Pan African Mathematical Olympiad 2000

Paper 1 Solutions

1. Solve the trigonometric equation

$$\sin^3 x(1+\cot x)+\cos^3 x(1+\tan x)=\cos 2x.$$

Solution

$$\begin{array}{ll} & \sin^3 x(1+\cot x)+\cos^3 x(1+\tan x)=\cos 2x \\ \Rightarrow & \sin^2 x(\sin x+\cos x)+\cos^2 x(\cos x+\sin x)=\cos^2 x-\sin^2 x \\ \Leftrightarrow & (\sin^2 x+\cos^2 x)(\sin x+\cos x)=(\cos x+\sin x)(\cos x-\sin x) \\ \Leftrightarrow & 0=(\cos x+\sin x)(\cos x-\sin x-1) \\ \Leftrightarrow & 0=(\sin\pi/4\cdot\cos x+\cos\pi/4\cdot\sin x)(\sin\pi/4\cdot\cos x-\cos\pi/4\cdot\sin x-\sqrt{2}/2) \\ \Leftrightarrow & 0=[\sin(\pi/4+x)][\sin(\pi/4-x)-\sqrt{2}/2]. \end{array}$$

Thus $\sin(\pi/4+x)=0$ *or* $\sin(\pi/4-x)=\sqrt{2}/2$ *The sine function takes the value 0 at and only at multiples of* π*, so the solutions to the first case are of the form* $k\pi-\pi/4$*, for* $k\in\mathbb{Z}$*. These solutions give* $1+\cot x=0$, $1+\tan x=0$ *and* $\cos 2x=0$*, so they clearly satisfy the original problem.*

We now examine the other candidate solutions. Now $\sin x=\sqrt{2}/2$ *at and only at* $\pi/4$ *and* $5\pi/4$ *if* $x\in[0,2\pi]$*. Thus the solutions of* $\sin(\pi/4-x)=\sqrt{2}/2$ *are of the form* $2k\pi$ *and* $2k\pi-\pi/2$ *for* $k\in\mathbb{Z}$*. None of these are solutions to the original problem, since they give* $\cot x$ *or* $\tan x$ *as undefined.*

2. Let the polynomials $P_0, P_1, P_2, \ldots$ be defined by

$$P_0(x)=x^3+213x^2-67x-2000$$

and

$$P_n(x)=P_{n-1}(x-n) \text{ for } n=1,2,3,\ldots$$

What is the coefficient of x in $P_{21}(x)$?

Solution

$$\begin{array}{rcl} P_{21}(x) & = & P_{20}(x-21) \\ & = & P_{19}(x-21-20) \\ & \vdots & \\ & = & P_0(x-21-20-\cdots-1) \\ & = & P_0(x-231) \\ & = & (x-231)^3+213(x-231)^2-67(x-231)-2000. \end{array}$$

The coefficient of x in each term can be found from the binomial theorem, so the coeffiient of x is

$$3 \cdot 231^2 - 213 \cdot 2 \cdot 213 - 67 = 61610.$$

3. Prove that if

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1334} + \frac{1}{1335}$$

where p and q are natural numbers, then 2003 divides p.
Solution *Firstly note that 2003 is a prime number (easily checked by trial division by all prime numbers less than $\sqrt{2003}$, i.e. all prime number up to 43.*

$$\begin{aligned}
1 - 1/2 + \cdots + 1/1335 &= 1 + 1/2 + \cdots + 1/1335 - 2(1/2 + 1/4 + \cdots + 1/1334) \\
&= 1 + 1/2 + \cdots + 1/1335 - (1 + 1/2 + \cdots + 1/667) \\
&= 1/668 + 1/669 + \cdots + 1/1335 \\
&= (1/668 + 1/1335) + (1/669 + 1/1334) + \cdots + (1/1001 + 1/1002) \\
&= 2003(1/(668 \times 1335) + 1/(669 \times 1334) + \cdots + 1/(1001 \times 1002)).
\end{aligned}$$

Now the denomominator of

$$1/(668 \times 1335) + 1/(669 \times 1334) + \cdots + 1/(1001 \times 1002)$$

in simplest form is coprime to 2003. This is because 2003 is prime and the numbers 668 to 1335 are all smaller than 2003, so their product cannot be a multiple of 2003. Thus the sum is p/q in simplest form where $2003 \mid p$.