

MA10209 Algebra 1A

Sheet 3 Problems and Solutions v2: GCS

24-x-11

The course website is <http://people.bath.ac.uk/masgcs/diary.html>

Hand in work to your tutor by 13:00, Monday Oct 24.

1. (a) Suppose that $f : X \rightarrow \mathbb{N}$ is an injective map. Prove that X is a countable set.
Solution Let $Y = \text{Im}f$, so f induces a bijection between X and Y . It suffices to prove that Y is countable. If Y is finite, we are done. Thus we may assume that Y is infinite. List the elements y_1, y_2, y_3, \dots of Y in ascending size, then the map $\zeta : \mathbb{N} \rightarrow Y$ defined by $\zeta(i) = y_i$ for each $i \in \mathbb{N}$ is a bijection. If you wish, we can be a little more formal, and instead define a map $\eta : Y \rightarrow \mathbb{N}$ via, for each $y \in Y$, letting $\eta(y) = |\{z \mid z \in Y, z \leq y\}|$ and then verify that η is a bijection.
 - (b) Prove that a subset of a countable set is countable.
Solution Suppose that Y is countable and that $X \subseteq Y$. If Y is finite, then X is finite and so countable. If Y is infinite, there is a bijection α from Y to \mathbb{N} . There is an injection ‘inclusion’ map $\beta : X \rightarrow Y$ defined by $\beta(x) = x \forall x \in X$. The map $\alpha \circ \beta : X \rightarrow \mathbb{N}$ is a composition of injections and so is injective. By part (a), X is countable.
 - (c) Suppose that $g : \mathbb{N} \rightarrow X$ is a surjective map. Prove that X is a countable set.
Solution g is surjective, and so has a right inverse $f : X \rightarrow \mathbb{N}$ such that $g \circ f = \text{Id}_X$. Therefore f has a left-inverse and so is injective. Now we are done by part (a).
 - (d) Suppose that A and B are countable sets. Prove that $A \times B$ is a countable set.
Solution From 1(a) it follows that a set S is countable if, and only if, there is an injection from S to \mathbb{N} . Thus there are injections $\alpha : A \rightarrow \mathbb{N}$ and $\beta : B \rightarrow \mathbb{N}$. Define a map $\gamma : A \times B \rightarrow \mathbb{N}$ via $(a, b) \mapsto 2^{\alpha(a)}3^{\beta(b)}$. It is easy to verify that γ is injective, for if $u_1, u_2, v_1, v_2 \in \mathbb{N}$ and $2^{u_1}3^{v_1} = 2^{u_2}3^{v_2}$ then $2^{u_1-u_2} = 3^{v_2-v_1}$ and without loss of generality $u_1 \geq u_2$ and $v_1 \leq v_2$. Therefore $2^{u_1-u_2} = 1 = 3^{v_2-v_1}$ and $(u_1, v_1) = (u_2, v_2)$.
2. Recall that if X is a set, then $P(X)$ is the power set of X , so the elements of $P(X)$ are the subsets of X .
 - (a) Determine the size of the set $P(P(P(P(P(\emptyset))))))$.
Solution Let us define $P^0(\emptyset) = \emptyset$ and for n a natural number let $P^n(\emptyset) = P(P^{n-1}(\emptyset))$. Now $|P^0(\emptyset)| = 0$, $|P^1(\emptyset)| = 2^0 = 1$, $|P^2(\emptyset)| = 2^1 = 2$, $|P^3(\emptyset)| = 2^2 = 4$, $|P^4(\emptyset)| = 2^4 = 16$ and $|P^5(\emptyset)| = 2^{16} = 65,536$ which answers the question.
 - (b) Determine the number of maps from $P(P(\emptyset))$ to $P(P(P(\emptyset)))$.
Solution This is the question, how many maps are there from a set of size 2 to a set of size 4. Since each element of the domain can map (independently) to any one of 4 elements of the codomain, the answer is $4^2 = 16$. The answer $2^4 = 16$ is incorrect, because the method is wrong!
 - (c) When X is a set, we let S_X denote the set of bijections from X to X . Suppose that $|X| = n \in \mathbb{N}$. Give a formula for $|S_{S_X}|$. In this spirit, what is the next term of the sequence 1, 2, 720, ...?

