February 2002 Exam 2 Solutions

A four hour examination. Iranian Olympiad 2001.

1. Let $n = 2^m + 1$ and suppose that $f_1, \ldots, f_n$ are increasing functions defined on $[0, 1]$ with values in $[0, 1]$ which satisfy:

$$|f_i(x) - f_i(y)| \leq |x - y| \ \forall x, y \in [0, 1], \ 1 \leq i \leq n$$

and $f_i(0) = 0$ for $1 \leq i \leq n$. Prove that there exist $i \neq j$ such that for all $x \in [0, 1]$ we have $|f_i(x) - f_j(x)| \leq 1/m$.
Solution *We call a function* $\phi : [0, \infty) \to [0, \infty)]$ simple *if for any integer* $n \geq 1$ *we have* $\phi(n) - \phi(n-1) \in \{0, 1\}$ *and for any* $n \geq 0$*, the retriction of* ϕ *to* $[n, n+1]$ *is a linear function.*

LEMMA Suppose that $g : [0, \infty) \to \mathbb{R}$ *is an increasing function such that* $|g(0)| \leq 1/2$ *and for every* $x, y, \geq 0$ *we have* $|g(x) - g(y)| \leq |x - y|$*, then there exists a simple function* ϕ *such that for every* $x \geq 0$ *we have* $|\phi(x) - g(x)| \leq 1/2$.

PROOF We will show that either Case 1 *For every* $x \in [0, 1]$ *we have* $|g(x) \leq 1/2$ *or* Case 2 *for every* $x \in [0, 1]$ *we have* $|g(x) - x| \leq 1/2$.

If we are not in Case 1, then there exists $x_0 \in [0, 1]$ *such that* $|g(x_r 0)| > 1/2$*. Since* g *is an increasing function.* $g(x_0) > 1/2$*. If Case 2 failed to hold, there would be* $x_1 \in [0, 1]$ *such that* $|g(x_1) - x_1| > 1/2$*. We know that* $g(x_1) - g(0) \leq x_1$ *so* $g(x_1) - x_1 \leq 1/2$ *which implies that* $x_1 - g(x_1) > 1/2$*. These two inequalties yield that*

$$g(x_0) + x_1 - g(x_1) > 1$$

or

$$g(x_0) - g(x_1) > 1 - x_1 \geq 0.$$

Since g *is an increasing function,* $x_0 \geq x_1$ *and it follows from* $g(x_0) - g(x_1) \leq x_0 - x_1$ *that* $x_0 > 1$ *which is absurd and the Lemma is proved.*

Now from this lemma it follows that there is a simple function ϕ_0 *such that* $|g(x) - \phi_0(x)| \leq 1/2$*. It is easy to see that* ϕ_0 *can be extended to the whole positive half-line. Now define*

$$g_i(x) = \begin{cases} m f_i(x/m) & 0 \leq x \leq m \\ m f_i(1) & x \geq m. \end{cases}$$

If we apply the Lemma to the functions g_i, then we see that there exists a simple function ϕ_i such that $g_i(x) - \phi_i(x) \leq 1/2$. However, there are precisely 2^m simple functions when restricted to the integral intervals $[0, m]$ so there exist $i \neq j$ such that $\phi_i(x) = \phi_j(x)$ for all $x \in [0, m]$ which shows that $f_i(x) - f_j(x)| \leq 1/m$ for all $x \in [0, 1]$.

2. In ΔABC let I and I_a denote the incentre and the excentre corresponding to the side BC. Suppose that II_a meets BC and the circumcircle of ΔABC at A' and M respectively. Let N be the midpoint[1] of the arc MBA. Let S, T be the respective intersection points of NI, NI_a with the circumcircle of ΔABC. Prove that S, T, A' are colinear.
   Solution *We first prove a Lemma.*
   LEMMA. Suppose that the circles Γ_1, Γ_2 are tangent at T. Let I be a point of Γ_1 suppose that the tangent at I meets Γ_2 at A and M. If TI meets Γ_2 at K, then K is the midpoint of arc AKM.
   PROOF Let Δ be a line parallel to AM passing through I. Since there is a homothety centred at T which maps Γ_1 to Γ_2 it follows that Δ is tangent to Γ_1 at K. So K is indeed the midpoint of arc AKM and the lemma is proved.

   Now let Γ denote the circumcircle of triangle ABC and C_1 be a circle which is tangent to AI and Γ. By the lemma, T is the tangency point of C_1 and Γ. Let C_2 be a circle which is tangent to Γ and passes through I_a. Apply the lemma to deduce that C_1, C_2 intersect at S. What! S? Yes, it was in the question but has been quiet lately. Now invert centred at N sending I to I_a. This inversion swaps C_1, C_2 and we deduce that S, T', A are colinear.

3. We define an *n-variable formula* to mean a function of n variables $x_1, \ldots, x_n$ which can be expressed as a composition of the functions $\max\{a, b, c, \ldots\}$ and $\min\{a, b, c, \ldots\}$. (For example, $\max\{x_2, x_3, \min\{x_1, x_2, \max\{x_4, x_5\}\}\}$). Suppose that $P(x_1, \ldots, x_n)$ and $Q(x_1, \ldots, x_n)$ are two n-variable formulas, and assume that if $x_i \in \{0, 1\}$ for every i, then

   $$P(x_1, \ldots, x_n) = Q(x_1, \ldots, x_n).$$

   Prove that $P \equiv Q$ (i.e. P and Q agree at all real arguments $x_1, x_2, \ldots, x_n$).
   Solution *Suppose (for contradiction) that the result is not true. Thus there are real numbers $x_1 < \cdots < x_n$ such that $P(x_1, x_2, \ldots, x_n) < Q(x_1, x_2, \ldots, x_n)$. By perturbing and relabelling we may assume that $x_1 < x_2 < \cdots < x_n$ since min and max are continuous functions. Thus there exist $p \neq q$ such that $x_p = P(x_1, \ldots, x_n) < Q(x_1, \ldots, x_n) = x_q$. Now if we replace $x_1, \ldots, x_p$ by 0 and $x_{p+1}, \ldots, x_n$ by 1, and induction shows that $P(x'_1, \ldots, x'_n) = 0$ and $Q(x'_1, \ldots, x'_n) = 1$ which contradicts our assumption.*

[1]The original says mindpoint, but I feel uneasy about this.