A projective Simson line

Geoff Smith, University of Bath

15th October 2014

Introduction

Let ABC be a triangle with circumcircle Γ. Let P be a point in the plane of the triangle. Simson's theorem [1] is that the feet of the perpendiculars dropped from P to the side lines of the triangle are collinear if, and only if, P is on Γ, in which case this line meets the line segment joining P to the orthocentre H of ABC at its midpoint. It seems that this result was misattributed to Robert Simson by Jean-Victor Poncelet, and the result was first published by William Wallace.

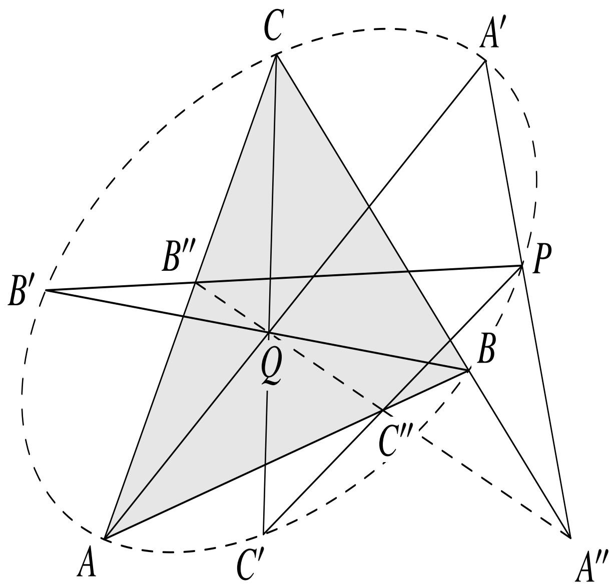
Instead of working with the feet of the perpendiculars, we can recast the statement as being that the three reflections of P in the sides of the triangle are collinear if, and only if, P lies on Γ, in which case this line passes through H.

It is known that the three reflections H_A, H_B and H_C of H in the sides of ABC are on Γ, so we can restate the result again as being that the three lines which are the reflection of PH_A in BC, the reflection of PH_B in CA and the reflection of PH_C in AB, are the same line if, and only if, P lies on Γ.

Result

We are now in a position to state a generalization of this result which is a theorem of projective geometry. Note that we introduce the points A'', B'' and C'' to obviate the need for Euclidean reflections, and the arbitrary point Q plays the role of H. To see how this theorem specializes to the classical Simson configuration, first enlarge the Simson line with scale factor 2 from the distinguished point on the circumcircle, to obtain the doubled Simson line which passes through H, and then consider the reflections of the doubled Simson line in each of the three sides of triangle ABC. The points A'', B'' and C'' are where the doubled Simson line meets BC, CA and AB respectively.

Theorem *Let ABC be a triangle and let P and Q be other points in the plane of this triangle. Let Γ be a conic which passes through A, B and C. Let AQ meet Γ again at A', and let PA' meet BC at A''. Points B', B'', C' and C'' are similarly defined. Then $A''Q, B''Q$ and $C''Q$ are the same line if, and only if, P lies on Γ.*



Proof When the six points A, A', P, C', C and B lie on the conic Γ, Pascal's Hexagram Theorem asserts that opposite sides of this figure meet in three collinear points: AA' meets CC' at Q, $A'P$ meets BC at A'', and finally $C'P$ meets AB at C'' so A'', C'' and Q are collinear. By cyclic change of letters (or another hexagram) it follows that A'', B'', C'' and Q are collinear.

Pascal's Hexagram Theorem is reversible, so if P does not lie on Γ, then A'', C'' and Q will not be collinear. ■

The non-projective variation of this result, where Γ is a circle and A'', B'' and C'' are instead the reflections of A', B' and C' in BC, CA and AB respectively, is the classical theorem of Hagge [2] that A'', B'', C'' and H lie on a circle.

History

Several authors have written papers about generalizations of the Simson line, including the Bradleys [3], Geiring [4] and Pech [5], but this author has not been able to find the stated result there, nor in the standard reference works [6] nor [7]. The geometric literature is so diverse and scattered, that it would be very rash to insist that a result of elementary geometry is new, so we simply ask where, if anywhere, this result has previously appeared?

References

[1] G. Leversha, The Geometry of the Triangle, UKMT, 2013.

[2] K. Hagge, Der Fuhrmannsche Kreis und der Brocardsche Kreis als Sonderfälle eines allgemeineren Kreises, *Zeitschrift für Math. Unterricht*, 38 (1907) 257–269.

[3] C J Bradley and J T Bradley, Countless Simson line configurations, *Math. Gazette*, Vol. 80, No. 488, Jul., 1996.

[4] O. Giering, Affine and Projective Generalization of Wallace Lines, *Journal for Geometry and Graphics* Vol 1 (1997), No. 2, 119–133.

[5] P. Pech, On the Simson-Wallace Theorem and its Generalizations, *Journal for Geometry and Graphics*, Vol 9 (2005), No. 2, 141–153.

[6] N. Altshiller-Court, College Geometry, Dover, 2007.

[7] R. Johnson, Advanced Euclidean Geometry, Dover, 2007.