MA10209 Algebra 1A

Sheet 4 Problems v0: GCS

22-x-11

The course website is http://people.bath.ac.uk/masgcs/diary.html

Hand in work to your tutor by 13:00, Monday Oct 31.

1. Suppose that p is a prime number and that $p > 3$. Prove that there is a natural number m such that $p = 6m + 1$ or $p = 6m - 1$.

2. Prove that there infinitely many prime numbers of the form $4n + 3$, where n is a natural number. *Hint: modify Euclid's argument (given in lectures) that there are infinitely many prime numbers.*

3. There are n glasses, and each glass contains the same amount of water. The glasses are big enough so that each one could hold all the water. It is allowed to pour from any glass to any second glass exactly as much water as the second glass held before the pouring began. For what values of n is it possible to collect all the water in one glass? *Experimenting with small values of n should give you some useful ideas.*

4. Let n be a natural number. Show that the sum of the largest odd divisors of $n+1$, $n+2$, $\ldots$, $2n$ is a perfect square.

5. Suppose that you have ten distinct two-digit numbers. Is it necessarily true that one may choose two disjoint non-empty subsets so that their elements have the same sum? *Hint:* $10 \times 99 = 990 < 1024 = 2^{10}$.

6. Suppose that we have a set S of 15 positive integers x in the range $1 < x \leq 2011$. Suppose also that each pair of elements of S is coprime. Prove that S contains a prime number.

7. Suppose that $m, n \in \mathbb{N}$. The *lowest common multiple* of m and n is the smallest positive integer into which they both divide. It is written $\operatorname{lcm}(m, n)$. Prove that
$$\gcd(m, n) \cdot \operatorname{lcm}(m, n) = mn.$$
Hint: what happens when m and n are powers of the same prime number?

8. (Interesting) Suppose that n is a positive integer. Show that $f(n) = 2^{2^n} + 2^{2^{n-1}} + 1$ has at least n different prime factors. *Hint: the polynomial $x^4 + x^2 + 1$ has a non-trivial factorization.*

9. Let $F_0 = 0$ and $F_1 = 1$. Let $F_n = F_{n-1} + F_{n-2}$ for all integers $n > 1$. This is the *Fibonacci sequence.* Prove that $\gcd(F_n, F_{n-1}) = 1$ for all $n \in \mathbb{N}$.

10. (Harder) Using the Fibonacci sequence defined in Question 9, prove that
$$\gcd(F_m, F_n) = F_{\gcd(m,n)}$$
for all $m, n \in \mathbb{N}$.