BMOC Mentoring Scheme

Advanced Level – Sheet 4

January 2002

1. (a) (**The Angle Bisector Theorem**) Let ABC be a triangle, L a point on the interior of the side BC. Show that AL is the internal bisector of the angle CAB if and only if
$$\frac{BL}{LC} = \frac{AB}{AC}$$
What is the corresponding theorem for external angle bisectors?

   (b) In quadrilateral ABCD, the internal bisectors of angles A and C meet on BD; prove that the internal bisectors of angles B and D meet on AC.

2. (a) (**Ceva's theorem**) Let ABC be a triangle, and suppose L,M,N are points on sides BC, CA, AB respectively. Prove that AL, BM, CN being concurrent is equivalent to
$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1$$
and to
$$\frac{\sin BAL}{\sin LAC} \cdot \frac{\sin CBM}{\sin MBA} \cdot \frac{\sin ACN}{\sin NCB} = 1$$

   (b) Let ABC be a triangle and D,E,F be points on sides BC, CA, AB respectively such that AD, BE, CF are concurrent; if D', E', F' are the reflections of D,E,F in the midpoints of BC, CA, AB respectively, prove that AD', BE', CF' are concurrent. Prove also that the reflection of AD in the internal bisector of angle A, the reflection of BE in the internal bisector of angle B, and the reflection of CF in the internal bisector of angle C are concurrent.

3. (a) If α, β, γ, δ are complex numbers, show that
$$(\alpha - \beta)(\gamma - \delta) + (\alpha - \delta)(\beta - \gamma) = (\alpha - \gamma)(\beta - \delta)$$

   (b) (**The Ptolemy–Euler Inequality**) Deduce that if A,B,C,D are any four points in the plane then
$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD$$
When does equality occur?

   (c) If P lies on the arc AB of the circle circumscribed about a regular pentagon ABCDE, show that
$$PC + PE = PA + PB + PD$$

4. (a) (**Miquel's Theorem**) If the points L,M,N lie on sides BC,CA,AB respectively of triangle ABC (possibly extended), prove that the circles AMN, BNL, CLM are concurrent at a point P.

   (b) Prove that $\measuredangle CPB = \measuredangle CAB + \measuredangle NLM$.

   (c) (**The Simson Line**) Prove that the feet of the perpendiculars to the sides of a triangle from a point are colinear if and only if the point lies on the circumcircle of the triangle.

   (d) Given four lines in the plane, prove that the circumcircles of the four triangles they form are concurrent.

5. Triangle ABC has incentre I, and L, M, N are the midpoints of BC, CA, AB respectively. Prove that I lies inside triangle LMN.

6. Let ABC be a triangle. Suppose X,Y are points on side BC such that $\measuredangle BAX = \measuredangle YAC$. If the incircle of triangle ABX touches BX at L and the incircle of triangle ACY touches CY at M, prove that

$$\frac{1}{BL} + \frac{1}{LX} = \frac{1}{CM} + \frac{1}{MY}$$

7. The triangle ABC is right–angled at A. A line through the midpoint D of BC meets AB at X and AC at Y (AB and AC being extended as necessary). The point P is taken on this line so that PD and XY have the same midpoint M. The perpendicular from P to BC meets BC at T. Prove that AM bisects angle TAD.

8. In a scalene triangle ABC, D is the foot of the perpendicular from A to BC, E and F are the midpoints of AD and BC respectively, and G is the foot of the perpendicular from B to AF. Prove that EF is the tangent at F to the circle GFC.

9. The acute–angled triangle ABC has circumcentre O and orthocentre H. The altitude BH meets circle ABC again at P and OP meets CA at Q. The altitude CH meets the circle again at R and OR meets AB at S. Prove that the lines PQ, QH, HS, SR touch a circle. What happens if $A = 45°$?