MA10209 Algebra 1A

Sheet 6 Problems v1: GCS

8-xi-11

The course website is http://people.bath.ac.uk/masgcs/diary.html

Hand in work to your tutor by 13:00, Monday Nov 15.

1. Find all integers x such that $x \equiv 3 \bmod 7$ and $x \equiv 4 \bmod 9$.

2. Find all integers y such that 9 divides $2y + 1$ and 11 divides $3y + 6$.

3. Find all integers z such that $z \equiv 10 \bmod 11$, $z \equiv 12 \bmod 13$, $z \equiv 17 \bmod 18$. *Hint: this is much easier than it looks. There is a very short method.*

4. Show that there are 1000 consecutive positive integers, each of which is divisible by at least 1000 different prime numbers.

5. (a) Suppose that u, v, d are integers, with u and d coprime. Show that if d divides uv, then d divides v.

   (b) Suppose that m is an odd natural number. Prove that there is a natural number n such that m divides $2^n - 1$.

6. Suppose that $m, n \in \mathbb{N}$. Consider the map $\pi_{mn} : \mathbb{Z}_{mn} \longrightarrow \mathbb{Z}_m \oplus \mathbb{Z}_n$ defined by $[x]_{mn} \mapsto ([x]_m, [x]_n)$ for each $x \in \mathbb{Z}$, where $[x]_k$ denotes the equivalence class of x under the relation $\sim_k$. Determine $|\mathrm{Im}\ \pi_{mn}|$.

7. Let G be a group with 'multiplication' $*$ and identity element 1.

   (a) Suppose that $x, y, z \in G$ and $x * y = x * z$. Prove that $y = z$.

   (b) Suppose that $x, y, z \in G$ and $y * x = z * x$. Prove that $y = z$.

   (c) Suppose that $e \in G$ and $e * x = x$ for some $x \in G$. Prove that $e = 1$, so that $e * y = y = y * e$ for every $y \in G$.

   (d) Suppose that G is a finite group and that $x \in G$. Prove that there is $k \in \mathbb{N}$ such that $x^k = 1$.

8. Suppose that $m \in \mathbb{N}$.

   (a) Let i be an an integer in the range $1 \leq i \leq m$. Prove that $[i]$ has a multiplicative inverse in $\mathbb{Z}_m$ if, and only if, i and m are coprime.

   (b) For each $n \in \mathbb{N}$, let $\varphi(n)$ denote the number of integers i in the range $1 \leq i \leq n$ which are coprime to n. Calculate $\varphi(n)$ for $1 \leq n \leq 12$.

   (c) Suppose that m, n are coprime natural numbers. By considering the natural map $\pi_{mn} : \mathbb{Z}_{mn} \longrightarrow \mathbb{Z}_m \oplus \mathbb{Z}_n$ defined in Problem 6 (or otherwise), prove that $\varphi(mn) = \varphi(m)\varphi(n)$.

   (d) How many integers in the range $1 \leq i \leq 990$ are coprime to 990?

9. Let d be a positive integer. A d-arithmetic set is defined to be a set of the form $\{a + md \mid m = 0, 1, 2, \ldots\}$ for some positive integer a. Suppose that $N > 1$ is a positive integer and that we have a p-arithmetic set S_p for each prime number $p \leq N$. Show that there are $2N + 1$ consecutive positive integers, all except two of which are in the union S of our sets S_p. *Hint: CRT & Eratosthenes*

10. (Tutor pacifier) A mathematical tree (i.e. a vertical unit interval) grows at each point of an infinite plane with integral co-ordinates except for the origin $(0, 0)$ where an observer, of height 1, stands. Many trees are visible, including those at $(1, 0), (7, 8)$ and $(45, -7)$. Other trees are invisible, because the view of them from the origin is obstructed by other trees. For example, the view of the tree at $(-14, 91)$ is obstructed by the tree at $(-2, 13)$.

    Show that it is possible for a *Tunguska event* of diameter 10^{10} to happen, yet be unknown to the observer. In other words, show that there is a circle in the plane of diameter 10^{10} which has only invisible trees in its interior.