HMO2 Solutions by

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1. *The graphs of $xy = 1$ and $xy = -1$ are given in the same co-ordinate system. We have a circle around the origin with radius R. The common points of the circle and the two hyperbolas are the vertices of a regular n-gon. Show that the area of the n-gon is R^4.*

   Solution The union of the graphs of the hyperbolas has equation $x^2y^2 = 1$. Solving this against $x^2 + y^2 = R^2$ we obtain $x^2 = (R^2 \pm \sqrt{R^4 - 2})/2$. Thus there are either 0, 4 or 8 points of intersection of the circle with the union of the hyperbolas. We investigate for which values of R these points form the vertices of a square or a regular octagon.

   The square is obtained when $R = \sqrt{2}$, and then $x^2 = 1$. The area of the square is $4x^2 = 4 = r^4$.

   Let $z = \tan \pi/8$ so
   $$1 = \tan \pi/4 = \frac{2z}{1 - z^2}$$
   so $z = \sqrt{2} - 1$. The condition that the vertices form a regular octagon amounts to the condition that the circle, the hyperbola $xy = 1$ and the ray $y = zx$ $(x > 0)$ should have a point of intersection. Solving $y = zx$ and the hyperbola yields $x^2 = \sqrt{2}+1$ so $x = \sqrt{\sqrt{2}+1}$ and $y = \sqrt{\sqrt{2}-1}$. Now $R^2 = x^2 + y^2 = 2\sqrt{2}$ so $R^4 = 8$ but using "half height times base", the area of the octagon is
   $$16 \cdot x \cdot y = 16 \cdot \frac{1}{2} \cdot \sqrt{\sqrt{2}+1}\sqrt{\sqrt{2}-1} = 8 = R^4.$$

2. *Find $x, y, z, t \in \mathbb{R} \setminus \{0, -1\}$ for which both (a) and (b) hold:*

   (a) $x + y + z = 3/2$

   (b) $\sqrt{4x-1} + \sqrt{4y-1} + \sqrt{4z-1} \geq 2 + 3^{\sqrt{t-2}}$.

   Solution Notice that x, y, z must all be at least 1/4 so that the left-hand side will make sense. Apply the Cauchy-Schwarz inequality:
   $$(1,1,1) \cdot (\sqrt{4x-1}, \sqrt{4y-1}, \sqrt{4z-1}) \leq \sqrt{3} \cdot \sqrt{4(x+y+z)-3} = 3.$$

Thus the given condition holds if and only if $t = 2$, and the vectors in the dot product are parallel, so $x = y = z = 1/2$. **Comment: This method is obvious. I was amazed to see other (more complicated) methods in Hungary (Jensen indeed!).**

3. *$f(x)$ is defined on $\mathbb{R} \setminus \{0, -1\}$ and takes real values. Determine $f(x)$ if for every x ($\neq 0, -1$)*
$$f(x) = kx^2 f\left(\frac{1}{x}\right) = \frac{x}{x+1}.$$
(here k is a constant, $0 < k^2 \neq 1$). Find those values of x for which $f(x) = 0$.
Solution Replace x by $1/x$ and solve the linear equations. In the 'unknowns' $f(x)$ and $f(1/x)$ and deduce that
$$f(x) = \frac{x(1-kx)}{(x+1)(1-k^2)}.$$
Notice how the side conditions keep the denominator under control. To obtain full credit, you must substitute back into the original equation to verify that this is indeed a solution. Now $f(x) = 0$ exactly when $x = 1/k$ (a sucker will allow $x = 0$ but this violates a side condition).

