor.

The number for $g = 5$ is explained by forthcoming work of the first author and A. Ortega [BO], in which it is shown that such a curve is unique up to translation. The situation is similar to the $g = 3$ case: the kernel has order 16, but hyperelliptic curves are invariant under a subgroup of order 4.

Now we will prove the main result of this paper.

Theorem 3.8. *Let* C *be a smooth hyperelliptic genus* 4 *curve. Then* JC *contains a pair of complementary* $(1,3)$*-polarised surfaces* A *and* $\hat{A}$ *if and only if* C *can be embedded in* A *as the* $(1,3)$ *theta divisor.*

Proof. One implication follows from Proposition 3.6.

Now, recall from [BN] that a (desingularised) Kummer surface of a general $(1,3)$-polarised abelian surface A, denoted $\mathrm{Km}(A)$, can be embedded as a H_{22}-invariant quartic surface in $\mathbb{P}^3$ containing a $(32)_{(10)}$ configuration of lines. That means that there are 16 pairwise disjoint lines, called odd lines, and 16 pairwise disjoint lines, called even lines, such that every odd line intersects exactly 10 even lines and every even line intersects exactly 10 odd ones. It is easy to see that the two groups of lines are exactly images of hyperelliptic curves C_A translated by 2-torsion points, and 2-torsion points blown-up.

Moreover, $\mathrm{Km}(A) = \mathrm{Km}(\hat{A})$ and the surfaces can be reconstructed by choosing which set of 16 disjoint lines are to be blown down. Now, the fact that $C_A \cong C_{\hat{A}}$ follows from the fact that there exists an involution in the extended Heisenberg group that changes odd lines to even ones: see [BN, Section 4]. Restricting the involution to a line with 10 points marked, we obtain the desired isomorphism. As $C_A \cong C_{\hat{A}}$, we have $JC_A = JC_{\hat{A}}$ and, as A can be chosen to be simple, we get the assertion. □

Remark 3. Theorem 3.8 shows that the kernel K^0 that occurs in Diagram (3) is in fact isomorphic to $\hat{A}$. We can draw two diagrams for embeddings in A and $\hat{A}$ respectively.

(4)

From the construction, we see that $k_A = \hat{f}_A$ and $k_{\hat{A}} = \hat{f}_{\hat{A}}$, so the diagrams are dual to each other. In particular, the rational map

$$\Phi\colon \mathcal{A}_{(1,3)} \dashrightarrow \mathcal{A}_{(1,3)}, \text{ given by } \hat{A} \longrightarrow K^0,$$

defined by the construction and Diagram (3), is the dualisation in the moduli space.

This construction raises several questions. We should like more information about the locus $\mathrm{Is}^4_{1,3}$ and more generally about the loci Is^g_D in the moduli spaces of abelian varieties. Some answers are given in [Bo]. Alternatively, one could look at the loci in $\mathcal{M}_4$ or $\overline{\mathcal{M}}_4$ determined by the curves C_A, and ask for the class of the corresponding cycle in cohomology or the Chow ring. One could also ask for a description of the sets of ten points in $\mathbb{P}^1$ that form branch loci of the hyperelliptic involutions of the C_A.

One could also ask about the genus 4 curves whose Jacobians contain $(1,3)$-polarised surfaces. What are their properties? The question seems to be interesting, because on the side of Jacobians, the locus is naturally defined, whereas on $\mathcal{M}_4$, it contains disjoint subloci of hyperelliptic curves and covers of genus 2 curves. Those loci seems to be completely differently treated in the moduli of curves theory.

Finally, we ask whether it is possible to find genus 4 curves whose Jacobian contains a pair of complementary abelian subvarieties that are isomorphic to each other. If so, what are their properties?

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