December Exam 1

This is a 4 hour paper

Various questions from Ukraine

1. Consider the collection of 5-digit (base 10) integers with digits increasing from left to right. Is it possible to erase a digit (and close up if necessary) from each of these 5-digit integers, and obtain the collection of all 4-digit integers with digits increasing from left to right?
   Solution *Each of the sets in question has size $\binom{9}{5} = \binom{9}{4}$ so such a bijection is not out of the question. Jacob Shepherd uniquely supplied an answer to this question, and he also supplied an explicit bijection in the form of a matrix:*

$$\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & X & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & 0 & X & \cdot & \cdot & \cdot \\
0 & 0 & 0 & X & \cdot & 0 & \cdot & \cdot & \cdot \\
0 & 0 & 0 & X & \cdot & \cdot & 0 & \cdot & \cdot \\
0 & 0 & 0 & X & \cdot & \cdot & \cdot & 0 & \cdot \\
0 & 0 & \cdot & 0 & 0 & X & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & \cdot & 0 & 0 & X & \cdot & \cdot \\
0 & 0 & \cdot & 0 & \cdot & 0 & X & \cdot & \cdot \\
0 & 0 & \cdot & 0 & X & \cdot & 0 & \cdot & \cdot \\
0 & 0 & \cdot & 0 & X & \cdot & \cdot & 0 & \cdot \\
0 & 0 & X & \cdot & 0 & \cdot & 0 & \cdot & \cdot \\
0 & 0 & X & \cdot & 0 & \cdot & \cdot & 0 & \cdot \\
0 & 0 & X & \cdot & \cdot & 0 & \cdot & 0 & \cdot \\
0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & X & \cdot
\end{array}$$

   which says it all really. The first row of this matrix corresponds to the following 4-digit numbers (focus on the zeros):

$$1234, 2345, 3456, 4567, 5678, 6789, 1789, 1289, 1239$$

   where the digits from 1 to 9 are written in cyclic order starting in each of the 9 possible places, and the digits falling in positions marked by a zero are put in ascending order. There are 14 rows giving rise to all $14 \times 9 = \binom{9}{4}$ ascending strings of digits of length 4. If you read each X as a 0 you get the $\binom{9}{5}$ ascending strings of digits of length 5. Erasing the digit in position X yields the required bijection.

2. Let I denote the incentre of triangle ABC. Let the bisector of $\angle BAC$ meet BC at A_1, and the bisector of $\angle BCA$ meet AB at C_1. Let M be an

arbitrary point on the line segment AC. Lines through M parallel to the given bisector lines meet AA_1, CC_1, AB, CB at points H, N, P and Q respectively. Let the lengths of BC, AC and AB be a, b and c respectively. Let d_1, d_2 and d_3 be the respective distances from H, I, N to the line PQ. Prove the following inequality.

$$\frac{d_1}{d_2} + \frac{d_2}{d_3} + \frac{d_3}{d_1} \geq \frac{2ab}{a^2+bc} + \frac{2ca}{c^2+ab} + \frac{2bc}{b^2+ca}.$$

So far no solution has been offered.

3. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n \geq n^2$, $a_1^2 + a_2^2 + \cdots + a_n^2 \leq n^3 + 1$. Prove that $n - 1 \leq a_k \leq n + 1$ for all k.
Solution We are given that $\sum_i a_i \geq n^2$ and $\sum_i a_i^2 \leq n^3 + 1$. Now

$$\sum_i (a_i - n)^2 = \sum_i a_i^2 - 2n \sum_i a_i + n^3 \leq n^3 + 1 - 2n^3 + n^3 = 1.$$

Thus $|a_i - n| \leq 1$ for every i in the range $1 \leq i \leq n$. **Notice that this is a mean and variance question, and uses the same trick which gives rise to the method of variance in proving geometric theorems.**

4. There are n mathematicians in each of three countries. Each mathematician corresponds with at least $n + 1$ foreign mathematicians. Prove that there exist three mathematicians who correspond with each other.
Solution *We assume (for contradiction) that no triple of mutually corresponding mathematicians. We prove that for all k in the range $1 \leq k \leq n$, each mathematician corresponds with at least k mathematicians from each other country. The hypothesis of the question ensures that the result holds when $k = 1$ and we proceed by induction. Suppose the result holds when $1 \leq k = r < n$ and consider a mathematician from country A and the situation regarding his (or her) correspondents in country B. There must be at least 1 of them, and each one of them corresponds with at least k mathematicians from country C, none of which can be a correspondent of A else wwould form a triangle. Thus A corresponds with at most $n - k$ mathematicians from country C, and therefore must correspond with at least $k + 1$ mathematicians from country B. We are done. Thus every mathematician corresponds with every mathematician outside his or her own country, and this violates the* no triangle *condition.* **There are other ways to cast this, but this version is very clear and has comedy value.**