MA10209 Algebra 1A

Sheet 9 Problems v1: GCS

28-xi-11

The course website is http://people.bath.ac.uk/masgcs/diary.html

Hand in work to your tutor by 13:00, Monday December 5^{th}. *In questions which involve the theory of groups, the default notation is to juxtapose group elements to denote the application of the group operation, just as we omit the multiplication sign in many contexts. Some students may find it helpful to delay starting work on the group theoretic problems until after the 9:15 lecture on Tuesday November* 29^{th}.

1. Consider $A, B \in \mathbb{R}_{2,2}$, where

$$A = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (a) Calculate $\det A$ and $\det B$.

   (b) Calculate A^2, A^3 and B^2.

   (c) Show that $BA = A^2B$.

   (d) Show that the smallest group which contains both A and B, and where the group operation is multiplication of matrices, is finite.

   (e) Discuss the determinants the elements of G.

   (f) What is the smallest positive integer n such that $BABA^2BA^3BA^4BA^5B = A^n$?

2. Suppose that m, n are coprime integers. Show that there are infinitely $(u, v) \in \mathbb{Z}^2$ such that the matrix

$$X = \begin{pmatrix} m & n \\ u & v \end{pmatrix}$$

   is invertible, and X^{-1} is a matrix with integral entries.

3. Let G be the set of $X = (x_{ij}) \in \mathbb{R}_{3,3}$ which have the properties that (i) for each $i \in \{1, 2, 3\}$ there is a unique $j \in \{1, 2, 3\}$ such that $x_{ij} \neq 0$, (ii) for each $j \in \{1, 2, 3\}$ there is a unique $i \in \{1, 2, 3\}$ such that $x_{ij} \neq 0$ and (iii) if $x_{ij} \neq 0$, then $x_{ij} = 1$.

   (a) List the elements of G.

   (b) Calculate the determinant of each element of G.

   (c) Show that G is a group under matrix multiplication.

4. Let T denote the set of 2 by 2 matrices with real entries $X = (x_{ij})$ such that $x_{21} = 0$ and $x_{11}x_{22} \neq 0$. Prove that T is a group under matrix multiplication.

5. Let G denote the set of matrices of the form

$$\rho_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $\theta \in \mathbb{R}$.

(a) Prove that $\rho_\alpha \rho_\beta = \rho_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{R}$.

(b) Prove that each ρ_θ is an invertible matrix, and that $(\rho_\theta)^{-1} = \rho_{-\theta}$.

(c) Prove that G is a group under multiplication of matrices.

(d) Give a geometric interpretation of the linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ induced by ρ_θ.

(e) What are the fibres of the map $f : \mathbb{R} \longrightarrow G$ defined by $f(\theta) = \rho_\theta$ for each $\theta \in \mathbb{R}$.

6. For each natural number n, let $\Omega_n = \{1, 2, 3, \ldots, n\} \subseteq \mathbb{N}$. Write $S_n = \{f \mid f : \Omega_n \longrightarrow \Omega_n, f \text{ is bijective } \}$. The set S_n is a group using composition of maps as the operation.

(a) Determine the values of n for which S_n abelian.

(b) Determine the orders of elements of S_n for $1 \leq n \leq 6$.

(c) Show that S_{10} has an abelian subgroup of order 32.

7. Suppose that G is a group and $h \in G$. Let $T = \{g \mid g \in G, gh = hg\}$. Prove that T is a subgroup of G.

8. If H, K are subgroups of G, let $HK = \{hk \mid h \in H, k \in K\}$. Give an example to show that HK need not be a subgroup of G.

9. Suppose that H, K are subgroups of G. Prove that $H \cup K$ is a subgroup of G if, and only if, either $H \subseteq K$ or $K \subseteq H$.

10. (Challenging) Suppose that $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in \mathbb{Z}^2 \subseteq \mathbb{R}^2$. Giving $\mathbb{R}^2$ the usual geometric interpretation as the Euclidean plane with Cartesian co-ordinates, let the given co-ordinates correspond to the geometric points A, B and C respectively. It so happens that there is no point inside triangle ABC with integral co-ordinates, nor is there any point with integral co-ordinates in the interior of the line segments AB, BC and CA. Determine the area of triangle ABC. *You may assume that ABC is an anticlockwise triangle, in order to avoid getting involved with signed area issues.*