December Exam 2

This is a 4 hour paper.

Various questions from Ukraine

1. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds:

$$f(xy) = \max\{f(x), y\} + \min\{f(y), x\}?$$

Solution *Put $x = y = 1$ to obtain that*

$$f(1) = \max\{f(1), 1\} + \min\{f(1), 1\} = f(1) + 1$$

which is absurd.

2. Positive integers a and n are such that n divides $a^2 + 1$. Prove that there is a positive integer b such that $n(n^2 + 1)$ divides $b^2 + 1$.
Solution *We have $a^2 \equiv -1 \mod n$ and $n^2 \equiv -1 \mod n^2+1$. Now $n, n^2 + 1$ are coprime (a common divisor would divide 1) so we may apply the Chinese Remainder Theorem to find an integer b such that $b \equiv a \mod n$ and $b \equiv n \mod n^2+1$ so $b^2 \equiv a^2 \equiv -1 \mod n$ and $b^2 \equiv n^2 \equiv -1 \mod n^2 + 1$. Thus $b^2 \equiv -1 \mod n(n^2+1)$ and therefore $n(n^2+1) \mid b^2+1$. Of course, if you are Tom Coker or Paul Jefferys, then you are at liberty to side-step the Chinese Remainder Theorem and simply hurl a bolt of lightning by letting $b = an^2 + a + n$ which visibly has the appropriate divisibility properties!*

3. An acute angled triangle ABC is such that AC and BC are of different lengths, and is inscribed in a circle ω. Let N be the midpoint of the arc AC (and B does not lie on this arc). Let M be the midpoint of the arc BC (and A does not lie on this arc). Let D be the point on the arc MN such that $DC \parallel NM$. Let K be an arbitrary point on the arc AB (and C does not lie on this arc). Let O, O_1 and O_2 be the incentres of the triangles ABC, ACK, CBK respectively. Suppose that L is the intersection of the line DO and the circle ω such that $L \neq D$. Prove that the points K, O_1, O_2, L are concyclic.
So far no solution has ben offered.

4. Let a, b, c and α, β, γ be positive real numbers such that $\alpha + \beta + \gamma = 1$. Prove the inequality

$$\alpha a + \beta b + \gamma c + 2\sqrt{(\alpha\beta + \beta\gamma + \gamma\alpha)(ab + bc + ca)} \leq a + b + c.$$

Solution (Thanks to Tim Austin and Paul Jefferys) *Replacing* a *by* $a/(a+b+c)$, b *by* $b/(a+b+c)$ *and* c *by* $c/(a+b+c)$ *we see that it suffices to assume that* $a+b+c=1$ *and to show that*

$$\alpha a+\beta b+\gamma c+2\sqrt{(\alpha\beta+\beta\gamma+\gamma\alpha)(ab+bc+ca)}\leq 1.$$

Now by the Cauchy-Schwarz inequality,

$$\alpha a+\beta b+\gamma c\leq\sqrt{\alpha^2+\beta^2+\gamma^2}\sqrt{a^2+b^2+c^2}\qquad(\dagger)$$

Let $p=\alpha\beta+\beta\gamma+\gamma\alpha$ *and* $q=ab+bc+ca$. *Now* $\alpha^2+\beta^2+\gamma^2=1-2p$ *and* $a^2+b^2+c^2=1-2q$. *Now* $(\dagger)$ *implies that*

$$\alpha a+\beta b+\gamma c+2\sqrt{pq}\leq\sqrt{1-2p}\sqrt{1-2q}+2\sqrt{pq}$$

$$\leq(1-2p+1-2q)/2+p+q=1$$

using the GM-AM inequality.