BMOC Mentoring Scheme
Advanced Level – Sheet 2
Possible Solutions

1. *Show that there are no non-zero integer solutions to the equation*

$$x^2 + y^2 + z^2 = 2xyz$$

Looking at the equation modulo 2 we see that either x,y and z are all even or two are odd and one even. In the second case, looking at the equation modulo 4 shows this isn't possible. So, we can write $x = 2x'$,$y = 2y'$ and $z = 2z'$. This gives a new equation

$$x'^2 + y'^2 + z'^2 = 4x'y'z'$$

Even though the 2 has become a 4 the same reasoning as before works (it isn't too hard to see that it will work for any even number appearing on the right hand side). So, we can remove another factor of two. Hence, if there is a non-zero solution then we will be able to repeatedly remove factors of 2 from it. This will eventually lead to non-integer solutions which is a contradiction. So, there are no non-zero solutions. This idea of repeatedly removing factors is known as **Fermat's method of infinite descent**.

2. *Find infinitely many integer solutions to the equation*

$$x^2 + y^2 + z^2 = 3xyz$$

Note that the number on the right hand side is odd, so the reasoning for question 1 will not work. There is an obvious solution: $x = y = z = 1$. Because we have to find infinitely many solutions we really have only two choices: find some easy to describe values of x,y and z which give a solution or find a way to get new solutions from old ones. We use the second method.

Suppose that x,y and z satisfy $x^2 + y^2 + z^2 = 3xyz$. Let us fix two of the variables, say y and z. This gives a quadratic equations that x satisfies: $x^2 - (3yz)x + (y^2 + z^2) = 0$. Call the other solution to the quadratic x'. Because $x + x' = 3yz$ we see that x' is an integer. So, if (x, y, z) is a solution we have seen that $(3yz - x, y, z)$ is a solution too (you can check this by multiplying out if you are skeptical!). The only remaining thing to check is that this method gives infinitely many solutions.

If we always fix the bigger two numbers then it is easy to see that $3yz - x$ is strictly bigger than x. Hence, the minimum of x, y and z can always be increased using this method and so we can generate infinitely many solutions.

3. *Let M lie on AB between A and B, and construct semicircles on diameters AM, MB, AB, all on the same side of the line AB. Let the inscribed circle of these three semicircles be S. S is tangent to semicircle AM at b, to semicircle MB at a, and to semicircle AB at m. Construct three circles orthogonal to AB passing through a and b, a and m, b and m, respectively. Show that the angles between pairs of these circles (at a, b and m) are equal.*

As there are lots of circles and lines around, trying inversion seems like a good idea. If we invert about A then we get a (horizontal) line BM with two perpendicular lines, one through B and one through M. On BM is drawn a semicircle. S is constructed to be tangent to the semicircle and two vertical lines. Points m, a and b are given as before. If we invert about B then we get an identical diagram apart from the order of m, a and b. Similarly if we invert about M. Because inversion preserves angles the angles at m, a and b must be the same.

4. (a) *Show that 3^n divides $2^{3^n} + 1$ for all n.*

   Let's do this by induction. For $n = 1$ it is true because 3 divides $8 + 1$. Now, suppose that for some n we know 3^n divides $2^{3^n} + 1$. Looking at this modulo 3^{n+1} we see

   $$2^{3^n} \equiv -1, 3^n - 1 \text{ or } 2 \cdot 3^n - 1 \bmod 3^{n+1}$$

   Raising both sides to the power 3 (using $(x-1)^3 = x^3 - 3x^2 + 3x - 1$ on the right hand side) we see

   $$2^{3^{n+1}} \equiv -1 \bmod 3^{n+1}$$

   This is what we needed to show to finish the induction.

   (b) *Find an n which is not a power of 3 such that n divides $2^n + 1$.*

   Let us hope that things are not too far from a power of 3. So, we will first try to see if there is a solution with $n = 3p$ (p a prime not equal to 3). Substituting this in gives two congruences

   $$\left(2^3\right)^p + 1 \equiv 0 \bmod 3 \qquad \left(2^p\right)^3 + 1 \equiv 0 \bmod p$$

   The first is true for any odd p and the second shows (after using Fermat's little theorem) that p divides $2^3 + 1 = 9$. Unfortunately this isn't possible as p is not 3. So, now we try $n = 9p$. The first congruence is again satisfied for odd p and the second shows that p divides $2^9 + 1 = 513$. Hence $p = 19$ will do and so $n = 9 * 19 = 171$ will do.

5. *Show that the distance from the circumcentre to the orthocentre of a triangle is less than three times the circumradius.*

   We do this question with vectors. WLOG we can assume that the circumcentre is at the origin and the circumradius is 1. The three vertices of the triangle are now represented by three unit vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. A little experimentation then gives (you can write down the equations for the orthocentre, intelligently guess, ...) the orthocentre as being at $\mathbf{a} + \mathbf{b} + \mathbf{c}$. The inequality is now obvious.

6. *If $x,y,z \geq 0$ and $x + y + z = 1$ prove that*

$$4(xy + yz + zx) - 9xyz \leq 1$$

When does equality occur?

Method 1. After a little fiddling the following (almost) factorisation of the inequality can be found:

$$(2x - 1)(2y - 1)(2z - 1) \geq -xyz$$

As the RHS is negative the inequality trivially holds when the LHS is positive. It is easy to see that this is when exactly one of x,y,z is at least $\frac{1}{2}$. So, assume that $0 \leq x, y, z \leq \frac{1}{2}$. Setting $u = 1 - 2x$, $v = 1 - 2y$, $w = 1 - 2z$ we get the following

$$8uvw \leq (1 - u)(1 - v)(1 - w) \quad \text{given that} \quad 0 \leq u, v, w \leq 1, u + w + v = 1$$

Expanding this gives

$$9uvw \leq (1 - u - v - w) + (uv + vw + wu) = uv + vw + wu$$

Provided that $uvw \neq 0$ we can divide by uvw and use the **arithmetic–harmonic mean inequality** to finish. If $uvw = 0$ then WLOG we can assume $u = 0$. This gives $0 \leq vw$ which is clearly true.

Putting all this together we see that equality holds if $x = y = z = \frac{1}{3}$ or $x = y = 0, z = 1$ (and cyclic permutations thereof).

Method 2. From the factorisation

$$(2x - 1)(2y - 1)(2z - 1) \geq -xyz$$

we can substitute in the boundary equation $x + y + z = 1$ to obtain

$$(x + y - z)(y + z - x)(z + x - y) \leq xyz$$

As before, we can assume everything in sight is positive (otherwise the inequality is trivial). We can now use AM–GM on the pairs of terms on the LHS to prove the inequality.