Solution If a set Y has size n , then $|S_Y| = n!$. Thus $|S_{S_X}| = n!!$. Note that $1!! = 1$, $2!! = 2$ and $3!! = 720$. For those who are interested (possibly the empty set of readers), the next term of the sequence is $4!!$ which is exactly $24!$ and approximately $6.20448402 \times 10^{23}$.

3. Let $I_n = \{1, 2, 3, \dots, n\}$.

(a) For $n = 1, 2, 3, 4$ and 5 , determine the number of partitions of $I_n = \{1, 2, 3, \dots, n\}$.

Solution

$n = 1$ There is a unique partition $\{1\}$.

$n = 2$ There are two partitions: $\{1, 2\}; \{1\}, \{2\}$

$n = 3$ There are 5 partitions: one of shape 1,1,1; 3 of shape 2,1 and 1 of shape 3.

$n = 4$ There are 15 partitions: one of shape 1,1,1,1; 6 of shape 2,1,1; 3 of shape 2,2; 4 of shape 3,1 and 1 of shape 4.

$n = 5$ There are 52 partitions: one of shape 1,1,1,1,1; 10 of shape 2,1,1,1; 15 of shape 2,2,1; 10 of shape 3,1,1; 10 of shape 3,2; 5 of shape 4,1 and one of shape 5.

(b) How many ways are there to partition I_n into two subsets?

Solution You can choose an unordered pair X, X' consisting of a subset of I_n and its complement in 2^{n-1} ways. You must discard the unordered pair I_n, \emptyset because empty sets are banned from partitions. Therefore the number of partitions of I_n into two sets is $2^{n-1} - 1$. Note how this works beautifully when $n = 1$.

4. Discuss whether the following relations are reflexive, symmetric or transitive.

(a) The relation $|$ (pronounced 'divides') on the set \mathbb{N} . (Here $m | n$ if, and only if, there is $l \in \mathbb{N}$ such that $lm = n$.)

Solution This relation is reflexive because $n = 1 \times n \forall n \in \mathbb{N}$. It is transitive because if $u | v$ and $v | w$ then there are $x, y \in \mathbb{N}$ such that $ux = v$ and $vy = w$. Therefore $u(xy) = w$ and $xy \in \mathbb{N}$. This relation fails to be symmetric because there is no natural number k such that $2k = 1$ but in fact $1 | 2$ because $2 = 2$.

(b) The (usual) relation \leq on \mathbb{R} .

Solution The reflexive and transitive conditions clearly hold. However, this relation is not symmetric because $1 \leq 2$ but it is not the case that $2 \leq 1$.

(c) The (usual) relation $=$ on \mathbb{Z} .

Solution This relation is the inspiration for the definition of an equivalence relation, and the reflexive, symmetric and transitive conditions all hold.

(d) Let S_X be the set of bijections from the set X to the set X . The relation \sim is defined on S_X as follows: when $f, g \in S_X$ we write $f \sim g$ if, and only if, $f \circ g = g \circ f$.

Solution If $|X| < 3$, then bijections from X to X commute (in the sense of composition), so all maps are related to all other maps, and the reflexive, symmetric and transitive conditions all hold. Now suppose that $|X| \geq 3$, so X contains three distinct elements a, b and c . The reflexive and symmetric laws hold, but the transitive law does not. Let $f \in S_X$ swap a with b , and fix everything else. Let $g \in S_X$ swap b with c , and fix everything else. Then $g \sim \text{Id}_X \sim f$ but $f(g(a)) = b$ but $g(f(a)) = c$ so $f \circ g \neq g \circ f$.

5. Fix a Euclidean plane. Consider the set L of all lines in this plane (a *line* is of infinite extent in both directions). Which of the following relations on L is an equivalence relation? In the case of equivalence relations, select a *natural* (i.e. sensible) transversal for the associated partition of L into equivalence classes.

(a) \parallel (is parallel to).

Solution This is an equivalence relation. We might fix a point P in the plane, and let T be the set of lines through P . Since any line in the plane is parallel to a unique element of T , the set T must be a transversal for the partition associated with \parallel .