4. *In triangle ABC the bisector of $\angle BAC$ meets the incircle first at O_A. (O_A is closer to A than the other common point of the bisector and the incircle.) Let k_A be the circle with centre O_A which nis tangent to AB and AC. We get k_B and k_C similarly. The external common tangent of k_B, k_C which is not a side of ΔABC is t_A. We get t_B, t_C similarly. Prove that t_A, t_B, t_C are concurrent.*
Solution (outline) Recall the following important fact about the orthocentre and circumcircle of a triangle: *the reflection of the orthocentre in any side of a triangle is on the circumcircle.* A short proof is available by joining the orthocentre to one of the vertices on the reflecting side, and chasing angles. Now to the question. The point of concurrency turns out to be the orthocentre H of $\Delta O_A O_B O_C$. In order to verify this fact, it suffices to do the following: drop the perpendicular IF_A to BC. Draw $O_A H$ and HF_A. It suffices to show that $O_B O_C \perp HF_A$ by the important fact which we recalled earlier, since then H will sit on one (and by symmetry all) of the internal common tangents mentioned in the question. Thus we have reduced the question to a single perpendicularity problem. Now draw another diagram which is less littered (omit the smaller circles and the internal common tangents). Let the angles of ΔABC be $2\alpha, 2\beta, 2\gamma$ respectively, then chase angles (nailing down the angles around I is helpful), and soon you will be done.

5. *In a folk dance the dancers are standing in two rows, 9 boys facing 9 girls. Each dancer gives his/her left hand to the person opposite, to his/her left neighbour, or to the person opposite his/her left neighbour. The analogous rule applies to right hands. Nobody gives both hands to the same person.*

(a) *Find the number of possible configurations.*

(b) *Is 2002 a divisor of the number of configurations if there are 2002 people (rather than 18).*

Solution

(a) Replace 9 by n and induct on it (inducting on 9 itself may confuse the casual reader). Let the number of configurations with n couples be u_n so $u_1 = 0$ and $u_2 = 2$ (the square or the infinity symbol). An induction argument for $n \geq 3$ goes as follows: either couple $n - 1$ hold hands or they don't. The first possibility gives rise to $2u_{n-2}$ configurations (letting couples $n-1$ and n do their own thing in two ways) and the second to $2u_{n-1}$ configurations (by slicing off the arms linking couples $n - 1$ and n, and then jamming the blokes and the blokesses together to avoid unsightly embarrassment and to create a replacement couple $n - 1$). Thus $u_n = 2u_{n-1} + 2u_{n-2}$ for every $n \geq 3$. Thus the sequence goes

$$0, 2, 4, 12, 32, 88, 240, 656, 1792$$

and the number of configurations with 9 couples is 1792 (which has a certain *je ne sais quoi*). **Note that if you are honest enough to allow $n = 0$ then there is a unique empty configuration for them, $u_0 = 1$ and the induction formula still works; such is the power of careful reasoning.**

(b) Note that $2002 = 2 \times 7 \times 11 \times 13$ **A fact which should be inscribed upon your soul.** Consider the sequence u_n (beginning $n = 1$) modulo each of the primes involved in the factorization of 2002. Of course all terms u_n are even so there is no problem there. You discover that the sequence is

$$0, 2, 4, 5, 4, 4, 2, 5, 0, 3, \ldots \mod 7$$

and since $2 \times 5 = 3 \mod 7$ we can write this as

$$0, 2, 4, 5, 4, 4, 2, 5, 5 \times (0, 2, 4, 5, 4, 4, 2, 5, \ldots)$$

which is an ostentatious way of showing that $u_n \equiv 0 \mod 7$ if and only if $n \cong 1 \mod 8$, so 7 divides u_{1001}. Now for 11. The series is

$$0, 2, 4, 1, 10, 0, 9, 7, 10, 1, 0, 2, \ldots$$

and this time we haven't the cheek to repeat the trick since the series repeats so quickly. Now $u_n \equiv 0 \mod 11$ if and only if $n \equiv 1 \mod 5$ so 11 divides u_{1001}. Next work modulo 13. The series is

$$0, 2, 4, 12, 6, 10, 6, 6, 11, 8, 12, 1, 0, 2 \ldots$$

and the series repeats. Thus $u_n \equiv 0 \mod 13$ if and only if $n \equiv 1 \mod 12$. But 12 is not a divisor of 1000, so u_{1001} is not divisible by 13 and therefore not divisible by 2002.