MA10209 Algebra 1A

Sheet 8 Problems v1: GCS

25-xi-11

The course website is http://people.bath.ac.uk/masgcs/diary.html

Hand in work to your tutor by 13:00, Monday Nov 28.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2 by 2 matrix with real entries. Define a map $f_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f_A((x, y)) = (u, v)$ where

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

for all $(x, y) \in \mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$. If $(x, y) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, we define $\lambda(x, y)$ to be $(\lambda x, \lambda y)$. We write $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as I_2, and more generally, the matrix I_n is an $n \times n$ matrix with each entry 0, except that the entries on the leading diagonal are 1.

1. (a) Prove that $f_A = \mathrm{Id}_{\mathbb{R}^2}$ if, and only if, $A = I_2$.
   (b) Suppose that $u, v \in \mathbb{R}^2$ and $\lambda, \mu \in \mathbb{R}$. Prove that $f_A(\lambda u + \mu v) = \lambda f_A(u) + \mu f_A(v)$.
   (c) Determine $f_A((0, 0))$, $f_A((1, 0))$, $f_A(1, 1))$ and $f_A((0, 1))$.
   (d) (accidentally omitted from original version): Determine the area of the polygon which has the points in part (c) as vertices. There is a helpful formula involving the vector (cross) product which, if you know it, will shorten your calculation.
   (e) (old (d)) Describe $\{f_A((x, 0)) \mid x \in \mathbb{R}\}$.
   (f) (old (e)) Describe $\{f_A((0, y)) \mid y \in \mathbb{R}\}$.

2. Let $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ be 2×2 matrices with real entries. Prove that $f_B \circ f_C = f_{BC}$ where BC denotes the product of the matrices B and C.

3. Suppose that $u, v \in \mathbb{R}^2$ and $u \neq v$. The straight line through u and v is $\{\lambda u + (1 - \lambda)v \mid \lambda \in \mathbb{R}\}$.

   (a) Suppose that f_A is injective. Prove that if $L \subseteq \mathbb{R}^2$ is a straight line, then $\{f_A(w) \mid w \in L\}$ is a straight line.
   (b) Suppose that f_A is injective. Choose three distinct points $u, v, w \in \mathbb{R}^2$ which are not collinear. Prove that $f_A(u), f_A(v)$ and $f_A(w)$ are not collinear.
   (c) Suppose that f_A is injective. Prove that it is bijective.
   (d) Prove that if $ad \neq bc$, then f is bijective.

(e) Suppose that f_A is surjective. Prove that $ad - bc \neq 0$.

4. Prove that the following are equivalent. (a) f_A is injective, (b) f_A is surjective, (c) $ad - bc \neq 0$, (d) f_A is bijective.

5. Suppose that f_A is bijective, that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A \begin{pmatrix} p \\ q \end{pmatrix}$, and that $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = A \begin{pmatrix} r \\ s \end{pmatrix}$.

   (a) Determine p, q, r and s in terms of a, b, c and d.

   (b) Deduce that if f_A is bijective, there is a unique matrix B such that $AB = I_2$, and moreover this matrix B is the unique matrix B such that $BA = I_2$.

6. Suppose that ε and θ are positive real numbers. Let $e_1 = (1, 0)$, $e_2 = (0, 1)$, and $v \in \mathbb{R}^2$ be an arbitrary vector. Compare the areas of the rectangle with vertices $v, v + \varepsilon e_1, v + \varepsilon e_1 + \theta e_2, v + \theta e_2$, the region obtained by applying f_A to each point on, or in the interior of, this rectangle.

7. Show that if $S \subseteq \mathbb{R}^2$ is a region with area $|S|$, then the region obtained by applying f_A to each element of S has area $|ad - bc| \cdot |S|$. *The punctilious may confine themselves to regions S bounded by horizontal and vertical line segments, but students with a relaxed attitude to analysis should be able to see that the same result applies to any "reasonable" region of $\mathbb{R}^2$: to circular disks and to other regions bounded by calm and reassuringly dull curves.*

8. Suppose that U, V, W are distinct collinear points in the plane. Identify the plane with $\mathbb{R}^2$, and suppose that the co-ordinates of U, V and W are u, v and w, respectively. Suppose that A is a 2×2 matrix with real entries, and that the map f_A is bijective. Let the geometric points U', V' and W' have co-ordinates $f_A(u), f_A(v)$ and $f_A(w)$. Show that $UV : VW = U'V' : V'W'$, in other words that V divides the interval UW in the same ratio that V' divides the interval $U'W'$.

9. Suppose that XYZ is a triangle in the plane. Show that we may endow the plane with co-ordinates to identify it with $\mathbb{R}^2$, and produce a 2×2 matrix A such that when we apply f_A to the region bounded by the triangle, the result is a region bounded by an equilateral triangle.

10. (Tutor pacifier) Let $ABCDEF$ be a convex hexagon whch has parallel opposite sides and area 1. The lines AB, CD and EF meet in pairs to determine the vertices of a triangle. Similarly, the lines BC, DE and FA meet in pairs to determine the vertices of another triangle. Show that the area of at least one of these two triangles is at least $3/2$. *An classical proof is possible, but you may find some of the ideas developed in this problem sheet will assist you to construct a conceptual proof (i.e. a proof consisting of a sequence of ideas, rather than algebraic or trigonometric manipulation).*