MA10209 Algebra 1A

Sheet 3 Problems v1: GCS

16-x-11

The course website is http://people.bath.ac.uk/masgcs/diary.html

Hand in work to your tutor by 13:00, Monday Oct 24.

1. (a) Suppose that $f : X \longrightarrow \mathbb{N}$ is an injective map. Prove that X is a countable set.
   (b) Prove that a subset of a countable set is countable.
   (c) Suppose that $g : \mathbb{N} \longrightarrow X$ is a surjective map. Prove that X is a countable set.
   (d) Suppose that A and B are countable sets. Prove that $A \times B$ is a countable set.

2. Recall that if X is a set, then $P(X)$ is the power set of X, so the elements of $P(X)$ are the subsets of X.
   (a) Determine the size of the set $P(P(P(P(P(\emptyset)))))$.
   (b) Determine the number of maps from $P(P(\emptyset))$ to $P(P(P(\emptyset)))$.
   (c) When X is a set, we let S_X denote the set of bijections from X to X. Suppose that $|X| = n \in \mathbb{N}$. Give a formula for $|S_{S_X}|$. In this spirit, what is the next term of the sequence $1, 2, 720, \ldots$?

3. Let $I_n = \{1, 2, 3, \ldots, n\}$.
   (a) For $n = 1, 2, 3, 4$ and 5, determine the number of partitions of $I_n = \{1, 2, 3, \ldots, n\}$.
   (b) How many ways are there to partition I_n into two subsets?

4. Discuss whether the following relations are reflexive, symmetric or transitive.
   (a) The relation $\mid$ (pronounced 'divides') on the set $\mathbb{N}$. (Here $m \mid n$ if, and only if, there is $l \in \mathbb{N}$ such that $lm = n$.)
   (b) The (usual) relation $\leq$ on $\mathbb{R}$.
   (c) The (usual) relation $=$ on $\mathbb{Z}$.
   (d) Let S_X be the set of bijections from the set X to the set X. The relation $\sim$ is defined on S_X as follows: when $f, g \in S_X$ we write $f \sim g$ if, and only if, $f \circ g = g \circ f$.

5. Fix a Euclidean plane. Consider the set L of all lines in this plane (a *line* is of infinite extent in both directions). Which of the following relations on L is an equivalence relation? In the case of equivalence relations, select a *natural* (i.e. sensible) transversal for the associated partition of L into equivalence classes.
   (a) $\parallel$ (is parallel to).
   (b) $\perp$ (is perpendicular to).

6. Define a relation $\sim$ on $\mathbb{C}$ as follows: for $\alpha, \beta \in \mathbb{C}$, write $\alpha \sim \beta$ if, and only if, there is a real number θ such that $\alpha = \beta e^{i\theta}$.

(a) Show that $\sim$ is an equivalence relation on $\mathbb{C}$.

(b) Describe the equivalence classes of this equivalence relation geometrically, in terms of the Argand diagram.

(c) Give an elegant transversal for this partition of $\mathbb{C}$.

7. Let $\mathbb{R}^\circ = \mathbb{R} \setminus \{0\}$. Define a relation $\sim$ on $\mathbb{R}^\circ$ by $r \sim s$ if, and only if, $r/s \in \mathbb{Q}$. Is $\sim$ an equivalence relation on $\mathbb{R}^\circ$?

8. Define a relation $\sim$ on $P(\mathbb{N})$ by writing $A \sim B$ if, and only if, there are finite subsets U, V of $\mathbb{N}$ such that $A \cup U = B \cup V$. Prove that $\sim$ is an equivalence relation on $P(\mathbb{N})$.

9. Suppose that n is a positive integer. Define a relation $\sim_n$ on $\mathbb{Z}$ by $x \sim_n y$ if, and only if, n divides $x - y$.

   (a) Prove that $\sim_n$ is an equivalence relation on $\mathbb{Z}$.

   (b) Describe the equivalence classes of $\sim_n$.

   (c) We write the set of equivalence classes as $\mathbb{Z}/\sim_n$. Determine $|\mathbb{Z}/\sim_n|$.

10. *(Tutor pacifier, and for enthusiasts)* Prove the *Schröder-Bernstein Theorem* **unaided** (no books, internet etc). The theorem states that if A, B are sets, and that there are injective maps $f : A \longrightarrow B$ and $g : B \longrightarrow A$, then there is a bijective map $h : A \longrightarrow B$. *By replacing B with an appropriate copy of itself if necessary, you can assume that $A \cap B = \emptyset$. This trick may well simplify writing up the proof.*