(b) \perp (is perpendicular to).

Solution This is symmetric, but is neither reflexive nor transitive.

6. Define a relation \sim on \mathbb{C} as follows: for $\alpha, \beta \in \mathbb{C}$, write $\alpha \sim \beta$ if, and only if, there is a real number θ such that $\alpha = \beta e^{i\theta}$.

(a) Show that \sim is an equivalence relation on \mathbb{C} .

Solution For each $z \in \mathbb{C}$ we have $z = z \times e^{i \times 0}$ (i.e. use $\theta = 0$) so we have reflexivity. If $z_1 = z_2 e^{i\theta}$, then $z_2 = z_1 e^{i(-\theta)}$ so symmetry is established. If $z_1 = z_2 e^{i\theta_1}$ and $z_2 = z_3 e^{i\theta_2}$, then $z_1 = z_3 e^{i(\theta_1 + \theta_2)}$, so transitivity is established.

(b) Describe the equivalence classes of this equivalence relation geometrically, in terms of the Argand diagram.

Solution The equivalence classes correspond, geometrically to the origin (corresponding to $\{0\}$) together with all circles with centre the origin in the Argand diagram.

(c) Give an elegant transversal for this partition of \mathbb{C} .

Solution The ray corresponding to the set of non-negative real numbers is an excellent natural transversal, because it intersects each set of the partition in a singleton set, and it has nice algebraic properties.

7. Let $\mathbb{R}^\circ = \mathbb{R} \setminus \{0\}$. Define a relation \sim on \mathbb{R}° by $r \sim s$ if, and only if, $r/s \in \mathbb{Q}$. Is \sim an equivalence relation on \mathbb{R}° ?

Solution This is an equivalence relation. Suppose that $r \in \mathbb{R}^\circ$, then $r/r = 1 \in \mathbb{Q}$ so reflexivity is established. Suppose that $r, s \in \mathbb{R}^\circ$ and $r/s = q \in \mathbb{Q} \setminus \{0\}$. Then $s/r = 1/q \in \mathbb{Q}$ so this relation is symmetric. Finally if $r, s, t \in \mathbb{R}^\circ$ and $r/s, s/t \in \mathbb{Q}$, then $r/t = r/s \times s/t \in \mathbb{Q}$. Transitivity is established.

8. Define a relation \sim on $P(\mathbb{N})$ by writing $A \sim B$ if, and only if, there are finite subsets U, V of \mathbb{N} such that $A \cup U = B \cup V$. Prove that \sim is an equivalence relation on $P(\mathbb{N})$.

Solution Reflexivity is immediate, using $U = V = \emptyset$. Symmetry is also immediate. Finally suppose that $A \sim B \sim C$ so there are finite subsets U_1, U_2, V_1, V_2 of \mathbb{N} such that $A \cup U_1 = B \cup V_1$ and $B \cup U_2 = C \cup V_2$. Now $A \cup (U_1 \cup U_2) = (A \cup U_1) \cup U_2 = (B \cup V_1) \cup U_2 = (B \cup U_2) \cup V_1 = (C \cup V_2) \cup V_1 = C \cup (V_1 \cup V_2)$. Transitivity is established. *Note that $A \sim B$ iff you can obtain B from A by adjoining finitely many natural numbers, and then discarding finitely many natural numbers.*

9. Suppose that n is a positive integer. Define a relation \sim_n on \mathbb{Z} by $x \sim_n y$ if, and only if, n divides $x - y$.

(a) Prove that \sim_n is an equivalence relation on \mathbb{Z} .

Solution If $z \in \mathbb{Z}$, then $z - z = 0 = 0 \times n$. Therefore this relation is reflexive. Now suppose that $a, b \in \mathbb{Z}$ and $a \sim_n b$. Therefore there is an integer t such that $a - b = tn$. Therefore $b - a = (-t)n$ so $b \sim_n a$ and symmetry is established. Finally suppose that $a, b, c \in \mathbb{Z}$ and $a \sim_n b \sim_n c$. Therefore there are integers u, v such that $a - b = un$ and $b - c = vn$. Therefore $a - c = a - b + b - c = un + vn = (u + v)n$ and so $a \sim_n c$. Transitivity is established.

