Sylvester's Coin Theorem

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Suppose that you have plenty of 2p coins and 5p coins. What is the largest sum of money which you cannot pay exactly?

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Answer: 3p

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Answer: there is no largest sum that you cannot pay. You can never pay an odd number of pence.

Suppose that you have plenty of 5p coins and 7p coins. What is the largest sum of money which you cannot pay exactly?

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Answer: 23p

5p, 10p, 15p, 20p, 25p, 30p, ...
7p, 12p, 17p, 22p, 27p, 32p, ...
14p, 19p, 24p, 29p, 34p, 39p, ...
21p, 26p, 31p, 36p, 41p, 46p, ...
28p, 33p, 38p, 43p, 48p, 53p, ...

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7p, 12p, 17p, 22p, 27p, 32p, ...
14p, 19p, 24p, 29p, 34p, 39p, ...
21p, 26p, 31p, 36p, 41p, 46p, ...
28p, 33p, 38p, 43p, 48p, 53p, ...

Suppose that you have plenty of xp coins and yp coins, and $\mathrm{hcf}(x, y) = \gcd(x, y) = 1$. What is the largest sum of money which you cannot pay exactly?

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Answer $xy - x - y$ pence

$$
\begin{array}{rrrr}
0 & x+0 & 2x+0 & 3x+0\ldots \\
y & x+y & 2x+y & 3x+y\ldots \\
2y & x+2y & 2x+2y & 3x+2y\ldots \\
\ldots & & & \\
\ldots & & & \\
(x-1)y & x+(x-1)y & 2x+(x-1)y & 3x+(x-1)y\ldots
\end{array}
$$

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0 & x+0 & 2x+0 & 3x+0\ldots \\
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(x-1)y & x+(x-1)y & 2x+(x-1)y & 3x+(x-1)y\ldots
\end{array}
$$

Claim: each row consists of numbers which give the same remainder on division by x. Different rows give rise to different remainders on division by x. If this is right, the largest inaccessible sum is $(x-1)y-x$ i.e. $xy-x-y$.

$$
\begin{array}{rrrr}
0 & x+0 & 2x+0 & 3x+0\ldots \\
y & x+y & 2x+y & 3x+3y\ldots \\
2y & x+2y & 2x+2y & 3x+2y\ldots \\
\ldots & & & \\
\ldots & & & \\
(x-1)y & x+(x-1)y & 2x+(x-1)y & 3x+(x-1)y\ldots
\end{array}
$$

The numbers in the first column are:

$$0 \cdot y, y, 2y, \ldots, iy, \ldots jy, \ldots, (x-1)y$$

with $0 \leq i < j \leq x-1$. We want them to have different remainders on division by x.

$$
\begin{array}{rrrr}
0 & x+0 & 2x+0 & 3x+0\ldots \\
y & x+y & 2x+y & 3x+3y\ldots \\
2y & x+2y & 2x+2y & 3x+2y\ldots \\
\ldots & & & \\
\ldots & & & \\
(x-1)y & x+(x-1)y & 2x+(x-1)y & 3x+(x-1)y\ldots
\end{array}
$$

The numbers in the first column are:

$$0 \cdot y, y, 2y, \ldots, iy, \ldots jy, \ldots, (x-1)y$$

with $0 \leq i < j \leq x-1$. We want them to have different remainders on division by x. What could go wrong?

If iy and jy leave the same remainder on division by x, then x divides $jy - iy$ so x divides $(j - i)y$ (and $0 < j - i < x$).
Now

$$\frac{(j - i)y}{x}$$

is an integer and x does not cancel with y (their hcf is 1) so x divides into $j - i$ but $0 < j - i < x$. This is absurd, so iy and jy do NOT leave the same remainder on division by x.

You have plenty of 9p and 11p coins. What is the largest sum which you cannot pay exactly?

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Answer: 99 - 20 = 79 pence.

Advice for young mathematicians who are interested in problem solving . . .

`http://people.bath.ac.uk/masgcs/advice.html`

or type Geoff Smith Bath advice into a search engine.

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