Pan African Mathematical Olympiad 2000

Paper 2 Solutions

4. Let a, b and c be real numbers such that
$$a^2 + b^2 = c^2.$$
Solve the system
$$\begin{aligned} x^2 + y^2 &= z^2 \\ (x+a)^2 + (y+b)^2 &= (z+c)^2. \end{aligned}$$
Solution *The third equation expands to*
$$(x^2 + y^2) + (a^2 + b^2) + 2ax + 2by = c^2 + z^2 + 2cz.$$
Subtracting the first two equations from this gives
$$2ax + 2by = 2cz \Leftrightarrow ax + by = cz \Leftrightarrow ax + by + cz = 2cz.$$
Also $a^2 + b^2 + c^2 = 2c^2$ *and* $x^2 + y^2 + z^2 = 2z^2$. *Now apply the Cauchy-Schwarz inequality to the vectors* (a, b, c) *and* (x, y, z)*:*
$$\begin{aligned} 2cz &= ax + by + cz \\ &\leq \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{x^2 + y^2 + z^2} = \sqrt{2c^2} \cdot \sqrt{2z^2} \ . \\ &= 2cz \end{aligned}$$
We have equality in the Cauchy-Schwarz inequality, which means that one vector must be a multiple of the other. This means that if $a = b = c = 0$, *then* x, y, z *could be any numbers such that* $x^2 + y^2 = z^2$ *(or equivalently* $z = \pm\sqrt{x^2 + y^2}$*), and such numbers clearly satisfy the conditions.*
If $(a, b, c) \neq (0, 0, 0)$, *then* x, y, z *must be a multiple of* (a, b, c). *Thus* $\exists \lambda \in \mathbb{R}$ *for which* $x = \lambda a$, $y = \lambda b$ *and* $z = \lambda c$. *We can check whether this satisfies the conditions for all* λ*:*
$$\begin{aligned} a^2 + b^2 = c^2 &\Rightarrow \lambda^2(a^2 + b^2) = \lambda^2(c^2) \Leftrightarrow x^2 + y^2 = z^2. \\ a^2 + b^2 = c^2 &\Rightarrow (\lambda + 1)^2(a^2 + b^2) = (\lambda + 1)^2 c^2 \\ &\Leftrightarrow (x + a)^2 + (y + b)^2 = (z + c)^2 \end{aligned}$$

5. From a point P outside a circle, tangents PA and PB are drawn. PQR is any secant[1], with Q and R on the circumference. Chord BS is parallel

[1] i.e. a straight line through P which intersects the circle twice

to PQR. Prove that SA bisects QR.

Solution *Let C be the intersection of AS and PQR. Then $\angle ACP = \angle SCR = \angle CSB = \angle ABP$ (tangent-chord theorem). Therefore $ACBP$ is a cyclic quadrilateral. Also $\angle SBC = \angle BCP = \angle BAP = \angle ABP = \angle BSC$ (since $PA = PB$) and thus ΔSCB is isosceles.*

It follows that the perpendicular bisector of BS passes through C. But oeroendicular bisectors of chords pass through the centre and $BS \parallel QR$, so BS and QR must share perpendicular bisectors. Thus C lies on the perpendicular bisector of QR, and therefore it is the midpoint of QR.

6. A company has five directors. The regulations of the company require that any majority (three or more) of the directors should be able to open the strongroom, but any minority (two or fewer) of the directors should not be able to do so. It is proposed to equip the strongroom with ten locks, so that it can only be opened when keys to all ten locks are available, and to give each director a set of keys to n different locks. Find all values of n for which there is a way to allocate the keys according to the regulations of the company.

   Solution *Directors A and B cannot open all the locks together, so there is a lock (say number 1), which neither can open, but all other directors must have key number 1. The same reasoning applies to all $\binom{5}{2} = 10$ unordered pairs of directors; each pair will be missing a particular key, and all other directors will own that key. This tells us how the 10 keys must be distributed to have the desired effect (the only choice being the particular labelling of the keys). Each director is missing a different key for each of the 4 directors with whom he or she can form a pair, so $n = 6$.*