(b) Describe the equivalence classes of \sim_n .

Solution If $z \in \mathbb{Z}$ then the equivalence class $[z]$ of z is $[z] = \{z + kn \mid k \in \mathbb{Z}\}$.

(c) We write the set of equivalence classes as \mathbb{Z}/\sim_n . Determine $|\mathbb{Z}/\sim_n|$.

Solution By design, here is a complete list of equivalence classes without repetition: $[1], [2], \dots, [n]$ so the number of equivalence classes is n .

10. (*Tutor pacifier, and for enthusiasts*) Prove the Schröder-Bernstein Theorem unaided. The theorem states that if A, B are sets, and there are injective maps $f : A \rightarrow B$ and $g : B \rightarrow A$, then there is a bijective map $h : A \rightarrow B$. *By replacing B with an appropriate copy of itself if necessary, you*

can assume that $A \cap B = \emptyset$. This trick may well simplify writing up the proof.

Solution Not yet, but here is a solution for Problem 10 of sheet 1.

IMO 2001/3 (Washington D.C., set by Christian Bey of Germany).

Twenty-one girls and twenty-one boys took part in a mathematical contest. Each contestant solved at most six problems. For each girl and each boy, at least one problem was solved by both of them. Prove that there was a problem that was solved by at least three girls and at least three boys.

In an IMO (international Mathematical Olympiad) there are six problems which the students must address, so this problem has an attractive self-referential quality. It is also a problem you could pose to almost anyone, with confidence that you would be understood. This problem does not state how many problems could be addressed in the competition, so we will have to leave that as unknown.

We say that a problem is girl-hard if at most two girls manage to solve it. Similarly we say that a problem is boy-hard if at most two boys manage to do it. We suppose, for contradiction, that every problem is either girl-hard or boy-hard.

Make a big table, like a chessboard, but 21×21 rather than 8×8 . Label the rows with the names of the girls, and the columns with the names of the boys. At the moment, each little square in the chessboard is blank, but now we are going to colour each one either red or blue. For each square, consider the girl and boy which label its row and column. This boy and girl must both have solved at least one problem in common. Pick one of those problems, it doesn't matter which one. If that problem is girl-hard, colour the square red. If it is not girl-hard, then it must be boy-hard, and we colour the square blue. Now every one of the $441 = 21 \times 21$ squares is coloured either red or blue.

Now, 441 is an odd number, so either (strictly) more than half the little squares are red, or (strictly) more than half the little squares are blue. The argument now splits into two cases, depending on which case we have on our hands. However, the two arguments are so similar that once you understand one of them, then you will see the other straight away.

Perhaps there are at least 221 blue squares. There are 21 rows, so it cannot be that every row contains at most 10 blue squares, so there is some row (corresponding to a particular girl, say Marina) which contains at least 11 blue squares. Pick exactly 11 of these blue squares and focus on them. These blue squares are blue because they correspond to a boy-hard problem which Marina solved. Any particular problem can be involved with at most two of these blue squares, because it is a boy-hard problem. There are 11 blue squares under consideration, and it cannot be that there are 5 or fewer problems associated with these squares. Now, Marina solved exactly 6 problems, and it turns out that every one of them is boy-hard. Therefore there are at most 12 boys who solved a problem in common with Marina. However, the conditions of the problem force every one of the 21 boys to solve a problem in common with Marina. This is impossible.

The other possibility is that there are 221 red squares. However, a similar argument will give rise to the same sort of problem. This time you observe that there must be a particular boy, say Max, whose name labels a column containing at least 11 red squares. Argue as before, to discover that there at most 12 girls who solve a problem in common with Max. The conditions of the problem force every one of the 21 girls to solve a problem in common with Max. This is impossible.

Now we have nowhere to run. We look back across the argument until we find the sentence "We suppose, for contradiction, that every problem is either girl-hard or boy-hard." Our arguments have shown that this assumption is false, because by making this assumption, we have deduced something which is impossible.

It follows that it is not the case that every problem is either girl-hard or boy-hard. Therefore there is a problem which is neither girl-hard nor boy-hard. That problem was solved by at least three girls and at least three boys.