Problem Sheet 2

UK IMO squad training camp, Bath 2001

1. (Israel) Find all solutions of the pair of simultaneous equations

$$\begin{cases} x_1 + x_2 + \cdots + x_{2000} &= 2000 \\ x_1^4 + x_2^4 + \cdots + x_{2000}^4 &= x_1^3 + x_2^3 + \cdots + x_{2000}^3 \end{cases}$$

2. (Israel) Given 2001 real numbers $x_1, x_2, \ldots, x_{2001}$ such that $0 \leq x_n \leq 1$ for each $n = 1, 2, \ldots, 2001$, find the maximum value of

$$\left(\frac{1}{2001} \sum_{n=1}^{2001} x_n^2 \right) - \left(\frac{1}{2001} \sum_{n=1}^{2001} x_n \right)^2 .$$

Where is this maximum attained?

3. (Hungary) Let t and r be the area and the inradius of the triangle ABC. Let r_a denote the radius of the circle touching the incircle, AB and AC. Define r_b and r_c in a similar fashion. The common tangent of the circles with radii r and r_a cuts a little triangle from ABC with area t_a. Quantities t_b and t_c are defined in a similar fashion. Prove that

$$\frac{t_a}{r_a} + \frac{t_b}{r_b} + \frac{t_c}{r_c} = \frac{t}{r}.$$

4. (Hungary-Israel competition) In this question, G_n is a simple undirected graph with n vertices, K_n is the complete graph with n vertices, and $e(G_n)$ is the number of edges of the graph G_n. Let C_n denote circle with n verticies.

   (i) The edges of K_n, $n \geq 3$ are coloured with n colours, and every colour appears at least once. Prove that one can find a triangle whose sides are coloured with 3 different colours.

(ii) Suppose that $n \geq 5$ is given. If $e(G_n) \geq n^2/4+2$, prove that there exist two triangles which have exactly one common vertex.

(iii) Suppose that $e(G_n) \geq \frac{n\sqrt{n}}{\sqrt{2}} + n$. Prove that G_n contains C_4.

5. (Israel) 2001 lines are given in the plane. No two of them are parallel and no three meet in a point. These lines partition the plane into some *regions* (not necessarily finite) bounded by segments of these lines. is called a *map*. Two regions on the map are called *neighbours* if they share a side. The set of intersection points of lines is called the set of *verticies*. Two vertices are called neighbours if they are found on the same side.

   A legal colouring of the regions (one colour per region) must have neighbouring regions coloured differently. A legal colouring of the vertices (one colour per vertex) must have neighbouring vertices coloured differently.

   (i) What is the minimum number of colours required for the legal colouring of any map?

   (ii) What is the minimum number of colours required for the legal colouring of the vertices of any map?

6. (Israel) Triangle ABC in the plane Π is said to be *good* if it has the following property: for any point D in space, out of the plane Π, it is possible to construct a triangle with sides of lengths $|AD|, |BD|$ and $|CD|$. Find all the good triangles.

7. (Israel) Find a pair of integers (x, y) such that $15x^2 + y^2 = 2^{2000}$. Does there exist such a pair (x, y) with x odd?