Problem Sheet 1

UK IMO squad training camp, Bath 2001

1. (Italy) In a hexagon with equal angles, the lengths of four consecutive edges are 5, 3 6 and 7 (in that order). Find the lengths of the remaining two edges.

2. (Italy) In a basketball tournament, every team plays twice versus each other team. A won game scores 2 points, a lost game scores 0 points, (and a drawn game is impossible in basketball). A single team won the tournament with 26 points, and exactly two teams were in last position with 20 points. How many teams participated in the tournament?

3. (Italy) Given the equation $x^{2001} = y^x$,

   (i) find all pairs (x, y) of solutions with x prime and y a positive integer;

   (ii) find all pairs (x, y) of positive integers satisfying the equation.

4. (Germany) Determine all real numbers q for which the equation

   $$x^4 - 40x^2 + q$$

   has four zeros which form an arithmetic progression.

5. (Germany) Determine the maximal number of points which can be placed in a rectangle with sides of lengths 14 and 28 such that the distance between any two of them is greater than 10.

6. (Italy) The incircle γ of triangle ABC touches the side AB at T. Let D be the point on γ diametrically opposite to T, and let S be the intersection of the line through C and D with the side AB. Show that $|AT| = |SB|$.

7. (Italy) A square is filled with 100 lamps, arranged in 10 rows and 10 columns. Some of them are alight. The others are out. To each lamp corresponds a push-button that, when pressed, switches (i.e. changes the on/off state) each lamp in the same row and columns (including the lamp itself).

   (i) Determine the states from which it is possible to light all the lamps.

   (ii) What is the answer if the square has 81 lamps in 9 rows and 9 columns?

8. (Germany) Wiebke and Stefan are playing the following game on a chess board-like sheet of paper with 60 rows and 40 columns. They alternately cut one rectangle into two smaller rectangles. A move of Stefan is a vertical cut, which divides an $n \times m$ rectangle into an $n \times k$ rectangle and a $n \times (m-k)$ rectangle where $k \in \{1, 2, \ldots, m-1\}$. A move of Wiebke is a corresponding horizontal cut. The person who cannot make a legal move loses the game.

   (i) Who has a winning strategy if Stefan starts the game?

   (ii) Who has a winning strategy if Wiebke starts the game?